Research Article

Some Interesting Inequalities for the Class of Generalized Convex Functions of Higher Order

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In this paper, we study a generalized version of strongly reciprocally convex functions of higher order. Firstly, we prove some basic properties for addition, scalar multiplication, and composition of functions. Secondly, we establish Hermite-Hadamard and Fejér type inequalities for the generalized version of strongly reciprocally convex functions of higher order. We also include some fractional integral inequalities concerning with this class of functions. Our results have applications in optimization theory and can be considered extension/generalization of many existing results.

1. Introduction

Convexity is a very simple and ordinary concept. Due to its massive applications in industry and business, convexity has a great influence on our daily life. In the solution of many real-world problems, the concept of convexity is very decisive. Problems faced in constrained control and estimation are convex. Geometrically, a real-valued function is said to be convex if the line segment joining any two of its points lies on or above the graph of the function in Euclidean space.

Convexity of a function in classical sense is defined as a function \( f_1 : M \rightarrow \mathbb{R} \) is convex if we have

\[
f_1(jx + (1 - j)y) \leq j f_1(x) + (1 - j)f_1(y), \forall j \in [0, 1]. \tag{1}
\]

If the above inequality is reversed, then the function is said to be concave.

Using different techniques, the notion of convexity is being extended day by day [1–3]. Many extensions and generalizations are made speedily due to its applications in modern engineering, optimization, economics, and nonlinear programming [4–7]. For recent generalizations, one can see [8, 9] and the references therein.

In this paper, we study a generalized version of strongly reciprocally convex functions of higher order. Firstly, we prove some basic properties for addition, scalar multiplication, and composition of functions. Secondly, we establish Hermite-Hadamard and Fejér type inequalities for the generalized version of strongly reciprocally convex functions of higher order. We also include some fractional integral inequalities concerning with this class of functions. Our results have applications in optimization theory and can be considered extension/generalization of many existing results.

In [9], using the weight function \( w(x) \), Fejér gave a generalization of the Hermite-Hadamard inequality as follows:

Let \( f_1 : [a_1, b_1] \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a convex function and \( w : [a_1, b_1] \rightarrow \mathbb{R} \) is nonnegative, integrable, and symmetric about \((a_1 + b_1)/2\), then we have

\[
f_1 \left( \frac{a_1 + b_1}{2} \right) \int_{a_1}^{b_1} w(x)dx \leq \int_{a_1}^{b_1} f_1(x)w(x)dx \leq \frac{f_1(a_1) + f_1(b_1)}{2} \int_{a_1}^{b_1} w(x)dx. \tag{3}
\]
In [10], the notations of \( p \)-convex set and \( p \)-convex functions are introduced. The strongly convex functions of modulus \( \mu \) are introduced in [11]. In [12, 13], the \( p \)-convex and harmonic convex functions were introduced, respectively. The \( p \)-harmonic convex set and \( p \)-harmonic convex functions were studied in [14], and in [15], the strongly reciprocally convex functions of modulus \( \mu \) are introduced. The strongly reciprocally \( p \)-convex and \( h \)-convex functions were introduced in [16, 17], respectively. The \(( p,h)\)-convex functions are introduced in [18], and the higher-order strongly convex with modulus \( \mu \) are introduced in [19]. Now, we present the notation of strongly reciprocally \(( p,h)\)-convex functions of higher order (SRHO).

**Definition 1.** Let \( \mu \in (0,\infty) \) and \( M \) is any interval. Then the function \( f_i : M = [a_i, b_i] \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \) is SRHO of modulus \( \mu \) on the interval \( M \), if we have

\[
\left( \frac{x^p y^p}{|x^p + (1-j)y^p|} \right)^{1/p} \leq h(1-j)f_i(x) + h(j)f_i(y) - \mu \phi(j) \left| \frac{1}{y} - \frac{1}{x} \right|,
\]

for all \( x, y \in M \), \( j \in [0,1] \), and \( l \geq 1 \), where \( \phi(j) = j(1-j) \).

**Remark 2.** Inserting \( l = 2 \) in Def. 1 with same \( \phi(j) \) as defined above, we obtain strongly reciprocally \(( p,h)\)-convex functions. Similarly, inserting \( l = 2 \) and \( h(j) = j \) in Def. 1, we obtain strongly reciprocally \( p \)-convex functions, and for \( l = 2 \), \( h(j) = j \), and \( p = 1 \), Def. 1 reduces to the strongly reciprocally convex function of modulus \( \mu \).

As we know that \( \mathbb{R} \) is a Norm space under the usual modulus norm, thus, for any \( x \in \mathbb{R} \),

\[
||x|| = |x|.
\]

Using (5), the inequality 1 can be written as

\[
f_i \left( \frac{x^p y^p}{|x^p + (1-j)y^p|} \right)^{1/p} \leq h(1-j)f_i(x) + h(j)f_i(y) - \mu \phi(j) \left| \frac{1}{y} - \frac{1}{x} \right|,
\]

\forall x, y \in M \text{ and } j \in [0,1] \text{ with } l \geq 1 \text{, where } \phi(j) \text{ is same as in Definition 1.}

The aim of this paper is to study a generalized version of strongly reciprocally convex functions of higher order and establish the Hermite-Hadamard and Fejér type inequalities for this new class of convex functions. We also presented fractional versions of the above mentioned inequalities for the strongly reciprocally \(( p,h)\)-convex of higher order. It is worthy to mention here that the results presented in this paper are more generalized and can be considered extensions of many existing results.

