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Research Article

Solving Differential Equation via Orthogonal Branciari Metric Spaces

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In this paper, we investigate an orthogonal L^*-contraction map concept and prove the fixed-point theorem in an orthogonal complete Branciari metric space (OCBMS). We also provide illustrative examples to support our theorems. We demonstrated the existence of a uniqueness solution to the fourth-order differential equation using a more orthogonal L^* contraction operator in OCBMS as an application of the main results.

1. Introduction

The Branciari metric (BM) concept was introduced by Branciari [1] in the year 2000. The generalization is via the fact that the triangle inequality is replaced by the rectangular inequality $\mathfrak{b}(\lambda_1, \lambda_2) \leq \mathfrak{b}(\lambda_1, \lambda_3) + \mathfrak{b}(\lambda_3, \lambda_4) + \mathfrak{b}(\lambda_4, \lambda_2)$ for all pairwise distinct points $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ of $\mathscr{P}$. Afterwards, many authors studied and elaborated the existence of old fixed-point theorems in the BMS (briefly Branciari metric spaces) [2–7]. The Θ-contraction concept was introduced by Jleli and Samet [8] in 2014. Later, some authors provided a variety of results based on Θ-contraction [9, 10]. Saleh et al. [11] introduced the concept of generalized L and L^*-contractions. And also proved fixed-point theorems in CBMS. Eshraghisamani et al. [12] initiated new contractive map and proved fixed-point theorem in BMS.

An orthogonality notion in metric spaces is presented by Gordji et al. in 2017 [13, 14]. Recently, many authors established a variety of fixed-point results in generalized orthogonal metric space (OMS). Nazam et al. [15] demonstrated the concept of (Ψ, Φ)-orthogonal interpolation contraction mappings. The notion of B metric-like space via a hybird pair of operators was introduced by Ali et al. [16] in 2022. In 2021, Hussain [17] presented another family of fractional symmetric α-η-contractions and builds up some new results for such contraction in the context of $\mathscr{F}$-metric space. Mukheimer et al. [18] initiated the concept of orthogonal L-contraction mapping and proved fixed-point results in OBMS.

From the above motivation, we prove some fixed-point results in the direction of OBMS. We also give some examples to argue that our results correctly generalize certain results in the literature.

In this article, we present basic definitions and examples in Section 2, prove some fixed-point theorems by orthogonal L^*-contractive mapping in an OCBMS in Section 3, and finally, obtain a unique solution of differential equation using orthogonal L^* contraction operator in Section 4.

2. Preliminaries

Throughout this article, we denote by $\mathscr{P}$, $\mathbb{N}$, and $\mathbb{R}_+$ the nonempty set, the set of positive integers, and the set of positive real numbers, respectively.

The Branciari metric space was introduced by Branciari [1] as follows.

Definition 1. Let $\mathscr{P} \neq \varnothing$ and a function $\mathfrak{b} : \mathscr{P} \times \mathscr{P} \longrightarrow \mathbb{R}_+$ s.t (briefly such that) $\forall \lambda_1, \lambda_2 \in \mathscr{P}$ and all $\lambda_3 \neq \lambda_4 \in \mathscr{P}/\{\lambda_1, \lambda_2\}$:

(BM1) $\mathfrak{b}(\lambda_1, \lambda_2) = 0$, iff $\lambda_1 = \lambda_2$;

(BM2) $\mathfrak{b}(\lambda_1, \lambda_2) = \mathfrak{b}(\lambda_2, \lambda_1)$;

(BM3) $\mathfrak{b}(\lambda_1, \lambda_2) \leq \mathfrak{b}(\lambda_1, \lambda_3) + \mathfrak{b}(\lambda_3, \lambda_4) + \mathfrak{b}(\lambda_4, \lambda_2)$.

The pair $(\mathscr{P}, \mathfrak{b})$ is called a BMS with Branciari metric $\mathfrak{b}$.

The following example is on the Branciari metric space (BMS).

Example 1. Let $\mathscr{P} = \{0, 2\} \cup \{(1/\imath): \imath \in \mathbb{N}\}$, where $E = \{0, 2\}$ and $G = \{(1/\imath): \imath \in \mathbb{N}\}$. Define $\mathfrak{b} : \mathscr{P} \times \mathscr{P} \longrightarrow \mathbb{R}_+$ as

$$\mathfrak{b}(\wp_1, \wp_2) = \begin{cases} 0, & \text{if } \wp_1 = \wp_2, \\ 1, & \text{if } \wp_1 \neq \wp_2 \text{ and } \{\wp_1, \wp_2\} \subset \mathrm{E} \text{ or } \{\wp_1, \wp_2\} \subset \mathrm{G}, \\ \wp_2, & \text{if } \wp_1 \in \mathrm{E} \text{ and } \wp_2 \in \mathrm{G}, \\ \wp_1, & \text{if } \wp_1 \in \mathrm{G} \text{ and } \wp_2 \in \mathrm{E}. \end{cases} \tag{1}$$

Then, $(\mathscr{P}, \mathfrak{b})$ is a CBMS (briefly complete Branciari metric space). However, we get

(1) $\lim_{\imath \longrightarrow \infty} \mathfrak{b}((1/\imath), (1/2)) \neq \mathfrak{b}(0, (1/2))$ although $\lim_{\imath \longrightarrow \infty} (1/\imath) = 0$, and hence, $\mathfrak{b}$ is discontinuous

(2) There is nonexistence $\ell > 0$ s.t $\mathrm{G}_\ell(0) \cap \mathrm{G}_\ell(2) = \phi$, and hence, the topology is not a Hausdorff

(3) $\mathrm{G}_{(2/3)} = \{0, 2, (1/3)\}$; however, there does not exist $\ell > 0$ s.t $\mathrm{G}_\ell(0) \subseteq \mathrm{G}_{(2/3)}(1/3)$, and thus, an open ball does not necessitate an open set

(4) $\{1/\imath\}_{\imath \in \mathbb{N}}$ is not a Cauchy sequence since it converges to both 0 and 2

Now, we give the following concepts, which are used in this paper.

Definition 2. Let $(\mathscr{P}, \mathfrak{b})$ be a BMS and $\{\alpha_\imath\}$ be a sequence in $\mathscr{P}$ and $\lambda_1 \in \mathscr{P}$.

(1) $\{\alpha_\imath\}$ is convergent to $\lambda_1 \Longleftrightarrow \mathfrak{b}(\alpha_\imath, \alpha_\ell) \longrightarrow 0$ as $\imath \longrightarrow \infty$. We denote this by $\alpha_\imath \longrightarrow \alpha$;

(2) $\{\alpha_\imath\}$ is Cauchy $\Longleftrightarrow \mathfrak{b}(\alpha_\imath, \alpha_\ell) \longrightarrow 0$ as $\imath, \ell \longrightarrow \infty$;

(3) $(\mathscr{P}, \mathfrak{b})$ is complete $\Longleftrightarrow$ every Cauchy sequence in $\mathscr{P}$ which converges to some element in $\mathscr{P}$.

Eshraghisamani et al. [12] introduced the concept of Θ-contraction as follows.

Definition 3. Let $(\mathscr{P}, \mathfrak{b})$ be a BMS. A map $\Phi : \mathscr{P} \longrightarrow \mathscr{P}$ is said to be Θ-contraction if there exist $\Theta \in \Gamma_{1,2,3}$ and $\nu \in (0, 1)$ s.t $(\forall \lambda_1, \lambda_2 \in \mathscr{P})$

$$\mathfrak{b}(\Phi\lambda_1, \Phi\lambda_2) > 0 \Longrightarrow \Theta(\mathfrak{b}(\Phi\lambda_1, \Phi\lambda_2)) \leq [\Theta(\mathfrak{b}(\lambda_1, \lambda_2))]^\nu, \tag{2}$$

where $\Gamma_{1,2,3}$ is the family of all functions $\Theta : (0, \infty) \longrightarrow (0,\infty)$ which satisfy the following axioms:

$(\Theta_1)\Theta$ is increasing

(Θ_2) For each sequence $\{\alpha_\imath\} \subset (0, \infty)$, $\lim_{\imath \longrightarrow \infty} \Theta(\alpha_i) = 1 \Longleftrightarrow \lim_{\imath \longrightarrow \infty} \alpha_\imath = 0^+$

$(\Theta_3)\Theta$ is continuous.

Using Definition 3, Eshraghisamani et al. [12] proved the following theorem.

Theorem 4. *Let $(\mathscr{P}, \mathfrak{b})$ be a CBMS and $\Phi : \mathscr{P} \longrightarrow \mathscr{P}$ a Θ-contraction function. Then, Φ has a* ***ufp*** *(briefly unique fixed point).*

The below example supports Theorem 4.

