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Research Article

Coefficient Estimates for Certain Families of Analytic Functions Associated with Faber Polynomial

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In this paper, we use the Faber polynomial expansion to obtain bounds for the general coefficients $|a_n|$ of bi-univalent functions in the family of analytic functions in the open unit disk. Estimation of the bound value for the initial coefficients of the functions in these classes is also established.

1. Introduction

Let A denote the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1)$$

which are analytic in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Also, let S be the class of all functions in A which are univalent in $\mathbb{U}$. For $f \in A$, Airault and Bouali ([1], page 184) used Faber polynomial to show that

$$\frac{zf'(z)}{f(z)} = 1 - \sum_{j=2}^{\infty} F_{j-1}(a_2, a_3, \cdots, a_j) z^{j-1}, \quad (2)$$

where $F_{j-1}(a_2, a_3, \cdots, a_j)$ is the Faber polynomial defined by

$$F_{j-1}(a_2, a_3, \cdots, a_j) = \sum_{i_1 + 2i_2 + \cdots + (j-1)i_{j-1} = j-1} A(i_1, i_2, \cdots, i_{j-1}) a_2^{i_1} a_3^{i_2} \cdots, a_j^{i_{j-1}}, \quad (3)$$

$$A(i_1, i_2, \cdots, i_{j-1}) := (-1)^{(j-1)+2i_1+\cdots+ji_{j-1}} \frac{(i_1 + i_2 + \cdots + i_{j-1} - 1)!(j-1)}{(i_1)!(i_2)!\cdots(i_{j-1})!}. \quad (4)$$

The first terms of the Faber polynomial F_{j-1}, $j \geq 2$, are given by (e.g., see [2], page 52)

$$\begin{aligned}
F_1 &= -a_2, \\
F_2 &= a_2^2 - 2a_3, \\
F_3 &= -a_2^3 + 3a_2 a_3 - 3a_4, \\
F_4 &= a_2^4 - 4a_2^2 a_3 + 4a_2 a_4 + 2a_3^2 - 4a_5, \\
F_5 &= -a_2^5 + 5a_2^3 a_3 + 5a_2^2 a_4 - 5a_2(a_3^2 - a_5) + 5a_3 a_4 - 5a_6.
\end{aligned} \quad (5)$$

The Koebe one-quarter theorem ([3], page 31) ensured that the range of each function of the class S contains the disc $\{w : |w| < 1/4\}$. Therefore, the univalent function $f \in S$ has an inverse f^{-1}, which is defined by

$$\begin{aligned}
f^{-1}(f(z)) &= z \, (z \in \mathbb{U}), \\
f(f^{-1}(\omega)) &= \omega \left(|\omega| < \frac{1}{4} \right).
\end{aligned} \quad (6)$$

The inverse $g := f^{-1}$ of the function $f \in S$ has Taylor expansion given by (see [1], page 185)

$$g(\omega) = f^{-1}(\omega) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \cdots, a_n)\omega^n = w - a_2\omega^2 + \left(2a_2^2 - a_3\right)\omega^3 - \left(5a_2^2 - 5a_2a_3 + a_4\right)\omega^4 + \cdots, \tag{7}$$

where the coefficients $K_n^p(a_2, a_3, \cdots, a_n)$ are given by

$$\begin{aligned}
K_1^p &= pa_2, K_2^p = \frac{p(p-1)}{2}a_2^2 + pa_3,\\
K_3^p &= p(p-1)a_2a_3 + pa_4 + \frac{p(p-1)(p-2)}{3!}a_2^3,\\
K_4^p &= p(p-1)a_2a_4 + pa_5 + \frac{p(p-1)}{2}a_3^2 + \frac{p(p-1)(p-2)}{2}a_2^2a_3 + \frac{p!}{(p-4)!4!}a_2^4,\\
&\vdots\\
K_n^p &= \frac{p!}{(p-n)!n!}a_2^n + \frac{p!}{(p-n+1)!(n-2)!}a_2^{n-2}a_3 + \frac{p!}{(p-n+2)!(n-3)!}a_2^{n-3}a_4\\
&\quad + \frac{p!}{(p-n+3)!(n-4)!}a_2^{n-4}\left[a_5 + \frac{p-n+3}{2}a_3^2\right] + \frac{p!}{(p-n+4)!(n-5)!}a_2^{n-4}[a_6 + (p-n+3)a_3a_4] + \sum_{j\geq 6}^{\infty} a_2^{n-j}V_j,
\end{aligned} \tag{8}$$

and V_j is homogeneous polynomial of degree j in the variables $a_3, \cdots, a_n$ (see [4], page 349 and [1], pages 183 and 205).

