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Research Article

A Study on the New Class of Inequalities of Midpoint-Type and Trapezoidal-Type Based on Twice Differentiable Functions with Conformable Operators

Hasan Kara,[1] **Hüseyin Budak**,[1] **Sina Etemad**,[2] **Shahram Rezapour**,[2,3,4] **Hijaz Ahmad**,[5] **and Mohammed K. A. Kaabar**[6,7]

[1]*Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce 81620, Turkey*
[2]*Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran*
[3]*Department of Mathematics, Kyung Hee University, 26 Kyungheedae-ro, Dongdaemun-gu, Seoul, Republic of Korea*
[4]*Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan*
[5]*Section of Mathematics, International Telematic University Uninettuno, Corso Vittorio Emanuele II, 00186 Rome, Italy*
[6]*Gofa Camp, Near Gofa Industrial College and German Adebabay, Nifas Silk-Lafto, Addis Ababa 26649, Ethiopia*
[7]*Department of Engineering Problems of Mining, The Academy of Engineering Sciences of Ukraine, Dnipro 49005, Ukraine*

Correspondence should be addressed to Shahram Rezapour; rezapourshahram@yahoo.ca
and Mohammed K. A. Kaabar; mohammed.kaabar@wsu.edu

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This paper derives some equalities via twice differentiable functions and conformable fractional integrals. With the help of the obtained identities, we present new trapezoid-type and midpoint-type inequalities via convex functions in the context of the conformable fractional integrals. New inequalities are obtained by taking advantage of the convexity property, power mean inequality, and Hölder's inequality. We show that this new family of inequalities generalizes some previous research studies by special choices. Furthermore, new other relevant results with trapezoid-type and midpoint-type inequalities are obtained.

1. Introduction

Fractional calculus and the theory of inequalities, which have recently received a lot of attention, have been the subject of many investigations in the mathematics. Mathematical modeling is one of the most important fields of this theory in which fractional operators are defined to design different fractional differential equations for describing the phenomena. For instance, one can mention to the third-order BVP with multistrip multipoint conditions [1], hybrid version and the Hilfer type of thermostat model [2, 3], fractional HIV model with the Mittag-Leffler-type kernel [4], mathematical fractional model of Q fever [5], fractional dynamics of mumps virus [6], fractional p-Laplacian equations [7], fractal-fractional version of AH1N1/09 virus along with the fractional Caputo-type version [8], etc.

In the last century, the Hermite–Hadamard inequality along with the midpoint and trapezoidal inequalities arising from this inequality has attracted many researchers. In addition, RL-fractional (Riemann-Liouville) integrals, conformable integrals, and many types of such integrals have been defined in these inequalities and have gained an important place in the literature.

More precisely, fractional calculus is a big part of mathematics in which the mathematicians develop and extend the existing classical ideas of integration and differentiation operators to noninteger orders. Recently, it has received the attention of many researchers from different areas like mathematicians, physicists, and engineers [9, 10]. For example, if we consider a fluid-dynamic traffic model, then we see that one can simulate the irregular oscillation of earthquakes via fractional derivatives. These operators are also utilized for modeling a main part of chemical and physical processes, biological processes, and engineering problems. For instance, biological population model [11], electrical circuits [12], viscous fluid and their semianalytical solutions [13], fractional gas dynamics [14], and fractal modeling of traffic flow [15] are applied examples of the application of fractional operators. Further, it is stated that fractional systems provide some numerical outcomes that are more appropriate than those given by integer-order systems [16, 17].

New investigations have developed a category of fractional integration operators and their application in various scientific fields. Using only the idea of the fundamental limit formulation for derivatives, a novel well-behaved fractional derivative was defined, entitled as the conformable derivative, by Khalil et al. in [18]. Some applied properties that cannot be derived by the Riemann-Liouville and Caputo operators are obtained by the conformable derivative. However, in [19], Abdelhakim stated that the conformable structure in [18] cannot yield acceptable data compared to the Caputo idea for special functions. This flaw in the conformable definition was overcome by giving several extensions of the conformable operators [20, 21]. Moreover, with the help of the well-known exponential and Mittag-Leffler functions and using them in the kernels, several researchers defined newly expanded fractional operators such as exponential discrete kernel-type operators [22], fractal-fractional operators [23], and some other derivatives [24, 25].

Inequalities are one of the important topics of mathematics, and in this field, convex functions and their generalizations play an important role. In [26–28], the authors focused on Hermite–Hadamard inequalities by using the majorization and some properties of convex functions. Later, some other researchers combined these notions with monocity and boundedness [29–31]. Over the years, many mathematicians have concentrated on acquired trapezoidal and midpoint-type inequalities that yield specific bounds via the R.H.S. and L.H.S. of the Hermite–Hadamard inequality, respectively. For instance, at first, Dragomir and Agarwal derived trapezoid inequalities in relation to the convex functions in [32], whereas Kirmac derived inequalities of midpoint type with the help of the convex functions in [33]. In addition, in [34], Qaisar and Hussain established a number of generalized inequalities of midpoint type. Moreover, Sarikaya et al. and Iqbal et al. derived some fractional trapezoid and midpoint-type inequalities for a family of the convex mappings in [35, 36], respectively. In [37, 38], studies obtained some extensions from midpoint inequalities involving the Riemann-Liouville operators. In [39], similar results are derived by Hyder et al. under the generalized Reimann-Liouville operators.

Researches on the differentiable functions of these inequalities also have an important place in the literature. Many researchers have focused on twice differentiable functions to obtain many important inequalities. For example, Barani et al. proved some inequalities under twice differentiable mappings having the convexity property which is connected to Hadamard-type inequalities in [40, 41]. In [42], several novel extensions of integral fractional inequalities of midpoint-trapezoid type for the abovementioned twice differentiable functions are established. In [43], authors obtained other class of novel inequalities in the sense of the Simpson and Hermite–Hadamard for some special functions whose absolute values of derivatives are convex.

The main goal of this paper is to acquire some new trapezoid-type and midpoint-type inequalities with the help of the twice differentiable function including conformable fractional integrals. We also establish that the newly obtained inequalities are a generalization of the existing trapezoid-type and midpoint type inequalities. The ideas and strategies for our results concerning trapezoid type and midpoint-type inequalities via conformable fractional integrals may open other directions for more research in this area.

2. Preliminaries

This section discusses the basics for building our main results. Here, definitions of the Riemann-Liouville integrals and conformable integrals, which are well known in the literature, are given. From the fact of fractional calculus theory, mathematical preliminaries will be given.

For $x, y > 0$ (real numbers), the famous gamma function and incomplete beta function are

$$\begin{aligned} \Gamma(x) &:= \int_0^\infty t^{x-1} e^{-t} \mathrm{d}t, \\ \mathscr{B}(x, y, r) &:= \int_0^r t^{x-1} (1-t)^{y-1} \mathrm{d}t, \end{aligned} \tag{1}$$

respectively.

In 2006, Kilbas et al. [44] defined fractional integrals, also called the Riemann-Liouville integrals (RL-integral) as follows:

Definition 1 (see [44]). For $\hbar \in L^1[\nu, \omega]$, the Riemann-Liouville integrals $J_{\nu+}^{\varkappa} \hbar(x)$ and $J_{\omega-}^{\varkappa} \hbar(x)$ of order $\varkappa > 0$ are, respectively, given as

$$J_{\nu+}^{\varkappa} \hbar(x) = \frac{1}{\Gamma(\varkappa)} \int_\nu^x (x-t)^{\varkappa-1} \hbar(t) \mathrm{d}t, x > \nu, \tag{2}$$

$$J_{\omega-}^{\varkappa} \hbar(x) = \frac{1}{\Gamma(\varkappa)} \int_x^\omega (t-x)^{\varkappa-1} \hbar(t) \mathrm{d}t, x < \omega, \tag{3}$$

where $J_{\nu+}^0 \hbar(x) = J_{\omega-}^0 \hbar(x) = \hbar(x)$. By setting $\varkappa = 1$, the Riemann-Liouville integrals reduce to the classical integrals.

