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Research Article

New Exact Solutions of Landau-Ginzburg-Higgs Equation Using Power Index Method

Khalil Ahmad,[1] **Khudija Bibi**,[2] **Muhammad Shoaib Arif**,[3] **and Kamaleldin Abodayeh**[3]

[1]*Department of Mathematics, Air University, PAF-Complex E-9, Islamabad, Pakistan*
[2]*Department of Mathematics, Faculty of Basic and Applied Sciences, International Islamic University, Islamabad, Pakistan*
[3]*Department of Mathematics and Sciences, College of Humanities and Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia*

Correspondence should be addressed to Khalil Ahmad; khalilmalik@mail.au.edu.pk and Muhammad Shoaib Arif; marif@psu.edu.sa

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In the present study, the optimality approach is applied to find the exact solution of the Landau-Ginzburg-Higgs Equation (LGHE) using new transformations. This method is a direct algebraic method for obtaining exact solutions of nonlinear differential equations. We find suitable solutions of the LGHE in terms of elliptic Jacobi functions by applying transformations of basic functions. Exact solutions of the equations are obtained with the help of symbolic software (Maple) which allows the computation of equations with parameter constants. It is exposed that PIM is influential, suitable, and shortest and offers an exact solution of LGHE.

1. Introduction

Exact solutions of nonlinear partial differential equations play an important role in nonlinear science, especially in nonlinear physical science because they can carry substantial physical information and more understanding into the physical aspects of the problem. [1]. Recently, several methods have been used to find exact solutions of nonlinear model equations like Cahn-Allen equation [2], $(2+1)$-dimensional Date-Jimbo-Kashiwara-Miwa (DJKM) equation [3], Newell-Whitehead-Segel (NWS) equations [4], the Chaffee–Infante equation [5], DNA Peyrard–Bishop equation [6], Burger's equation [7], the $(2+1)$-dimensional nonlinear Sharma–Tasso–Olver equation [8], and Ablowitz-Kaup-Newell-Segur water wave equation [9]. Recently, a number of concrete techniques have been recognized for finding accurate and comprehensible solutions of nonlinear physical models with the help of computer algebra, such as Maple, MATLAB, and Mathematica. These include power ondex method [7, 9], lie symmetry groups [3, 4], new extended direct algebraic method, and the generalized Kudryashov method [10].

The Landau-Ginzburg-Higgs (LGH) equation was introduced by Lev Devidovich Landau and Vitaly Lazarevich Ginzburg having very wide range of applications in radially inhomogeneous plasma having a constant phase relation of ion-cyclotron waves. It demonstrated superconductivity and unidirectional wave propagation in nonlinear media [11]. Several approaches have been adopted to attain the close form and approximate solutions of the LGH equation. Khuri has investigated the close form solutions of LGH, Klein–Gordon, and Sine–Gordon equations by using unified approach [12]. In [13], the new modification method is proved by solving LGH equation and Cahn-Allen equation.

$$\frac{\partial^2}{\partial t^2} u(x,t) - \frac{\partial^2}{\partial x^2} u(x,t) - m^2 u(x,t) + n^2 u^3(x,t) = 0. \tag{1}$$

In this paper, we will examine the solutions of Landau-Ginzburg-Higgs equation by using polynomial and elementary functions. In Section 2, we designate temporarily the PIM and explain index powers by taking first partial derivative of dependent function with respect to time. In Section 3, we apply this method to the LGH equation by introducing new transformations. The PDE (1) is reduced in different form of differential equations. These ODEs are solved by using the elliptic JacobiSN (z, k) function and its differential equation, see (10).