Lemma 1 (Schwarz lemma ([3], page 3). *If $\omega(z)$ is analytic in the unit disc $\mathbb{U}$, with $\omega(0) = 0$ and $|\omega(z)| < 1$ in $\mathbb{U}$, then $|\omega(z)| < |z|$ and $|\omega'(0)| < 1$ in $\mathbb{U}$.*

Let f and g be analytic functions in $\mathbb{U}$; we say that the function f is subordinate to g, written as follows:

$$f \prec g \text{ or } f(z) \prec g(z), \tag{9}$$

if there exists a Schwarz function w, which (by definition) is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1 (z \in \mathbb{U})$, such that $f(z) = g(w(z))$ for all $z \in \mathbb{U}$. In particular, if the function $g \in S$ in $\mathbb{U}$, then we have the following equivalence relation (cf., e.g., [5, 6]; see also [3]):

$$f(z) \prec g(z) \Leftrightarrow f(0) \prec g(0), \quad f(\mathbb{U}) \subset g(\mathbb{U}). \tag{10}$$

Let ϕ be analytic function with positive real part in $\mathbb{U}$, satisfying $\phi(0) = 1, \phi'(0) > 0$. Also, let $\phi(\mathbb{U})$ be symmetric with respect to the real axis. Such a function has a Taylor series of the form

$$\phi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots (B_1 > 0). \tag{11}$$

Attiya [7] introduced the operator $H_{\alpha,\beta}^{\gamma,k}(f)$, where $H_{\alpha,\beta}^{\gamma,k}(f)\colon A \longrightarrow A$ is defined by

$$H_{\alpha,\beta}^{\gamma,k}(f) = \mu_{\alpha,\beta}^{\gamma,k} * f(z)(z \in \mathbb{U}), \tag{12}$$

with $\beta, \gamma \in \mathbb{C}$, $\mathrm{Re}\,(\alpha) > \max\{0, \mathrm{Re}\,(k) - 1\}$ and $\mathrm{Re}\,(k) > 0$. Also, $\mathrm{Re}\,(\alpha) = 0$ when $\mathrm{Re}\,(k) = 1$; $\beta \neq 0$. Here, $\mu_{\alpha,\beta}^{\gamma,k}$ is the generalized Mittag-Leffler function defined by [8] (see also [7]), and the symbol $(*)$ denotes the Hadamard product or convolution.

A detailed investigation of the Mittag-Leffler function has been studied by many authors (see, e.g., [8–12]).

Attiya [7] noted that

$$H_{\alpha,\beta}^{\gamma,k}(f)(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\gamma + nk)\Gamma(\alpha + \beta)}{\Gamma(\gamma + k)\Gamma(\beta + \alpha n)n!} a_n z^n. \tag{13}$$

From (13), it follows (see [7])

$$z\left(H_{\alpha,\beta}^{\gamma,k}(f)(z)\right)' = \left(\frac{\gamma + k}{k}\right)\left(H_{\alpha,\beta}^{\gamma+1,k}(f)(z)\right) - \frac{\gamma}{k}\left(H_{\alpha,\beta}^{\gamma,k}(f)(z)\right), \tag{14}$$

$$\alpha z\left(H_{\alpha,\beta+1}^{\gamma,k}(f)(z)\right)' = (\alpha + \beta)\left(H_{\alpha,\beta}^{\gamma,k}(f)(z)\right) - \beta\left(H_{\alpha,\beta+1}^{\gamma,k}(f)(z)\right). \tag{15}$$

Also, note the following:

(1) $H_{0,\beta}^{1,1}(f)(z) = f(z)$

(2) $H_{0,\beta}^{2,1}(f)(z) = (1/2)(f(z) + zf'(z))$

(3) $H_{0,\beta}^{0,1}(f)(z) = \int_0^z (1/t)f(t)\mathrm{d}t$

(4) $H_{1,0}^{1,1}(z/(1-z)) = \mathrm{z}e^z$

(5) $H_{1,1}^{1,1}(z/(1-z)) = e^z - 1$

(6) $H_{2,1}^{1,1}(z/(1-z)) = -2 + \cosh\left(\sqrt{z}\right)$

Definition 2. A function $f(z) \in A$ is said to be in the class $S(\alpha, \beta, \gamma, k, \phi)$ if it satisfies

$$\left(\frac{\gamma+k}{k}\right) \frac{H_{\alpha,\beta}^{\gamma+1,k}(f)(z)}{H_{\alpha,\beta}^{\gamma,k}(f)(z)} - \frac{\gamma}{k} \prec \phi(z), \tag{16}$$

where $\phi(z)$ is defined by (11).

Definition 3. A function $f(z) \in A$ is said to be in the class $T(\alpha, \beta, \gamma, k, \phi)$ if it satisfies

$$\left(\frac{\alpha+\beta}{\alpha}\right) \frac{H_{\alpha,\beta}^{\gamma,k}(f)(z)}{H_{\alpha,\beta+1}^{\gamma,k}(f)(z)} - \frac{\beta}{\alpha} \prec \phi(z), \tag{17}$$

where $\phi(z)$ is defined by (11).

With the virtue of (14) and (15), we find that $S(0, \beta, 1, 1, (1 + (1-2\tau)z)/(1-z))$ and $T(0, \beta, 1, 1, (1 + (1-2\tau)z)/(1-z))$ when $\tau \in 0, 1)$ is the well-known class of starlike function of order τ.

A single-valued function f analytic in a domain $\mathbb{D} \subset \mathbb{C}$ is said to be univalent, if it never takes the same value twice in $\mathbb{D}$, that is, if $f(z_1) \neq f(z_2)$ for all points z_1 and z_2 in $\mathbb{D}$ with $z_1 \neq z_2$ (see [3], page 26). Also, a function $f \in A$ is said to be bi-univalent in $\mathbb{U}$ if f and its inverse map f^{-1} are univalent in $\mathbb{U}$.

We denote by Σ the class of bi-univalent functions in $\mathbb{U}$ given by (1). The class of analytic bi-univalent functions was introduced and studied by Lewin [13] and showed that $|a_2| < 1.51$. Recently, many authors found nonsharp estimates on the first two Taylor–Maclaurin's coefficients $|a_2|$ and $|a_3|$.

The following are examples of bi-univalent functions in $\mathbb{U}$:

$$\begin{gathered} \frac{z}{1-z}, \\ -\log(1-z), \\ \frac{1}{2}\log\left(\frac{1-z}{1+z}\right). \end{gathered} \tag{18}$$

For various subclasses of bi-univalent functions, see, for example, [14–26].

Definition 4. A function $f \in \Sigma$ given by (1) is said to be in the class $S_\Sigma(\alpha, \beta, \gamma, k, \phi)$ if both f and its inverse map $g = f^{-1}$ are in $S(\alpha, \beta, \gamma, k, \phi)$. Also, a function $f \in \Sigma$ given by (1) is said to be in the class $T_\Sigma(\alpha, \beta, \gamma, k, \phi)$ if both f and its inverse map $g = f^{-1}$ are in $T(\alpha, \beta, \gamma, k, \phi)$.

In this paper, we use the Faber polynomial expansion to obtain bounds for the general coefficients $|a_n|$ of bi-univalent functions in $S_\Sigma(\alpha, \beta, \gamma, k, \phi)$ and $T_\Sigma(\alpha, \beta, \gamma, k, \phi)$ as well as we estimate the bounds of the initial coefficients of the functions in these classes.

Unless otherwise mentioned, we assume throughout this paper that $\beta, \gamma \in \mathbb{C}$, $\mathrm{Re}(\alpha) > \max\{0, \mathrm{Re}(k) - 1\}$ and $\mathrm{Re}(k) > 0$. Also, $\mathrm{Re}(\alpha) = 0$ when $\mathrm{Re}(k) = 1$; $\beta \neq 0$.