In 2017, Jarad et al. [25] formulated a novel fractional conformable integration operators. These researchers gave certain characteristics for these operators and some other fractional operators defined before. The fractional conformable integral operators are defined in the following definition:

Definition 2 (see [25]). For $\hbar \in L^1[\nu,\omega]$, the fractional conformable integral operator ${}^{\varkappa}\mathcal{J}^{\mu}_{\nu+}\hbar(x)$ and ${}^{\varkappa}\mathcal{J}^{\mu}_{\omega-}\hbar(x)$ of order $\varkappa \in C$, Re $(\varkappa) > 0$ and $\mu \in (0,1]$ are, respectively, given by

$$ {}^{\varkappa}\mathcal{J}^{\mu}_{\nu+}\hbar(x) = \frac{1}{\Gamma(\varkappa)}\int_{\nu}^{x}\left(\frac{(x-\nu)^{\mu}-(t-\nu)^{\mu}}{\mu}\right)^{\varkappa-1} \cdot \frac{\hbar(t)}{(t-\nu)^{1-\mu}}\mathrm{d}t, t>\nu, \tag{4} $$

$$ {}^{\varkappa}\mathcal{J}^{\mu}_{\omega-}\hbar(x) = \frac{1}{\Gamma(\varkappa)}\int_{x}^{\omega}\left(\frac{(\omega-x)^{\mu}-(\omega-t)^{\mu}}{\mu}\right)^{\varkappa-1} \cdot \frac{\hbar(t)}{(\omega-t)^{1-\mu}}\mathrm{d}t, t<\omega. \tag{5} $$

It is notable that the fractional integral in (4) coincides with the fractional RL-integral in (2) when $\mu = 1$. Moreover, the fractional integral in (5) coincides with the fractional RL-integral in (3) when $\mu = 1$. For more studies about several recent results in relation to fractional integral inequalities, we can mention some versions in the context of the Caputo-Fabrizio operators [45, 46], proportional generalized operators [47, 48], some inequalities in the Maxwell fluid modeling with nonsingular operators [49], conformable integral inequalities [50], some inequalities based on the Caputo-type operators [51], the Katugampola-type inequalities [52, 53], and the references cited therein.

3. Trapezoid-Type Inequalities Based on Conformable Fractional Integrals

In this section, inequalities of trapezoid type are obtained for twice differentiable functions. We use the conformable fractional integral operators to obtain these inequalities.

To acquire conformable fractional integrals trapezoid-type inequalities, we consider the following lemma.

Lemma 3. *Let $\hbar : [\nu,\omega] \longrightarrow \mathbb{R}$ be a twice differentiable mapping on (ν,ω) such that $\hbar'' \in L_1([\nu,\omega])$. In this case, the equality*

$$ \frac{\hbar(\nu)+\hbar(\omega)}{2} - \frac{2^{\mu\varkappa-1}\mu^{\varkappa}\Gamma(\varkappa+1)}{(\omega-\nu)^{\mu\varkappa}}\left[{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2-}\hbar(\nu) + {}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2+}\hbar(\omega)\right] = \frac{(\omega-\nu)^2\mu^{\varkappa}}{8}\left[\int_0^1\left(\int_0^t\left[\frac{1}{\mu^{\varkappa}} - \left(\frac{1-(1-s)^{\mu}}{\mu}\right)^{\varkappa}\right]ds\right) \cdot \hbar''\left(\frac{2-t}{2}\nu + \frac{t}{2}\omega\right)dt + \int_0^1\left(\int_0^t\left[\frac{1}{\mu^{\varkappa}} - \left(\frac{1-(1-s)^{\mu}}{\mu}\right)^{\varkappa}\right]ds\right)\hbar''\left(\frac{t}{2}\nu + \frac{2-t}{2}\omega\right)dt\right], \tag{6} $$

holds.

Proof. Employing integration by parts, it yields

$$ \begin{aligned} I_1 &= \int_0^1\left(\int_0^t\left[\frac{1}{\mu^{\varkappa}} - \left(\frac{1-(1-s)^{\mu}}{\mu}\right)^{\varkappa}\right]\mathrm{d}s\right)\hbar''\left(\frac{2-t}{2}\nu + \frac{t}{2}\omega\right)\mathrm{d}t \\ &= \frac{2}{\omega-\nu}\left(\int_0^t\left[\frac{1}{\mu^{\varkappa}} - \left(\frac{1-(1-s)^{\mu}}{\mu}\right)^{\varkappa}\right]\mathrm{d}s\right)\hbar'\left(\frac{2-t}{2}\nu + \frac{t}{2}\omega\right)\Bigg|_0^1 \\ &\quad - \frac{2}{\omega-\nu}\int_0^1\left[\frac{1}{\mu^{\varkappa}} - \left(\frac{1-(1-t)^{\mu}}{\mu}\right)^{\varkappa}\right]\hbar'\left(\frac{2-t}{2}\nu + \frac{t}{2}\omega\right)\mathrm{d}t \\ &= \frac{2}{\omega-\nu}\left(\int_0^1\left[\frac{1}{\mu^{\varkappa}} - \left(\frac{1-(1-s)^{\mu}}{\mu}\right)^{\varkappa}\right]\mathrm{d}s\right)\hbar'\left(\frac{\nu+\omega}{2}\right) \\ &\quad - \frac{2}{\omega-\nu}\left\{\frac{2}{\omega-\nu}\left[\frac{1}{\mu^{\varkappa}} - \left(\frac{1-(1-t)^{\mu}}{\mu}\right)^{\varkappa}\right]\hbar\left(\frac{2-t}{2}\nu + \frac{t}{2}\omega\right)\Bigg|_0^1 \right. \\ &\quad \left. + \frac{2\varkappa}{\omega-\nu}\int_0^1\left(\frac{1-(1-t)^{\mu}}{\mu}\right)^{\varkappa-1}(1-t)^{\mu-1}\hbar\left(\frac{2-t}{2}\nu + \frac{t}{2}\omega\right)\mathrm{d}t\right\} \\ &= \frac{2}{\omega-\nu}\left(\int_0^1\left[\frac{1}{\mu^{\varkappa}} - \left(\frac{1-(1-s)^{\mu}}{\mu}\right)^{\varkappa}\right]\mathrm{d}s\right)f'\left(\frac{\nu+\omega}{2}\right) \\ &\quad + \left(\frac{2}{\omega-\nu}\right)^2\frac{\hbar(\nu)}{\mu^{\varkappa}} - \left(\frac{2}{\omega-\nu}\right)^2\frac{\Gamma(\varkappa+1)}{\Gamma(\varkappa)}\int_{\nu}^{\nu+\omega/2} \\ &\quad \cdot\left(\frac{1-(2/\omega-\nu(\nu+\omega/2-x))^{\mu}}{\mu}\right)^{\varkappa-1} \\ &\quad \cdot\left(\frac{2}{\omega-\nu}\left(\frac{\nu+\omega}{2}-x\right)\right)^{\mu-1}\frac{2}{\omega-\nu}\hbar(x)\mathrm{d}x \\ &= \frac{2}{\omega-\nu}\left(\int_0^1\left[\frac{1}{\mu^{\varkappa}} - \frac{1-(1-s)^{\mu}}{\mu}\right]^{\varkappa}\mathrm{d}s\right)\hbar'\left(\frac{\nu+\omega}{2}\right) \\ &\quad + \left(\frac{2}{\omega-\nu}\right)^2\frac{\hbar(\nu)}{\mu^{\varkappa}} - \left(\frac{2}{\omega-\nu}\right)^{2+\mu\varkappa}\frac{\Gamma(\varkappa+1)}{\Gamma(\varkappa)} \\ &\quad \cdot\int_{\nu}^{\nu+\omega/2}\left(\frac{(\omega-\nu/2)^{\mu}-(\nu+\omega/2-x)^{\mu}}{\mu}\right)^{\varkappa-1} \\ &\quad \cdot\frac{\hbar(x)}{(\nu+\omega/2-x)^{1-\mu}}\hbar(x)\mathrm{d}x \\ &= \frac{2}{\omega-\nu}\left(\int_0^1\left[\frac{1}{\mu^{\varkappa}} - \frac{1-(1-s)^{\mu}}{\mu}\right]^{\varkappa}\mathrm{d}s\right)\hbar'\left(\frac{\nu+\omega}{2}\right) \\ &\quad + \left(\frac{2}{\omega-\nu}\right)^2\frac{\hbar(\nu)}{\mu^{\varkappa}} - \left(\frac{2}{\omega-\nu}\right)^{2+\mu\varkappa}\Gamma(\varkappa+1)\,{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2-}\hbar(\nu). \end{aligned} \tag{7} $$