Example 2. Let $\varsigma_{\Phi,\mathfrak{F}} : [1, \infty) \times [1, \infty) \longrightarrow R$ be two functions defined as below:

$$\varsigma_{\Phi,\mathfrak{F}}(\sigma, \sigma_1) = \frac{\mathfrak{F}(\sigma_1)}{\Phi(\sigma_1)}, \forall \sigma, \sigma_1 \geq 1, \tag{3}$$

where $\Phi, \mathfrak{F} : [1, \infty) \longrightarrow [1, \infty)$ are upper semicontinuous from the right s.t $\mathfrak{F}(\sigma) < \sigma \leq \Phi(\sigma)$, for all $\sigma > 1$. Then, $\varsigma_{\Phi,\mathfrak{F}} \in \mathrm{L}$.

In Theorem 4, by replacing the condition (Θ_3), we get the following remark.

Remark 5. Let $\{\mathfrak{u}_\imath\}, \{\mathfrak{x}_\imath\}, \{\mathfrak{y}_\imath\}$ be the sequence of $\mathbb{R}_+$ s.t $\lim_{\imath \longrightarrow \infty} \mathfrak{u}_\imath = \mathfrak{u}$, $\lim_{\imath \longrightarrow \infty} \mathfrak{x}_\imath = \mathfrak{x}$ and $\lim_{\imath \longrightarrow \infty} \mathfrak{y}_\imath = \mathfrak{y}$. Then,

(1) $\lim_{\imath \longrightarrow \infty} \max \{\mathfrak{u}_\imath, \mathfrak{x}_\imath, \mathfrak{y}_\imath\} = \max \{\mathfrak{u}, \mathfrak{x}, \mathfrak{y}\}$,

(2) $\lim_{\imath \longrightarrow \infty} \min \{\mathfrak{u}_\imath, \mathfrak{x}_\imath, \mathfrak{y}_\imath\} = \min \{\mathfrak{u}, \mathfrak{x}, \mathfrak{y}\}$.

In 2017, Gordji et al. [13] introduced the concept of an orthogonal set as follows.

Definition 6. Let $\mathscr{P} \neq \varnothing$ and $\perp \subseteq \mathscr{P} \times \mathscr{P}$ be a binary relation. If $\perp$ holds

$$\exists \lambda_{10} \in \mathscr{P} : (\forall \lambda_1 \in \mathscr{P}, \lambda_1 \perp \lambda_{10}) \text{ or } (\forall \lambda_1 \in \mathscr{P}, \lambda_{10} \perp \lambda_1), \tag{4}$$

then $(\mathscr{P}, \perp)$ is called an orthogonal set.

The following example and Figure 1 are satisfied by Definition 6.

Example 3. Let $\mathscr{P} = Z$ and define $\lambda_2 \perp \lambda_1$ if $\exists \nu \in Z : \lambda_2 = \nu\lambda_1$. It is clear that $0 \perp \lambda_1, \forall \lambda_1 \in Z$. Hence, $(\mathscr{P}, \perp)$ is an orthogonal set.

Example 4. A wheel graph $\mathscr{W}_\imath$ with $\imath$ edge for every $\imath \geq 4$, a node connect to each node to every edge of $(\imath - 1)$ -cycle. Let $\mathscr{P}$ be the set of all edge of $\mathscr{W}_\imath$ for every $\imath \geq 4$. Define $\lambda_1 \perp \lambda_2$ if there is a connection from λ_1 to λ_2. Then, $(\mathscr{P}, \perp)$ is an orthogonal set.

The following orthogonal sequence definition was introduced by Gordji et al. [13] which will be utilized in this paper to prove main results.

Definition 7. Let $(\mathscr{P}, \perp)$ be an orthogonal set. A sequence $\{\lambda_{1_\imath}\}$ is called an orthogonal sequence (shortly, O -sequence) if

$$(\forall \imath \in \mathbb{N}, \lambda_{1_\imath} \perp \lambda_{1_{\imath+1}}) \text{ or } (\forall \imath \in \mathbb{N}, \lambda_{1_{\imath+1}} \perp \lambda_{1_\imath}). \tag{5}$$

Again, the concepts of orthogonal continuous also introduced by Gordji et al. [13].

Definition 8. Let $(\mathscr{P}, \perp, \mathfrak{b})$ be a OMS. Then, a mapping $\Phi : \mathscr{P} \longrightarrow \mathscr{P}$ is called orthogonal continuous in $\lambda_1 \in \mathscr{P}$ if for every O -sequence $\{\lambda_{1_\imath}\}$ in $\mathscr{P}$ with $\lambda_{1_\imath} \longrightarrow \lambda_1$ as $\imath \longrightarrow \infty$, we have $\Phi(\lambda_{1_\imath}) \longrightarrow \Phi(\lambda_1)$ as $\imath \longrightarrow \infty$.

Definition 9. Let $(\mathscr{P}, \perp, \mathfrak{b})$ be a OBMS.

(1) $\{\lambda_{1_\imath}\}$, an orthogonal sequence in $\mathscr{P}$, converges at a point λ_1 if

$$\lim_{\imath \longrightarrow \infty} \Phi\left(\lambda_{1_\imath}, \lambda_1\right) = 0. \tag{6}$$

(2) $\{\lambda_{1_\imath}\}, \{\lambda_{1_m}\}$ are orthogonal sequences in $\mathscr{P}$ and are said to be orthogonal Cauchy sequence if

$$\lim_{\imath, m \longrightarrow \infty} \Phi\left(\lambda_{1_\imath}, \lambda_{1_m}\right) < \infty. \tag{7}$$

Gordji et al. [13] introduced the concept of an orthogonal complete as follows.

Definition 10. Let $(\mathscr{P}, \perp, \mathfrak{b})$ be a OMS. Then, $\mathscr{P}$ is called an orthogonal complete, if every orthogonal Cauchy sequence is convergent.

Finally, the following orthogonal-preserving concepts introduced by Gordji et al. [13] is of importance in this paper.

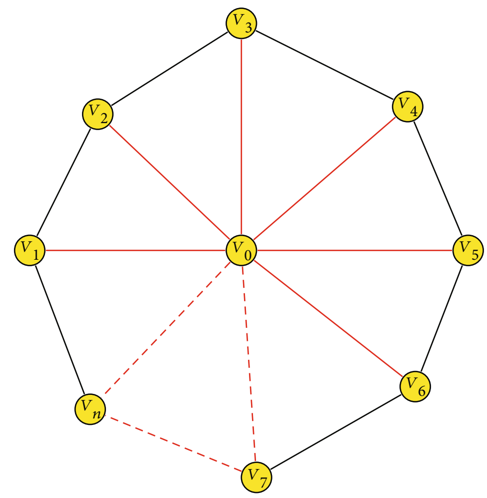

Figure 1: A wheel graph.

Definition 11. Let $(\mathscr{P}, \perp)$ be an orthogonal set. A function $\Phi : \mathscr{P} \longrightarrow \mathscr{P}$ is called a $\perp$ -preserving if $\Phi\lambda_1 \perp \Phi\lambda_2$ whenever $\lambda_1 \perp \lambda_2, \forall \lambda_1, \lambda_2 \in \mathscr{P}$.

Lemma 12. *Let $\{\lambda_{1_\imath}\}$ be an orthogonal Cauchy sequence in BMS $(\mathscr{P}, \mathfrak{b})$ s.t $\lim_{\imath \longrightarrow \infty} \mathfrak{b}(\lambda_{1_\imath}, \lambda_1) = 0$, for some $\lambda_1 \in \mathscr{P}$. Then, $\lim_{\imath \longrightarrow \infty} \mathfrak{b}(\lambda_{1_\imath}, \lambda_2) = \mathfrak{b}(\lambda_1, \lambda_2)$, for all $\lambda_1, \lambda_2 \in \mathscr{P}$, with $\lambda_1 \perp \lambda_2$.*

Eshraghisamani et al. [12] proved fixed-point result on Branciari metric space as follows.

Theorem 13. *Let $(\mathscr{P}, \mathfrak{b})$ be a complete generalized metric space and a map $\Phi : \mathscr{P} \longrightarrow \mathscr{P}$. Suppose that there exist $\ell \in (0, 1)$ and function $\pi : \mathscr{R}_+ \longrightarrow \mathscr{R}_+$, satisfying the following conditions:*

(i) For every $\{\beta_\imath\} \subset (0, \infty)$ and nonconstant

$$\lim_{\imath \longrightarrow \infty} \pi(\beta_\imath) = 0 \Longleftrightarrow \lim_{\imath \longrightarrow \infty} \beta_\imath = 0. \tag{8}$$

(ii) For every $\{\beta_\imath\} \subset (0,\infty)$ that $\beta_\imath \longrightarrow 0^+$, $\limsup_{\imath \longrightarrow \infty} \sqrt[\imath]{\pi(\beta_\imath)} < 1 \Longrightarrow \sum_1^\infty \beta_\imath < \infty$, such that

$$\pi(\mathfrak{b}(\Phi\lambda_1, \Phi\lambda_2)) \leq \ell\pi(\mathfrak{b}(\lambda_1, \lambda_2)), \tag{9}$$

*then ϕ has a **ufp**.*

3. Main Results

Before presenting our main result of this section, we are inspired by the concept of L^* contraction mapping defined by Saleh et al. [11]; we introduce a new concept of an orthogonal L^*-contraction mapping. Then, we prove a fixed-point results in OCBMS.