### 2. Basic Results

Now, we present some basic properties for strongly reciprocally \(( p,h)\)-convex of higher order.

**Proposition 3.** For any two SRHO \( f_1, g_1 : M \rightarrow \mathbb{R} \) with modulus \( \mu \) on the interval \( M \), the \( f_1 + g_1 : M \rightarrow \mathbb{R} \) is also SRHO with modulus \( \mu^* \) on the interval \( M \), where \( 1/2\mu^* = \mu \).

**Proof.** By definition, we have

\[
f_1 + g_1 \left[ \frac{x^p y^p}{|x^p + (1-j)y^p|} \right]^{1/p} = f_1 \left[ \frac{x^p y^p}{|x^p + (1-j)y^p|} \right]^{1/p} + g_1 \left[ \frac{x^p y^p}{|x^p + (1-j)y^p|} \right]^{1/p} \leq h(j)f_1(x) + h(1-j)f_1(y) + h(j)g_1(x) + h(1-j)g_1(y) - \mu \phi(j) \left| \frac{1}{y} - \frac{1}{x} \right|,
\]

which in turns implies that

\[
h(j)(f_1 + g_1)(x) + h(1-j)(f_1 + g_1)(y) - 2\mu j(1-j) \left| \frac{1}{y} - \frac{1}{x} \right| = h(j)(f_1 + g_1)(x) + h(1-j)(f_1 + g_1)(y) - \mu^* \phi(j) \left| \frac{1}{y} - \frac{1}{x} \right|,
\]

where \( \mu^* = 2\mu, \mu \geq 0 \) and \( \phi(j) = j(1-j) \).

This completes the proof.

**Proposition 4.** For any SRHO \( f_1 : M \rightarrow \mathbb{R} \) with modulus \( \mu \geq 0 \) and any \( \lambda \geq 0, \lambda f_1 \) is also SRHO with modulus \( \nu^* \) on the interval \( M \), where \( 1/\lambda \nu^* = \mu \).

**Proof.** Let \( \lambda \geq 0 \), then by definition of \( f_1 \), we obtain

\[
\lambda f_1 \left[ \frac{x^p y^p}{|x^p + (1-j)y^p|} \right]^{1/p} = \lambda f_1 \left[ \frac{x^p y^p}{|x^p + (1-j)y^p|} \right]^{1/p} \leq \lambda h(j)f_1(x) + h(1-j)f_1(y) - \lambda \mu \phi(j) \left| \frac{1}{y} - \frac{1}{x} \right|,
\]

where \( \nu^* = \lambda \mu, \mu \geq 0 \) and \( \phi(j) = j(1-j) \). This completes the proof.

**Proposition 5.** Consider a sequence of SRHO, \( f_{i} \), are defined on an interval \( M \), provide \( 1 \leq i \leq n \), then for positive constants \( \lambda_i \), the function \( \sum_{i=1}^{n} \lambda_i f_i \) is SRHO with nonnegative modulus \( \gamma \sum_{i=1}^{n} \lambda_i \mu \).
Proof. For a $p$-harmonic convex set $M$, we have

$$f_1 \left[ \frac{2 \lambda_i b_i^j}{a_i^j + b_i^j} \right]^{1/p} \leq \frac{1}{2h(1/2)} \left[ f_1 \left( \frac{2 \lambda_i b_i^j}{a_i^j + b_i^j} \right) \right]^{1/p} + \mu \frac{1}{2} \left( \frac{b_i^j - a_i^j}{a_i^j b_i^j} \right) \left\{ \frac{1 - (-1)^{2r+1}}{2(l+1)} \right\}$$

and integrating (13)

$$f_1 \left[ \frac{2 \lambda_i b_i^j}{a_i^j + b_i^j} \right]^{1/p} \leq h \left( \frac{1}{2} \right) f_1 \left[ \frac{a_i^j b_i^j}{a_i^j + (1 - j) b_i^j} \right]^{1/p}$$

and $y = \frac{a_i^j b_i^j}{a_i^j + (1 - j) b_i^j}$ and integrating (13)

$$f_1 \left[ \frac{2 \lambda_i b_i^j}{a_i^j + b_i^j} \right]^{1/p} \leq h \left( \frac{1}{2} \right) f_1 \left[ \frac{a_i^j b_i^j}{a_i^j + (1 - j) b_i^j} \right]^{1/p}$$

This completes the proof.

3. Hermite-Hadamard Type Inequality

In this section, we establish Hermite-Hadamard’s type inequality for the function belonging to $SR(p)h$.