2. Coefficient Estimates of $S_\Sigma(\alpha, \beta, \gamma, k, \phi)$ and $T_\Sigma(\alpha, \beta, \gamma, k, \phi)$

Theorem 5. *Let the function $f \in \Sigma$ given by (1) be in the class $S_\Sigma(\alpha, \beta, \gamma, k, \phi)$. Also, let $a_m = 0$ and $a_p \neq 0$ for $2 \leq m, p \leq n$, where $(p-1)$ is a divisor of $(n-1)$. Then,*

$$|a_p| \leq \left|\frac{\Gamma(\gamma+pk)\Gamma(\alpha+\beta)}{\Gamma(\gamma+k)\Gamma(\beta+\alpha p)}\right| \frac{1}{p!} \left(\frac{B_1}{p-1}\right)^{((p-1)/(n-1))}, p \geq 3, \tag{19}$$

where B_1 is defined in (11).

Proof. If we set $F(z) = H_{\alpha,\beta}^{\gamma,k}(f)(z)$, then $F(z) = z + \sum_{n=2}^{\infty} \delta_n z^n$, with $\delta_n = ((\Gamma(\gamma+nk)\Gamma(\alpha+\beta))/(\Gamma(\gamma+k)\Gamma(\beta+\alpha n)n!))a_n$; using relation (14), we have

$$f \in S_\Sigma(\alpha, \beta, \gamma, k, \phi) \text{ if and only if } \frac{zF'(z)}{F(z)} \prec \phi(z). \tag{20}$$

Since f and its inverse $g = f^{-1}$ are in $S_\Sigma(\alpha, \beta, \gamma, k, \phi)$, $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, therefore, there are analytic functions u and v with $u(0) = v(0) = 0$, $|u(z)| < 1$ and $|v(z)| < 1$, such that

$$\begin{aligned} \frac{zF'(z)}{F(z)} &= \phi(u(z)) \ (z \in \mathbb{U}), \\ \frac{wG'(w)}{G(w)} &= \phi(v(w)) \ (z \in \mathbb{U}), \end{aligned} \tag{21}$$

where $G(w) = H_{\alpha,\beta}^{\gamma,k}(g)(w)$; then, $G(w) = w + \sum_{n=2}^{\infty} \zeta_n w^n$, with $\zeta_n = ((\Gamma(\gamma+nk)\Gamma(\alpha+\beta))/(\Gamma(\gamma+k)\Gamma(\beta+\alpha n)n!))b_n$; using (7), we have $b_n = (1/n)K_{n-1}^{-n}(a_2, a_3, \cdots, a_n)$.

The functions $u(z)$ and $v(z)$ are defined by

$$\begin{aligned} u(z) &= \sum_{n=1}^{\infty} c_n z^n, \\ v(z) &= \sum_{n=1}^{\infty} d_n z^n (z \in \mathbb{U}). \end{aligned} \tag{22}$$

It is well known that (see Duren [3], page 265)

$$\begin{gathered} |c_n| \leq 1, \\ |d_n| \leq 1, \\ n = 2, 3, \cdots. \end{gathered} \tag{23}$$

Also, we have

$$\phi(u(z)) = 1 - B_1 \sum_{n=1}^{\infty} K_n^{-1}\left(c_1,-c_2,\cdots,(-1)^{n+1}c_n,B_1,B_2,\cdots,B_n\right) \cdot z^n (z \in \mathbb{U}), \tag{24}$$

$$\phi(v(\omega)) = 1 - B_1 \sum_{n=1}^{\infty} K_n^{-1}\left(d_1,-d_2,\cdots,(-1)^{n+1} \cdot d_n,B_1,B_2,\cdots,B_n\right) w^n (z \in \mathbb{U}). \tag{25}$$

In general (see [21], page 649), the coefficients $K_n^p := \mathscr{K}_n^p(k_1,k_2,\cdots,k_n,B_1,B_2,B_3,\cdots,B_n)$ are given by

$$\begin{aligned}\mathscr{K}_n^p = {} & \frac{p!}{(p-n)!n!}k_1^n\frac{B_n}{B_1} + \frac{p!}{(p-n+1)!(n-2)!}k_1^{n-2}k_2\frac{B_{n-1}}{B_1} \\ & + \frac{p!}{(p-n+2)!(n-3)!}k_1^{n-3}k_3\frac{B_{n-2}}{B_1} \\ & + \frac{p!}{(p-n+3)!(n-4)!}k_1^{n-4}\left[k_4\frac{B_{n-3}}{B_1} + \frac{p-n+3}{2}k_2^2k_3\frac{B_{n-2}}{B_1}\right] \\ & + \sum_{j\geq 5}^{\infty} k_1^{n-j}X_j,\end{aligned} \tag{26}$$

where X_j is a homogeneous polynomial of degree j in the variables $k_2,\cdots,k_n$.