2. Power Index Method

Considering the PDE (1), and we want to find its exact solution, we introduce the variables η as

$$\begin{aligned} \eta = q(x,t) &:= k_1 \exp(a_1 x + a_2 t) + k_2 \ln(b_1 x + b_2 t), \\ \eta = q(x,t) &:= (b_1 x + b_2 t)^s, \end{aligned} \tag{2}$$

or

$$\eta = q(x,t) := \tan^m(a_1 x + a_2 t), \tag{3}$$

and the function transforms

$$u(t,x) = (b_1 x + b_2 t)^r f(\eta), \tag{4}$$

or

$$u(t,x) = \tan^n(a_1 x + a_2 t) f(\eta). \tag{5}$$

Now, we differentiate (4) according to PDE (1), and we can find the relation of indexes of x and t in each term as

$$\frac{\partial}{\partial t} u(x,t) = r(b_1 x + b_2 t)^{r-1} f(\eta) + s(b_1 x + b_2 t)^{r+s-1} f'(\eta), \tag{6}$$

if we choose $\eta = (b_1 x + b_2 t)^s$.

We observe the coefficient functions of $f(\eta)$ and its derivative terms so that the PDE equation may be changed into ODE. In (6), the coefficient function of $f(\eta)$ is $(b_1 x + b_2 t)^{r-1}$, and the coefficient function of $f'(\eta)$ is $(b_1 x + b_2 t)^{r+s-1}$. The optimal indices x and t of the independent variables are chosen such that only two indices are varied at a time and the rest are fixed. We continue this process with different indices x and t so that we find all well-defined transformation. Our objective in this method is to get the required ODE.

$$Q\left(\eta, f, ff', f'', \cdots\right) = 0. \tag{7}$$

Solve the ODE (7) by using computerized symbolic package like Maple. The exact solution of (1) can be obtained from (4) to (7) after replacing an unknown function $f(\eta)$. Figure 1 shows the procedure of transformation of PDE (1) to ODE (7).

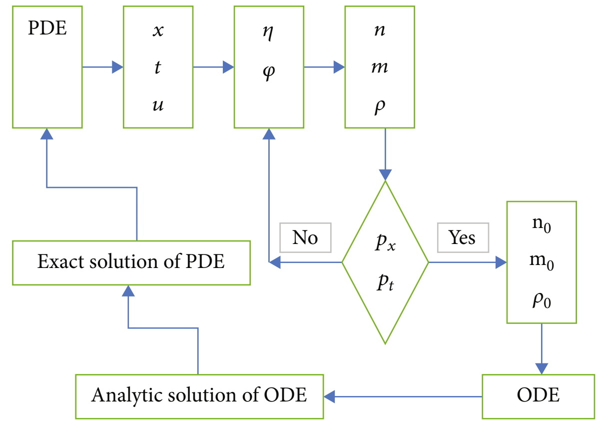

Figure 1: Flow chart of the power index method.

3. Exact Solutions of Landau-Ginzburg-Higgs Equation

Case 1. We choose anew variable ξ in exponential form and choosing new transformation $h(\xi)$ as

$$\xi = \exp(a_1 x + a_2 t), u(t,x) = \frac{a_1 a_2}{n} h(\xi). \tag{8}$$

The PDE (1) reduces to the nonlinear ODE

$$\begin{aligned} &\left(a_1^2 - a_2^2\right)\xi^2 h''(\xi) + \left(a_1^2 - a_2^2\right)\xi h'(\xi) \\ &\quad - a_1^2 a_2^2 \xi h^3(\xi) + m^2 h(\xi) = 0. \end{aligned} \tag{9}$$

Since the ODE (9) is second-order nonlinear ODE, to find the solution of ODE (9), we need the following second-order differential equation.

$$h''(z) - 2k^2 h^3(\xi) + \left(1 + k^2\right) h(\xi) = 0. \tag{10}$$

The elliptic function JacobiSN (z, k) is the solution of (10). The InverseJacobiSN(z, k) is defined as

$$\text{InverseJacobiSN}(z,k) = \int_0^z \frac{d\eta}{\sqrt{\left(1-\eta^2\right)\left(1-k^2\eta^2\right)}}. \tag{11}$$

Jacobi elliptic functions are named for the famous mathematician Jacobi, and then, Gauss gave some attention to JacobiSN (z, k). Applications of Jacobi elliptic functions include closed-form solutions for nonlinear integrable equations.