Theorem 7. Consider an interval $M$ not containing zero and $SRH f_i : M \rightarrow \mathbb{R}$ of nonnegative modulus $\mu$ and $f_1 \in L^p[a_i, b_i]$, then for $h(1/2) \neq 0, \mu$ we have

$$f_1 \left[ \frac{2 \lambda_i b_i^j}{a_i^j + b_i^j} \right]^{1/p} \leq 2h \left( \frac{1}{2} \right) \frac{p(a_i^j b_i^j)}{a_i^j b_i^j} \int_{a_i}^{b_i} \frac{f_1(x)}{a_i^j b_i^j} dx$$
which is left side of the inequality (12).

Finally, for the right side of the inequality (12), setting \( x = a_1 \) and \( y = b_1 \) in Definition 1 gives

\[
\int_{a_1}^{b_1} \left[ \frac{a_1^r b_1^r}{d_1^r + (1-t)b_1^r} \right]^{1/p} \ dx \\
\leq h(1 - j)f_1(a_1) + h(j)f_1(b_1) \\
- \mu \phi(j) \left[ \frac{1}{a_1^r} - \frac{1}{b_1^r} \right].
\]

Integrating (15)

\[
\int_{0}^{1} \left[ \frac{a_1^r b_1^r}{d_1^r + (1-t)b_1^r} \right]^{1/p} \ dx \\
\leq \int_{0}^{1} h(1 - j)f_1(a_1) \ dx \\
+ \int_{0}^{1} h(j)f_1(b_1) \ dx - \mu \phi(j) \int_{0}^{1} \left[ \frac{1}{a_1^r} - \frac{1}{b_1^r} \right] \ dx.
\]

\[
P \left( \frac{a_1^r b_1^r}{b_1^r - a_1^r} \right) \int_{a_1}^{b_1} f_1(x) \ dx \\
\leq \int_{0}^{1} h(1 - j)f_1(a_1) + h(j)f_1(b_1) \ dx \\
- \mu \int_{0}^{1} \left[ \frac{b_1^r - a_1^r}{a_1^r b_1^r} \right] \ dx.
\]

that is right hand side of (12) and proof is completed. \( \square \)

4. Fejér Type Inequality

Now, we are going to develop the Fejér type inequality for the function belonging to \( SR(\phi(h)) \).

**Theorem 8.** Consider an interval \( M \) not containing zero and real-valued SRHO \( f_1 \) defined on \( M \) of nonnegative modulus \( \phi \), then for \( h(1/2) \neq 0 \), we have

\[
\frac{1}{2h(1/2)} \left[ f_1 \left( \frac{2a_1^r b_1^r}{d_1^r + b_1^r} \right)^{1/p} + \mu \phi \left( \frac{1}{2} \right) \left| \frac{b_1^r - a_1^r}{d_1^r b_1^r} \right| \frac{1 - (-1)^{2l+1}}{2l+1} \right] \\
\leq \int_{a_1}^{b_1} f_1(x) \ dx + \frac{\mu}{|a_1^r b_1^r|} \phi \left( \frac{1}{2} \right) \left| b_1^r - a_1^r \right| \phi \left( \frac{1}{2} \right) \frac{|x|^p}{|x|^{N-p}} \ dx
\]

\[
\leq \int_{a_1}^{b_1} f_1(x) \ dx \\
\leq \left[ f_1(a_1) + f_1(b_1) \right] \\
- \mu \left[ \frac{b_1^r - a_1^r}{a_1^r b_1^r} \right] \int_{a_1}^{b_1} \phi \left( \frac{1}{2} \right) \left| b_1^r - a_1^r \right| \ dx.
\]

holds for \( a_1, b_1 \in M \) with \( a_1 \leq b_1 \) and \( f_1 \in L[a_1, b_1] \), where the nonnegative real-valued function \( w \) defined on \( M \) satisfies

\[
w \left( \frac{a_1^r b_1^r}{x^p} \right)^{1/p} = w \left( \frac{a_1^r b_1^r}{a_1^r + b_1^r - x^p} \right)^{1/p}.
\]

Proof. Substituting \( j = 1/2 \) in Definition 1, yields

\[
f_1 \left( \frac{2a_1^r b_1^r}{x^p + y^p} \right)^{1/p} \leq h \left( \frac{1}{2} \right) f_1(x) + h \left( \frac{1}{2} \right) f_1(y) - \mu \phi \left( \frac{1}{2} \right) \left| \frac{1}{x^p} - \frac{1}{y^p} \right|.
\]

Considering \( x = [(a_1^r b_1^r/jd_1^r + (1-j)b_1^r)^{1/p}] \) and \( y = [(a_1^r b_1^r/jb_1^r + (1-j)a_1^r)^{1/p}] \) and integrating (19),

\[
f_1 \left( \frac{2a_1^r b_1^r}{a_1^r + b_1^r} \right)^{1/p} \leq h \left( \frac{1}{2} \right) f_1 \left( \frac{a_1^r b_1^r}{jd_1^r + (1-j)b_1^r} \right)^{1/p} \\
+ h \left( \frac{1}{2} \right) f_1 \left( \frac{a_1^r b_1^r}{jb_1^r + (1-j)a_1^r} \right)^{1/p} \\
- \mu \phi \left( \frac{1}{2} \right) \left| \frac{1}{jdb_1^r + (1-j)b_1^r} - \frac{1}{jba_1^r + (1-j)a_1^r} \right|.
\]