Using the Faber polynomial expansion, (2) yields the following identities:

$$\frac{zF'(z)}{F(z)} = 1 - \sum_{n=2}^{\infty} F_{n-1}(\delta_2,\delta_3,\cdots,\delta_n)z^{n-1}, \tag{27}$$

$$\frac{wG'(w)}{G(w)} = 1 - \sum_{n=2}^{\infty} F_{n-1}(\zeta_2,\zeta_3,\cdots,\zeta_n)w^{n-1}. \tag{28}$$

Comparing the corresponding coefficients of (27) and (24) yields

$$\begin{aligned}&F_{n-1}(\delta_2,\delta_3,\cdots,\delta_n) \\ &\quad = B_1K_{n-1}^{-1}(c_1,-c_2,\cdots,(-1)^n c_{n-1},B_1,B_2,B_3,\cdots,B_{n-1}),\end{aligned} \tag{29}$$

and similarly, from (28) and (25), we have

$$\begin{aligned}&F_{n-1}(\zeta_2,\zeta_3,\cdots,\zeta_n) \\ &\quad = B_1K_{n-1}^{-1}(d_1,-d_2,\cdots,(-1)^n d_{n-1},B_1,B_2,B_3,\cdots,B_{n-1}).\end{aligned} \tag{30}$$

Since $a_m = 0$ and $a_p \neq 0$ for $2 \leq m, p \leq n$, and $(p-1)$ is a divisor of $(n-1)$, then by using (29) and (30), we have

$$(p-1)(-1)^{((n-1)/(p-1))}\left(\delta_p\right)^{((n-1)/(p-1))} = -B_1(-1)^p c_{p-1}, \tag{31}$$

$$(p-1)(-1)^{((n-1)/(p-1))}\left(\zeta_p\right)^{((n-1)/(p-1))} = -B_1(-1)^p d_{p-1}, \tag{32}$$

also, under the condition $a_m = 0$ and $a_p \neq 0$ for $2 \leq m, p \leq n$ and using the relation between δ_p and ζ_p, we have $\zeta_p = -\delta_p$. Then, (31) yields

$$(p-1)(-1)^{((n-1)/(p-1))}\left(-\delta_p\right)^{((n-1)/(p-1))} = -B_1(-1)^p d_{p-1}. \tag{33}$$

Using either (31) or (33), we have $|\delta_p| \leq (B_1/(p-1))^{((p-1)/(n-1))}$, substitute of $\delta_p = ((\Gamma(\gamma+pk)\Gamma(\alpha+\beta))/(\Gamma(\gamma+k)\Gamma(\beta+\alpha p)p!))a_p$. This completes the proof of the theorem. □

Putting $p = n$ in Theorem 5, we have the following:

Corollary 6. *Let the function $f \in \Sigma$ given by (1) be in the class $S_\Sigma(\alpha,\beta,\gamma,k,\phi)$; if $a_m = 0$ for $2 \leq m \leq n-1$, then*

$$|a_n| \leq \left|\frac{\Gamma(\gamma+nk)\Gamma(\alpha+\beta)}{\Gamma(\gamma+k)\Gamma(\beta+\alpha n)}\right|\frac{1}{n!}\left(\frac{B_1}{n-1}\right), n \geq 3, \tag{34}$$

where B_1 is defined in (11).

Using the same technique used in Theorem 5, we get the following theorem:

Theorem 7. *Let the function $f \in \Sigma$ given by (1) be in the class $T_\Sigma(\alpha,\beta,\gamma,k,\phi)$. Also, let $a_m = 0$ and $a_p \neq 0$ for $2 \leq m, p \leq n$, where $(p-1)$ is a divisor of $(n-1)$. Then,*

$$|a_p| \leq \left|\frac{\Gamma(\gamma+pk)\Gamma(\alpha+\beta+1)}{\Gamma(\gamma+k)\Gamma(\beta+1+\alpha p)}\right|\frac{1}{p!}\left(\frac{B_1}{p-1}\right)^{((p-1)/(n-1))}, p \geq 3, \tag{35}$$

where B_1 is defined in (11).

