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Research Article

Some Interesting Inequalities for the Class of Generalized Convex Functions of Higher Order

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In this paper, we study a generalized version of strongly reciprocally convex functions of higher order. Firstly, we prove some basic properties for addition, scalar multiplication, and composition of functions. Secondly, we establish Hermite-Hadamard and Fejér type inequalities for the generalized version of strongly reciprocally convex functions of higher order. We also include some fractional integral inequalities concerning with this class of functions. Our results have applications in optimization theory and can be considered extension/generalization of many existing results.

1. Introduction

Convexity is a very simple and ordinary concept. Due to its massive applications in industry and business, convexity has a great influence on our daily life. In the solution of many real-world problems, the concept of convexity is very decisive. Problems faced in constrained control and estimation are convex. Geometrically, a real-valued function is said to be convex if the line segment joining any two of its points lies on or above the graph of the function in Euclidean space.

Convexity of a function in classical sense is defined as a function $f_1 : M \longrightarrow \mathbb{R}, f_1$ is convex if we have

$$f_1(\mathrm{j}x + (1-j)y) \leq \mathrm{j}f_1(x) + (1-j)f_1(y), \forall j \in [0,1]. \quad (1)$$

If the above inequality is reversed, then the function is said to be concave.

Using different techniques, the notion of convexity is being extended day by day [1–3]. Many extensions and generalizations are made speedily due to its applications in modern engineering, optimization, economics, and nonlinear programming [4–7]. For recent generalizations, one can see [8, 9] and the references therein.

Using the definition of convex functions, several important inequalities can be proved, and the Hermite-Hadamard inequality is one of them. The Hermite-Hadamard inequality is for any convex function $f_1 : M \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ with $a_1, b_1 \in M$ and $a_1 b_1$, the Hermite-Hadamard double inequality is

$$f_1\left(\frac{a_1+b_1}{2}\right) \leq \frac{1}{b_1-a_1}\int_{a_1}^{b_1} f_1(x)dx \leq \frac{f_1(a_1)+f_1(b_1)}{2}. \quad (2)$$

In [9], using the weight function $w(x)$, Fejér gave a generalization of the Hermite-Hadamard inequality as follows:

Let $f_1 : [a_1, b_1] \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function and $w : [a_1, b_1] \longrightarrow \mathbb{R}$ is nonnegative, integrable, and symmetric about $(a_1+b_1)/2$, then we have

$$f_1\left(\frac{a_1+b_1}{2}\right)\int_{a_1}^{b_1} w(x)\mathrm{d}x \leq \int_{a_1}^{b_1} f_1(x)w(x)\mathrm{d}x \leq \frac{f_1(a_1)+f_1(b_1)}{2}\int_{a_1}^{b_1} w(x)\mathrm{d}x. \quad (3)$$

In [10], the notations of p-convex set and p-convex functions are introduced. The strongly convex functions of modulus μ are introduced in [11]. In [12, 13], the strongly p-convex and harmonic convex functions were introduced, respectively. The p-harmonic convex set and p-harmonic convex functions were studied in [14], and in [15], the strongly reciprocally convex of modulus μ are introduced. The strongly reciprocally p-convex and h-convex functions were introduced in [16, 17], respectively. The (p,h)-convex functions are introduced in [18], and the higher-order strongly convex with modulus μ are introduced in [19]. Now, we present the notation of strongly reciprocally (p,h)-convex functions of higher order (SRHO).

Definition 1. Let $\mu \in (0,\infty)$ and M is any interval. Then the function $f_1 : M = [a_1, b_1] \subseteq \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ is SRHO of modulus μ on the interval M, if we have

$$f_1\left[\left(\frac{x^p y^p}{jx^p + (1-j)y^p}\right)^{1/p}\right] \le h(1-j)f_1(x) + h(j)f_1(y) - \mu\phi(j)\left\|\frac{1}{x^p} - \frac{1}{y^p}\right\|^l, \tag{4}$$

for all $x, y \in M$, $j \in [0,1]$, and $l \ge 1$, where $\phi(j) = j(1-j)$.

Remark 2. Inserting $l = 2$ in Def. 1 with same $\phi(j)$ as defined above, we obtain strongly reciprocally (p,h)-convex functions. Similarly, inserting $l = 2$ and $h(j) = j$ in Def. 1, we obtain strongly reciprocally p-convex functions, and for $l = 2$, $h(j) = j$, and $p = 1$, Def. 1 reduces to the strongly reciprocally convex function of modulus μ.

As we know that $\mathbb{R}$ is a Norm space under the usual modulus norm, thus, for any $x \in \mathbb{R}$,

$$\|x\| = |x|. \tag{5}$$

Using (5), the inequality 1 can be written as

$$f_1\left[\left(\frac{x^p y^p}{jx^p + (1-j)y^p}\right)^{1/p}\right] \le h(1-j)f_1(x) + h(j)f_1(y) - \mu\phi(j)\left|\frac{1}{x^p} - \frac{1}{y^p}\right|^l, \tag{6}$$

$\forall x, y \in M$ and $j \in [0,1]$ with $l \ge 1$, where $\phi(j)$ is same as in Definition 1.

The aim of this paper is to study a generalized version of strongly reciprocally convex functions of higher order and establish the Hermite-Hadamard and Fejér type inequalities for this new class of convex functions. We also presented fractional versions of the above mentioned inequalities for the strongly reciprocally (p,h)-convex of higher order. It is worthy to mention here that the results presented in this paper are more generalized and can be considered extensions of many existing results.

2. Basic Results

Now, we present some basic properties for strongly reciprocally (p,h)-convex of higher order.

Proposition 3. *For any two SRHO $f_1, g_1 : M \longrightarrow \mathbb{R}$ with modulus μ on the interval M, the $f_1 + g_1 : M \longrightarrow \mathbb{R}$ is also SRHO with modulus μ^* on the interval M, where $1/2\mu^* = \mu$.*

Proof. By definition, we have

$$\begin{aligned} f_1 + g_1\left[\left(\frac{x^p y^p}{jx^p + (1-j)y^p}\right)^{1/p}\right] &= f_1\left[\left(\frac{x^p y^p}{jx^p + (1-j)y^p}\right)^{1/p}\right] \\ &+ g_1\left[\left(\frac{x^p y^p}{jx^p + (1-j)y^p}\right)^{1/p}\right] \le h(j)f_1(x) + h(1-j)f_1(y) \\ &- \mu\phi(j)\left\|\frac{1}{y^p} - \frac{1}{x^p}\right\|^l + h(j)g_1(x) + h(1-j)g_1(y) - \mu\phi(j)\left\|\frac{1}{y^p} - \frac{1}{x^p}\right\|^l, \end{aligned} \tag{7}$$

which in turns implies that

$$\begin{aligned} &= h(j)(f_1 + g_1)(x) + h(1-j)(f_1 + g_1)(y) - 2\mu j(1-j)\left\|\frac{1}{y^p} - \frac{1}{x^p}\right\|^l \\ &= h(j)(f_1 + g_1)(x) + h(1-j)(f_1 + g_1)(y) - \mu^*\phi(j)\left\|\frac{1}{y^p} - \frac{1}{x^p}\right\|^l, \end{aligned} \tag{8}$$

where $\mu^* = 2\mu$, $\mu \ge 0$ and $\phi(j) = j(1-j)$.