If we choose $z = \alpha_1(\sqrt{2}\alpha_2 \ln(\xi) + C_1)$ and $k = \alpha_3$ with

$$\alpha_1 = \sqrt{\frac{m^2}{C_2^2 a_1^2 a_2^2 + 2m^2}}, \quad \alpha_2 = \sqrt{\frac{(2m^2 - a_1^2 a_2^2)}{2(a_1^2 - a_2^2)}}, \quad \alpha_3 = \frac{C_2 a_1 a_2}{\sqrt{2m^2 - a_1^2 a_2^2}} \tag{12}$$

and using relations

$$h'(z) = \frac{\xi}{\sqrt{2}\alpha_1\alpha_2} h'(\xi), \quad h''(z) = \frac{\xi^2}{2(\alpha_1\alpha_2)^2} h''(\xi) + \frac{\xi}{2(\alpha_1\alpha_2)^2} h'(\xi), \tag{13}$$

then the ODE (10) transforms to ODE (9), and the solution of ODE (10) transforms to the solution of ODE (9) which has the form

$$h(\xi) = C_2\alpha_1\sqrt{2}\mathrm{JacobiSN}\left[\alpha_1\left(\sqrt{2}\alpha_2 \ln(\xi) + C_1\right), \alpha_3\right]. \tag{14}$$

Using (8) and (14), we get the exact solution of the PDE (1) which is

$$u(t,x) = \frac{a_1 a_2}{n} C_2\alpha_1\sqrt{2}\mathrm{JacobiSN} \cdot \left[\alpha_1\sqrt{2}(\alpha_2(a_1x + a_2t) + C_1), \alpha_3\right], \tag{15}$$

where $\alpha_1, \alpha_2, \alpha_3$ are defined above.

Figure 2 demonstrates solution (15) of PDE (1). Figure 3 demonstrates contrasts of $u(x,t)$ in (15).

Case 2. Next, we define a new variable ξ in logarithmic function and choosing new transformation $h(\xi)$ as

$$\xi = \ln(b_1x + b_2t), u(t,x) = h(\xi). \tag{16}$$

The PDE (1) can be reduced to the ODE

$$(b_1^2 - b_2^2)e^{-2\xi}h''(\xi) + (b_1^2 - b_2^2)e^{-2\xi}h'(\xi) - n^3h^3(\xi) + m^2h(\xi) = 0. \tag{17}$$

Since the ODE (17) can be obtained by using transformation $z = \alpha_4(\alpha_5 e^{\xi} + C_1)$, $k = \alpha_6 C_2$, and relations,

$$h'(z) = \frac{e^{-\xi}}{\alpha_4\alpha_5} h'(\xi), \quad h''(z) = \frac{e^{-2\xi}}{(\alpha_4\alpha_5)^2} h''(\xi) - \frac{e^{-2\xi}}{(\alpha_4\alpha_5)^2} h'(\xi), \tag{18}$$

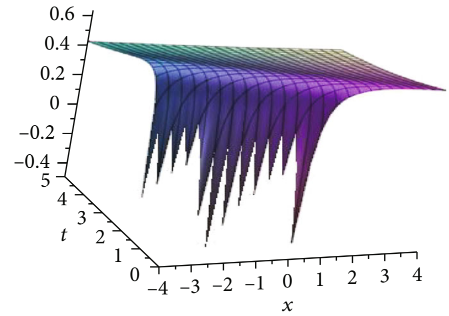

Figure 2: 3D plot of solution (15) of PDE (1).