Definition 14. Let $(\mathscr{P},\perp,\mathfrak{b})$ be a OBMS and $\Phi:\mathscr{P}\longrightarrow\mathscr{P}$. Then, Φ is called an orthogonal L^* -contraction *w.r.t* $\zeta\in L$ if $\exists\Theta\in\Omega_{1,2,3}$ s.t.

$$\begin{aligned}&\forall\lambda_1,\lambda_2\in\mathscr{P}\text{ with }\lambda_1\perp\lambda_2,\mathfrak{b}(\Phi\lambda_1,\Phi\lambda_2)\\&\quad>0\Longrightarrow\zeta[\Theta(\mathfrak{b}(\Phi\lambda_1,\Phi\lambda_2)),\Theta(\mathscr{M}(\lambda_1,\lambda_2))]\geq 1,\end{aligned}\tag{10}$$

where $\mathscr{M}(\lambda_1,\lambda_2)=\max\{\mathfrak{b}(\lambda_1,\lambda_2),\mathfrak{b}(\lambda_1,\Phi\lambda_1),\mathfrak{b}(\lambda_2,\Phi\lambda_2)\}$.

Motivated by Theorem 13, we prove the below theorem.

Theorem 15. *Let $(\mathscr{P},\perp,\mathfrak{b})$ be a OCBMS and Φ is a self-map on $\mathscr{P}$. Suppose that $\exists\ell\in(0,1)$ and a function $\pi:\mathscr{R}_+\longrightarrow\mathscr{R}_+$ hold the axioms:*

(i) Φ is orthogonal-preserving

(ii) For every $\{\beta_\imath\}\subset(0,\infty)$ and nonconstant

$$\lim_{\imath\longrightarrow\infty}\pi(\beta_\imath)=0\Longleftrightarrow\lim_{\imath\longrightarrow\infty}\beta_\imath=0.\tag{11}$$

(iii) $\Phi_\perp$ with for every $\{\beta_\imath\}\subset(0,\infty)$ that $\beta_\imath\longrightarrow 0^+$, $\lim\sup_{\imath\longrightarrow\infty}\sqrt[\imath]{\pi(\beta_\imath)}<1\Longrightarrow\sum_1^\infty\beta_\imath<\infty$such that

$$\forall\lambda_1,\lambda_2\in\mathscr{P}\text{ with }\lambda_1\perp\lambda_2\Longrightarrow\pi(\mathfrak{b}(\Phi\lambda_1,\Phi\lambda_2))\leq\ell\pi(\mathfrak{b}(\lambda_1,\lambda_2)),\tag{12}$$

*then Φ has a **ufp**.*

Proof. Since $(\mathscr{P},\perp)$ is orthogonal set,

$$\exists\lambda_2\in\mathscr{P}:(\forall\lambda_1\in\mathscr{P},\lambda_1\perp\lambda_2)\text{ or }(\forall\lambda_1\in\mathscr{P},\lambda_2\perp\lambda_1).\tag{13}$$

It follows that $\lambda_2\perp\Phi\lambda_2$ or $\Phi\lambda_2\perp\lambda_2$. Let

$$\begin{aligned}\lambda_{1_1}&=\Phi\lambda_2,\lambda_{1_2}=\Phi\lambda_{1_1}=\Phi^2\lambda_2\cdots\cdots,\lambda_{1_{\imath+1}}\\&=\Phi\lambda_{1_\imath}=\Phi^{\imath+1}\lambda_2,\forall\imath\in\mathbb{N}\cup\{0\}.\end{aligned}\tag{14}$$

If $\lambda_{1_{\imath_0}}=\lambda_{1_{\imath_0+1}}$ for any $\imath\in\mathbb{N}\cup\{0\}$, then it is easy to see that λ_{10} is a fixed point of Φ. Consider that $\lambda_{1_{\imath_0}}\neq\lambda_{1_{\imath_0+1}}$ for all $\imath\in\mathbb{N}\cup\{0\}$. Since Φ is $\perp$-preserving, we have

$$\lambda_{1_{\imath_0}}\perp\lambda_{1_{\imath_0+1}}\text{or }\lambda_{1_{\imath_0+1}}\perp\lambda_{1_{\imath_0}}\forall\imath\in\mathbb{N}\cup\{0\}.\tag{15}$$

This implies that $\{\mathfrak{b}(\lambda_\imath,\lambda_{\imath+1})\}>0$ is an O-sequence.

First, we show that $\lim_{\imath\longrightarrow\infty}\mathfrak{b}(\lambda_\imath,\lambda_{\imath+1})=0$. Since Φ satisfies (12), for all $\imath\in\mathbb{N}$, we have

$$\pi(\mathfrak{b}(\lambda_{1_\imath},\lambda_{1_{\imath+1}}))\leq\ell\pi(\mathfrak{b}(\lambda_{1_{\imath-1}},\lambda_{1_\imath})).\tag{16}$$

Since $\ell\in(0,1)$, we have

$$\pi(\mathfrak{b}(\lambda_{1_\imath},\lambda_{1_{\imath+1}}))\leq\ell\pi(\mathfrak{b}(\lambda_{1_{\imath-1}},\lambda_{1_\imath}))\leq\pi(\mathfrak{b}(\lambda_{1_{\imath-1}},\lambda_{1_\imath})),\forall\imath\in\mathbb{N}.\tag{17}$$

Thus, $\{\pi(\mathfrak{b}(\lambda_{\imath+1},\lambda_\imath))\}$ is a decreasing sequence; hence, it is convergent and

$$\lim_{\imath\longrightarrow\infty}\pi(\mathfrak{b}(\lambda_{1_{\imath+1}},\lambda_{1_\imath}))=\mathfrak{u}\geq 0.\tag{18}$$

Now, we show that $\mathfrak{u}=0$. From (17), we have

$$\pi(\mathfrak{b}(\lambda_{1_{\imath+1}},\lambda_{1_\imath}))\leq\ell\pi(\mathfrak{b}(\lambda_{1_\imath},\lambda_{1_{\imath-1}}))\leq\cdots\cdots\leq\ell^\imath\pi\left(\mathfrak{b}\left(\lambda_{1_1},\lambda_{10}\right)\right),\tag{19}$$

since $0<\ell<1$; therefore, $\lim_{\imath\longrightarrow\infty}\pi(\mathfrak{b}(\lambda_{1_{\imath+1}},\lambda_{1_\imath}))=0$. So, $\lim_{\imath\longrightarrow\infty}\mathfrak{b}(\lambda_{1_{\imath+1}},\lambda_{1_\imath})=0$ by (ii).

On the other hand from (19), we have

$$\pi(\mathfrak{b}(\lambda_{1_{\imath+1}},\lambda_{1_\imath}))\leq\ell^\imath\pi\left(\mathfrak{b}\left(\lambda_{1_1},\lambda_{10}\right)\right),\forall\imath\in\mathbb{N}.\tag{20}$$

Then,

$$\sqrt[\imath]{\pi(\mathfrak{b}(\lambda_{1_{\imath+1}},\lambda_{1_\imath}))}\leq\ell^\imath\sqrt[\imath]{\pi\left(\mathfrak{b}\left(\lambda_{1_1},\lambda_{10}\right)\right)},\forall\imath\in\mathbb{N}.\tag{21}$$

Thus,

$$\lim_{\imath\longrightarrow\infty}\sqrt[\imath]{\pi(\mathfrak{b}(\lambda_{1_{\imath+1}},\lambda_{1_\imath}))}\leq\ell<1.\tag{22}$$

Put $\beta_\imath=\mathfrak{b}(\lambda_{1_{\imath+1}},\lambda_{1_\imath})$; using (22), and condition (iii) of π, we get

$$\sum_1^\infty\beta_\imath<\infty\text{ and also }\beta_\imath\longrightarrow 0.\tag{23}$$

Now, we will show that $\mathfrak{b}(\lambda_{1_\imath},\lambda_{1_{\imath+2}})\longrightarrow 0$ as $\imath\longrightarrow\infty$.

$$\begin{aligned}0&<\pi(\mathfrak{b}(\lambda_{1_{\imath+2}},\lambda_{1_\imath}))\leq\ell\pi(\mathfrak{b}(\lambda_{1_{\imath+1}},\lambda_{1_{\imath-1}}))\\&\leq\cdots\leq\ell^\imath\pi\left(\mathfrak{b}\left(\lambda_{1_2},\lambda_{10}\right)\right).\end{aligned}\tag{24}$$

Therefore, $\mathfrak{b}(\lambda_{1_{\imath+2}},\lambda_{1_\imath})\longrightarrow 0$, as $\imath\longrightarrow\infty$. □

Now, to prove that the sequence $\{\lambda_{1_\imath}\}$ is Cauchy, we consider two cases.