This completes the proof. □

Proposition 4. *For any SRHO $f_1 : M \longrightarrow \mathbb{R}$ with modulus $\mu \ge 0$ and any $\lambda \ge 0$, λf_1 is also SRHO with modulus ν^* on the interval M, where $1/\lambda\nu^* = \mu$.*

Proof. Let $\lambda \ge 0$, then by definition of f_1, we obtain

$$\begin{aligned} \lambda f_1&\left[\left(\frac{x^p y^p}{jx^p + (1-j)y^p}\right)^{1/p}\right] \\ &= \lambda\left[f_1\left[\left(\frac{x^p y^p}{jx^p + (1-j)y^p}\right)^{1/p}\right]\right] \\ &\le \lambda\left[h(j)f_1(x) + h(1-j)f_1(y) - \mu\phi(j)\left\|\frac{1}{y^p} - \frac{1}{x^p}\right\|^l\right] \\ &= h(j)\lambda f_1(x) + h(1-j)\lambda f_1(y) - \lambda\mu\phi(j)\left\|\frac{1}{y^p} - \frac{1}{x^p}\right\|^l \\ &= h(j)\lambda f_1(x) + h(1-j)\lambda f_1(y) - \nu^*\phi(j)\left\|\frac{1}{y^p} - \frac{1}{x^p}\right\|^l, \end{aligned} \tag{9}$$

where $\nu^* = \lambda\mu$, $\mu \ge 0$ and $\phi(j) = j(1-j)$. This completes the proof. □

Proposition 5. *Consider a sequence of SRHO, f_{1i} are defined on an interval M, provide $1 \le i \le n$, then for positive constants λ_i, the function $f_1 = \sum_{i=1}^n \lambda_i f_{1i}$ is SRHO with nonnegative modulus $\gamma\sum_{i=1}^n \lambda_i \mu$.*

Proof. For a p-harmonic convex set M, we have

$$\begin{aligned} f_1\left[\left(\frac{x^p y^p}{jx^p+(1-j)y^p}\right)^{1/p}\right] &= \sum_{i=1}^{n}\lambda_i f_{1i}\left[\left(\frac{x^p y^p}{jx^p+(1-j)y^p}\right)^{1/p}\right] \\ &\le \sum_{i=1}^{n}\lambda_i\left[h(j)f_{1i}(x)+h(1-j)f_{1i}(y)-\mu\phi(j)\left\|\frac{1}{y^p}-\frac{1}{x^p}\right\|^l\right] \\ &= h(j)\sum_{i=1}^{n}\lambda_i f_{1i}(x)+h(1-j)\sum_{i=1}^{n}\lambda_i f_{1i}(y)-\sum_{i=1}^{n}\lambda_i\left[\mu\phi(j)\left\|\frac{1}{y^p}-\frac{1}{x^p}\right\|^l\right] \\ &= h(j)f_1(x)+h(1-j)f_1(y)-\gamma\phi(j)\left\|\frac{1}{y^p}-\frac{1}{x^p}\right\|^l, \end{aligned} \tag{10}$$

$\forall x, y \in M$ and $j \in [0,1]$, where $\gamma=\sum_{i=1}^{n}\lambda_i\mu$. Hence, the result is proved. □

Proposition 6. *Consider a sequence of SRHO, f_{1i} are defined on an interval M, provide $1 \le i \le n$, then for positive constants λ_i, the function $f_1=\max\{f_{1i}, i=1,2,\cdots,n\}$, is SRHO of modulus μ.*

Proof. Let M be a p-harmonic convex set. Then, $\forall x, y \in M$ and $j \in [0,1]$, we have

$$\begin{aligned} f_1\left[\left(\frac{x^p y^p}{jx^p+(1-j)y^p}\right)^{1/p}\right] &= \max\left\{f_{1i}\left[\left(\frac{x^p y^p}{jx^p+(1-j)y^p}\right)^{1/p}\right], i=1,2,3,\cdots,n\right\} \\ &= f_c\left(\left[\frac{x^p y^p}{jx^p+(1-j)y^p}\right]\right) \\ &\le h(j)f_c(x)+h(1-j)f_c(y)-\mu\phi(j)\left\|\frac{1}{y^p}-\frac{1}{x^p}\right\|^l \\ &= h(j)\max\{f_{1i}(x)\}+h(1-j)\max\{f_{1i}(y)\}-\mu\phi(j)\left\|\frac{1}{y^p}-\frac{1}{x^p}\right\|^l \\ &= h(j)f_1(x)+h(1-j)f_1(y)-\mu\phi(j)\left\|\frac{1}{y^p}-\frac{1}{x^p}\right\|^l. \end{aligned} \tag{11}$$

This completes the proof. □

3. Hermite-Hadamard Type Inequality

In this section, we establish Hermite-Hadamard's type inequality for the function belonging to $SR(ph)$.

Theorem 7. *Consider an interval M not containing zero and SRHO $f_1: M \longrightarrow \mathbb{R}$ of nonnegative modulus μ and $f_1 \in L[a_1, b_1]$, then for $h(1/2) \neq 0$, we have*

$$\begin{aligned} &\frac{1}{2h(1/2)}\left[f_1\left(\frac{2a_1^p b_1^p}{a_1^p+b_1^p}\right)^{1/p}+\mu\phi\left(\frac{1}{2}\right)\left|\frac{b_1^p-a_1^p}{a_1^p b_1^p}\right|^l\left[\frac{1-(-1)^{2l+1}}{2(l+1)}\right]\right] \\ &\le \frac{p\left(a_1^p b_1^p\right)}{b_1^p-a_1^p}\int_{a_1}^{b_1}\frac{f_1(x)}{x^{1+p}}dx \\ &\le \int_0^1[h(1-j)f_1(a_1)+h(j)f_1(b_1)]dj-\mu\left|\frac{b_1^p-a_1^p}{a_1^p b_1^p}\right|^l\int_0^1\phi(j)dj. \end{aligned} \tag{12}$$

Proof. Substituting $j=1/2$ in Definition 1, gives

$$f_1\left[\left(\frac{2x^p y^p}{x^p+y^p}\right)^{1/p}\right] \le h\left(\frac{1}{2}\right)f_1(x)+h\left(\frac{1}{2}\right)f_1(y)-\mu\phi\left(\frac{1}{2}\right)\left\|\frac{1}{x^p}-\frac{1}{y^p}\right\|^l. \tag{13}$$