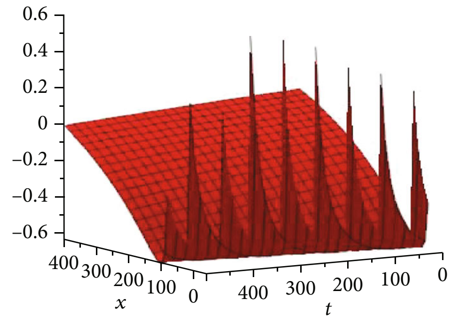

Figure 3: 3D plot of solution (15) of PDE (1).

in (10). By using the solution of ODE (10), the analytic solution of ODE (17) can be expressed in the form

$$h(\xi) = C_2\alpha_4\mathrm{JacobiSN}\left[\alpha_4\left(\alpha_5 e^{\xi} + C_1\right), \alpha_6 C_2\right], \tag{19}$$

where

$$\alpha_4 = \sqrt{\frac{2m^2}{(C_2^2 - 1)n^2 + 2m^2}}, \quad \alpha_5 = \sqrt{\frac{(2m^2 - n^2)}{2(b_1^2 - b_2^2)}}, \quad \alpha_6 = \frac{n}{\sqrt{2m^2 - n^2}}. \tag{20}$$

Using (19) and (16), we get the exact solution of the PDE (1) that is

$$u(t,x) = C_2\alpha_4\sqrt{2}\mathrm{JacobiSN}[\alpha_4(\alpha_5(b_1x + b_2t) + C_1), \alpha_6 C_2], \tag{21}$$

where $\alpha_4, \alpha_5, \alpha_6$ are defined above in this case. Figure 4 presents graphical diagram of the analytical solution of (21) which is periodic solution of PDE (1).

Case 3. We choose a new variable ξ in polynomial form and choosing new transformation $h(\xi)$ as

$$\xi = (a_1x + a_2t)^s, u(t,x) = (a_1x + a_2t)^r h(\xi). \tag{22}$$

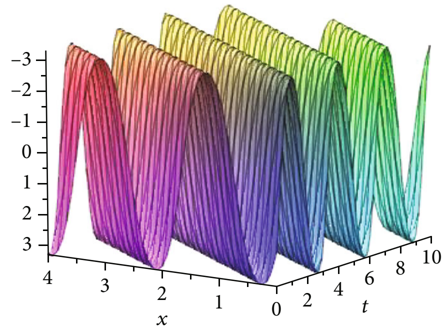

Figure 4: 3D plot of solution (21) of PDE (1).

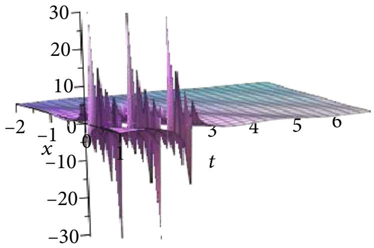

Figure 5: 3D plot of solution (28) of PDE (1).

The PDE (1) reduces to the algebraic form

$$(a_1 x + a_2 t)^{\beta_i} F\left(h(\xi), h'(\xi), h''(\xi)\right) = 0, \tag{23}$$

where

$$\beta_i \in \{r-2, r+s-2, r+2s-2, r, 3r\}. \tag{24}$$

To convert PDE (1) into nonlinear ODE, the index values of β_i must be chosen in such a way so that there are multiples of index s. To select suitable values of β_i, its members must have the following relation $r = s - 1$ or $r = s/2 - 1$. If we choose the first relation, the algebraic equation (23) can be changed to the following ODE, successfully.

$$\begin{aligned}&\left[A\left(s^2 - 3s + 2\right) - m^2 \xi^{2/s}\right] h(\xi) + 3sA(s-1)\xi h'(\xi)\\&\quad + n^2 \xi^2 h^3(\xi) + m^2 s^2 \left(a_2^2 - a_1^2\right) h''(\xi) = 0,\end{aligned} \tag{25}$$

where $A = (a_2^2 - a_1^2)$. Since the ODE (25) can be achieved by using transformation $z = \alpha_8(1/2\alpha_8 \xi + C_1)$, $k = \alpha_9 C_2$, and chain rule in (10). Figure 5 presents graphical diagram of the analytical solution of (28).