Case 1. If $\mathfrak{m}=2\mathfrak{p}+1$, $\mathfrak{p}\geq 1$, then

$$\begin{aligned}\mathfrak{b}(\lambda_{1_\imath},\lambda_{1_{\imath+\mathfrak{m}}})&\leq\mathfrak{b}(\lambda_{1_\imath},\lambda_{1_{\imath+1}})+\mathfrak{b}(\lambda_{1_{\imath+1}},\lambda_{1_{\imath+2}})\\&\quad+..\cdots+\mathfrak{b}\left(\lambda_{1_{\imath+2\mathfrak{p}}},\lambda_{1_{\imath+2\mathfrak{p}+1}}\right)\\&\leq\sum_\imath^{\imath+2\mathfrak{p}+1}\beta_\imath<\sum_\imath^\infty\beta_\imath.\end{aligned}\tag{25}$$

Case 2. If $\mathfrak{m} = 2\mathfrak{p}$, $\mathfrak{p} \geq 2$, then

$$\mathfrak{b}(\lambda_{1_\imath}, \lambda_{1_{\imath+\mathfrak{m}}}) \leq \mathfrak{b}(\lambda_{1_\imath}, \lambda_{1_{\imath+2}}) + \mathfrak{b}(\lambda_{1_{\imath+2}}, \lambda_{1_{\imath+3}}) + \ldots \cdots + \mathfrak{b}\left(\lambda_{1_{\imath+2\mathfrak{p}-1}}, \lambda_{1_{\imath+2\mathfrak{p}}}\right) \leq \sum_{\imath}^{\imath+2\mathfrak{p}+1} \beta_\imath < \sum_{\imath}^{\infty} \beta_\imath. \tag{26}$$

Thus, combining these two cases and using (23), when $\imath \longrightarrow \infty$, we have

$$\mathfrak{b}(\lambda_{1_\imath}, \lambda_{1_{\imath+\mathfrak{m}}}) \leq \sum_{\imath}^{\infty} \beta_\imath \longrightarrow 0, \text{ as } \imath \longrightarrow \infty. \tag{27}$$

Thus, we deduce that $\{\Phi^\imath \lambda_1\}$ is an orthogonal Cauchy sequence.

Completeness of $(\mathscr{P},\perp,\mathfrak{b})$ ensures $\lim_{\mathfrak{m}\longrightarrow\infty}\lambda_{1_\imath} = \mathfrak{z}$ for some $\mathfrak{z} \in \mathscr{P}$.

Now, we want to show that $\mathfrak{z}$ is a fixed point of $\mathscr{P}$. From (12), we have

$$\pi(\mathfrak{b}(\Phi\lambda_{1_\imath}, \Phi\mathfrak{z})) \leq \pi(\mathfrak{b}(\lambda_{1_\imath}, \mathfrak{z})). \tag{28}$$

Hence, $\mathfrak{b}(\lambda_{1_\imath}, \mathfrak{z}) \longrightarrow 0$, and $\pi(\mathfrak{b}(\lambda_{1_\imath}, \mathfrak{z})) \longrightarrow 0$, and therefore, $\lim_{\imath\longrightarrow\infty}\pi(\mathfrak{b}(\lambda_{1_{\imath+1}}, \Phi\mathfrak{z})) = 0$ as $\imath \longrightarrow \infty$. Again,

$$\lim_{\imath\longrightarrow\infty} \mathfrak{b}(\lambda_{1_{\imath+1}}, \Phi\mathfrak{z}) = 0, \tag{29}$$

by using (ii).

$$\mathfrak{b}(\mathfrak{z}, \Phi\mathfrak{z}) \leq \mathfrak{b}(\mathfrak{z}, \lambda_{1_\imath}) + \mathfrak{b}(\lambda_{1_\imath}, \lambda_{1_{\imath+1}}) + \mathfrak{b}(\lambda_{1_{\imath+1}}, \Phi\mathfrak{z}). \tag{30}$$

Thus, $\mathfrak{z} = \Phi\mathfrak{z}$, and hence, $\mathfrak{z}$ is a fixed point on Φ.

Now, we prove that Φ is unique. Conversely, assume that any two fixed points s.t $\mathfrak{b}(\lambda_1, \mathfrak{z}) = \mathfrak{b}(\Phi\lambda_1, \Phi\mathfrak{z}) > 0$. From (12), since Φ is preserving, $\forall \Phi\lambda_1 \perp \Phi\mathfrak{z}$, we have

$$\begin{aligned}&(\Phi^\imath\lambda_1 \perp \Phi^\imath\lambda_2 \text{ and } \Phi^\imath\lambda_1 \perp \Phi^\imath\mathfrak{z}) \text{ or}\\&(\Phi^\imath\lambda_1 \perp \Phi^\imath\mathfrak{z} \text{ and } \Phi^\imath\lambda_1 \perp \Phi^\imath\lambda_2), \forall \imath \in \mathbb{N}.\end{aligned} \tag{31}$$

Now,

$$\mathfrak{b}(\lambda_2, \mathfrak{z}) = \mathfrak{b}(\Phi^\imath\lambda_2, \Phi^\imath\mathfrak{z}) \leq \mathfrak{b}(\Phi^\imath\lambda_2, \Phi^\imath\lambda_1) + \mathfrak{b}(\Phi^\imath\lambda_1, \Phi^\imath\mathfrak{z}). \tag{32}$$

This implies that

$$\pi(\mathfrak{b}(\lambda_2, \mathfrak{z})) < \pi(\mathfrak{b}(\lambda_2, \mathfrak{z})). \tag{33}$$

This is a contradiction. Then Φ has a **ufp**.

The below example validates the proof of Theorem 15.

Example 5. Let $\mathscr{P} = [-2,-1] \cup [1, 2]$ and $\mathfrak{b} : \mathscr{P} \times \mathscr{P} \longrightarrow [0,\infty)$ defined as follow $\mathfrak{b}(\lambda_1, \lambda_1) = 0$, for all $\lambda_1 \in \mathscr{P}$

$$\begin{aligned}\mathfrak{b}(1, 2) = \mathfrak{b}(2, 1) = 3, \mathfrak{b}(1,-1) = \mathfrak{b}(-1, 1)\\ = \mathfrak{b}(-1, 2) = \mathfrak{b}(2,-1) = 1,\end{aligned} \tag{34}$$

we define the relation $\lambda_1 \perp \lambda_2$ and $\mathfrak{b}(\lambda_1, \lambda_2) = |\lambda_1 - \lambda_2|$, otherwise.

We observe that

$$\mathfrak{b}(1, 2) > \mathfrak{b}(1,-1) + \mathfrak{b}(-1, 2). \tag{35}$$

Hence, $\Phi_\perp$-preserving, $\mathfrak{b}(\lambda_1, \lambda_2)$ is not a BMS. It is obvious that $\mathfrak{b}(\lambda_1, \lambda_2)$ is a OCBMS.

Let $\Phi : \mathscr{P} \longrightarrow \mathscr{P}$ be a map defined by

$$\Phi\lambda_1 = \begin{cases} \dfrac{3}{4}\lambda_1, & \lambda_1 \in \left[-2,-\dfrac{3}{2}\right) \cup \left(\dfrac{3}{2}, 2\right], \\ 0, & \text{otherwise}. \end{cases} \tag{36}$$

Now, we define $\pi : [0,\infty) \longrightarrow [0,\infty)$ by $\pi(\beta) = \sqrt{\beta}$.

Easily, we can show that π satisfies conditions (ii) and (iii) of Theorem 15, Φ satisfies (12), and $\lambda_1^* = 0$ is fixed point of Φ.

Saleh et al. [11] proved a new contractive maps and their fixed points on BMS as follows:

Theorem 16. *Let $(\mathscr{P}, \mathfrak{b})$ be a BMS and $\Phi : \mathscr{P} \longrightarrow \mathscr{P}$ be an L^* -contraction w.r.t (briefly with respect to) $\zeta \in L$. Then, Φ has a **ufp**.*

In the following theorem, we are going to prove fixed-point theorem on an orthogonal L^*-contraction mapping using continuity hypothesis of Φ.

Theorem 17. *Let $(\mathscr{P},\perp,\mathfrak{b})$ be a OCBMS with an orthogonal element λ_2 and a function $\Phi : \mathscr{P} \longrightarrow \mathscr{P}$, orthogonal L^* -contraction w.r.t $\zeta \in L$, the following axioms are satisfy:*

(i) Φ is orthogonal-preserving.

(ii) Φ is $\Phi_\perp$ with L^*-contraction mapping.

*Then, Φ has a **ufp**.*

Proof. Since $(\mathscr{P}, \perp)$ is orthogonal set,

$$\exists \lambda_2 \in \mathscr{P} : (\forall \lambda_1 \in \mathscr{P}, \lambda_1 \perp \lambda_2) \text{ or } (\forall \lambda_1 \in \mathscr{P}, \lambda_2 \perp \lambda_1). \tag{37}$$

It follows that $\lambda_2 \perp \Phi\lambda_2$ or $\Phi\lambda_2 \perp \lambda_2$. Let

$$\lambda_{1_1} = \Phi\lambda_2, \lambda_{1_2} = \Phi\lambda_{1_1} = \Phi^2\lambda_2 \cdots \cdots, \lambda_{1_{\imath+1}} = \Phi\lambda_{1_i} = \Phi^{\imath+1}\lambda_2, \tag{38}$$

for all $\imath \in \mathbb{N} \cup \{0\}$.