By the properties of \( w \),

\[
f_1 \left( \frac{2a_1^r b_1^r}{a_1^r + b_1^r} \right)^{1/p} w \left[ \left( \frac{a_1^r b_1^r}{jd_1^r + (1-j)b_1^r} \right)^{1/p} \right] \\
\leq h \left( \frac{1}{2} \right) f_1 \left( \frac{a_1^r b_1^r}{jd_1^r + (1-j)b_1^r} \right)^{1/p} w \left( \frac{a_1^r b_1^r}{jd_1^r + (1-j)b_1^r} \right)^{1/p} \\
+ h \left( \frac{1}{2} \right) f_1 \left( \frac{a_1^r b_1^r}{jb_1^r + (1-j)a_1^r} \right)^{1/p} w \left( \frac{a_1^r b_1^r}{jb_1^r + (1-j)a_1^r} \right)^{1/p}.
\]
Integrating inequality (21),

\[
\int_0^1 f_i \left( \frac{2 a_i b_i'}{a_i' + b_i'} \right)^{1/p} \, w \left( \frac{a_i b_i'}{a_i' + (1 - j) b_i'} \right) \, \frac{1}{a_i'} \, dx \\
\leq \frac{h}{2} \int_0^1 f_i \left( \frac{2 a_i b_i'}{a_i' + b_i'} \right)^{1/p} \, w \left( \frac{a_i b_i'}{a_i' + (1 - j) b_i'} \right) \, \frac{1}{a_i'} \, dx \\
+ \frac{\mu}{|a_i'|} \phi \left( \frac{1}{5} \right) \int_0^1 f_i \left( \frac{2 a_i b_i'}{a_i' + b_i'} \right)^{1/p} \, w \left( \frac{a_i b_i'}{a_i' + (1 - j) b_i'} \right) \, \frac{1}{a_i'} \, dx \\
\leq \frac{h}{2} \int_0^1 f_i (w(x)) \, \frac{1}{a_i'} \, dx + \frac{\mu}{|a_i'|} \phi \left( \frac{1}{5} \right) \int_0^1 f_i (w(x)) \, \frac{1}{a_i'} \, dx \\
\leq \frac{h}{2} \int_0^1 f_i (w(x)) \, \frac{1}{a_i'} \, dx + \frac{\mu}{|a_i'|} \phi \left( \frac{1}{5} \right) \int_0^1 f_i (w(x)) \, \frac{1}{a_i'} \, dx
\]

(23)

which is left side of the inequality (17).

Finally, for the right side of the inequality (17), setting \( x = a_i \) in Definition 1 gives

\[
\left( 1 - \frac{1}{j} \right) f_i \left( \frac{a_i b_i'}{a_i' + (1 - j) b_i'} \right)^{1/p} \leq h(1 - j) f_i (a_i) + h(j) f_i (b_i) - \mu \phi(j) \left( \frac{1}{a_i'} - \frac{1}{b_i'} \right).
\]

(24)

By the properties of \( w \),

\[
f_i \left( \frac{a_i b_i'}{a_i' + (1 - j) b_i'} \right)^{1/p} \, w \left( \frac{a_i b_i'}{a_i' + (1 - j) b_i'} \right)^{1/p} \\
\leq h(1 - j) f_i (a_i) \left( \frac{a_i b_i'}{a_i' + (1 - j) b_i'} \right)^{1/p} \\
+ h(j) f_i (b_i) \left( \frac{a_i b_i'}{a_i' + (1 - j) b_i'} \right)^{1/p} \\
- \mu \phi(j) \left( \frac{1}{a_i'} - \frac{1}{b_i'} \right) \left( \frac{a_i b_i'}{a_i' + (1 - j) b_i'} \right)^{1/p}.
\]

(25)

Integrating inequality (25),

\[
\int_0^1 f_i \left[ \left( \frac{a_i b_i'}{a_i' + (1 - j) b_i'} \right)^{1/p} \right] \, w \left( \frac{a_i b_i'}{a_i' + (1 - j) b_i'} \right)^{1/p} \, \frac{1}{a_i'} \, dx \\
\leq \int_0^1 h(1 - j) f_i (a_i) \left( \frac{a_i b_i'}{a_i' + (1 - j) b_i'} \right)^{1/p} \, \frac{1}{a_i'} \, dx \\
+ \int_0^1 h(j) f_i (b_i) \left( \frac{a_i b_i'}{a_i' + (1 - j) b_i'} \right)^{1/p} \, \frac{1}{a_i'} \, dx \\
- \mu \int_0^1 \phi(j) \left( \frac{1}{a_i'} - \frac{1}{b_i'} \right) \left( \frac{a_i b_i'}{a_i' + (1 - j) b_i'} \right)^{1/p} \, \frac{1}{a_i'} \, dx.
\]