Considering $x=[(a_1^p b_1^p/ja_1^p+(1-j)b_1^p)^{1/p}]$ and $y=[(a_1^p b_1^p/jb_1^p+(1-j)a_1^p)^{1/p}]$ and integrating (13)

$$\begin{aligned} f_1\left(\frac{2a_1^p b_1^p}{a_1^p+b_1^p}\right)^{1/p} &\le h\left(\frac{1}{2}\right)f_1\left[\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right] \\ &\quad + h\left(\frac{1}{2}\right)f_1\left[\left(\frac{a_1^p b_1^p}{jb_1^p+(1-j)a_1^p}\right)^{1/p}\right] \\ &\quad - \mu\phi\left(\frac{1}{2}\right)\left|\frac{b_1^p-a_1^p}{a_1^p b_1^p}\right|^l|1-2j|^l, \end{aligned}$$

$$\begin{aligned} \int_0^1 f_1\left(\frac{2a_1^p b_1^p}{a_1^p+b_1^p}\right)^{1/p}dj &\le \int_0^1 h\left(\frac{1}{2}\right)f_1\left[\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]dj \\ &\quad + \int_0^1 h\left(\frac{1}{2}\right)f_1\left[\left(\frac{a_1^p b_1^p}{jb_1^p+(1-j)a_1^p}\right)^{1/p}\right]dj \\ &\quad - \mu\phi\left(\frac{1}{2}\right)\left|\frac{b_1^p-a_1^p}{a_1^p b_1^p}\right|^l\int_0^1|1-2j|^l dj, \end{aligned}$$

$$\begin{aligned} f_1\left(\frac{2a_1^p b_1^p}{a_1^p+b_1^p}\right)^{1/p} &\le 2h\left(\frac{1}{2}\right)\frac{p\left(a_1^p b_1^p\right)}{b_1^p-a_1^p}\int_{a_1}^{b_1}\frac{f_1(x)}{x^{1+p}}dx \\ &\quad - \mu\phi\left(\frac{1}{2}\right)\left|\frac{b_1^p-a_1^p}{a_1^p b_1^p}\right|^l\left[\frac{1-(-1)^{2l+1}}{2(l+1)}\right], \end{aligned}$$

$$\begin{aligned} &f_1\left(\frac{2a_1^p b_1^p}{a_1^p+b_1^p}\right)^{1/p}+\mu\phi\left(\frac{1}{2}\right)\left|\frac{b_1^p-a_1^p}{a_1^p b_1^p}\right|^l\left[\frac{1-(-1)^{2l+1}}{2(l+1)}\right] \\ &\le 2h\left(\frac{1}{2}\right)\frac{p\left(a_1^p b_1^p\right)}{b_1^p-a_1^p}\int_{a_1}^{b_1}\frac{f_1(x)}{x^{1+p}}dx, \end{aligned}$$

$$\frac{1}{2h(1/2)}\left[f_1\left(\frac{2a_1^pb_1^p}{a_1^p+b_1^p}\right)^{1/p}+\mu\phi\left(\frac{1}{2}\right)\left|\frac{b_1^p-a_1^p}{a_1^pb_1^p}\right|^l\left[\frac{1-(-1)^{2l+1}}{2(l+1)}\right]\right] \leq \frac{p\left(a_1^pb_1^p\right)}{b_1^p-a_1^p}\int_{a_1}^{b_1}\frac{f_1(x)}{x^{1+p}}dx, \tag{14}$$

which is left side of the inequality (12).

Finally, for the right side of the inequality (12), setting $x=a_1$ and $y=b_1$ in Definition 1 gives

$$f_1\left[\left(\frac{a_1^pb_1^p}{ta_1^p+(1-t)b_1^p}\right)^{1/p}\right] \leq h(1-j)f_1(a_1)+h(j)f_1(b_1) - \mu\phi(j)\left\|\frac{1}{a_1^p}-\frac{1}{b_1^p}\right\|^l. \tag{15}$$

Integrating (15)

$$\int_0^1 f_1\left[\left(\frac{a_1^pb_1^p}{ta_1^p+(1-t)b_1^p}\right)^{1/p}\right]dj \leq \int_0^1 h(1-j)f_1(a_1)dj + \int_0^1 h(j)f_1(b_1)dj-\mu\left|\frac{b_1^p-a_1^p}{a_1^pb_1^p}\right|^l\int_0^1\phi(j)dj,$$

$$\frac{p\left(a_1^pb_1^p\right)}{b_1^p-a_1^p}\int_{a_1}^{b_1}\frac{f_1(x)}{x^{1+p}}dx \leq \int_0^1[h(1-j)f_1(a_1)+h(j)f_1(b_1)]dj - \mu\left|\frac{b_1^p-a_1^p}{a_1^pb_1^p}\right|^l\int_0^1\phi(j)dj, \tag{16}$$

that is right hand side of (12) and proof is completed. □

4. Fejér Type Inequality

Now, we are going to develop the Fejér type inequality for the function belonging to $SR(ph)$.

Theorem 8. *Consider an interval M not containing zero and real-valued SRHO f_1 defined on M of nonnegative modulus μ, then for $h(1/2)\neq 0$, we have*

$$\frac{1}{2h(1/2)}\left[f_1\left(\frac{2a_1^pb_1^p}{a_1^p+b_1^p}\right)^{1/p}\int_{a_1}^{b_1}\frac{w(x)}{x^{1+p}}dx+\frac{\mu}{\left|a_1^pb_1^p\right|^l}\phi\left(\frac{1}{2}\right) \cdot\int_{a_1}^{b_1}\frac{\left|2a_1^pb_1^p-\left(a_1^p+b_1^p\right)x^p\right|^l w(x)}{\left|x^p\right|^l x^{1+p}}dx\right] \leq \int_{a_1}^{b_1}\frac{f_1(x)w(x)}{x^{1+p}}dx \leq [f_1(a_1)+f_1(b_1)]\int_{a_1}^{b_1}h\left(\frac{a_1^p\left(b_1^p-x^p\right)}{x^p\left(b_1^p-a_1^p\right)}\right)\frac{w(x)}{x^{1+p}}dx - \mu\left\|\frac{b_1^p-a_1^p}{a_1^pb_1^p}\right\|^l\int_{a_1}^{b_1}\phi\left(\frac{a_1^p\left(b_1^p-x^p\right)}{x^p\left(b_1^p-a_1^p\right)}\right)\frac{w(x)}{x^{1+p}}dx, \tag{17}$$

holds for $a_1, b_1\in M$ with $a_1\leq b_1$ and $f_1\in L[a_1,b_1]$, where the nonnegative real-valued function w defined on M satisfies

$$w\left(\frac{a_1^pb_1^p}{x^p}\right)^{1/p} = w\left[\left(\frac{a_1^pb_1^p}{a_1^p+b_1^p-x^p}\right)^{1/p}\right]. \tag{18}$$