The analytic solution of ODE (25) for $s = 1$ can attained from ODE (10) in the form

$$h(\xi) = C_2 \alpha_7 \mathrm{JacobiSN}\left[\alpha_8\left(\frac{1}{2}\alpha_8 \xi + C_1\right), \alpha_9 C_2\right], \tag{26}$$

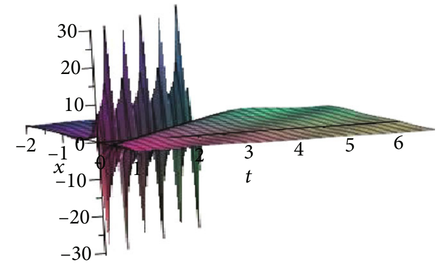

Figure 6: 3D plot of solution (30) of PDE (1).

where

$$\begin{aligned}\alpha_7 &= \sqrt{\frac{2m^2}{\left(C_2^2 - 1\right)n^2 + 2m^2}},\\ \alpha_8 &= \sqrt{\frac{2\left(2m^2 - n^2\right)}{\left(a_1^2 - a_2^2\right)}},\\ \alpha_9 &= \frac{n}{\sqrt{2m^2 - n^2}}.\end{aligned} \tag{27}$$

Using (22) and (26), we get the exact solution of the PDE (1) that is

$$u(t,x) = C_2 \alpha_7 \mathrm{JacobiSN}\left[\alpha_8\left(\frac{1}{2}\alpha_8 (a_1 x + a_2 t) + C_1\right), \alpha_9 C_2\right]. \tag{28}$$

The analytic solution of ODE (25) for $s = 2$ is

$$h(\xi) = \frac{C_2 \alpha_7}{\sqrt{\xi}} \mathrm{JacobiSN}\left[\alpha_7\left(\frac{1}{2}\alpha_8 \sqrt{\xi} + C_1\right), \alpha_9 C_2\right]. \tag{29}$$

Using (22) and (25), we get the exact solution of the PDE (1) that is

$$\begin{aligned}u(t,x) = {}& \frac{C_2 \alpha_7}{(a_1 x + a_2 t))} \mathrm{JacobiSN}\\ &\cdot \left[\alpha_8\left(\frac{1}{2}\alpha_8 (a_1 x + a_2 t)\right) + C_1), \alpha_9 C_2\right],\end{aligned} \tag{30}$$

where $\alpha_7, \alpha_8, \alpha_9$ are defined above. If we choose the second relation $r = s/2 - 1$, then the transformation (22) reduces of the form

$$\xi = (a_1 x + a_2 t)^s, u(t,x) = (a_1 x + a_2 t)^{(1/s)-1} h(\xi). \tag{31}$$

Figure 6 shows graphical representation of solution of (28) which is periodic solution of PDE (1).

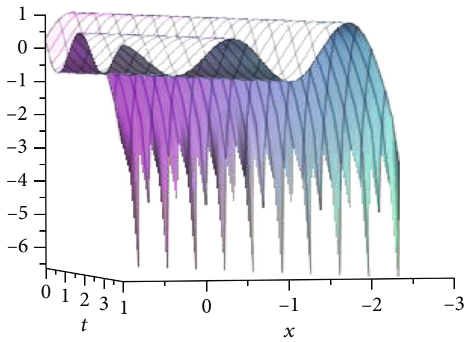

FIGURE 7: 3D plot of solution (34) of PDE (1).