(26)

that is right hand side of (17) and the proof is completed. \( \square \)

5. Fractional Integral Inequalities

**Lemma 9** ([20], Lemma 2.1). Let \( f_i : M = [a_i, b_i] \subseteq \mathbb{R} \) be a differentiable function on the interior \( M \) of \( M \). If \( f_i' \in L[a_i, b_i] \) and \( \lambda \in [0, 1] \), then

\[
(1 - \lambda) f_i \left[ \left( \frac{a_i b_i'}{a_i' + b_i'} \right)^{1/p} \right] + (\lambda f_i (a_i) + f_i (b_i)) \\
- \frac{p (a_i b_i')}{2p (a_i b_i')} \int_0^{1/2} (2j - \lambda) \left( \frac{a_i b_i'}{a_i' + (1 - j) b_i'} \right)^{1+1/p} \, dx \\
- \frac{f_i' \left[ \left( \frac{a_i b_i'}{a_i' + b_i'} \right)^{1/p} \right]}{2p (a_i b_i')} \int_0^{1/2} (2j - 2 + \lambda) \left( \frac{a_i b_i'}{a_i' + (1 - j) b_i'} \right)^{1+1/p} \, dx \\
+ \frac{f_i' \left[ \left( \frac{a_i b_i'}{a_i' + b_i'} \right)^{1/p} \right]}{2p (a_i b_i')} \int_0^{1/2} (2j - 2 + \lambda) \left( \frac{a_i b_i'}{a_i' + (1 - j) b_i'} \right)^{1+1/p} \, dx.
\]

(27)

**Theorem 10.** Let \( M = [a_i, b_i] \subseteq \mathbb{R} \setminus \{0\} \) be a p-harmonic convex set and let \( f_i : M = [a_i, b_i] \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \) be a differentiable function on the interior \( M \) of \( M \). If \( f_i' \in L[a_i, b_i] \) and \( \left| f_i' \right|^{q} \) are strongly reciprocally \((p,h)\)-convex function of higher order on \( M \), \( q \geq 1 \), and \( \lambda \in [0, 1] \), then
\[
(1 - \lambda) f_i \left[ \left( \frac{2a_i b_i}{a_i + b_i} \right)^{\frac{1}{p'}} \right] + \lambda \left( f_i(a_i) + f_i(b_i) \right)\frac{1}{2} - \frac{p \left( a_i b_i \right)}{b_i - a_i} \int \limits_{a_i}^{b_i} f_i(x) x^{i+1} dx \leq \left( \frac{b_i - a_i}{2p(a_i b_i)} \right) \left( C_1(p, a_i, b_i) \right)^{1 - \frac{1}{q}} \left[ C_3(p, a_i, b_i) \right] f_i^q \left( a_i \right) + C_7(p, a_i, b_i) \mu_{1-p} \left[ C_6(p, a_i, b_i) \right] f_i^q \left( a_i \right) + C_8(p, a_i, b_i) \mu_{1-p} \left[ C_9(p, a_i, b_i) \right] f_i^q \left( a_i \right),
\]

where

\[
C_1(p, a_i, b_i) = \int \limits_{0}^{1/2} (2j - \lambda) \left( \frac{a_i b_i}{a_i + (1 - j) b_i} \right)^{1 + \frac{1}{p'}} dj,
\]

\[
C_2(p, a_i, b_i) = \int \limits_{1/2}^{1} (2j - 2 + \lambda) \left( \frac{a_i b_i}{a_i + (1 - j) b_i} \right)^{1 + \frac{1}{p'}} dj,
\]

\[
C_3(p, a_i, b_i) = \int \limits_{0}^{1/2} h(j) (2j - \lambda) \left( \frac{a_i b_i}{a_i + (1 - j) b_i} \right)^{1 + \frac{1}{p'}} dj,
\]

\[
C_4(p, a_i, b_i) = \int \limits_{1/2}^{1} h(j) (2j - 2 + \lambda) \left( \frac{a_i b_i}{a_i + (1 - j) b_i} \right)^{1 + \frac{1}{p'}} dj.
\]

\[
\phi(p, a_i, b_i) = \int \limits_{0}^{1/2} \phi(j) (2j - \lambda) \left( \frac{a_i b_i}{a_i + (1 - j) b_i} \right)^{1 + \frac{1}{p'}} dj,
\]

\[
\phi(p, a_i, b_i) = \int \limits_{1/2}^{1} \phi(j) (2j - 2 + \lambda) \left( \frac{a_i b_i}{a_i + (1 - j) b_i} \right)^{1 + \frac{1}{p'}} dj.
\]

\[
\lambda \left( f_i(a_i) + f_i(b_i) \right) \frac{1}{2} - \frac{p \left( a_i b_i \right)}{b_i - a_i} \int \limits_{a_i}^{b_i} f_i(x) x^{i+1} dx \leq \left( \frac{b_i - a_i}{2p(a_i b_i)} \right) \left( \frac{1}{b_i} - \frac{1}{a_i} \right)^{1/2} dj.
\]