Proof. Substituting $j=1/2$ in Definition 1, yields

$$f_1\left[\left(\frac{2x^py^p}{x^p+y^p}\right)^{1/p}\right]\leq h\left(\frac{1}{2}\right)f_1(x)+h\left(\frac{1}{2}\right)f_1(y)-\mu\phi\left(\frac{1}{2}\right)\left\|\frac{1}{x^p}-\frac{1}{y^p}\right\|^l. \tag{19}$$

Considering $x=[(a_1^pb_1^p/ja_1^p+(1-j)b_1^p)^{1/p}]$ and $y=[(a_1^pb_1^p/jb_1^p+(1-j)a_1^p)^{1/p}]$ and integrating (19),

$$f_1\left(\frac{2a_1^pb_1^p}{a_1^p+b_1^p}\right)^{1/p}\leq h\left(\frac{1}{2}\right)f_1\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right] + h\left(\frac{1}{2}\right)f_1\left[\left(\frac{a_1^pb_1^p}{jb_1^p+(1-j)a_1^p}\right)^{1/p}\right] - \mu\phi\left(\frac{1}{2}\right)\left|\frac{ja_1^p+(1-j)b_1^p}{a_1^pb_1^p}-\frac{jb_1^p+(1-j)a_1^p}{a_1^pb_1^p}\right|^l. \tag{20}$$

By the properties of w,

$$f_1\left(\frac{2a_1^pb_1^p}{a_1^p+b_1^p}\right)^{1/p}w\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right] \leq h\left(\frac{1}{2}\right)f_1\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]w\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right] + h\left(\frac{1}{2}\right)f_1\left[\left(\frac{a_1^pb_1^p}{jb_1^p+(1-j)a_1^p}\right)^{1/p}\right]w\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right], \tag{21}$$

$$-\mu\phi\left(\frac{1}{2}\right)\left|\frac{ja_1^p+(1-j)b_1^p}{a_1^pb_1^p}-\frac{jb_1^p+(1-j)a_1^p}{a_1^pb_1^p}\right|^l w\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]. \tag{22}$$

Integrating inequality (21),

$$\begin{aligned}&\int_0^1 f_1\left(\frac{2a_1^pb_1^p}{a_1^p+b_1^p}\right)^{1/p} w\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]dj\\ &\le \int_0^1 h\left(\frac{1}{2}\right)f_1\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]w\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]dj\\ &+\int_0^1 h\left(\frac{1}{2}\right)f_1\left[\left(\frac{a_1^pb_1^p}{jb_1^p+(1-j)a_1^p}\right)^{1/p}\right]w\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]dj\\ &-\mu\phi\left(\frac{1}{2}\right)\left|\frac{ja_1^p+(1-j)b_1^p}{a_1^pb_1^p}-\frac{jb_1^p+(1-j)a_1^p}{a_1^pb_1^p}\right|^l w\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]dj,\end{aligned}$$

$$\begin{aligned}&f_1\left(\frac{2a_1^pb_1^p}{a_1^p+b_1^p}\right)^{1/p}\int_{a_1}^{b_1}\frac{w(x)}{x^{1+p}}dx+\frac{\mu}{\left|a_1^pb_1^p\right|^l}\phi\left(\frac{1}{2}\right)\int_{a_1}^{b_1}\frac{\left|2a_1^pb_1^p-\left(a_1^p+b_1^p\right)x^p\right|^l w(x)}{|x^p|^l x^{1+p}}dx\\ &\le 2h\left(\frac{1}{2}\right)\int_{a_1}^{b_1}\frac{f_1(x)w(x)}{x^{1+p}}dx\,\frac{1}{2h(1/2)}\left[f_1\left(\frac{2a_1^pb_1^p}{a_1^p+b_1^p}\right)^{1/p}\int_{a_1}^{b_1}\frac{w(x)}{x^{1+p}}dx\right.\\ &\left.+\frac{\mu}{\left|a_1^pb_1^p\right|^l}\phi\left(\frac{1}{2}\right)\int_{a_1}^{b_1}\frac{\left|2a_1^pb_1^p-\left(a_1^p+b_1^p\right)x^p\right|^l w(x)}{|x^p|^l x^{1+p}}dx\right]\\ &\le\int_{a_1}^{b_1}\frac{f_1(x)w(x)}{x^{1+p}}dx,\end{aligned} \tag{23}$$

which is left side of the inequality (17).

Finally, for the right side of the inequality (17), setting $x=a_1$ in Definition 1 gives

$$f_1\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]\le h(1-j)f_1(a_1)+h(j)f_1(b_1)-\mu\phi(j)\left\|\frac{1}{a_1^p}-\frac{1}{b_1^p}\right\|^l. \tag{24}$$

By the properties of w,

$$\begin{aligned}&f_1\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]w\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]\\ &\le h(1-j)f_1(a_1)w\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]\\ &+h(j)f_1(b_1)w\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]\\ &-\mu\phi(j)\left\|\frac{1}{a_1^p}-\frac{1}{b_1^p}\right\|^l w\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right].\end{aligned} \tag{25}$$

Integrating inequality (25),

$$\begin{aligned}&\int_0^1 f_1\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]w\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]dj\\ &\le\int_0^1 h(1-j)f_1(a_1)w\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]dj\\ &+\int_0^1 h(j)f_1(b_1)w\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]dj\\ &-\mu\int_0^1\phi(j)\left\|\frac{1}{a_1^p}-\frac{1}{b_1^p}\right\|^l w\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]dj,\end{aligned}$$

$$\begin{aligned}\int_{a_1}^{b_1}\frac{f_1(x)w(x)}{x^{1+p}}dx&\le[f_1(a_1)+f_1(b_1)]\int_{a_1}^{b_1}h\left(\frac{a_1^p\left(b_1^p-x^p\right)}{x^p\left(b_1^p-a_1^p\right)}\right)\frac{w(x)}{x^{1+p}}dx\\ &-\mu\left\|\frac{b_1^p-a_1^p}{a_1^pb_1^p}\right\|^l\int_{a_1}^{b_1}\phi\left(\frac{a_1^p\left(b_1^p-x^p\right)}{x^p\left(b_1^p-a_1^p\right)}\right)\frac{w(x)}{x^{1+p}}dx,\end{aligned} \tag{26}$$

that is right hand side of (17) and the proof is completed. □

5. Fractional Integral Inequalities

Lemma 9 ([20], Lemma 2.1). *Let $f_1 : M=[a_1,b_1]\subseteq\mathbb{R}$ be a differentiable function on the interior M of M. If $f_1{}'\in L[a_1, b_1]$ and $\lambda\in[0,1]$, then*

$$\begin{aligned}&(1-\lambda)f_1\left[\left(\frac{2a_1^pb_1^p}{a_1^p+b_1^p}\right)^{1/p}\right]+\lambda\left(\frac{f_1(a_1)+f_1(b_1)}{2}\right)\\ &-\frac{p\left(a_1^pb_1^p\right)}{b_1^p-a_1^p}\int_{a_1}^{b_1}\frac{f_1(x)}{x^{1+p}}dx\\ &=\frac{\left(b_1^p-a_1^p\right)}{2p\left(a_1^pb_1^p\right)}\left[\int_0^{1/2}(2j-\lambda)\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1+1/p}\right.\\ &\cdot f_1'\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]dj\\ &+\int_{1/2}^1(2j-2+\lambda)\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1+1/p}\\ &\left.\cdot f_1'\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]dj\right].\end{aligned} \tag{27}$$