By using (31), the algebraic equation can be changed to the following ODE.

$$\begin{aligned} &s^2\left(a_2^2-a_1^2\right)\xi^2 h''(\xi)+s(2s-3)\left(a_2^2-a_1^2\right)\xi h'(\xi)\\ &\quad+\left[\left(a_2^2-a_1^2\right)\left(\frac{1}{4}s^2-\frac{3}{2}s+2\right)-m^2\xi^{2/s}\right]h(\xi)\\ &\quad+n^2\xi h^3(\xi)=0. \end{aligned} \tag{32}$$

Using transformation $z=\alpha_7((1/2)\alpha_8(\xi)^{1/s}+C_1)$, $k=\alpha_9 C_2$ in (1), we found the analytic solution of the ODE (32) in the form

$$h(\xi)=C_2\alpha_7\xi^{(2-s)/2s}\text{JacobiSN}\left[\alpha_7\left(\frac{1}{2}\alpha_8(\xi)^{1/s}+C_1\right),\alpha_9 C_2\right]. \tag{33}$$

The PDE (1) has the solution

$$\begin{aligned} u(t,x)&=C_2\alpha_7(a_1x+a_2t)^{(2-s)/2s}\text{JacobiSN}\\ &\quad\cdot\left[\alpha_7\left(\frac{1}{2}\alpha_8(a_1x+a_2t)^{1/s}+C_1\right),\alpha_9 C_2\right]. \end{aligned} \tag{34}$$

Figure 7 is the 3D representation of solution (30).

Case 4. We choose a new variable ξ in polynomial form and choosing new transformation $h(\xi)$ as

$$\xi=\tan^2(a_1x+a_2t),\ u(t,x)=\tan(a_1x+a_2t)h(\xi), \tag{35}$$

$$\begin{aligned} &\left(a^2-b^2\right)\xi(1+\xi)^2h''(\xi)+\frac{5}{2}\left(a^2-b^2\right)(1+\xi)(\xi+3/5)h'(\xi)\\ &\quad+\frac{1}{2}h(\xi)\left[\frac{-1}{2}n^2\xi+\left(a^2-b^2\right)(\xi+1)+2\right]=0. \end{aligned} \tag{36}$$

Using suitable transformation in (1), we can get analytic solution of (36) as

$$\begin{aligned} h(\xi)&=\frac{1}{\sqrt{\xi}}\left[2C_2\text{JacobiSN}\left[n^2\left(\frac{-2}{\alpha_{10}}\arctan\left(\frac{\alpha_{11}}{\alpha_{10}}\sqrt{\xi}\right)\right)\right.\right.\\ &\quad\left.\left.+\frac{-2}{\alpha_{10}}\arctan\left(\frac{\alpha_{11}}{\alpha_{10}}\sqrt{\xi}+2C_1\right)\frac{1}{\alpha_{12}},\frac{n}{\alpha_{13}}C_2\right]\frac{1}{\alpha_{12}}\right], \end{aligned} \tag{37}$$

where

$$\begin{aligned} \alpha_{10}&=\sqrt{-2a^2n^2+2b^2n^2+16a^2-16b^2},\\ \alpha_{11}&=\sqrt{2\left(a^2-b^2\right)\left(8-n^2\right)},\ \alpha_{12}=\sqrt{\frac{2}{C_2^2n^2-n^2+8}},\\ \alpha_{13}&=\sqrt{-n^2+8}. \end{aligned} \tag{38}$$

The PDE (1) has the solution of the form

$$\begin{aligned} u(t,x)&=\frac{1}{\tan(a_1x+a_2t)}\left[2C_2\text{JacobiSN}\left[n^2\left(\frac{-2}{\alpha_{10}}\arctan\right.\right.\right.\\ &\quad\left.\left.\left.\cdot\left(\frac{\alpha_{11}}{\alpha_{10}}\tan(a_1x+a_2t)\right)\right)\right]\right.\\ &\quad+\frac{1}{\tan(a_1x+a_2t)}\frac{-2}{\alpha_{10}}+\left[\arctan\left(\frac{\alpha_{11}}{\alpha_{10}}\tan\right.\right.\\ &\quad\left.\left.\cdot(a_1x+a_2t)+2C_1\right)\frac{1}{\alpha_{12}},\frac{n}{\alpha_{13}}C_2\right]\frac{1}{\alpha_{12}}. \end{aligned} \tag{39}$$

4. Conclusion

In this paper, PIM has been used to construct exact solutions of LGH equation by using function conversion. The exact solutions of the LGH equation that we have obtained in this article are useful for understanding physical phenomena from many aspects. The performance of this method is found to be reliable and effective.