Putting $p = n$ in Theorem 7, we have the following:

Corollary 8. *Let the function $f \in \Sigma$ given by (1) be in the class $T_\Sigma(\alpha,\beta,\gamma,k,\phi)$; if $a_m = 0$ for $2 \leq m \leq n-1$, then*

$$|a_n| \leq \left|\frac{\Gamma(\gamma+nk)\Gamma(\alpha+\beta+1)}{\Gamma(\gamma+k)\Gamma(\beta+1+\alpha n)}\right|\frac{1}{n!}\left(\frac{B_1}{n-1}\right), n \geq 3, \tag{36}$$

where B_1 is defined in (11).

To prove our next theorem, we shall need the following lemma.

Lemma 9 [3, 21]. *Let the function $\Phi(z)=\sum_{n=1}^{\infty}\Phi_n z^n$ be a Schwarz function with $|\Phi(z)|<1$, $z\in\mathbb{U}$. Then for $-\infty<\eta<\infty$,*

$$\left|\Phi_2+\eta\Phi_1^2\right| \le \begin{cases} 1-(1-\eta)\left|\Phi_1^2\right| & \eta>0,\\ 1-(1+\eta)\left|\Phi_1^2\right| & \eta\le 0.\end{cases} \tag{37}$$

Theorem 10. *Let the function $f\in\Sigma$ given by (1) be in the class $S_\Sigma(\alpha,\beta,\gamma,k,\phi)$. Then,*

$$|a_2| \le \begin{cases} \dfrac{2\sqrt{B_1}\,|\Gamma(\gamma+k)\Gamma(\beta+2\alpha)|}{|\Gamma(\gamma+2k)\Gamma(\alpha+\beta)|} & ((B_1\ge |B_2|)),\\ \dfrac{2\sqrt{|B_2|}\,|\Gamma(\gamma+k)\Gamma(\beta+2\alpha)|}{|\Gamma(\gamma+2k)\Gamma(\alpha+\beta)|} & (B_1<|B_2|),\end{cases} \tag{38}$$

$$\left|a_3-a_2^2\right| \le \begin{cases} B1\left(\left|\dfrac{3}{4R_3}-\dfrac{1}{2R_2^2}\right|+\left|\dfrac{1}{4R_3}-\dfrac{1}{2R_2^2}\right|\right) & (B_1\ge |B_2|),\\ |B2|\left(\left|\dfrac{3}{4R_3}-\dfrac{1}{2R_2^2}\right|+\left|\dfrac{1}{4R_3}-\dfrac{1}{2R_2^2}\right|\right) & (B_1<|B_2|),\end{cases} \tag{39}$$

where $R_2=1/2|(\Gamma(\gamma+2k)\Gamma(\alpha+\beta))/(\Gamma(\gamma+k)\Gamma(\beta+2\alpha))|$ and $R_3=1/6|(\Gamma(\gamma+3k)\Gamma(\alpha+\beta))/(\Gamma(\gamma+k)\Gamma(\beta+3\alpha))|$.

Proof. Putting $n=2$ and $n=3$ in (29) and (30), respectively, we find that

$$\begin{aligned}\delta_2&=c_1B_1,\\ -\zeta_2&=d_1B_1,\end{aligned} \tag{40}$$

$$\delta_2^2-2\delta_3=c_1^2B_2+c_2B_1, \tag{41}$$

$$\zeta_2^2-2\zeta_3=d_1^2B_2+d_2B_1. \tag{42}$$

Moreover, we have $\zeta_2=-\delta_2$ and $\zeta_3=2\delta_2^2-\delta_2$. Therefore, (41) and (42) imply

$$a_2^2=\frac{-B_1}{2R_2^2}\left(\left(c_2+\frac{B_2}{B_1}c_1^2\right)+\left(d_2+\frac{B_2}{B_1}d_1^2\right)\right). \tag{43}$$

Apply Lemma 9.