Theorem 10. *Let $M=[a_1,b_1]\subset\mathbb{R}\setminus\{0\}$ be a p-harmonic convex set and let $f_1 : M=[a_1,b_1]\subseteq\mathbb{R}\setminus\{0\}\longrightarrow\mathbb{R}$ be a differentiable function on the interior M of M. If $f_1{}'\in L[a_1,b_1]$ and $|f_1{}'|^q$ are strongly reciprocally (p,h)-convex function of higher order on M, $q\ge 1$, and $\lambda\in[0,1]$, then*

$$\left|(1-\lambda)f_1\left[\left(\frac{2a_1^pb_1^p}{a_1^p+b_1^p}\right)^{1/p}\right]+\lambda\left(\frac{f_1(a_1)+f_1(b_1)}{2}\right)-\frac{p\left(a_1^pb_1^p\right)}{b_1^p-a_1^p}\int_{a_1}^{b_1}\frac{f_1(x)}{x^{1+p}}dx\right|\le\frac{\left(b_1^p-a_1^p\right)}{2p\left(a_1^pb_1^p\right)}\left[C_1(p,a_1,b_1)^{1-1/q}\left[C_3(p,a_1,b_1)\left|f_1'(a_1)\right|^q+C_5(p,a_1,b_1)\left|f_1'(b_1)\right|^q+C_7(p,a_1,b_1)\mu\right]^{1/q}+C_2(p,b_1,a_1)^{1-1/q}\left[C_6(p,b_1,a_1)\left|f_1'(a_1)\right|^q+C_4(p,b_1,a_1)\left|f_1'(b_1)\right|^q+C_8(p,b_1,a_1)\mu\right]^{1/q}\right], \tag{28}$$

where

$$C_1(p,a_1,b_1)=\int_0^{1/2}|2j-\lambda|\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1+1/p}dj, \tag{29}$$

$$C_2(p,b_1,a_1)=\int_{1/2}^{1}|2j-2+\lambda|\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1+1/p}dj, \tag{30}$$

$$C_3(p,a_1,b_1)=\int_0^{1/2}h(1-j)|2j-\lambda|\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1+1/p}dj, \tag{31}$$

$$C_4(p,b_1,a_1)=\int_{1/2}^{1}h(j)|2j-2+\lambda|\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1+1/p}dj, \tag{32}$$

$$C_5(p,a_1,b_1)=\int_0^{1/2}h(j)|2j-\lambda|\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1+1/p}dj, \tag{33}$$

$$C_6(p,b_1,a_1)=\int_{1/2}^{1}h(1-j)|2j-2+\lambda|\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1+1/p}dj, \tag{34}$$

$$C_7(p,a_1,b_1)=-\int_0^{1/2}\phi(j)|2j-\lambda|\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1+1/p}\cdot\left\|\frac{1}{b_1^p}-\frac{1}{a_1^p}\right\|^l dj, \tag{35}$$

$$C_8(p,b_1,a_1)=-\int_{1/2}^{1}\phi(j)|2j-2+\lambda|\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1+1/p}\cdot\left\|\frac{1}{b_1^p}-\frac{1}{a_1^p}\right\|^l dj. \tag{36}$$

Proof. Using Lemma 9, we have

$$\left|(1-\lambda)f\left[\left(\frac{2a_1^pb_1^p}{a_1^p+b_1^p}\right)^{1/p}\right]+\lambda\left(\frac{f(a_1)+f(b_1)}{2}\right)-\frac{p\left(a_1^pb_1^p\right)}{b_1^p-a_1^p}\int_{a_1}^{b_1}\frac{f(x)}{x^{1+p}}dx\right|\le\frac{\left(b_1^p-a_1^p\right)}{2p\left(a_1^pb_1^p\right)}\left[\int_0^{1/2}\left|(2j-\lambda)\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1+1/p}\right|\left|f_1'\cdot\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]\right|dj+\int_{1/p}^{1}|(2j-2+\lambda)\cdot\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1+1/p}\left|\left|f_1'\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]\right|dj\right]. \tag{37}$$

Using power mean inequality,

$$\le\frac{\left(b_1^p-a_1^p\right)}{2p\left(a_1^pb_1^p\right)}\left[\left(\int_0^{1/2}|(2j-\lambda)|\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1+1/p}dj\right)^{1-1/q}\cdot\left(\int_0^{1/2}|(2j-\lambda)|\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1+1/p}\left|f_1'\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]\right|^q dj\right)^{1/q}+\left(\int_{1/2}^{1}|(2j-2+\lambda)|\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1+1/p}dj\right)^{1-1/q}\cdot\left(\int_{1/2}^{1}|(2j-2+\lambda)|\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1+1/p}\left|f_1'\left[\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]\right|^q dj\right)^{1/q}\right]. \tag{38}$$