If $\lambda_{1_{\imath_0}} = \lambda_{1_{\imath_0+1}}$ for any $\imath \in \mathbb{N} \cup \{0\}$, then it is easy to see that λ_{10} is a fixed point of Φ. Consider $\lambda_{1_{\imath_0}} \neq \lambda_{1_{\imath_0+1}}, \forall \imath \in \mathbb{N} \cup \{0\}$. Since Φ is $\perp$-preserving, we have

$$\lambda_{1_{\imath_0}} \perp \lambda_{1_{\imath_0+1}} \text{ or } \lambda_{1_{\imath_0+1}} \perp \lambda_{1_{\imath_0}}, \tag{39}$$

for all $\imath \in \mathbb{N} \cup \{0\}$. Which implies that $\{\lambda_{1_\imath}\}$ is a O-sequence. □

Using equation (10) and (ζ_2^*), we have

$$\begin{aligned} 1 &\le \zeta[\Theta(\mathfrak{b}(\Phi\lambda_{1_{\imath-1}}, \Phi\lambda_{1_\imath})), \Theta(\mathscr{M}(\lambda_{1_{\imath-1}}, \lambda_{1_\imath}))] \\ &= \zeta[\Theta(\mathfrak{b}(\lambda_{1_\imath}, \lambda_{1_{\imath+1}})), \Theta(\mathscr{M}(\lambda_{1_{\imath-1}}, \lambda_{1_\imath}))] \\ &< \frac{\Theta(\mathscr{M}(\lambda_{1_{\imath-1}}, \lambda_{1_\imath}))}{\Theta(\mathfrak{b}(\lambda_{1_\imath}, \lambda_{1_{\imath+1}}))}. \end{aligned} \tag{40}$$

Consequently, we obtain that

$$\Theta(\mathfrak{b}(\lambda_{1_\imath}, \lambda_{1_{\imath+1}})) < \Theta(\mathscr{M}(\lambda_{1_{\imath-1}}, \lambda_{1_i})), \forall \imath \in \mathbb{N}, \tag{41}$$

where

$$\begin{aligned} &\mathscr{M}(\lambda_{1_{\imath-1}}, \lambda_{1_\imath}) \\ &\quad = \max\{\mathfrak{b}(\lambda_{1_{\imath-1}}, \lambda_{1_\imath}), \mathfrak{b}(\lambda_{1_{\imath-1}}, \Phi\lambda_{1_{\imath-1}}), \mathfrak{b}(\lambda_{1_\imath}, \Phi\lambda_{1_\imath})\} \\ &\quad = \max\{\mathfrak{b}(\lambda_{1_{\imath-1}}, \lambda_{1_\imath}), \mathfrak{b}(\lambda_{1_\imath}, \lambda_{1_{\imath+1}})\}. \end{aligned} \tag{42}$$

If $\mathscr{M}(\lambda_{1_{\imath-1}}, \lambda_{1_\imath}) = \mathfrak{b}(\lambda_{1_\imath}, \lambda_{1_{\imath+1}})$, then inequality (41) becomes

$$\Theta(\mathfrak{b}(\lambda_{1_\imath}, \lambda_{1_{\imath+1}})) < \Theta(\mathfrak{b}(\lambda_{1_\imath}, \lambda_{1_{\imath+1}})), \forall \imath \in \mathbb{N}. \tag{43}$$

This is a contradiction. Hence, we must have $\mathscr{M}(\lambda_{1_{\imath-1}}, \lambda_{1_\imath}) = \mathfrak{b}(\lambda_{1_{\imath-1}}, \lambda_{1_\imath})$, for all $\imath \in \mathbb{N}$. Therefore, inequality (41) becomes

$$\Theta(\mathfrak{b}(\lambda_{1_\imath}, \lambda_{1_{\imath+1}})) < \Theta(\mathfrak{b}(\lambda_{1_{\imath-1}}, \lambda_{1_\imath})), \forall \imath \in \mathbb{N}, \tag{44}$$

which implies from (Θ_1) that

$$\mathfrak{b}(\lambda_{1_\imath}, \lambda_{1_{\imath+1}}) < \mathfrak{b}(\lambda_{1_{\imath-1}}, \lambda_{1_\imath}), \forall \imath \in \mathbb{N}. \tag{45}$$

Thus, $\{\mathfrak{b}(\lambda_{1_{\imath-1}}, \lambda_{1_\imath})\}$ is decreasing sequence and boundary below by 0, so $\exists \mathfrak{r} \ge 0$ s.t $\lim_{\imath \longrightarrow \infty} \mathfrak{b}(\lambda_{1_{\imath-1}}, \lambda_{1_\imath}) = \mathfrak{r}$. Suppose that $\mathfrak{r} \neq 0$, then from (Θ_2)

$$\lim_{\imath \longrightarrow \infty} \Theta(\mathfrak{b}(\lambda_{1_{\imath-1}}, \lambda_{1_\imath})) > 1. \tag{46}$$

Taking $\alpha_\imath = \Theta(\mathfrak{b}(\lambda_{1_\imath}, \lambda_{1_{\imath+1}}))$ and $\mathfrak{b}_\imath = \Theta(\mathfrak{b}(\lambda_{1_{\imath-1}}, \lambda_{1_\imath}))$, $\forall \imath \in \mathbb{N}$, it is clear from (44), (46), and (Θ_3) that $\alpha_\imath < \mathfrak{b}_\imath$, $\forall \imath \in \mathbb{N}$, and $\lim_{\imath \longrightarrow \infty} \alpha_\imath = \lim_{\imath \longrightarrow \infty} \mathfrak{b}_\imath > 1$. Hence, using (ζ_3^*), we get

$$1 \le \limsup_{\imath \longrightarrow \infty} \zeta(\alpha_\imath, \mathfrak{b}_\imath) < 1. \tag{47}$$

This is a contradiction. Therefore, $\mathfrak{r} = 0$, we have

$$\lim_{\imath \longrightarrow \infty} \mathfrak{b}(\lambda_{1_{\imath-1}}, \lambda_{1_\imath}) = 0, \forall \imath \in \mathbb{N}. \tag{48}$$

Now, let us assume that $\lambda_{1_\mathfrak{m}} = \lambda_{1_\imath}$, for some $\mathfrak{m} > \imath$. Then, we have $\lambda_{1_{\mathfrak{m}+1}} = \lambda_{1_{\imath+1}}$. Using (44), we get

$$\begin{aligned} &\Theta(\mathfrak{b}(\lambda_{1_\mathfrak{m}}, \lambda_{1_{\mathfrak{m}+1}})) \\ &\quad < \Theta(\mathfrak{b}(\lambda_{1_{\mathfrak{m}-1}}, \lambda_{1_\mathfrak{m}})) < \Theta(\mathfrak{b}(\lambda_{1_{\mathfrak{m}-2}}, \lambda_{1_{\mathfrak{m}-1}})) \\ &\quad < \cdots \cdots < \Theta(\mathfrak{b}(\lambda_{1_\imath}, \lambda_{1_{\imath+1}})) = \Theta(\mathfrak{b}(\lambda_{1_\mathfrak{m}}, \lambda_{1_{\mathfrak{m}+1}})). \end{aligned} \tag{49}$$

This is a contradiction. To summarize $\lambda_{1_\mathfrak{m}} \neq \lambda_{1_\imath}$, for all $\mathfrak{m} \neq \imath$.