Likewise,

$$ \begin{aligned} I_2 &= \int_0^1\left(\int_0^t\left[\frac{1}{\mu^{\varkappa}} - \left(\frac{1-(1-s)^{\mu}}{\mu}\right)^{\varkappa}\right]\mathrm{d}s\right)\hbar''\left(\frac{t}{2}\nu + \frac{2-t}{2}\omega\right)\mathrm{d}t \\ &= -\frac{2}{\omega-\nu}\left(\int_0^1\left[\frac{1}{\mu^{\varkappa}} - \left(\frac{1-(1-s)^{\mu}}{\mu}\right)^{\varkappa}\right]\mathrm{d}s\right)\hbar'\left(\frac{\nu+\omega}{2}\right) \\ &\quad + \left(\frac{2}{\omega-\nu}\right)^2\frac{\hbar(\omega)}{\mu^{\varkappa}} - \left(\frac{2}{\omega-\nu}\right)^{2+\mu\varkappa}\Gamma(\varkappa+1)\,{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2+}\hbar(\omega). \end{aligned} \tag{8} $$

Then, it follows that

$$\frac{(\omega-\nu)^2\mu^\varkappa}{8}[I_1+I_2] = \frac{\hbar(\nu)+\hbar(\omega)}{2} - \frac{2^{\mu\varkappa-1}\mu^\varkappa\Gamma(\varkappa+1)}{(\omega-\nu)^{\mu\varkappa}} \cdot \left[{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2-}\hbar(\nu) + {}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2+}\hbar(\omega)\right]. \tag{9}$$

So, the proof is accomplished. □

Theorem 4. *Consider $\hbar : [\nu, \omega] \longrightarrow \mathbb{R}$ as a twice differentiable mapping on (ν, ω) s.t. $\hbar'' \in L_1([\nu, \omega])$. If $|\hbar''|$ is convex on $[\nu, \omega]$, then*

$$\left|\frac{\hbar(\nu)+\hbar(\omega)}{2} - \frac{2^{\mu\varkappa-1}\mu^\varkappa\Gamma(\varkappa+1)}{(\omega-\nu)^{\mu\varkappa}}\left[{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2-}\hbar(\nu) + {}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2+}\hbar(\omega)\right]\right| \le \frac{(\omega-\nu)^2\mu^\varkappa}{8}\Phi_1(\mu,\varkappa)\left(\left|\hbar''(\nu)\right| + \left|\hbar''(\omega)\right|\right), \tag{10}$$

where

$$\Phi_1(\mu,\varkappa) = \int_0^1\left|\int_0^t\left[\frac{1}{\mu^\varkappa} - \left(\frac{1-(1-s)^\mu}{\mu}\right)^\varkappa\right]ds\right|dt = \frac{1}{\mu^\varkappa}\int_0^1\left|t - \frac{1}{\mu}\mathcal{B}\left(\varkappa+1, \frac{1}{\mu}, 1-(1-t)^\mu\right)\right|dt. \tag{11}$$

Proof. Taking the absolute value of both sides of (6), we derive

$$\begin{aligned}&\left|\frac{\hbar(\nu)+\hbar(\omega)}{2} - \frac{2^{\mu\varkappa-1}\mu^\varkappa\Gamma(\varkappa+1)}{(\omega-\nu)^{\mu\varkappa}}\left[{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2-}\hbar(\nu) + {}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2+}\hbar(\omega)\right]\right| \\ &\le \frac{(\omega-\nu)^2\mu^\varkappa}{8}\left[\int_0^1\left|\int_0^t\left[\frac{1}{\mu^\varkappa} - \left(\frac{1-(1-s)^\mu}{\mu}\right)^\varkappa\right]\mathrm{d}s\right| \cdot \left|\hbar''\left(\frac{2-t}{2}\nu + \frac{t}{2}\omega\right)\right|\mathrm{d}t + \int_0^1\left|\int_0^t\left[\frac{1}{\mu^\varkappa} - \left(\frac{1-(1-s)^\mu}{\mu}\right)^\varkappa\right]\mathrm{d}s\right| \cdot \left|\hbar''\left(\frac{t}{2}\nu + \frac{2-t}{2}\omega\right)\right|\mathrm{d}t\right].\end{aligned} \tag{12}$$

By using the convexity property of the $|\hbar''|$, we establish

$$\begin{aligned}&\left|\frac{\hbar(\nu)+\hbar(\omega)}{2} - \frac{2^{\mu\varkappa-1}\mu^\varkappa\Gamma(\varkappa+1)}{(\omega-\nu)^{\mu\varkappa}}\left[{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2-}\hbar(\nu) + {}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2+}\hbar(\omega)\right]\right| \\ &\le \frac{(\omega-\nu)^2\mu^\varkappa}{8}\left[\int_0^1\left|\int_0^t\left[\frac{1}{\mu^\varkappa} - \left(\frac{1-(1-s)^\mu}{\mu}\right)^\varkappa\right]\mathrm{d}s\right|\left[\frac{2-t}{2} \cdot \left|\hbar'(\nu)\right| + \frac{t}{2}\left|\hbar'(\omega)\right|\right]\mathrm{d}t + \int_0^1\left|\int_0^t\left[\frac{1}{\mu^\varkappa} - \left(\frac{1-(1-s)^\mu}{\mu}\right)^\varkappa\right]\mathrm{d}s\right| \cdot \left[\frac{t}{2}\left|\hbar'(\nu)\right| + \frac{2-t}{2}\left|\hbar'(\omega)\right|\right]\mathrm{d}t\right] \\ &= \frac{(\omega-\nu)^2\mu^\varkappa}{8}\left(\int_0^1\left|\int_0^t\left[\frac{1}{\mu^\varkappa} - \left(\frac{1-(1-s)^\mu}{\mu}\right)^\varkappa\right]\mathrm{d}s\right|\mathrm{d}t\right) \cdot \left(\left|\hbar'(\nu)\right| + \left|\hbar'(\omega)\right|\right).\end{aligned} \tag{13}$$

The proof is ended. □

Remark 5. In Theorem 11, we have the inequalities as follows:

(i) If we set $\mu = 1$ in (10), then Theorem 4 leads to [42], Corollary 7.

(ii) If we take $\mu = 1$ and $\varkappa = 1$ in (10), then Theorem 4 leads to [43], Proposition 2.

Theorem 6. *Assume that $\hbar : [\nu, \omega] \longrightarrow \mathbb{R}$ is a twice differentiable function on (ν, ω) s.t. $\hbar'' \in L_p([\nu, \omega])$ with $\nu < \omega$. Let $|\hbar''|^q$ be convex on $[\nu, \omega]$ with $q > 1$. Then, the inequality*

$$\begin{aligned}&\left|\frac{\hbar(\nu)+\hbar(\omega)}{2} - \frac{2^{\mu\varkappa-1}\mu^\varkappa\Gamma(\varkappa+1)}{(\omega-\nu)^{\mu\varkappa}}\left[{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2-}\hbar(\nu) + {}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2+}\hbar(\omega)\right]\right| \\ &\le \frac{(\omega-\nu)^2\mu^\varkappa}{8}\Theta^\varkappa_\mu(p)\left[\left(\frac{3\left|\hbar''(\nu)\right|^q + \left|\hbar''(\omega)\right|^q}{4}\right)^{1/q} + \left(\frac{\left|\hbar''(\nu)\right|^q + 3\left|\hbar''(\omega)\right|^q}{4}\right)^{1/q}\right] \\ &\le \frac{(\omega-\nu)^2\mu^\varkappa}{2^{3-2/p}}\Theta^\varkappa_\mu(p)\left[\left|\hbar''(\nu)\right|^q + \left|\hbar''(\omega)\right|^q\right],\end{aligned} \tag{14}$$

holds, where $1/q + 1/p = 1$ and

$$\Theta^\varkappa_\mu(p) = \left(\int_0^1\left|\int_0^t\left[\frac{1}{\mu^\varkappa} - \left(\frac{1-(1-s)^\mu}{\mu}\right)^\varkappa\right]ds\right|^p dt\right)^{1/q}. \tag{15}$$