Since $\left|f_1'(x)\right|^q$ is in $SR(ph)$, so

$$\begin{aligned}
&\le\frac{\left(b_1^p-a_1^p\right)}{2p\left(a_1^pb_1^p\right)}\left[\left(\int_0^{1/2}|(2j-\lambda)|^r dj\right)^{1/r}\left(\int_0^{1/2}\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{q+q/p}\cdot\left[h(1-j)\left|f_1'(a_1)\right|^q+h(j)\left|f_1'(b_1)\right|^q-\mu\phi(j)\left\|\frac{1}{b_1^p}-\frac{1}{a_1^p}\right\|^l\right]dj\right)^{1/q}\right.\\
&\quad\left.+\left(\int_{1/2}^{1}|(2j-2+\lambda)|^r dj\right)^{1/r}\left(\int_{1/2}^{1}\left(\frac{a_1^pb_1^p}{ja_1^p+(1-j)b_1^p}\right)^{q+q/p}\cdot\left[h(1-j)\left|f_1'(a_1)\right|^q+h(j)|(b_1)|^q-\mu\phi(j)\left\|\frac{1}{b_1^p}-\frac{1}{a_1^p}\right\|^l\right]dj\right)^{1/q}\right]\\
&\le\frac{\left(b_1^p-a_1^p\right)}{2p\left(a_1^pb_1^p\right)}\left[\left(\int_0^{1/2}|(2j-\lambda)|^r dj\right)^{1/r}\cdot\left(C_9(q,p;a_1,b_1)\left|f_1'(a_1)\right|^q+C_{11}(q,p;a_1,b_1)\left|f_1'(b_1)\right|^q+C_{13}(q,p;a_1,b_1)\mu\right)^{1/q}\right.\\
&\quad\left.+\left(\int_0^{1/2}|(2j-2+\lambda)|^r dj\right)^{1/r}\cdot\left(C_{12}(q,p;b_1,a_1)\left|f_1'(a_1)\right|^q+C_{10}(q,p;b_1,a_1)\left|f_1'(b_1)\right|^q+C_{14}(q,p;b_1,a_1)\mu\right)^{1/q}\right]\\
&\le\frac{\left(b_1^p-a_1^p\right)}{2p\left(a_1^pb_1^p\right)}\times\left(\frac{\lambda^{r+1}+(1-\lambda)^{r+1}}{2(r+1)}\right)^{1/r}\cdot\left[\left(C_9(q,p;a_1,b_1)\left|f_1'(a_1)\right|^q+C_{11}(q,p;a_1,b_1)\left|f_1'(b_1)\right|^q+C_{13}(q,p;a_1,b_1)\mu\right)^{1/q}\right.\\
&\quad\left.+\left(C_{12}(q,p;b_1,a_1)\left|f_1'(a_1)\right|^q+C_{10}(q,p;b_1,a_1)\left|f_1'(b_1)\right|^q+C_{14}(q,p;b_1,a_1)\mu\right)^{1/q}\right].
\end{aligned} \tag{39}$$

Hence, the desired result is obtained. □

For $q = 1$, Theorem 10 reduces to the following result.

Corollary 11. *Let $M = [a_1, b_1] \subset \mathbb{R} \setminus \{0\}$ be a p-harmonic convex and let $f_1 : M = [a_1, b_1] \subseteq \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ be a differentiable function on the interior M of M. If $f'_1 \in L[a_1, b_1]$ and $|f'_1|^q$ are in SR(ph) on M and $\lambda \in [0, 1]$, then*

$$\begin{aligned}&\left|(1-\lambda)f_1\left[\left(\frac{2a_1^p b_1^p}{a_1^p + b_1^p}\right)^{1/p}\right] + \lambda\left(\frac{f_1(a_1) + f_1(b_1)}{2}\right) - \frac{p\left(a_1^p b_1^p\right)}{b_1^p - a_1^p}\int_{a_1}^{b_1}\frac{f_1(x)}{x^{1+p}}dx\right| \\ &\leq \frac{\left(b_1^p - a_1^p\right)}{2p\left(a_1^p b_1^p\right)}\Big[\left(C_3(p, a_1, b_1) + C_6(p, b_1, a_1)\right)\left|f'_1(a_1)\right| \\ &\quad + \left(C_5(p, b_1, a_1) + C_4(p, a_1, b_1)\right)\left|f'_1(b_1)\right| + \left(C_7(p, a_1, b_1) + C_8(p, b_1, a_1)\right)\mu\Big],\end{aligned} \tag{40}$$

where C_3, C_4, C_5, C_6, C_7, and C_8 are given by (31) to (36).

Theorem 12. *Let $M = [a_1, b_1] \subset \mathbb{R} \setminus \{0\}$ be a p-harmonic convex and let $f_1 : M = [a_1, b_1] \subseteq \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ be a differentiable function on the interior M of M. If $f'_1 \in L[a_1, b_1]$ and $|f'_1|^q$ are strongly reciprocally (p,h)-convex function of higher order on M, $r, q > 1$, $1/r + 1/q = 1$ and $\lambda \in [0, 1]$, then*

$$\begin{aligned}&\left|(1-\lambda)f_1\left(\left[\frac{2a_1^p b_1^p}{a_1^p + b_1^p}\right]^{1/p}\right) + \lambda\left(\frac{f_1(a_1) + f_1(b_1)}{2}\right) - \frac{p\left(a_1^p b_1^p\right)}{b_1^p - a_1^p}\int_{a_1}^{b_1}\frac{f_1(x)}{x^{1+p}}dx\right| \\ &\leq \frac{\left(b_1^p - a_1^p\right)}{2p\left(a_1^p b_1^p\right)} \times \left(\frac{\lambda^{r+1} + (1-\lambda)^{r+1}}{2(r+1)}\right)^{1/r} \\ &\quad \cdot \Big[\Big(C_9(q, p; a_1, b_1)\left|f'_1(a_1)\right|^q + C_{11}(q, p; a_1, b_1)\left|f'_1(b_1)\right|^q + C_{13}(q, p; a_1, b_1)\mu\Big)^{1/q} + \Big(C_{12}(q, p; b_1, a_1)\left|f'_1(a_1)\right|^q \\ &\quad + C_{10}(q, p; b_1, a_1)\left|f'_1(b_1)\right|^q + C_{14}(q, p; b_1, a_1)\mu\Big)^{1/q}\Big],\end{aligned} \tag{41}$$

where

$$C_9(q, p; a_1, b_1) = \int_0^{1/2} h(1-j)\left(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\right)^{q+q/p} dj, \tag{42}$$

$$C_{10}(q, p; b_1, a_1) = \int_{1/2}^1 h(j)\left(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\right)^{q+q/p} dj, \tag{43}$$

$$C_{11}(q, p; a_1, b_1) = \int_0^{1/2} h(j)\left(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\right)^{q+q/p} dj, \tag{44}$$

$$C_{12}(q, p; b_1, a_1) = \int_{1/2}^1 h(1-j)\left(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\right)^{q+q/p} dj, \tag{45}$$

$$C_{13}(q, p; a_1, b_1) = -\int_0^{1/2} \phi(j)\left(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\right)^{q+q/p}\left\|\frac{1}{b_1^p} - \frac{1}{a_1^p}\right\|^l dj, \tag{46}$$

$$\begin{aligned}C_{14}(q, p; b_1, a_1) = -\int_{1/2}^1 \phi(j)|2j - 2 + \lambda|\left(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\right)^{q+q/p} \cdot \left\|\frac{1}{b_1^p} - \frac{1}{a_1^p}\right\|^l dj.\end{aligned} \tag{47}$$

Proof. Using Lemma 9, we have

$$\begin{aligned}&\left|(1-\lambda)f_1\left(\left[\frac{2a_1^p b_1^p}{a_1^p + b_1^p}\right]\right)^{1/p} + \lambda\left(\frac{f_1(a_1) + f_1(b_1)}{2}\right) - \frac{p\left(a_1^p b_1^p\right)}{b_1^p - a_1^p}\int_{a_1}^{b_1}\frac{f_1(x)}{x^{1+p}}dx\right| \\ &\leq \frac{\left(b_1^p - a_1^p\right)}{2p\left(a_1^p b_1^p\right)}\left[\int_0^{1/2}\left|(2j-\lambda)\left(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\right)^{1+1/p}\right|\left|f'_1 \cdot \left[\left(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\right)^{1/p}\right]\right|dj + \int_{1/2}^1\left|(2j-2+\lambda) \cdot \left(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\right)^{1+1/p}\right|\left|f'_1\left[\left(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\right)^{1/p}\right]\right|dj\right].\end{aligned} \tag{48}$$