Next, to prove $\{\lambda_{1_\imath}\}$ is a orthogonal Cauchy sequence in $(\mathscr{P}, \perp, \mathfrak{b})$. Now, we consider it as not an orthogonal Cauchy; then, we can find two subsequences $\{\lambda_{1_{\imath_\ell}}\}$, and $\{\lambda_{1_{\mathfrak{m}_\ell}}\}$ of $\{\lambda_{1_\imath}\}$ s.t $\imath_\ell$ is the smallest integer for which

$$\begin{gathered} \imath_\ell > \mathfrak{m}_\ell > \ell, \\ \mathfrak{b}\left(\lambda_{1_{\mathfrak{m}_\ell}}, \lambda_{1_{\imath_\ell}}\right) \ge \varepsilon, \\ \mathfrak{b}\left(\lambda_{1_{\mathfrak{m}_\ell}}, \lambda_{1_{\imath_\ell - 2}}\right) < \varepsilon. \end{gathered} \tag{50}$$

By using a similar argument, we obtain

$$\lim_{\ell \longrightarrow \infty} \mathfrak{b}\left(\lambda_{1_{\mathfrak{m}_\ell}}, \lambda_{1_{\imath_\ell}}\right) = \varepsilon = \lim_{\ell \longrightarrow \infty} \mathfrak{b}\left(\lambda_{1_{\mathfrak{m}_\ell - 1}}, \lambda_{1_{\imath_\ell - 1}}\right). \tag{51}$$

Now, using (10) and (ζ_2^*), we have

$$\begin{aligned} 1 &\le \zeta\left[\Theta\left(\mathfrak{b}\left(\Phi\lambda_{1_{\mathfrak{m}_\ell - 1}}, \Phi\lambda_{1_{\imath_\ell - 1}}\right)\right), \Theta\left(\mathscr{M}\left(\lambda_{1_{\mathfrak{m}_\ell - 1}}, \lambda_{1_{\imath_\ell - 1}}\right)\right)\right] \\ &= \zeta\left[\Theta\left(\mathfrak{b}\left(\lambda_{1_{\mathfrak{m}_\ell}}, \lambda_{1_{\imath_\ell}}\right)\right), \Theta\left(\mathscr{M}\left(\lambda_{1_{\mathfrak{m}_\ell - 1}}, \lambda_{1_{\imath_\ell - 1}}\right)\right)\right] \\ &< \frac{\Theta\left(\mathscr{M}\left(\lambda_{1_{\mathfrak{m}_\ell - 1}}, \lambda_{1_{\imath_\ell - 1}}\right)\right)}{\Theta\left(\mathfrak{b}\left(\lambda_{1_{\mathfrak{m}_\ell}}, \lambda_{1_{\imath_\ell}}\right)\right)}, \end{aligned} \tag{52}$$

which implies that

$$\Theta\left(\mathfrak{b}\left(\lambda_{1_{\mathfrak{m}_\ell}}, \lambda_{1_{\imath_\ell}}\right)\right) < \Theta\left(\mathscr{M}\left(\lambda_{1_{\mathfrak{m}_\ell - 1}}, \lambda_{1_{\imath_\ell - 1}}\right)\right), \forall \ell \in \mathbb{N}, \tag{53}$$

where

$$\begin{aligned} \mathscr{M}\left(\lambda_{1_{\mathfrak{m}_\ell - 1}}, \lambda_{1_{\imath_\ell - 1}}\right) &= \max\left\{\mathfrak{b}\left(\lambda_{1_{\mathfrak{m}_\ell - 1}}, \lambda_{1_{\imath_\ell - 1}}\right), \mathfrak{b}\right. \\ &\quad \left. \cdot \left(\lambda_{1_{\mathfrak{m}_\ell - 1}}, \lambda_{1_{\mathfrak{m}_\ell}}\right), \mathfrak{b}\left(\lambda_{1_{\imath_\ell - 1}}, \lambda_{1_{\imath_\ell}}\right)\right\}. \end{aligned} \tag{54}$$

From (48), (51), and Remark 5, we get

$$\lim_{\ell \longrightarrow \infty} \mathscr{M}\left(\lambda_{1_{\mathfrak{m}_\ell - 1}}, \lambda_{1_{\imath_\ell - 1}}\right) = \max\{\varepsilon, 0, 0\} = \varepsilon. \tag{55}$$

Now, let $\alpha_\ell = \Theta(\mathfrak{b}(\lambda_{1\mathfrak{m}_\ell}, \lambda_{1_{\iota_\ell}}))$, and $\mathfrak{b}_\ell = \Theta(\mathscr{M}(\lambda_{1\mathfrak{m}_\ell - 1}, \lambda_{1_{\iota_\ell - 1}}))$, for all $\ell \in \mathbb{N}$. In view of (51), (53), (55), and (Θ_3), we have $\alpha_\ell < \mathfrak{b}_\ell$, for all $\ell \in \mathbb{N}$ and $\lim_{\ell \longrightarrow \infty} \alpha_\ell = \lim_{\ell \longrightarrow \infty} \mathfrak{b}_\ell > 1$. Therefore, using (ζ_3^*), we obtain

$$1 \leq \limsup_{\ell \longrightarrow \infty} \zeta(\alpha_\ell, \mathfrak{b}_\ell) < 1, \tag{56}$$

which is contradiction. Hence, $\{\lambda_{1_\iota}\} \in (\mathscr{P}, \perp, \mathfrak{b})$ is orthogonal Cauchy sequence. As $(\mathscr{P}, \perp, \mathfrak{b})$ is complete, then there exists $\ell \in \mathscr{P}$ s.t

$$\lim_{\iota \longrightarrow \infty} (\mathfrak{b}(\lambda_{1_\iota}, \ell)) = 0. \tag{57}$$

Without loss of generality, we consider $\lambda_{1_\iota} \neq \ell$ and $\Phi\lambda_{1_\iota} \neq \Phi\ell$, for all $\iota \in \mathbb{N}$. Suppose that $\mathfrak{b}(\ell, \Phi\ell) > 0$, it follows from (10) and ζ_2^* that

$$\begin{aligned} 1 &\leq \zeta[\Theta(\mathfrak{b}(\Phi\lambda_{1_\iota}, \Phi\ell)), \Theta(\mathscr{M}(\lambda_{1_\iota}, \ell))] \\ &= \zeta[\Theta(\mathfrak{b}(\lambda_{1_{\iota+1}}, \Phi\ell)), \Theta(\mathscr{M}(\lambda_{1_\iota}, \ell))] \\ &< \frac{\Theta(\mathscr{M}(\lambda_{1_\iota}, \ell))}{\Theta(\mathfrak{b}(\lambda_{1_{\iota+1}}, \Phi\ell))}, \end{aligned} \tag{58}$$

where $\mathscr{M}(\lambda_{1_\iota}, \ell) = \max\{\mathfrak{b}(\lambda_{1_\iota}, \ell), \mathfrak{b}(\lambda_{1_\iota}, \lambda_{1_{\iota+1}}), \mathfrak{b}(\ell, \Phi\ell)\}$, which implies that

$$\Theta(\mathfrak{b}(\lambda_{1_{\iota+1}}, \Phi\ell)) < \Theta(\mathscr{M}(\lambda_{1_\iota}, \ell)). \tag{59}$$

From Remark 5 and Lemma 12, we have

$$\lim_{\iota \longrightarrow \infty} \mathfrak{b}(\lambda_{1_{\iota+1}}, \Phi\ell) = \lim_{\iota \longrightarrow \infty} \mathscr{M}(\lambda_{1_\iota}, \ell) = \mathfrak{b}(\ell, \Phi\ell) > 0. \tag{60}$$

Let $\alpha_\iota = \Theta(\mathfrak{b}(\lambda_{1_{\iota+1}}, \Phi\ell))$, and $\mathfrak{b}_\iota = \Theta(\mathscr{M}(\lambda_{1_\iota}, \ell))$, for all $\iota \in \mathbb{N}$; it follows from (10) and ζ_3^* that

$$1 \leq \limsup_{\iota \longrightarrow \infty} \zeta(\alpha_\iota, \mathfrak{b}_\iota) < 1. \tag{61}$$

This is a contradiction. Therefore, summarize $\ell = \Phi\ell$, that is, ℓ is a fixed point of Φ. Finally, prove that Φ is **ufp**.

Consider two different fixed points ℓ and $\mathfrak{z}$ in $\mathscr{P}$.

Then, $\mathfrak{b}(\ell, \mathfrak{z}) = \mathfrak{b}(\Phi\ell, \Phi\mathfrak{z}) > 0$, since Φ is an orthogonal-preserving, $\forall \Phi\ell \perp \Phi\mathfrak{z}$.

Using (10) and ζ_2^*, we deduce that

$$\begin{aligned} 1 &\leq \zeta[\Theta(\mathfrak{b}(\Phi\ell, \Phi\mathfrak{z})), \Theta(\mathscr{M}(\ell, \mathfrak{z}))] \\ &= \zeta[\Theta(\mathfrak{b}(\ell, \mathfrak{z})), \Theta(\mathscr{M}(\ell, \mathfrak{z}))] < \frac{\Theta(\mathscr{M}(\ell, \mathfrak{z}))}{\Theta(\mathfrak{b}(\ell, \mathfrak{z}))}, \end{aligned} \tag{62}$$

where $\mathscr{M}(\ell, \mathfrak{z}) = \max\{\mathfrak{b}(\ell, \mathfrak{z}), \mathfrak{b}(\ell, \Phi\ell), \mathfrak{b}(\mathfrak{z}, \Phi\mathfrak{z})\} = \mathfrak{b}(\ell, \mathfrak{z})$, which implies that

$$\Theta(\mathfrak{b}(\ell, \mathfrak{z})) < \Theta(\mathscr{M}(\ell, \mathfrak{z})) = \Theta(\mathfrak{b}(\ell, \mathfrak{z})). \tag{63}$$

This is a contradiction. Therefore, Φ has a **ufp**.