Case I: if $B_2>0$, then both $|c_2+(B_2/B_1)c_1^2|$ and $|d_2+(B_2/B_1)d_1^2|$ have maximum value at 1 when $B_1>B_2$. Also, $|c_2+(B_2/B_1)c_1^2|$ and $|d_2+(B_2/B_1)d_1^2|$ have maximum value at B_2/B_1 when $B_1\le B_2$, where $|c_1|\le 1$ and $|d_1|\le 1$

Case II: if $B_2\le 0$, then both $|c_2+(B_2/B_1)c_1^2|$ and $|d_2+(B_2/B_1)d_1^2|$ have maximum value at 1 when $B_1>-B_2$ and $|c_2+(B_2/B_1)c_1^2|$ and $|d_2+(B_2/B_1)d_1^2|$ have maximum value at $-B_2/B_1$ when $B_1\le -B_2$. Then, we have (38)

Moreover, from (41) and (42), we get

$$a_3=-\left(\frac{3c_2+d_2}{4R_3}\right)B_1-\left(\frac{3c_1^2+d_1^2}{4R_3}\right)B_2. \tag{44}$$

It follows from (43) and (44) that

$$\begin{aligned}a_3-a_2^2=&-\left(\frac{3}{4R_3}-\frac{1}{2R_2^2}\right)\left(c_2+\frac{B_2}{B_1}c_1^2\right)B_1\\ &-\left(\frac{1}{4R_3}-\frac{1}{2R_2^2}\right)\left(d_2+\frac{B_2}{B_1}d_1^2\right)B_1.\end{aligned} \tag{45}$$

Applying the cases mentioned above for $B_2>0$ and $B_2\le 0$, we have (39). This completes the proof of the theorem. □

Using the same technique used in Theorem 10, we get the following theorem:

Theorem 11. *Let the function $f\in\Sigma$ given by (1) be in the class $T_\Sigma(\alpha,\beta,\gamma,k,\phi)$. Then,*

$$|a_2| \le \begin{cases} \dfrac{2\sqrt{B_1}\,|\Gamma(\gamma+k)\Gamma(\beta+1+2\alpha)|}{|\Gamma(\gamma+2k)\Gamma(\alpha+\beta+1)|} & ((B_1\ge |B_2|)),\\ \dfrac{2\sqrt{|B_2|}\,|\Gamma(\gamma+k)\Gamma(\beta+1+2\alpha)|}{|\Gamma(\gamma+2k)\Gamma(\alpha+\beta+1)|} & (B_1<|B_2|),\end{cases} \tag{46}$$

$$\left|a_3-a_2^2\right| \le \begin{cases} B1\left(\left|\dfrac{3}{4R_3}-\dfrac{1}{2R_2^2}\right|+\left|\dfrac{1}{4R_3}-\dfrac{1}{2R_2^2}\right|\right) & (B_1\ge |B_2|),\\ |B2|\left(\left|\dfrac{3}{4R_3}-\dfrac{1}{2R_2^2}\right|+\left|\dfrac{1}{4R_3}-\dfrac{1}{2R_2^2}\right|\right) & (B_1<|B_2|),\end{cases} \tag{47}$$

where $R_2=1/2|(\Gamma(\gamma+2k)\Gamma(\alpha+\beta+1))/(\Gamma(\gamma+k)\Gamma(\beta+1+2\alpha))|$ and $R_3=1/6|(\Gamma(\gamma+3k)\Gamma(\alpha+\beta+1))/(\Gamma(\gamma+k)\Gamma(\beta+1+3\alpha))|$.

3. Conclusion

By using the Faber polynomial expansion, we obtain bounds for the general coefficients $|a_n|$ of bi-univalent functions for functions in the classes $S_\Sigma(\alpha,\beta,\gamma,k,\phi)$ and $T_\Sigma(\alpha,\beta,\gamma,k,\phi)$ in $\mathbb{U}=\{z: z\in\mathbb{C}\,\text{and}\,|z|<1\}$; also, estimation of the bound value for the initial coefficients of the functions in the classes $S_\Sigma(\alpha,\beta,\gamma,k,\phi)$ and $T_\Sigma(\alpha,\beta,\gamma,k,\phi)$ is established.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

The authors contributed equally to the writing of this paper. All authors approved the final version of the manuscript.

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