Applying Hölder's integral inequality,

$$\begin{aligned}&\leq \frac{\left(b_1^p - a_1^p\right)}{2p\left(a_1^p b_1^p\right)}\left[\left(\left(\int_0^{1/2}|(2j-\lambda)|^r dj\right)^{1/r}\left(\int_0^{1/2}\left|\left(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\right)^{1+1/p} f'_1 \cdot \left[\left(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\right)^{1/p}\right]\right|^q dj\right)^{1/q} + \left(\int_{1/2}^1|(2j-2+\lambda)|^r dj\right)^{1/r} \cdot \left(\int_{1/2}^1\left|\left(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\right)^{1+1/p} f'_1 \cdot \left[\left(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\right)^{1/p}\right]\right|^q dj\right)^{1/q}\right] = \frac{\left(b_1^p - a_1^p\right)}{2p\left(a_1^p b_1^p\right)} \\ &\cdot \left[\left(\int_0^{1/2}|(2j-\lambda)|^r dj\right)^{1/r}\left(\int_0^{1/2}\left(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\right)^{q+q/p}\left|f'_1 \cdot \left[\left(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\right)^{1/p}\right]\right|^q dj\right)^{1/q} + \left(\int_{1/2}^1|(2j-2+\lambda)|^r dj\right)^{1/r} \cdot \left(\int_{1/2}^1\left(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\right)^{q+q/p}\left|f'_1\left[\left(\frac{a_1^p b_1^p}{ja_1^p + (1-j)b_1^p}\right)^{1/p}\right]\right|^q dj\right)^{1/q}\right].\end{aligned} \tag{49}$$

Since $|f_1'(x)|^q$ is in $SR(ph)$, so

$$\leq \frac{(b_1^p - a_1^p)}{2p(a_1^p b_1^p)} \left[\left(\int_0^{1/2} |(2j-\lambda)|^r dj \right)^{1/r} \left(\int_0^{1/2} \left(\frac{a_1^p b_1^p}{j a_1^p + (1-j) b_1^p} \right)^{q+q/p} \right. \right.$$
$$\cdot \left. \left[h(1-j)|f_1'(a_1)|^q + h(j)|f_1'(b_1)|^q - \mu\phi(j) \left\| \frac{1}{b_1^p} - \frac{1}{a_1^p} \right\|^l \right] dj \right)^{1/q}$$
$$+ \left(\int_{1/2}^1 |(2j-2+\lambda)|^r dj \right)^{1/r} \left(\int_{1/2}^1 \left(\frac{a_1^p b_1^p}{j a_1^p + (1-j) b_1^p} \right)^{q+q/p} \right.$$
$$\cdot \left. \left. \left[h(1-j)|f_1'(a_1)|^q + h(j)|(b_1)|^q - \mu\phi(j) \left\| \frac{1}{b_1^p} - \frac{1}{a_1^p} \right\|^l \right] dj \right)^{1/q} \right]$$
$$\leq \frac{(b_1^p - a_1^p)}{2p(a_1^p b_1^p)} \left[\left(\int_0^{1/2} |(2j-\lambda)|^r dj \right)^{1/r} \left(C_9(q,p;a_1,b_1)|f_1'(a_1)|^q \right. \right.$$
$$\left. + C_{11}(q,p;a_1,b_1)|f_1'(b_1)|^q + C_{13}(q,p;a_1,b_1)\mu \right)^{1/q}$$
$$+ \left(\int_0^{1/2} |(2j-2+\lambda)|^r dj \right)^{1/r} \left(C_{12}(q,p;b_1,a_1)|f_1'(a_1)|^q \right.$$
$$\left. \left. + C_{10}(q,p;b_1,a_1)|f_1'(b_1)|^q + C_{14}(q,p;b_1,a_1)\mu \right)^{1/q} \right]$$
$$\leq \frac{(b_1^p - a_1^p)}{2p(a_1^p b_1^p)} \times \left(\frac{\lambda^{r+1} + (1-\lambda)^{r+1}}{2(r+1)} \right)^{1/r}$$
$$\cdot \left[\left(C_9(q,p;a_1,b_1)|f_1'(a_1)|^q + C_{11}(q,p;a_1,b_1)|f_1'(b_1)|^q \right. \right.$$
$$\left. + C_{13}(q,p;a_1,b_1)\mu \right)^{1/q} + \left(C_{12}(q,p;b_1,a_1)|f_1'(a_1)|^q \right.$$
$$\left. \left. + C_{10}(q,p;b_1,a_1)|f_1'(b_1)|^q + C_{14}(q,p;b_1,a_1)\mu \right)^{1/q} \right]. \tag{50}$$

Hence, proved. □

For $\lambda = 0$, Theorem 12 reduces to the following result.

Corollary 13. *Let $M = [a_1, b_1] \subset \mathbb{R} \setminus \{0\}$ be a p-harmonic convex set and let $f_1 : M = [a_1, b_1] \subseteq \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ be a differentiable function on the interior M of M. If $f'_1 \in L[a_1, b_1]$ and $|f'_1|^q$ are in $SR(ph)$ on M, $r, q > 1$, $1/r + 1/q = 1$, and $\lambda \in [0, 1]$, then*

$$\left| f_1 \left[\left(\frac{2a_1^p b_1^p}{a_1^p + b_1^p} \right)^{1/p} \right] - \frac{p(a_1^p b_1^p)}{b_1^p - a_1^p} \int_{a_1}^{b_1} \frac{f_1(x)}{x^{1+p}} dx \right|$$
$$\leq \frac{(b_1^p - a_1^p)}{2p(a_1^p b_1^p)} \times \left(\frac{1}{2(r+1)} \right)^{1/r} \left[\left(C_9(q,p;a_1,b_1)|f_1'(a_1)|^q \right. \right.$$
$$\left. + C_{11}(q,p;a_1,b_1)|f_1'(b_1)|^q + C_{14}(q,p;a_1,b_1)\mu \right)^{1/q}$$
$$+ \left(C_{12}(q,p;b_1,a_1)|f_1'(a_1)|^q + C_{10}(q,p;b_1,a_1)|f_1'(b_1)|^q \right.$$
$$\left. \left. + C_{14}(q,p;b_1,a_1)\mu \right)^{1/q} \right], \tag{51}$$

where C_9, C_{10}, C_{11}, C_{12}, C_{14},and C_{14} are given by (42)–(47). For $\lambda = 1$, Theorem 12 reduces to the following result.