Corollary 18. *Let $(\mathscr{P}, \perp, \mathfrak{b})$ be a OCBMS and $\Phi : \mathscr{P} \longrightarrow \mathscr{P}$. Assume that (for all $\lambda_1, \lambda_2 \in \mathscr{P}$ with $\lambda_1 \perp \lambda_2$):*

(i) Φ is orthogonal-preserving

(ii) $\mathfrak{b}(\Phi\lambda_1, \Phi\lambda_2) > 0 \Longrightarrow$

$$\begin{aligned} \Theta(\mathfrak{b}(\Phi\lambda_1, \Phi\lambda_2)) \leq \mathscr{M}(\lambda_1, \lambda_2) - \varphi(\mathscr{M}(\lambda_1, \lambda_2)), \forall \lambda_1, \lambda_2 \\ \in \mathscr{P} \text{ with } \lambda_1 \perp \lambda_2, \end{aligned} \tag{64}$$

where $\mathscr{M}(\lambda_1, \lambda_2) = \max\{\mathfrak{b}(\lambda_1, \lambda_2, \mathfrak{b}(\lambda_1, \Phi\lambda_1), \mathfrak{b}(\lambda_2, \Phi\lambda_2))\}$, and $\varphi : [0, \infty) \longrightarrow [0, \infty)$ is nondecreasing and lower semicontinuous s.t $\varphi^{-1}(\{0\}) = 0$. Then, Φ has a ***ufp****.*

Proof. Let $\Theta(\alpha) = e^\alpha$, for all $\alpha > 0$. From (64), we have

$$\begin{aligned} \Theta(\mathfrak{b}(\Phi\lambda_1, \Phi\lambda_2)) = e^{(\mathfrak{b}(\Phi\lambda_1, \Phi\lambda_2))} \leq e^{\mathscr{M}(\lambda_1, \lambda_2) - \varphi(\mathscr{M}(\lambda_1, \lambda_2))} \\ = \frac{\Theta(\mathscr{M}(\lambda_1, \lambda_2)}{e^{\varphi(\mathscr{M}(\lambda_1, \lambda_2))}}, \end{aligned} \tag{65}$$

for all $\lambda_1, \lambda_2 \in \mathscr{P}$ with $\lambda_1 \perp \lambda_2$, and $\mathfrak{b}(\Phi\lambda_1, \Phi\lambda_2) > 0$. Therefore, Φ is orthogonal-preserving.

Now, we define $\varphi(\alpha) = \mathrm{In}(\Phi(\Theta(\alpha)))$, for all $\alpha > 0$, where $\Phi : [1, \infty) \longrightarrow [1, \infty)$ is nondecreasing and lower semicontinuous s.t $\Phi^{-1}(\{1\}) = 1$.

From (65), we have

$$\Theta(\mathfrak{b}(\Phi\lambda_1, \Phi\lambda_2)) \leq \frac{\Theta(\mathscr{M}(\lambda_1, \lambda_2))}{\Phi(\Theta(\mathscr{M}(\lambda_1, \lambda_2)))}. \tag{66}$$

Taking $\zeta(\alpha, \mathfrak{b}) = ((\mathfrak{b}/\alpha)\Phi(\mathfrak{b}))$ and using (66), we have

$$\begin{aligned} 1 &\leq \frac{\Theta(\mathscr{M}(\lambda_1, \lambda_2))}{\Theta(\mathfrak{b}(\Phi\lambda_1, \Phi\lambda_2))\Phi(\Theta(\mathscr{M}(\lambda_1, \lambda_2)))} \\ &= \zeta[\Theta(\mathfrak{b}(\Phi\lambda_1, \Phi\lambda_2)), \Theta(\mathscr{M}(\lambda_1, \lambda_2))]. \end{aligned} \tag{67}$$

Therefore, all conditions are satisfied in Theorem 17, and hence, Φ has a **ufp**. □

In the following example, validate the proof of Theorem 17.

Example 6. Let $\mathscr{P} = \Pi \cup \Psi$, where $\Pi = [1, 2]$ and $\Psi = \{(1/\iota): \iota = 2, 3, 4, 5\}$. Define a map $\mathfrak{b} : \mathscr{P} \times \mathscr{P} \longrightarrow [0, \infty)$ as follows:

(1) $\mathfrak{b}(1/2, 1/3) = \mathfrak{b}(1/4, 1/5) = 3/10$,

(2) $\mathfrak{b}(1/2, 1/5) = \mathfrak{b}(1/3, 1/4) = 2/10$,

(3) $\mathfrak{b}(1/2, 1/4) = \mathfrak{b}(1/5, 1/3) = 6/10$,

(4) $\mathfrak{b}(\lambda_1, \lambda_1) = 0, \mathfrak{b}(\lambda_1, \lambda_2) = \mathfrak{b}(\lambda_2, \lambda_1), \forall \lambda_1, \lambda_2 \in \Psi$, and

(5) $\mathfrak{b}(\lambda_1, \lambda_2) = |\lambda_1 - \lambda_2|$ if $\lambda_1, \lambda_2 \in \Pi$ or $\lambda_1 \in \Pi, \lambda_2 \in \Psi$ or $\lambda_1 \in \Psi, \lambda_2 \in \Pi$.

Here, the triangle inequality is not satisfied, so $\mathfrak{b}$ is not a metric on $\mathscr{P}$; we have

$$\frac{6}{10} = \mathfrak{b}\left(\frac{1}{5}, \frac{1}{3}\right) > \mathfrak{b}\left(\frac{1}{5}, \frac{1}{4}\right) + \mathfrak{b}\left(\frac{1}{4}, \frac{1}{3}\right) = \frac{5}{10}. \tag{68}$$

It is easy to verify that $(\mathscr{P}, \mathfrak{b})$ is a OCBMS. Let $\Phi : \mathscr{P} \longrightarrow \mathscr{P}$ be defined as an orthogonality relation $\perp$ on $\mathscr{P}$ by

$$\Phi\lambda_1 = \begin{cases} \frac{1}{5}, & \text{if } \lambda_1 \in \left[1, \frac{3}{2}\right], \\ \frac{1}{4}, & \text{if } \lambda_1 \in \left(\frac{3}{2}, 2\right] \cup \Psi. \end{cases} \tag{69}$$

Since Φ is not continuous at $\lambda_1 = (3/2)$, and $\Phi - \perp$ is not continuous, then Φ is neither orthogonal Θ-contraction nor an orthogonal L^*-contraction.

Declare that Φ is an orthogonal L^*-contraction w.r.t $\zeta : [1, \infty) \times [1, \infty) \longrightarrow \mathbb{R}$, where

$$\zeta_\ell(\alpha, \mathfrak{b}) = \frac{\mathfrak{b}^\ell}{\alpha}, \forall \alpha, \mathfrak{b} \in [1,\infty), \ell \in \left[\frac{3}{8}, 1\right), \tag{70}$$

and $\Theta : (0, \infty) \longrightarrow (1, \infty)$, s.t $\Theta(\alpha) = \mathfrak{e}^\alpha, \forall \alpha \in (0, \infty)$.

Indeed, for $\lambda_1 \in [1, (3/2)]$, and $\lambda_2 \in [(3/2), 2] \cup \Psi$, we have

$$\begin{aligned} &\mathfrak{b}(\Phi\lambda_1, \Phi\lambda_2) = \mathfrak{b}\left(\frac{1}{4}, \frac{1}{5}\right) = \frac{3}{10} > 0, \\ &\zeta[\Theta(\mathfrak{b}(\Phi\lambda_1, \Phi\lambda_2)), \Theta(\mathscr{M}(\lambda_1, \lambda_2))] \\ &\quad = \frac{[\Theta\mathscr{M}(\lambda_1, \lambda_2)]^\ell}{\Theta(\mathfrak{b}(\Phi\lambda_1, \Phi\lambda_2))} \geq \frac{\mathfrak{e}^{4\ell/5}}{\mathfrak{e}^{3/10}} = \mathfrak{e}^{(1/5)(4\ell-(3/2))} \\ &\quad \geq 1, \text{for any } \ell \in \left[\frac{3}{8}, 1\right). \end{aligned} \tag{71}$$

Hence, all the hypotheses are satisfied in Theorem 17, and $\ell = 1/4$ is the **ufp** of Φ.

4. An Application

The following BVP of a fourth-order differential equation is taken from Saleh et al. [11].