Proof. By employing the Hölder inequality on (12), we have

$$\begin{aligned}&\left|\frac{\hbar(\nu)+\hbar(\omega)}{2} - \frac{2^{\mu\varkappa-1}\mu^\varkappa\Gamma(\varkappa+1)}{(\omega-\nu)^{\mu\varkappa}}\left[{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2-}\hbar(\nu) + {}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2+}\hbar(\omega)\right]\right| \\ &\le \frac{(\omega-\nu)^2\mu^\varkappa}{8}\left[\left(\int_0^1\left|\int_0^t\left[\frac{1}{\mu^\varkappa} - \left(\frac{1-(1-s)^\mu}{\mu}\right)^\varkappa\right]\mathrm{d}s\right|^p\mathrm{d}t\right)^{1/p} \cdot \left(\int_0^1\left|\hbar''\left(\frac{2-t}{2}\nu + \frac{t}{2}\omega\right)\right|^q\mathrm{d}t\right)^{1/q} + \left(\int_0^1\left|\int_0^t\left[\frac{1}{\mu^\varkappa} - \left(\frac{1-(1-s)^\mu}{\mu}\right)^\varkappa\right]\mathrm{d}s\right|^p\mathrm{d}t\right)^{1/p} \cdot \left(\int_0^1\left|\hbar''\left(\frac{t}{2}\nu + \frac{2-t}{2}\omega\right)\right|^q\mathrm{d}t\right)^{1/q}\right].\end{aligned} \tag{16}$$

For the sake of the convexity of $|\hbar''|^q$ on $[\nu, \omega]$, we get

$$\begin{aligned}\int_0^1\left|\hbar''\left(\frac{2-t}{2}\nu + \frac{t}{2}\omega\right)\right|^q\mathrm{d}t &\le \int_0^1\left[\frac{2-t}{2}\left|\hbar''(\nu)\right|^q + \frac{t}{2}\left|\hbar''(\omega)\right|^q\right]\mathrm{d}t \\ &= \frac{3\left|\hbar''(\nu)\right|^q + \left|\hbar''(\omega)\right|^q}{4},\end{aligned} \tag{17}$$

and similarly

$$\int_0^1 \left|\hbar''\left(\frac{t}{2}\nu + \frac{2-t}{2}\omega\right)\right|^q \mathrm{d}t \le \frac{|\hbar''(\nu)|^q + 3|\hbar''(\omega)|^q}{4}. \tag{18}$$

If we substitute the inequalities (17) and (18) in (16), the first inequality of (14) will be established.

The next inequality is derived directly if we let $\varpi_1 = 3|\hbar''(\nu)|^q, \rho_1 = |\hbar''(\omega)|^q, \varpi_2 = |\hbar''(\nu)|^q$, and $\rho_2 = 3|\hbar''(\omega)|^q$ and apply the inequality

$$\sum_{k=1}^{n} (\varpi_k + \rho_k)^s \le \sum_{k=1}^{n} \varpi_k^s + \sum_{k=1}^{n} \rho_k^s, 0 \le s < 1. \tag{19}$$

Thus, our deduction is ended. □

Corollary 7. *In Theorem 6, we have the inequalities as follows:*

(i) *If we set $\mu = 1$ in Theorem 6, we derive*

$$\begin{aligned}
&\left|\frac{\hbar(\nu)+\hbar(\omega)}{2} - \frac{2^{\varkappa-1}\Gamma(\varkappa+1)}{(\omega-\nu)^{\varkappa}}\left[J^{\varkappa}_{\nu+\omega/2-}\hbar(\nu) + J^{\varkappa}_{\nu+\omega/2+}\hbar(\omega)\right]\right| \\
&\le \frac{(\omega-\nu)^2}{8}\left(\frac{1}{p+1} - \frac{1}{(\varkappa+1)^p(\varkappa p+p+1)}\right) \\
&\times \left[\left(\frac{3|\hbar''(\nu)|^q + |\hbar''(\omega)|^q}{4}\right)^{1/q} + \left(\frac{|\hbar''(\nu)|^q + 3|\hbar''(\omega)|^q}{4}\right)^{1/q}\right] \\
&\le \frac{(\omega-\nu)^2}{2^{3-2/p}}\left(\frac{1}{p+1} - \frac{1}{(\varkappa+1)^p(\varkappa p+p+1)}\right) \cdot \left[|\hbar''(\nu)|^q + |\hbar''(\omega)|^q\right].
\end{aligned} \tag{20}$$

Proof. For the proof, it will be sufficient to write down the solution of the integral below.

$$\begin{aligned}
\Theta^{\varkappa}_{\mu}(p) = \Theta^{\varkappa}_{1}(p) &= \left(\int_0^1\left|\int_0^t (1-s^{\varkappa})\mathrm{d}s\right|^p \mathrm{d}t\right)^{1/p} \\
&= \left(\int_0^1\left|t - \frac{t^{\varkappa+1}}{\varkappa+1}\right|^p \mathrm{d}t\right)^{1/p}.
\end{aligned} \tag{21}$$

Under conditions $A > B > 0$ and $p > 1$, the inequality

$$|A-B|^p \le A^p - B^p \tag{22}$$

is satisfied.

From the inequality (22), $A = t$ and $B = t^{\varkappa+1}/\varkappa+1$, we have

$$\begin{aligned}
\Theta^{\varkappa}_{1}(p) &\le \left(\int_0^1 t^p \mathrm{d}t - \int_0^1\left(\frac{t^{\varkappa+1}}{\varkappa+1}\right)^p \mathrm{d}t\right)^{1/p} \\
&= \left(\frac{1}{p+1} - \frac{1}{(\varkappa+1)^p(\varkappa p+p+1)}\right)^{1/p}.
\end{aligned} \tag{23}$$

When the solution of $\Theta^{\varkappa}_{\mu}(p)$ is substituted for (14), the proof is clear.

(ii) If we take $\mu = 1$ and $\varkappa = 1$ in Theorem 6, then

$$\begin{aligned}
&\left|\frac{\hbar(\nu)+\hbar(\omega)}{2} - \frac{1}{(\omega-\nu)}\int_\nu^\omega \hbar(x)dx\right| \\
&\le \frac{(\omega-\nu)^2}{8}\left(\frac{1}{p+1} - \frac{1}{2^p(2p+1)}\right) \cdot \left[\left(\frac{3|\hbar''(\nu)|^q + |\hbar''(\omega)|^q}{4}\right)^{1/q} + \left(\frac{|\hbar''(\nu)|^q + 3|\hbar''(\omega)|^q}{4}\right)^{1/q}\right] \\
&\le \frac{(\omega-\nu)^2}{2^{3-2/p}}\left(\frac{1}{p+1} - \frac{1}{2^p(2p+1)}\right) \cdot \left[|\hbar''(\nu)|^q + |\hbar''(\omega)|^q\right].
\end{aligned} \tag{24}$$

□

Theorem 8. *Consider $\hbar : [\nu, \omega] \longrightarrow \mathbb{R}$ as a twice differentiable mapping on (ν, ω) s.t. $\hbar'' \in L_q([\nu, \omega])$. Assume that $|\hbar''|^q$ admits the convexity property on $[\nu, \omega]$ with $q \ge 1$. Then,*

$$\begin{aligned}
&\left|\frac{\hbar(\nu)+\hbar(\omega)}{2} - \frac{2^{\mu\varkappa-1}\mu^{\varkappa}\Gamma(\varkappa+1)}{(\omega-\nu)^{\mu\varkappa}}\left[{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2-}\hbar(\nu) + {}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2+}\hbar(\omega)\right]\right| \\
&\le \frac{(\omega-\nu)^2\mu^{\varkappa}}{8}(\Phi_1(\mu,\varkappa))^{1-1/q} \times \left[\left(\frac{(2\Phi_1(\mu,\varkappa) - \Phi_2(\mu,\varkappa))}{2} \cdot |\hbar''(\nu)|^q + \frac{\Phi_2(\mu,\varkappa)}{2}|\hbar''(\omega)|^q\right)^{1/q} + \left(\frac{\Phi_2(\mu,\varkappa)}{2} \cdot |\hbar''(\nu)|^q + \frac{(2\Phi_1(\mu,\varkappa) - \Phi_2(\mu,\varkappa))}{2}|\hbar''(\omega)|^q\right)^{1/q}\right],
\end{aligned} \tag{25}$$

holds, where

$$\begin{aligned}
\Phi_2(\mu,\varkappa) &= \int_0^1 t\left|\int_0^t\left[\frac{1}{\mu^{\varkappa}} - \left(\frac{1-(1-s)^{\mu}}{\mu}\right)^{\varkappa}\right]ds\right|dt \\
&= \frac{1}{\mu^{\varkappa}}\int_0^1 t\left|t - \frac{1}{\mu}\mathscr{B}\left(\varkappa+1, \frac{1}{\mu}, 1-(1-t)^{\mu}\right)\right|dt.
\end{aligned} \tag{26}$$