**Proof.** Using Lemma 9, we have

\[
\lambda \left( f_i(a_i) + f_i(b_i) \right) \frac{1}{2} - \frac{p \left( a_i b_i \right)}{b_i - a_i} \int \limits_{a_i}^{b_i} f_i(x) x^{i+1} dx \leq \left( \frac{b_i - a_i}{2p(a_i b_i)} \right) \left( \frac{1}{b_i} - \frac{1}{a_i} \right)^{1/2} dj.
\]

Using power mean inequality,

\[
\lambda \left( f_i(a_i) + f_i(b_i) \right) \frac{1}{2} - \frac{p \left( a_i b_i \right)}{b_i - a_i} \int \limits_{a_i}^{b_i} f_i(x) x^{i+1} dx \leq \left( \frac{b_i - a_i}{2p(a_i b_i)} \right) \left( \frac{1}{b_i} - \frac{1}{a_i} \right)^{1/2} dj.
\]

Since \( f_i^q(x) \) is in \( SR(p, h) \), so

\[
\lambda \left( f_i(a_i) + f_i(b_i) \right) \frac{1}{2} - \frac{p \left( a_i b_i \right)}{b_i - a_i} \int \limits_{a_i}^{b_i} f_i(x) x^{i+1} dx \leq \left( \frac{b_i - a_i}{2p(a_i b_i)} \right) \left( \frac{1}{b_i} - \frac{1}{a_i} \right)^{1/2} dj.
\]

Hence, the desired result is obtained.
For $q = 1$, Theorem 10 reduces to the following result.

**Corollary 11.** Let $M = \{a_j, b_j\} \subset \mathbb{R} \setminus \{0\}$ be a $p$-harmonic convex and let $f_j : M = \{a_j, b_j\} \subset \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ be a differentiable function on the interior of $M$. If $f'_{j} \in L[a_j, b_j]$ and $|f_{j}'|^q$ are in SR(pH) on $M$ and $\lambda \in [0, 1]$, then

$$
(1 - \lambda)f_j \left[\left(\frac{2a_j^p b_j^p}{a_j^p + b_j^p}\right)^{\frac{1}{p'}} + \lambda \frac{f_j(a_j) + f_j(b_j)}{2}\right] - \frac{p(a_j^p b_j^p)}{b_j^p - a_j^p} \int_{a_j}^{b_j} x^{\frac{1}{1+q'}} dx \\
\leq \frac{(b_j^p - a_j^p)}{2p(a_j^p b_j^p)} \left[\left(\frac{1}{(1 - \lambda)^{1+q}} \right)^{\frac{1}{r}} + (C_{9}(q, p; a_j, b_j) |f_{j}'|^q) + C_{9}(q, p; a_j, b_j) |f_{j}'|^q + C_{14}(q, p; b_j, a_j) \mu \right]^{\frac{1}{1+q'}}
$$

where $C_{9}$, $C_{10}$, $C_{14}$ are given by (31) to (36).

**Theorem 12.** Let $M = \{a_j, b_j\} \subset \mathbb{R} \setminus \{0\}$ be a $p$-harmonic convex and let $f_j : M = \{a_j, b_j\} \subset \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ be a differentiable function on the interior of $M$. If $f'_{j} \in L[a_j, b_j]$ and $|f_{j}'|^q$ are strongly reciprocally $(p, h)$-convex function of higher order on $M$, $q > 1$, $1/r + 1/lq = 1$ and $\lambda \in [0, 1]$, then

$$
(1 - \lambda)f_j \left[\left(\frac{2a_j^p b_j^p}{a_j^p + b_j^p}\right)^{\frac{1}{p'}} + \lambda \frac{f_j(a_j) + f_j(b_j)}{2}\right] - \frac{p(a_j^p b_j^p)}{b_j^p - a_j^p} \int_{a_j}^{b_j} x^{\frac{1}{1+q'}} dx \\
\leq \frac{(b_j^p - a_j^p)}{2p(a_j^p b_j^p)} \left[\left(\frac{1}{(1 - \lambda)^{1+q}} \right)^{\frac{1}{r}} + (C_{9}(q, p; a_j, b_j) |f_{j}'|^q) + C_{9}(q, p; a_j, b_j) |f_{j}'|^q + C_{14}(q, p; b_j, a_j) \mu \right]^{\frac{1}{1+q'}}
$$

where

$$
C_{9}(q, p; a_j, b_j) = \int_0^{\frac{1}{2}} h(1 - j) \left(\frac{a_j^p b_j^p}{ja_j^p + (1 - j)b_j^p}\right)^{q^{+p'}} dj,
$$

$$
C_{10}(q, p; b_j, a_j) = \int_{\frac{1}{2}}^{1} h(j) \left(\frac{a_j^p b_j^p}{ja_j^p + (1 - j)b_j^p}\right)^{q^{+p'}} dj,
$$

$$
C_{11}(q, p; a_j, b_j) = \int_0^{\frac{1}{2}} h(j) \left(\frac{a_j^p b_j^p}{ja_j^p + (1 - j)b_j^p}\right)^{q^{+p'}} dj,
$$