Corollary 14. *Let $M = [a_1, b_1] \subset \mathbb{R} \setminus \{0\}$ be a p -harmonic convex set and let $f_1 : M = [a_1, b_1] \subseteq \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ be a differentiable function on the interior M of M. If $f'_1 \in L[a_1, b_1]$ and $|f'_1|^q$ are strongly reciprocally (p,h)-convex function of higher order on M, $r, q > 1$, $1/r + 1/q = 1$ and $\lambda \in [0, 1]$ then,*

$$\left| \frac{f_1(a_1) + f_1(b_1)}{2} - \frac{p(a_1^p b_1^p)}{b_1^p - a_1^p} \int_{a_1}^{b_1} \frac{f_1(x)}{x^{1+p}} dx \right|$$
$$\leq \frac{(b_1^p - a_1^p)}{2p(a_1^p b_1^p)} \times \left(\frac{1}{2(r+1)} \right)^{1/r}$$
$$\cdot \left[\left(C_9(q,p;a_1,b_1)|f_1'(a_1)|^q + C_{11}(q,p;a_1,b_1)|f_1'(b_1)|^q \right. \right.$$
$$\left. + C_{14}(q,p;a_1,b_1)\mu \right)^{1/q} + \left(C_{12}(q,p;b_1,a_1)|f_1'(a_1)|^q \right.$$
$$\left. \left. + C_{10}(q,p;b_1,a_1)|f_1'(b_1)|^q + C_{14}(q,p;b_1,a_1)\mu \right)^{1/q} \right], \tag{52}$$

where C_9, C_{10}, C_{11}, C_{12}, C_{14},and C_{14} are given by (42)–(47). For $\lambda = 1/3$, Theorem 12 reduces to the following result.

Corollary 15. *Let $M = [a_1, b_1] \subset \mathbb{R} \setminus \{0\}$ be a p-harmonic convex set and let $f_1 : M = [a_1, b_1] \subseteq \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ be a differentiable function on the interior M of M. If $f'_1 \in L[a_1, b_1]$ and $|f'_1|^q$ is strongly reciprocally (p,h)-convex function of higher order on M, $r, q > 1$, $1/r + 1/q = 1$ and $\lambda \in [0, 1]$ then,*

$$\left| \frac{1}{6} \left[f_1(a) + 4f_1 \left[\left(\frac{2a_1^p b_1^p}{a_1^p + b_1^p} \right)^{1/p} \right] + f_1(b) \right] - \frac{p(a_1^p b_1^p)}{b_1^p - a_1^p} \int_{a_1}^{b_1} \frac{f_1(x)}{x^{1+p}} dx \right|$$
$$\leq \frac{(b_1^p - a_1^p)}{2p(a_1^p b_1^p)} \times \left(\frac{1 + 2^{r+1}}{6.3^r (r+1)} \right)^{1/r}$$
$$\cdot \left[\left(C_9(q,p;a_1,b_1)|f_1'(a_1)|^q + C_{11}(q,p;a_1,b_1)|f_1'(b_1)|^q \right. \right.$$
$$\left. + C_{14}(q,p;a_1,b_1)\mu \right)^{1/q} + \left(C_{12}(q,p;b_1,a_1)|f_1'(a_1)|^q \right.$$
$$\left. \left. + C_{10}(q,p;b_1,a_1)|f_1'(b_1)|^q + C_{14}(q,p;b_1,a_1)\mu \right)^{1/q} \right], \tag{53}$$

where C_9, C_{10}, C_{11}, C_{12}, C_{14}, and C_{14} are given by (42)–(47). For $\lambda = 1/2$, Theorem 12 reduces to the following result.

Corollary 16. *Let $M = [a_1, b_1] \subset \mathbb{R} \setminus \{0\}$ be a p-harmonic convex set and let $f_1 : M = [a_1, b_1] \subseteq \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ be a differentiable function on the interior M of M. If $f'_1 \in L[a_1, b_1]$ and $|f'_1|^q$ is strongly reciprocally (p,h)-convex function of higher order on M, $r, q > 1$, $1/r + 1/q = 1$ and $\lambda \in [0, 1]$ then,*

$$\left|\frac{1}{4}\left[f_1(a)+2f_1\left[\left(\frac{2a_1^p b_1^p}{a_1^p+b_1^p}\right)^{1/p}\right]+f_1(b)\right]-\frac{p\left(a_1^p b_1^p\right)}{b_1^p-a_1^p}\int_{a_1}^{b_1}\frac{f_1(x)}{x^{1+p}}dx\right|$$
$$\leq\frac{\left(b_1^p-a_1^p\right)}{2p\left(a_1^p b_1^p\right)}\times\left(\frac{2}{4.2^r(r+1)}\right)^{1/r}$$
$$\cdot\left[\left(C_9(q,p;a_1,b_1)\left|f_1'(a_1)\right|^q+C_{11}(q,p;a_1,b_1)\left|f_1'(b_1)\right|^q\right.\right.$$
$$\left.+C_{14}(q,p;a_1,b_1)\mu\right)^{1/q}+\left(C_{12}(q,p;b_1,a_1)\left|f_1'(a_1)\right|^q\right.$$
$$\left.\left.+C_{10}(q,p;b_1,a_1)\left|f_1'(b_1)\right|^q+C_{14}(q,p;b_1,a_1)\mu\right)^{1/q}\right], \tag{54}$$

where C_9, C_{10}, C_{11}, C_{12}, C_{14}, and C_{14} are given by (42)–(47).

Remark 17. Inserting $h(j)=j$, $\mu=0$ and $l=2$ with $\phi(j)=j(1-j)$ in Corollary 16, we obtained ([20], Corollary 3.8).

6. Conclusion

In this paper, a new definition of convex functions "namely strongly reciprocally (p,h)-convex functions of higher order" is introduced. This new definition extends almost all the existing versions of convex functions. For the strongly reciprocally (p,h)-convex functions of higher order, we established several interesting inequalities which have applications in optimization theory, probability theory, as well as pure and applied mathematics. We also established several fractional versions of the Hermite-Hadamard type inequalities. The remarks presented in the paper justify the validity of our results and prove that our results are more general than almost every existing result.

Data Availability

All data required for this research is included within this paper.

Conflicts of Interest

The authors declare that they do not have any competing interests.

Authors' Contributions

Limei Liu proposed and proved the main results, verified and analyzed the results, and arranged the funding. Muhammad Shoaib Saleem also verified and analyzed the results, arranged the funding for this paper, and prepared the final version of the manuscript. Faisal Yasin wrote the introduction and related the results of this paper to the existing literature. In particular, he showed that several results in the literature can be obtained directly from the results of this paper by suitable substitution of the involved parameters. Kiran Naseem Aslam supervised this work and proved some of its main results. Pengfei Wang proposed the problem and supervised this work.

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