Proof. By employing the power-mean inequality in (12), we have

$$\begin{aligned}&\left|\frac{\hbar(\nu)+\hbar(\omega)}{2}-\frac{2^{\mu\varkappa-1}\mu^{\varkappa}\Gamma(\varkappa+1)}{(\omega-\nu)^{\mu\varkappa}}\left[{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2-}\hbar(\nu)+{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2+}\hbar(\omega)\right]\right|\\&\quad\le\frac{(\omega-\nu)^2\mu^{\varkappa}}{8}\left[\left(\int_0^1\left|\int_0^t\left[\frac{1}{\mu^{\varkappa}}-\left(\frac{1-(1-s)^{\mu}}{\mu}\right)^{\varkappa}\right]\mathrm{d}s\right|\mathrm{d}t\right)^{1-1/q}\right.\\&\qquad\times\left(\int_0^1\left|\int_0^t\left[\frac{1}{\mu^{\varkappa}}-\left(\frac{1-(1-s)^{\mu}}{\mu}\right)^{\varkappa}\right]\mathrm{d}s\right|\right.\\&\qquad\left.\cdot\left|\hbar''\left(\frac{2-t}{2}\nu+\frac{t}{2}\omega\right)\right|^q\mathrm{d}t\right)^{1/q}\\&\qquad+\left(\int_0^1\left|\int_0^t\left[\frac{1}{\mu^{\varkappa}}-\left(\frac{1-(1-s)^{\mu}}{\mu}\right)^{\varkappa}\right]\mathrm{d}s\right|\mathrm{d}t\right)^{1-1/q}\\&\qquad\times\left(\int_0^1\left|\int_0^t\left[\frac{1}{\mu^{\varkappa}}-\left(\frac{1-(1-s)^{\mu}}{\mu}\right)^{\varkappa}\right]\mathrm{d}s\right|\right.\\&\qquad\left.\left.\cdot\left|\hbar''\left(\frac{t}{2}\nu+\frac{2-t}{2}\omega\right)\right|^q\mathrm{d}t\right)^{1/q}\right].\end{aligned}\tag{27}$$

We know that $|\hbar'|^q$ is convex. Thus,

$$\begin{aligned}&\int_0^1\left|\int_0^t\left[\frac{1}{\mu^{\varkappa}}-\left(\frac{1-(1-s)^{\mu}}{\mu}\right)^{\varkappa}\right]\mathrm{d}s\right|\left|\hbar''\left(\frac{2-t}{2}\nu+\frac{t}{2}\omega\right)\right|^q\mathrm{d}t\\&\quad\le\int_0^1\left|\int_0^t\left[\frac{1}{\mu^{\varkappa}}-\left(\frac{1-(1-s)^{\mu}}{\mu}\right)^{\varkappa}\right]\mathrm{d}s\right|\\&\qquad\cdot\left[\frac{2-t}{2}\left|\hbar''(\nu)\right|^q+\frac{t}{2}\left|\hbar''(\omega)\right|^q\right]\mathrm{d}t\\&\quad=\frac{(2\Phi_1(\mu,\varkappa)-\Phi_2(\mu,\varkappa))}{2}\left|\hbar''(\nu)\right|^q\\&\qquad+\frac{\Phi_2(\mu,\varkappa)}{2}\left|\hbar''(\omega)\right|^q,\end{aligned}\tag{28}$$

and similarly

$$\begin{aligned}&\int_0^1\left|\int_0^t\left[\frac{1}{\mu^{\varkappa}}-\left(\frac{1-(1-s)^{\mu}}{\mu}\right)^{\varkappa}\right]\mathrm{d}s\right|\left|\hbar''\left(\frac{t}{2}\nu+\frac{2-t}{2}\omega\right)\right|^q\mathrm{d}t\\&\quad\le\frac{\Phi_2(\mu,\varkappa)}{2}\left|\hbar''(\nu)\right|^q+\frac{(2\Phi_1(\mu,\varkappa)-\Phi_2(\mu,\varkappa))}{2}\left|\hbar''(\omega)\right|^q.\end{aligned}\tag{29}$$

Substituting the inequalities (28) and (29) in (27), we derive the desired result. □

Corollary 9. *In Theorem 8, we have the inequalities as follows:*

(i) By choosing $\mu=1$ in Theorem 8, we derive

$$\begin{aligned}&\left|\frac{\hbar(\nu)+\hbar(\omega)}{2}-\frac{2^{\varkappa-1}\Gamma(\varkappa+1)}{(\omega-\nu)^{\varkappa}}\left[J^{\varkappa}_{\nu+\omega/2-}\hbar(\nu)+J^{\varkappa}_{\nu+\omega/2+}\hbar(\omega)\right]\right|\\&\quad\le\frac{(\omega-\nu)^2}{8}\left(\frac{1}{2}-\frac{1}{(\varkappa+1)(\varkappa+2)}\right)^{1-1/q}\\&\qquad\times\left[\left(\left(\frac{1}{3}-\frac{\varkappa+4}{2(\varkappa+1)(\varkappa+2)(\varkappa+3)}\right)\left|\hbar''(\nu)\right|^q\right.\right.\\&\qquad\left.+\left(\frac{1}{6}-\frac{1}{2(\varkappa+1)(\varkappa+3)}\right)\left|\hbar''(\omega)\right|^q\right)^{1/q}\\&\qquad+\left(\left(\frac{1}{6}-\frac{1}{2(\varkappa+1)(\varkappa+3)}\right)\left|\hbar''(\nu)\right|^q\right.\\&\qquad\left.\left.+\left(\frac{1}{3}-\frac{\varkappa+4}{2(\varkappa+1)(\varkappa+2)(\varkappa+3)}\right)\left|\hbar''(\omega)\right|^q\right)^{1/q}\right].\end{aligned}\tag{30}$$

(ii) If we take $\mu=1$ and $\varkappa=1$ in Theorem 8, we derive

$$\begin{aligned}&\left|\frac{\hbar(\nu)+\hbar(\omega)}{2}-\frac{1}{(\omega-\nu)}\int_{\nu}^{\omega}\hbar(x)dx\right|\\&\quad\le\frac{(\omega-\nu)^2}{24}\left[\left(\frac{11}{16}\left|\hbar''(\nu)\right|^q+\frac{5}{16}\left|\hbar''(\omega)\right|^q\right)^{1/q}\right.\\&\qquad\left.+\left(\frac{5}{16}\left|\hbar''(\nu)\right|^q+\frac{11}{16}\left|\hbar''(\omega)\right|^q\right)^{1/q}\right].\end{aligned}\tag{31}$$

4. Midpoint-Type Inequalities Based on Conformable Fractional Integrals

In this section, midpoint-type inequalities are created for twice differentiable functions with the help of conformable fractional integrals. To formulate these inequalities, let us first set up the following identity.

Lemma 10. *Let $\hbar:[\nu,\omega]\longrightarrow\mathbb{R}$ be a twice differentiable map on (ν,ω) with $\hbar''\in L_1([\nu,\omega])$. Then, the equality*

$$\begin{aligned}&\frac{2^{\mu\varkappa-1}\mu^{\varkappa}\Gamma(\varkappa+1)}{(\omega-\nu)^{\mu\varkappa}}\left[{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2+}\hbar(\omega)+{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2-}\hbar(\nu)\right]-\hbar\left(\frac{\nu+\omega}{2}\right)\\&\quad=\frac{(\omega-\nu)^2\mu^{\varkappa}}{8}\left[\int_0^1\left(\int_0^t\left[\frac{1-(1-s)^{\mu}}{\mu}\right]^{\varkappa}ds\right)\hbar''\left(\frac{2-t}{2}\nu+\frac{t}{2}\omega\right)dt\right.\\&\qquad\left.+\int_0^1\left(\int_0^t\left[\frac{1-(1-s)^{\mu}}{\mu}\right]^{\varkappa}ds\right)\hbar''\left(\frac{t}{2}\nu+\frac{2-t}{2}\omega\right)dt\right]\end{aligned}\tag{32}$$

is valid.