Data Availability

Please contact the authors for data requests.

Conflicts of Interest

The authors declare that they have no competing interest.

Authors' Contributions

All authors contributed equally to the manuscript and read and approved the final manuscript.

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References

[1] W. Malfliet, "Solitary wave solutions of nonlinear wave equations," *American Journal of Physics*, vol. 60, no. 7, pp. 650–654, 1992.

[2] M. Ekramul Islam, H. K. Barman, and M. Ali Akbar, "Stable soliton solutions to the nonlinear low-pass electrical transmission lines and the Cahn-Allen equation," *Heliyon*, vol. 7, no. 5, article e06910, 2021.

[3] K. Bibi and K. Ahmad, "New exact solutions of Date Jimbo Kashiwara Miwa equation using lie symmetry groups," *Mathematical Problems in Engineering.*, vol. 2021, article 5533983, pp. 1–8, 2021.

[4] K. Bibi and K. Ahmad, "Exact solutions of Newell-Whitehead-Segel equations using symmetry transformations," *Journal of Function Spaces*, vol. 2021, Article ID 6658081, 8 pages, 2021.

[5] S. Kumar, H. Almusawa, I. Hamid, M. A. Akbar, and M. A. Abdou, "Abundant analytical soliton solutions and Evolutionary behaviors of various wave profiles to the Chaffee-Infante equation with gas diffusion in a homogeneous medium," *Results in Physics*, vol. 30, article 104866, 2021.

[6] M. A. Loubna Ouahid, S. K. Abdou, S. Owyed, and S. Saha Ray, "A plentiful supply of soliton solutions for DNA Peyrard–Bishop equation by means of a new auxiliary equation strategy," *International Journal of Modern Physics B*, vol. 35, no. 26, 2021.

[7] K. Ahmad and K. Bibi, "New exact solutions of Burgers' equation using power index method," *Discrete Dynamics in Nature and Society*, vol. 2022, Article ID 5211625, 9 pages, 2022.

[8] S. Kumar, I. Khan, S. Rani, and B. Ghanbari, "Lie symmetry analysis and dynamics of exact solutions of the (2+1)-dimensional nonlinear Sharma–Tasso–Olver equation," *Mathematical Problems in Engineering*, vol. 2021, Article ID 9961764, 12 pages, 2021.

[9] K. Ahmad and K. Bibi, "New function solutions of Ablowitz-Kaup-Newell-Segur water wave equation via power index method," *Journal of Function Spaces*, vol. 2022, Article ID 9405644, 10 pages, 2022.

[10] M. Bilal, U. Younas, and J. Ren, "Dynamics of exact soliton solutions in the double-chain model of deoxyribonucleic acid," *Mathematical Methods in the Applied Sciences*, vol. 44, no. 17, pp. 13357–13375, 2021.

[11] W. P. Hu, Z. C. Deng, S. M. Han, and W. Fa, "Multi-symplectic Runge-Kutta methods for Landau-Ginzburg-Higgs equation," *Applied Mathematics and Mechanics*, vol. 30, no. 8, pp. 1027–1034, 2009.

[12] S. A. Khuri, "Exact solutions for a class of nonlinear evolution equations, a unified ansätze approach," *Chaos, Solitons Fractals*, vol. 36, no. 5, pp. 1181–1188, 2008.

[13] A. Irshad, S. T. Mohyud-Din, N. Ahmed, and U. Khan, "A new modification in simple equation method and its applications on nonlinear equations of physical nature," *Results in Physics*, vol. 7, pp. 4232–4240, 2017.