$$
C_{12}(q, p; b_j, a_j) = \int_{\frac{1}{2}}^{1} h(j - 1 - j) \left(\frac{a_j^p b_j^p}{ja_j^p + (1 - j)b_j^p}\right)^{q^{+p'}} dj,
$$

$$
C_{13}(q, p; a_j, b_j) = -\int_0^{\frac{1}{2}} \phi(j) \left(\frac{a_j^p b_j^p}{ja_j^p + (1 - j)b_j^p}\right)^{q^{+p'}} \left|\frac{1}{b_j^p - a_j^p}\right|^\frac{1}{1+q'} dj,
$$

$$
C_{14}(q, p; b_j, a_j) = -\int_{\frac{1}{2}}^{1} \phi(j)(2j + 2 + \lambda) \left(\frac{a_j^p b_j^p}{ja_j^p + (1 - j)b_j^p}\right)^{q^{+p'}} \left|\frac{1}{b_j^p - a_j^p}\right|^\frac{1}{1+q'} dj.
$$
Since \( |f_j'(x)|^q \) is in \( \text{SR}(p,h) \), so

\[
\leq \frac{b_j^q - a_j^q}{2p(a_j^q b_j^q)} \left( \int_0^{1/2} |2j - \lambda|^q d\lambda \right)^{1/q} \left( \int_0^{1/2} \left( \frac{a_j^q b_j^q}{ja_j^q + (1 - j)b_j^q} \right)^{q+q/p} d\lambda \right)^{1/q} \\
\cdot \left[ h(1 - j)|f_j'(a_j)|^q + h(j)|f_j'(b_j)|^q - \mu \phi(j) \left| \frac{1}{b_j^q} - \frac{1}{a_j^q} \right| \right] d\lambda \]

\[
\left( \int_1^{1/2} |2j - 2 + \lambda|^q d\lambda \right)^{1/q} \left( \int_1^{1/2} \left( \frac{a_j^q b_j^q}{ja_j^q + (1 - j)b_j^q} \right)^{q+q/p} d\lambda \right)^{1/q} \\
\cdot \left[ h(1 - j)|f_j'(a_j)|^q + h(j)|f_j'(b_j)|^q - \mu \phi(j) \left| \frac{1}{b_j^q} - \frac{1}{a_j^q} \right| \right] d\lambda \]

\[
\leq \frac{b_j^q - a_j^q}{2p(a_j^q b_j^q)} \left( \int_0^{1/2} |2j - \lambda|^q d\lambda \right)^{1/q} \left( C_9(q, p, a_j, b_j)|f_j'(a_j)|^q \\
+ C_{11}(q, p, a_j, b_j)|f_j'(b_j)|^q + C_{12}(q, p, a_j, b_j)\mu \right)^{1/q} \\
+ \left( \int_1^{1/2} |2j - 2 + \lambda|^q d\lambda \right)^{1/q} \left( C_9(q, p, a_j, b_j)|f_j'(a_j)|^q \\
+ C_{11}(q, p, a_j, b_j)|f_j'(b_j)|^q + C_{12}(q, p, a_j, b_j)\mu \right)^{1/q} \]

\[
\leq \frac{b_j^q - a_j^q}{2p(a_j^q b_j^q)} \times \left( \frac{1}{2} + \left( \frac{\lambda^q}{1 - \lambda^q} \right) \right)^{1/r} \\
\cdot \left[ \left( C_9(q, p, a_j, b_j)|f_j'(a_j)|^q + C_{11}(q, p, a_j, b_j)|f_j'(b_j)|^q \\
+ C_{12}(q, p, a_j, b_j)\mu \right)^{1/q} + \left( C_9(q, p, a_j, b_j)|f_j'(a_j)|^q \\
+ C_{11}(q, p, a_j, b_j)|f_j'(b_j)|^q + C_{12}(q, p, a_j, b_j)\mu \right)^{1/q} \right].
\]

\[
(50)
\]

Hence, proved.

For \( \lambda = 0 \), Theorem 12 reduces to the following result.

**Corollary 13.** Let \( M = [a_j, b_j] \subset \mathbb{R} \setminus \{0\} \) be a p-harmonic convex set and let \( f_j : M = [a_j, b_j] \subset \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \) be a differentiable function on the interior M of M. If \( f_j' \in L[a_j, b_j] \) and \( |f_j'|^q \) are in \( \text{SR}(p,h) \) on M, \( r, q > 1 \), \( 1/r + 1/q = 1 \), and \( \lambda \in [0, 1] \), then