Proof. With the help of the integration by parts

$$\begin{aligned} I_3 &= \int_0^1 \left(\int_0^t \left[\frac{1-(1-s)^\mu}{\mu} \right]^\varkappa \mathrm{ds} \right) \hbar'' \left(\frac{2-t}{2}\nu + \frac{t}{2}\omega \right) \mathrm{dt} \\ &= \frac{2}{\omega-\nu} \left(\int_0^t \left[\frac{1-(1-s)^\mu}{\mu} \right]^\varkappa \mathrm{ds} \right) \hbar' \left(\frac{2-t}{2}\nu + \frac{t}{2}\omega \right) \Big|_0^1 \\ &\quad - \frac{2}{\omega-\nu} \int_0^1 \left[\frac{1-(1-t)^\mu}{\mu} \right]^\varkappa \hbar' \left(\frac{2-t}{2}\nu + \frac{t}{2}\omega \right) \mathrm{dt} \\ &= \frac{2}{\omega-\nu} \left(\int_0^t \left[\frac{1-(1-s)^\mu}{\mu} \right]^\varkappa \mathrm{ds} \right) \hbar' \left(\frac{\nu+\omega}{2} \right) \\ &\quad - \frac{2}{\omega-\nu} \left\{ \frac{2}{\omega-\nu} \left(\frac{1-(1-t)^\mu}{\mu} \right)^\varkappa \hbar \left(\frac{2-t}{2}\nu + \frac{t}{2}\omega \right) \Big|_0^1 \right. \\ &\quad \left. - \frac{2\varkappa}{\omega-\nu} \int_0^1 \left(\frac{1-(1-t)^\mu}{\mu} \right)^{\varkappa-1} (1-t)^{\mu-1} \mathrm{dt}. \right. \end{aligned} \tag{33}$$

By using variable change, equality is obtained as follows:

$$\begin{aligned} I_3 &= \frac{2}{\omega-\nu} \left(\int_0^1 \left[\frac{1-(1-s)^\mu}{\mu} \right]^\varkappa \mathrm{ds} \right) \hbar' \left(\frac{\nu+\omega}{2} \right) \\ &\quad - \left(\frac{2}{\omega-\nu} \right)^2 \frac{1}{\mu^\varkappa} \hbar \left(\frac{\nu+\omega}{2} \right) + \left(\frac{2}{\omega-\nu} \right)^{2+\mu\varkappa} \frac{\Gamma(\varkappa+1)}{\Gamma(\varkappa)} \int_\nu^{\nu+\omega/2} \\ &\quad \cdot \left(\frac{(\omega-\nu/2)^\mu - (\nu+\omega/2-x)^\mu}{\mu} \right)^{\varkappa-1} \frac{\hbar(x)}{(\nu+\omega/2-x)^{1-\mu}} \hbar(x) \mathrm{dx} \\ &= \frac{2}{\omega-\nu} \left(\int_0^1 \left[\frac{1-(1-s)^\mu}{\mu} \right]^\varkappa \mathrm{ds} \right) \hbar' \left(\frac{\nu+\omega}{2} \right) - \left(\frac{2}{\omega-\nu} \right)^2 \\ &\quad \cdot \frac{1}{\mu^\varkappa} \hbar \left(\frac{\nu+\omega}{2} \right) + \left(\frac{2}{\omega-\nu} \right)^{2+\mu\varkappa} \Gamma(\varkappa+1)^\varkappa \mathcal{J}^\mu_{\nu+\omega/2-} \hbar(\nu). \end{aligned} \tag{34}$$

In the same way,

$$\begin{aligned} I_4 &= \int_0^1 \left(\int_0^t \left[\frac{1-(1-s)^\mu}{\mu} \right]^\varkappa \mathrm{ds} \right) \hbar'' \left(\frac{t}{2}\nu + \frac{2-t}{2}\omega \right) \mathrm{dt} \\ &= -\frac{2}{\omega-\nu} \left(\int_0^1 \left[\frac{1-(1-s)^\mu}{\mu} \right]^\varkappa \mathrm{ds} \right) \hbar' \left(\frac{\nu+\omega}{2} \right) \\ &\quad - \left(\frac{2}{\omega-\nu} \right)^2 \frac{1}{\mu^\varkappa} \hbar \left(\frac{\nu+\omega}{2} \right) \\ &\quad + \left(\frac{2}{\omega-\nu} \right)^{2+\mu\varkappa} \Gamma(\varkappa+1)^\varkappa \mathcal{J}^\mu_{\nu+\omega/2+} \hbar(\omega). \end{aligned} \tag{35}$$

If (34) and (35) are added together and then multiplied by $(\omega-\nu)^2\mu^\varkappa/8$, the proof is completed. □

Theorem 11. *Assume $\hbar : [\nu, \omega] \longrightarrow \mathbb{R}$ as a twice differentiable function on (ν, ω) s.t. $\hbar'' \in L_1([\nu, \omega])$. By considering the convexity of $|\hbar''|$ on $[\nu, \omega]$, the inequality*

$$\begin{aligned} &\left| \frac{2^{\mu\varkappa-1}\mu^\varkappa \Gamma(\varkappa+1)}{(\omega-\nu)^{\mu\varkappa}} \left[{}^\varkappa\mathcal{J}^\mu_{\nu+\omega/2+} \hbar(\omega) + {}^\varkappa\mathcal{J}^\mu_{\nu+\omega/2-} \hbar(\nu) \right] - \hbar \left(\frac{\nu+\omega}{2} \right) \right| \\ &\quad \le \frac{(\omega-\nu)^2\mu^\varkappa}{8} Y_1(\mu, \varkappa) \left(|\hbar''(\nu)| + |\hbar''(\omega)| \right) \end{aligned} \tag{36}$$

is satisfied, where $\mathcal{B}$ denotes the beta function and

$$\begin{aligned} Y_1(\mu, \varkappa) &= \int_0^1 \left| \int_0^t \left[\frac{1-(1-s)^\mu}{\mu} \right]^\varkappa ds \right| dt \\ &= \frac{1}{\mu^\varkappa} \int_0^1 \left| \frac{1}{\mu} \mathcal{B} \left(\varkappa+1, \frac{1}{\mu}, 1-(1-t)^\mu \right) \right| dt. \end{aligned} \tag{37}$$

Proof. On both sides of (32), we take the absolute value and get

$$\begin{aligned} &\left| \frac{2^{\mu\varkappa-1}\mu^\varkappa \Gamma(\varkappa+1)}{(\omega-\nu)^{\mu\varkappa}} \left[{}^\varkappa\mathcal{J}^\mu_{\nu+\omega/2+} \hbar(\omega) + {}^\varkappa\mathcal{J}^\mu_{\nu+\omega/2-} \hbar(\nu) \right] - \hbar \left(\frac{\nu+\omega}{2} \right) \right| \\ &\quad \le \frac{(\omega-\nu)^2\mu^\varkappa}{8} \left[\int_0^1 \left| \int_0^t \left[\frac{1-(1-s)^\mu}{\mu} \right]^\varkappa \mathrm{ds} \right| \left| \hbar'' \left(\frac{2-t}{2}\nu + \frac{t}{2}\omega \right) \right| \mathrm{dt} \right. \\ &\quad \left. + \int_0^1 \left| \int_0^t \left[\frac{1-(1-s)^\mu}{\mu} \right]^\varkappa \mathrm{ds} \right| \left| \hbar'' \left(\frac{t}{2}\nu + \frac{2-t}{2}\omega \right) \right| \mathrm{dt} \right]. \end{aligned} \tag{38}$$

Since convexity of $|\hbar''|$, then we have

$$\begin{aligned} &\left| \frac{2^{\mu\varkappa-1}\mu^\varkappa \Gamma(\varkappa+1)}{(\omega-\nu)^{\mu\varkappa}} \left[{}^\varkappa\mathcal{J}^\mu_{\nu+\omega/2+} \hbar(\omega) + {}^\varkappa\mathcal{J}^\mu_{\nu+\omega/2-} \hbar(\nu) \right] - \hbar \left(\frac{\nu+\omega}{2} \right) \right| \\ &\quad \le \frac{(\omega-\nu)^2\mu^\varkappa}{8} \left[\int_0^1 \left| \int_0^t \left[\frac{1-(1-s)^\mu}{\mu} \right]^\varkappa \mathrm{ds} \right| \right. \\ &\quad \cdot \left(\frac{2-t}{2} |\hbar''(\nu)| + \frac{t}{2} |\hbar''(\omega)| \right) dt \\ &\quad \left. + \int_0^1 \left| \int_0^t \left[\frac{1-(1-s)^\mu}{\mu} \right]^\varkappa \mathrm{ds} \right| \left(\frac{t}{2} |\hbar''(\nu)| + \frac{2-t}{2} |\hbar''(\omega)| \right) \mathrm{dt} \right] \\ &\quad = \frac{(\omega-\nu)^2\mu^\varkappa}{8} \left(\int_0^1 \left| \int_0^t \left[\frac{1-(1-s)^\mu}{\mu} \right]^\varkappa \mathrm{ds} \right| \mathrm{dt} \right) \\ &\quad \cdot \left(|\hbar''(\nu)| + |\hbar''(\omega)| \right). \end{aligned} \tag{39}$$

□

Remark 12. In Theorem 11:

(i) If we set $\mu = 1$, then we lead to [42], Theorem 1.5.