In this section, as an application of Theorem 17, we present the following result which provides an existence and uniqueness solution to the BVP of a fourth-order differential equation through an orthogonal L^*-contraction.

$$\begin{cases} \lambda_1''''(\alpha) = \mathfrak{g}\left(\alpha, \lambda_1(\alpha), \lambda_1'(\alpha), \lambda_1''(\alpha), \lambda_1'''(\alpha)\right), \alpha \in [0, 1], \\ \lambda_1(0) = \lambda_1'(0) = \lambda_1''(1) = \lambda_1'''(1) = 0. \end{cases} \tag{72}$$

Let $\mathfrak{g} : [0, 1] \times \mathbb{R}^4 \longrightarrow \mathbb{R}$ is a continuous function. Let $\mathscr{P} = \mathscr{C}[0, 1]$ represent the space of all continuous functions defined on the interval [0,1]. Define a metric $\Phi : \mathscr{P} \times \mathscr{P} \longrightarrow \mathbb{R}$ by

$$\Phi(\lambda_1, \lambda_2) = \max_{\alpha\in[0,1]} |\lambda_1(\alpha) - \lambda_2(\alpha)|, \text{for all } \lambda_1, \lambda_2 \in \mathscr{P}. \tag{73}$$

It is known that $(\mathscr{P}, \Phi)$ is a complete BMS. Define the green function associated with (72)

$$\mathrm{G}(\mathfrak{b}, \alpha) = \begin{cases} \frac{1}{6}\alpha^2(3\mathfrak{b} - \alpha), & 0 \leq \alpha \leq \mathfrak{b} \leq 1, \\ \frac{1}{6}\mathfrak{b}^2(3\alpha - \mathfrak{b}), & 0 \leq \mathfrak{b} \leq \alpha \leq 1. \end{cases} \tag{74}$$

Now, we provide the following result regarding the BVP (72) solution.

Theorem 19. *Assume that the following axioms are satisfied:*

(P1) $\mathfrak{g} : [0, 1] \times \mathbb{R}^4 \longrightarrow \mathbb{R}$ *is orthogonal continuous function*

(P2) there exist $\tau > 0$ and s.t, for all $\lambda_1, \lambda_2 \in \mathscr{P}, \lambda_1 \perp \lambda_2$, and $\mathfrak{b} \in [0, 1]$

$$\begin{aligned} &\left|\mathfrak{g}\left(\mathfrak{b}, \lambda_1, \lambda_1'\right) - \mathfrak{g}\left(\mathfrak{b}, \lambda_2, \lambda_2'\right)\right| \\ &\quad \leq 8e^{-\tau}[\max\{|\lambda_1(\mathfrak{b}) - \lambda_2(\mathfrak{b})|, |\lambda_1(\mathfrak{b}) \\ &\quad - \Phi\lambda_1(\mathfrak{b})|, |\lambda_2(\mathfrak{b}) - \Phi\lambda_2(\mathfrak{b})|\}], \end{aligned} \tag{75}$$

where $\Phi : \mathscr{P} \longrightarrow \mathscr{P}$ is defined by

$$\Phi\lambda_1(\alpha) = \int_0^1 G(\alpha, \mathfrak{b})\mathfrak{g}\left(\mathfrak{b}, \lambda_1(\mathfrak{b}), \lambda_1'(\mathfrak{b})\right) ds. \tag{76}$$

Then, (72) has a unique solution in $\mathscr{P}$.

Proof. Define the binary relation $\perp$ on $\mathscr{P}$ by

$$\begin{aligned} \lambda_1 \perp \lambda_2 \Leftarrow \lambda_1(\sigma)\lambda_2(\sigma) \geq \lambda_1(\sigma) \text{ or } \lambda_1(\sigma)\lambda_2(\sigma) \\ \geq \lambda_2(\sigma), \forall \sigma \in [0, 1]. \end{aligned} \tag{77}$$

Observe that $\lambda_1 \in \mathscr{P}$ is a solution of (72) iff $\lambda_1 \in \mathscr{P}$ is a solution of the differential equation

$$\lambda_1(\alpha) = \int_0^1 \mathrm{G}(\alpha, \mathfrak{b})\mathfrak{g}\left(\mathfrak{b}, \lambda_1(\mathfrak{b}), \lambda_1'(\mathfrak{b})\right) ds, \forall \lambda_1 \in \mathscr{P}. \tag{78}$$

Then, Φ is an orthogonal-continuous.

Now, we show that Φ is orthogonal-preserving, in (P2), for all $\lambda_1, \lambda_2 \in \mathscr{P}$ with $\mathfrak{b}(\Phi\lambda_1, \Phi\lambda_2) > 0$ and for all $\alpha \in [0, 1]$. Then, Φ is an orthogonal-preserving.

Next, we claim that Φ is orthogonal $\mathscr{L}^{\star}$-contraction. We have

$$\begin{aligned}
&|\Phi\lambda_1(\alpha) - \Phi\lambda_2(\alpha)| \\
&\quad = \left| \int_0^1 \mathrm{G}(\alpha, \mathfrak{b}) \mathfrak{g}\left(\mathfrak{b}, \lambda_1(\mathfrak{b}), \lambda_1'(\mathfrak{b})\right) ds \right. \\
&\qquad \left. - \int_0^1 \mathrm{G}(\alpha, \mathfrak{b}) \mathfrak{g}\left(\mathfrak{b}, \lambda_2(\mathfrak{b}), \lambda_2'(\mathfrak{b})\right) ds \right| \\
&\quad \le \int_0^1 \mathrm{G}(\alpha, \mathfrak{b}) \left| \mathfrak{g}\left(\mathfrak{b}, \lambda_1(\mathfrak{b}), \lambda_1'(\mathfrak{b})\right) - \mathfrak{g}\left(\mathfrak{b}, \lambda_2(\mathfrak{b}), \lambda_2'(\mathfrak{b})\right) \right| ds \\
&\quad \le 8e^{-\tau} \int_0^1 \mathrm{G}(\alpha, \mathfrak{b}) [\max \{|\lambda_1 - \lambda_2|, |\lambda_1 - \Phi\lambda_1|, |\lambda_2 - \Phi\lambda_2|\}] ds \\
&\quad \le 8e^{-\tau} [\mathscr{M}(\lambda_1, \lambda_2)] \left(\sup_{\alpha \in [0,1]} \int_0^1 \mathrm{G}(\alpha, \mathfrak{b}) ds \right),
\end{aligned} \tag{79}$$

where $\mathscr{M}(\lambda_1, \lambda_2) = \max \{\mathfrak{b}(\lambda_1, \lambda_2), \mathfrak{b}(\lambda_1, \Phi\lambda_1), \mathfrak{b}(\lambda_2, \Phi\lambda_2)\}$. As $\int_0^1 \mathrm{G}(\alpha, \mathfrak{b}) ds = (\alpha^4/24) - (\alpha^3/6) + (\alpha^2/4)$, for all $\alpha \in [0, 1]$, $\sup_{\alpha \in [0,1]} \int_0^1 \mathrm{G}(\alpha, \mathfrak{b}) ds = 1/8$, we obtain

$$\begin{aligned}
\mathfrak{b}(\Phi\lambda_1, \Phi\lambda_2) &\le 8e^{-\tau} [\mathscr{M}(\lambda_1, \lambda_2)], \\
e^{\mathfrak{b}(\Phi\lambda_1, \Phi\lambda_2)} &\le 8e^{-\tau} \left(e^{\mathscr{M}(\lambda_1, \lambda_2)} \right)^{e^{\tau}}.
\end{aligned} \tag{80}$$

Observe that $e^{\tau} \in (0, 1)$ as $\tau > 0$. It follows that Φ is an orthogonal $\mathscr{L}^{\star}$-contraction. Therefore, for all $\lambda_1, \lambda_2 \in \mathscr{P}$, we obtain

$$\begin{aligned}
&\zeta[\Theta(\mathfrak{b}(\Phi\lambda_1, \Phi\lambda_2)), \Theta(\mathscr{M}(\lambda_1, \lambda_2))] \\
&\quad = \frac{[\Theta\mathscr{M}(\lambda_1, \lambda_2)]^{\ell}}{\Theta(\mathfrak{b}(\Phi\lambda_1, \Phi\lambda_2))} \ge \frac{\left(e^{\mathscr{M}(\lambda_1, \lambda_2)^{e^{-\tau}}} \right)}{e^{\mathfrak{b}(\Phi\lambda_1, \Phi\lambda_2)}} \ge 1,
\end{aligned} \tag{81}$$

where $\Theta(\alpha) = e^{\alpha}, \zeta(\alpha, \mathfrak{b}) = (\mathfrak{b}^{\ell}/\alpha)$, and $\ell = e^{\tau}$. Thus, all the axioms of Theorem 17 are fulfilled. Therefore, Φ has a **ufp** in $\mathscr{P}$ which is a solution of (72). □

5. Conclusion

In this paper, we proved the fixed-point results for orthogonal $\mathscr{L}^{\star}$-contraction map on OCBMS. Furthermore, we presented some examples to strengthen our main results. Also, we provided an application to the BVP of a fourth-order differential equation.

Khalehoghli et al. [19, 20] presented a real generalization of the mentioned Banach's contraction principle by introducing R-metric spaces, where R is an arbitrary relation on L. We note that in a special case, R can be considered as $R = \preceq$ [partially ordered relation], $R = \perp$ [orthogonal relation], etc. If one can find a suitable replacement for a Banach theorem that may determine the values of fixed points, then many problems can be solved in this R-relation. This will provide a structural method for finding a value of a fixed point. It is an interesting open problem to study the fixed-point results on $\mathbb{R}$-complete R-metric spaces.

Data Availability

This clause is not applicable to this paper.

Additional Points

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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