\[
\frac{1}{2} \left[ f_j(a_j) + 4f_j \left( \frac{2a_j^q b_j^q}{a_j^q + b_j^q} \right)^{1/p} \right] + f_j(b_j) - \frac{p(a_j^q b_j^q)}{b_j^q - a_j^q} \int_a^b f_j(x) dX \\
\leq \frac{(b_j^q - a_j^q)}{2p(a_j^q b_j^q)} \times \left( \frac{1}{2} + \left( \frac{\lambda^q}{1 - \lambda^q} \right) \right)^{1/r} \\
\cdot \left[ \left( C_9(q, p, a_j, b_j)|f_j'(a_j)|^q + C_{11}(q, p, a_j, b_j)|f_j'(b_j)|^q \\
+ C_{12}(q, p, a_j, b_j)\mu \right)^{1/q} + \left( C_9(q, p, a_j, b_j)|f_j'(a_j)|^q \\
+ C_{11}(q, p, a_j, b_j)|f_j'(b_j)|^q + C_{12}(q, p, a_j, b_j)\mu \right)^{1/q} \right].
\]

\[
(51)
\]

where \( C_9, C_{10}, C_{11}, C_{12}, C_{13}, \) and \( C_{14} \) are given by (42)-(47). For \( \lambda = 1/2 \), Theorem 12 reduces to the following result.

**Corollary 14.** Let \( M = [a_j, b_j] \subset \mathbb{R} \setminus \{0\} \) be a p-harmonic convex set and let \( f_j : M = [a_j, b_j] \subset \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \) be a differentiable function on the interior M of M. If \( f_j' \in L[a_j, b_j] \) and \( |f_j'|^q \) are strongly reciprocally \((p,h))\)-convex function of higher order on M, \( r, q > 1 \), \( 1/r + 1/q = 1 \) and \( \lambda \in [0, 1] \) then,

\[
\frac{1}{2} \left[ f_j(a_j) + 4f_j \left( \frac{2a_j^q b_j^q}{a_j^q + b_j^q} \right)^{1/p} \right] + f_j(b_j) - \frac{p(a_j^q b_j^q)}{b_j^q - a_j^q} \int_a^b f_j(x) dX \\
\leq \frac{(b_j^q - a_j^q)}{2p(a_j^q b_j^q)} \times \left( \frac{1 + 2\lambda^q}{6\lambda^q + (r + 1)} \right)^{1/r} \\
\cdot \left[ \left( C_9(q, p, a_j, b_j)|f_j'(a_j)|^q + C_{11}(q, p, a_j, b_j)|f_j'(b_j)|^q \\
+ C_{12}(q, p, a_j, b_j)\mu \right)^{1/q} + \left( C_9(q, p, a_j, b_j)|f_j'(a_j)|^q \\
+ C_{11}(q, p, a_j, b_j)|f_j'(b_j)|^q + C_{12}(q, p, a_j, b_j)\mu \right)^{1/q} \right].
\]

\[
(52)
\]

where \( C_9, C_{10}, C_{11}, C_{12}, C_{13}, \) and \( C_{14} \) are given by (42)-(47). For \( \lambda = 1/3 \), Theorem 12 reduces to the following result.

**Corollary 15.** Let \( M = [a_j, b_j] \subset \mathbb{R} \setminus \{0\} \) be a p-harmonic convex set and let \( f_j : M = [a_j, b_j] \subset \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \) be a differentiable function on the interior M of M. If \( f_j' \in L[a_j, b_j] \) and \( |f_j'|^q \) is strongly reciprocally \((p,h))\)-convex function of higher order on M, \( r, q > 1 \), \( 1/r + 1/q = 1 \) and \( \lambda \in [0, 1] \) then,

\[
\frac{1}{2} \left[ f_j(a_j) + 4f_j \left( \frac{2a_j^q b_j^q}{a_j^q + b_j^q} \right)^{1/p} \right] + f_j(b_j) - \frac{p(a_j^q b_j^q)}{b_j^q - a_j^q} \int_a^b f_j(x) dX \\
\leq \frac{(b_j^q - a_j^q)}{2p(a_j^q b_j^q)} \times \left( \frac{1 + 2\lambda^q}{6\lambda^q + (r + 1)} \right)^{1/r} \\
\cdot \left[ \left( C_9(q, p, a_j, b_j)|f_j'(a_j)|^q + C_{11}(q, p, a_j, b_j)|f_j'(b_j)|^q \\
+ C_{12}(q, p, a_j, b_j)\mu \right)^{1/q} + \left( C_9(q, p, a_j, b_j)|f_j'(a_j)|^q \\
+ C_{11}(q, p, a_j, b_j)|f_j'(b_j)|^q + C_{12}(q, p, a_j, b_j)\mu \right)^{1/q} \right].
\]

\[
(53)
\]
where $C_\mu$, $C_{1\mu}$, $C_{12}$, $C_{14}$, and $C_{14}$ are given by (42)–(47).

Remark 17. Inserting $h(j) = j, \mu = 0$ and $l = 2$ with $\phi(j) = j(1 - j)$ in Corollary 16, we obtained ([20], Corollary 3.8).

6. Conclusion

In this paper, a new definition of convex functions “namely strongly reciprocally $(p,h)$-convex functions of higher order” is introduced. This new definition extends almost all the existing versions of convex functions. For the strongly reciprocally $(p,h)$-convex functions of higher order, we established several interesting inequalities which have applications in optimization theory, probability theory, as well as pure and applied mathematics. We also established several fractional versions of the Hermite-Hadamard type inequalities.

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