(ii) If we allow $\mu = 1$ and $\varkappa = 1$, then Theorem 11 and [43], Proposition 1 are identical.

Theorem 13. *Let $\hbar : [\nu, \omega] \longrightarrow \mathbb{R}$ be a twice differentiable map on (ν, ω) s.t. $\hbar'' \in L_1([\nu, \omega])$. Let $|\hbar''|^q$ be convex on $[\nu, \omega]$ with $q > 1$. Then,*

$$\left|\frac{2^{\mu\varkappa-1}\mu^{\varkappa}\Gamma(\varkappa+1)}{(\omega-\nu)^{\mu\varkappa}}\left[{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2+}\hbar(\omega)+{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2-}\hbar(\nu)\right]-\hbar\left(\frac{\nu+\omega}{2}\right)\right|$$
$$\leq\frac{(\omega-\nu)^2}{2}\left(Y^{\varkappa}_{\mu}(p)\right)^{1/p}\left[\left(\frac{3|\hbar''(\nu)|^q+|\hbar''(\omega)|^q}{4}\right)^{1/q}+\left(\frac{|\hbar''(\nu)|^q+3|\hbar''(\omega)|^q}{4}\right)^{1/q}\right]$$
$$\leq\frac{(\omega-\nu)^2}{2}\left(4Y^{\varkappa}_{\mu}(p)\right)^{1/p}\left[|\hbar''(\nu)|+|\hbar''(\omega)|\right], \tag{40}$$

where $1/p+1/q=1$, and

$$Y^{\varkappa}_{\mu}(p)=\int_0^1\left|\int_0^t\left[\frac{1-(1-s)^{\mu}}{\mu}\right]^{\varkappa}ds\right|^p dt. \tag{41}$$

Proof. Using the Hölder inequality in (38), we have

$$\left|\frac{2^{\mu\varkappa-1}\mu^{\varkappa}\Gamma(\varkappa+1)}{(\omega-\nu)^{\mu\varkappa}}\left[{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2+}\hbar(\omega)+{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2-}\hbar(\nu)\right]-\hbar\left(\frac{\nu+\omega}{2}\right)\right|$$
$$\leq\frac{(\omega-\nu)^2\mu^{\varkappa}}{8}\left[\left(\int_0^1\left|\int_0^t\left[\frac{1-(1-s)^{\mu}}{\mu}\right]^{\varkappa}\mathrm{ds}\right|^p\mathrm{dt}\right)^{1/p}\cdot\left(\int_0^1\left|\hbar''\left(\frac{2-t}{2}\nu+\frac{t}{2}\omega\right)\right|^q\mathrm{dt}\right)^{1/q}+\left(\int_0^1\left|\int_0^t\left[\frac{1-(1-s)^{\mu}}{\mu}\right]^{\varkappa}\mathrm{ds}\right|^p\mathrm{dt}\right)^{1/p}\cdot\left(\int_0^1\left|\hbar''\left(\frac{t}{2}\nu+\frac{2-t}{2}\omega\right)\right|^q\mathrm{dt}\right)^{1/q}\right]. \tag{42}$$

Since $|\hbar''|^q$ is convex, we obtain

$$\left|\frac{2^{\mu\varkappa-1}\mu^{\varkappa}\Gamma(\varkappa+1)}{(\omega-\nu)^{\mu\varkappa}}\left[{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2+}\hbar(\omega)+{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2-}\hbar(\nu)\right]-\hbar\left(\frac{\nu+\omega}{2}\right)\right|$$
$$\leq\frac{(\omega-\nu)^2\mu^{\varkappa}}{8}\left[\left(\int_0^1\left|\int_0^t\left[\frac{1-(1-s)^{\mu}}{\mu}\right]^{\varkappa}\mathrm{ds}\right|^p\mathrm{dt}\right)^{1/p}\cdot\left(\int_0^1\left(\frac{2-t}{2}|\hbar''(\nu)|^q+\frac{t}{2}|\hbar''(\omega)|^q\right)dt\right)^{1/q}+\left(\int_0^1\left|\int_0^t\left[\frac{1-(1-s)^{\mu}}{\mu}\right]^{\varkappa}ds\right|^p dt\right)^{1/p}\cdot\left(\int_0^1\left(\frac{2-t}{2}|\hbar''(\nu)|^q+\frac{t}{2}|\hbar''(\omega)|^q\right)dt\right)^{1/q}\right]. \tag{43}$$

If we substitute the inequalities (17) and (18) in (43), we obtain the first inequality of (40).

The last inequality is established by letting $\varpi_1=3|\hbar''(\nu)|^q$, $\rho_1=|\hbar''(\omega)|^q$, $\varpi_2=|\hbar''(\nu)|^q$, and $\rho_2=3|\hbar''(\omega)|^q$ and with help of the inequality (19). □

Corollary 14. *In Theorem 13, we have the inequalities as follows:*

(i) If we set $\mu=1$ in Theorem 13, we derive

$$\left|\frac{2^{\varkappa-1}\Gamma(\varkappa+1)}{(\omega-\nu)^{\varkappa}}\left[J^{\varkappa}_{\nu+\omega/2+}\hbar(\omega)+J^{\varkappa}_{\nu+\omega/2-}\hbar(\nu)\right]-\hbar\left(\frac{\nu+\omega}{2}\right)\right|$$
$$\leq\frac{(\omega-\nu)^2}{2(\varkappa+1)}\left(\frac{1}{\varkappa p+p+1}\right)^{1/p}\left[\left(\frac{3|\hbar''(\nu)|^q+|\hbar''(\omega)|^q}{4}\right)^{1/q}+\left(\frac{|\hbar''(\nu)|^q+3|\hbar''(\omega)|^q}{4}\right)^{1/q}\right]$$
$$\leq\frac{(\omega-\nu)^2}{2(\varkappa+1)}\left(\frac{4}{\varkappa p+p+1}\right)^{1/p}\left[|\hbar''(\nu)|+|\hbar''(\omega)|\right]. \tag{44}$$

(ii) If we take $\mu=1$ and $\varkappa=1$ in Theorem 13, we have

$$\left|\frac{1}{\omega-\nu}\int_{\nu}^{\omega}\hbar(x)dx-\hbar\left(\frac{\nu+\omega}{2}\right)\right|$$
$$\leq\frac{(\omega-\nu)^2}{4}\left(\frac{1}{2p+1}\right)^{1/p}\left[\left(\frac{3|\hbar''(\nu)|^q+|\hbar''(\omega)|^q}{4}\right)^{1/q}+\left(\frac{|\hbar''(\nu)|^q+3|\hbar''(\omega)|^q}{4}\right)^{1/q}\right]$$
$$\leq\frac{(\omega-\nu)^2}{4}\left(\frac{4}{2p+1}\right)^{1/p}\left[|\hbar''(\nu)|+|\hbar''(\omega)|\right]. \tag{45}$$

Theorem 15. *Let $\hbar:[\nu,\omega]\longrightarrow\mathbb{R}$ be a twice differentiable map on (ν,ω) s.t. $\hbar''\in L_1([\nu,\omega])$. Suppose that $|\hbar'|^q$ is convex on $[\nu,\omega]$ with $q\geq 1$. Then,*

$$\left|\frac{2^{\mu\varkappa-1}\mu^{\varkappa}\Gamma(\varkappa+1)}{(\omega-\nu)^{\mu\varkappa}}\left[{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2+}\hbar(\omega)+{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2-}\hbar(\nu)\right]-\hbar\left(\frac{\nu+\omega}{2}\right)\right|$$
$$\leq\frac{(\omega-\nu)^2\mu^{\varkappa}}{8}(Y_1(\mu,\varkappa))^{1-1/q}\times\left[\left(\frac{2Y_1(\mu,\varkappa)-Y_2(\mu,\varkappa)}{2}\cdot|\hbar''(\nu)|^q+\frac{Y_2(\mu,\varkappa)}{2}|\hbar''(\omega)|^q dt\right)^{1/q}+\left(\frac{Y_2(\mu,\varkappa)}{2}\cdot|\hbar''(\nu)|^q+\frac{2Y_1(\mu,\varkappa)-Y_2(\mu,\varkappa)}{2}|\hbar''(\omega)|^q dt\right)^{1/q}\right], \tag{46}$$

in which $\mathscr{B}$ depicts the beta function, and $Y_1(\mu,\varkappa)$ is defined as in (37). Here,

$$\begin{aligned}Y_2(\mu,\varkappa)&=\int_0^1\left|\int_0^t t\left[\frac{1-(1-s)^{\mu}}{\mu}\right]^{\varkappa}ds\right|dt\\&=\frac{1}{\mu^{\varkappa}}\int_0^1 t\left|\frac{1}{\mu}\mathscr{B}\left(\varkappa+1,\frac{1}{\mu},1-(1-t)^{\mu}\right)\right|dt.\end{aligned}\tag{47}$$

Proof. By utilizing the power-mean inequality in (38), it becomes

$$\begin{aligned}&\left|\frac{2^{\mu\varkappa-1}\mu^{\varkappa}\Gamma(\varkappa+1)}{(\omega-\nu)^{\mu\varkappa}}\left[{}^{\varkappa}\mathscr{J}^{\mu}_{\nu+\omega/2+}\hbar(\omega)+{}^{\varkappa}\mathscr{J}^{\mu}_{\nu+\omega/2-}\hbar(\nu)\right]-\hbar\left(\frac{a+\omega}{2}\right)\right|\\&\le\frac{(\omega-\nu)^2\mu^{\varkappa}}{8}\left[\left(\int_0^1\left|\int_0^t\left[\frac{1-(1-s)^{\mu}}{\mu}\right]^{\varkappa}ds\right|dt\right)^{1-1/q}\right.\\&\quad\times\left(\int_0^1\left|\int_0^t\left[\frac{1-(1-s)^{\mu}}{\mu}\right]^{\varkappa}ds\right|\left|\hbar''\left(\frac{2-t}{2}\nu+\frac{t}{2}\omega\right)\right|^q dt\right)^{1/q}\\&\quad+\left(\int_0^1\left|\int_0^t\left[\frac{1-(1-s)^{\mu}}{\mu}\right]^{\varkappa}ds\right|dt\right)^{1-1/q}\\&\quad\left.\times\left(\int_0^1\left|\int_0^t\left[\frac{1-(1-s)^{\mu}}{\mu}\right]^{\varkappa}ds\right|\left|\hbar''\left(\frac{2-t}{2}\nu+\frac{t}{2}\omega\right)\right|^q dt\right)^{1/q}\right].\end{aligned}\tag{48}$$

Due to the convexity of $|\hbar''|^q$ on $[\nu,b]$, we may write

$$\begin{aligned}&\left|\frac{2^{\mu\varkappa-1}\mu^{\varkappa}\Gamma(\varkappa+1)}{(\omega-\nu)^{\mu\varkappa}}\left[{}^{\varkappa}\mathscr{J}^{\mu}_{\nu+\omega/2+}\hbar(\omega)+{}^{\varkappa}\mathscr{J}^{\mu}_{\nu+\omega/2-}\hbar(\nu)\right]-\hbar\left(\frac{\nu+\omega}{2}\right)\right|\\&\le\frac{(\omega-\nu)^2\mu^{\varkappa}}{8}\left(\int_0^1\left|\int_0^t\left[\frac{1-(1-s)^{\mu}}{\mu}\right]^{\varkappa}ds\right|dt\right)^{1-1/q}\\&\quad\times\left[\left(\int_0^1\left|\int_0^t\left[\frac{1-(1-s)^{\mu}}{\mu}\right]^{\varkappa}ds\right|\frac{2-t}{2}\left|\hbar''(\nu)\right|^q+\frac{t}{2}\left|\hbar''(\omega)\right|^q dt\right)^{1/q}\right.\\&\quad\left.+\left(\int_0^1\left|\int_0^t\left[\frac{1-(1-s)^{\mu}}{\mu}\right]^{\varkappa}ds\right|\frac{t}{2}\left|\hbar''(\nu)\right|^q+\frac{2-t}{2}\left|\hbar''(\omega)\right|^q dt\right)^{1/q}\right].\end{aligned}\tag{49}$$

It is clearly seen that

$$\begin{aligned}&\left|\frac{2^{\mu\varkappa-1}\mu^{\varkappa}\Gamma(\varkappa+1)}{(\omega-\nu)^{\mu\varkappa}}\left[{}^{\varkappa}\mathscr{J}^{\mu}_{\nu+\omega/2+}\hbar(\omega)+{}^{\varkappa}\mathscr{J}^{\mu}_{\nu+\omega/2-}\hbar(\nu)\right]-\hbar\left(\frac{\nu+\omega}{2}\right)\right|\\&\le\frac{(\omega-\nu)^2\mu^{\varkappa}}{8}\left(Y_1(\mu,\varkappa)\right)^{1-1/q}\\&\quad\times\left[\left(\frac{2Y_1(\mu,\varkappa)-Y_2(\mu,\varkappa)}{2}\left|\hbar''(\nu)\right|^q+\frac{Y_2(\mu,\varkappa)}{2}\left|\hbar''(\omega)\right|^q dt\right)^{1/q}\right.\\&\quad\left.+\left(\frac{Y_2(\mu,\varkappa)}{2}\left|\hbar''(\nu)\right|^q+\frac{2Y_1(\mu,\varkappa)-Y_2(\mu,\varkappa)}{2}\left|\hbar''(\omega)\right|^q dt\right)^{1/q}\right].\end{aligned}\tag{50}$$

The proof is ended. □

Corollary 16. *In Theorem 15,*

(i) *if we set $\mu=1$, then we acquire*

$$\begin{aligned}&\left|\frac{2^{\varkappa-1}\Gamma(\varkappa+1)}{(\omega-\nu)^{\varkappa}}\left[J^{\varkappa}_{\nu+\omega/2+}\hbar(\omega)+J^{\varkappa}_{\nu+\omega/2-}\hbar(\nu)\right]-\hbar\left(\frac{\nu+\omega}{2}\right)\right|\\&\le\frac{(\omega-\nu)^2}{8}\left(\frac{1}{(\varkappa+1)(\varkappa+2)}\right)^{1-1/q}\\&\quad\times\left[\left(\frac{\varkappa+4}{2(\varkappa+1)(\varkappa+2)(\varkappa+3)}\left|\hbar''(\nu)\right|^q\right.\right.\\&\quad\left.+\frac{1}{2(\varkappa+1)(\varkappa+3)}\left|\hbar''(\omega)\right|^q dt\right)^{1/q}\\&\quad+\left(\frac{1}{2(\varkappa+1)(\varkappa+3)}\left|\hbar''(\nu)\right|^q\right.\\&\quad\left.\left.+\left(\frac{\varkappa+4}{2(\varkappa+1)(\varkappa+2)(\varkappa+3)}\right)\left|\hbar''(\omega)\right|^q dt\right)^{1/q}\right],\end{aligned}\tag{51}$$

(ii) *if we take $\mu=1$ and $\varkappa=1$, we obtain*

$$\begin{aligned}&\left|\frac{1}{\omega-\nu}\int_{\nu}^{\omega}\hbar(x)dx-\hbar\left(\frac{\nu+\omega}{2}\right)\right|\\&\le\frac{(\omega-\nu)^2}{48}\left[\left(\frac{5}{8}\left|\hbar''(\nu)\right|^q+\frac{3}{8}\left|\hbar''(\omega)\right|^q dt\right)^{1/q}\right.\\&\quad\left.+\left(\frac{3}{8}\left|\hbar''(\nu)\right|^q+\frac{5}{8}\left|\hbar''(\omega)\right|^q dt\right)^{1/q}\right].\end{aligned}\tag{52}$$

5. Conclusion

In this research, we established new estimates of trapezoid type and midpoint-type inequalities via conformable fractional integrals under twice differentiable functions. These inequalities were proven to be generalizations of the Riemann-Liouville fractional integrals related to inequalities of trapezoid type and midpoint type. In future works, researchers can obtain likewise inequalities of midpoint type and trapezoid type via conformable fractional integrals for convex functions in the context of quantum calculus. Moreover, curious readers can investigate our obtained inequalities via different kinds of fractional integrals.

Data Availability

No data were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

Conceptualization was performed by H.K. and H.B.; formal analysis was contributed by H.K., H.B., S.E., S.R., and

M.K.A.K.; methodology was performed by H.K., H.B., S.E., S.R., and M.K.A.K.; H.B. and S.E. were assigned for the software. All authors have read and agreed to the published version of the manuscript.

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