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*Retraction*

# Retracted: Analysis of Option Butterfly Portfolio Models Based on Nonparametric Estimation Deep Learning Method

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This article has been retracted by Hindawi, as publisher, following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of systematic manipulation of the publication and peer-review process. We cannot, therefore, vouch for the reliability or integrity of this article.

Please note that this notice is intended solely to alert readers that the peer-review process of this article has been compromised.

Wiley and Hindawi regret that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

## References

[1] X. Ge, X. Zhu, G. Bi, H. Zheng, and Q. Li, "Analysis of Option Butterfly Portfolio Models Based on Nonparametric Estimation Deep Learning Method," *Journal of Function Spaces*, vol. 2023, Article ID 4989036, 17 pages, 2023.

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*Research Article*

# Analysis of Option Butterfly Portfolio Models Based on Nonparametric Estimation Deep Learning Method

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The option butterfly portfolio is the commonly option arbitrage strategy. In reality, because the distribution of the option state price density (SPD) function is not normal and unknown, so the nonparametric deep learning methods to estimate option butterfly portfolio returns are proposed. This paper constructs the single-index nonparametric option pricing model which contains multiple influencing factors and presents the nonparametric estimation form for option butterfly portfolio returns. The empirical analysis shows that the SPD function estimated by using single-index nonparametric option model can effectively calculate the option butterfly portfolio returns with the minimum option strike price interval and provide an effective reference tool for risk-averse investors with limited risk preferences.



## 1. Introduction

Since the listing of Shanghai Stock Exchange (SSE) 50ETF Fund options on February 9, 2015 has opened the prologue of options trading in mainland China, it becomes important to study the actual trading methods of various option spread combinations. Actually, options are of great significance to promote the healthy development of the capital market, which not only can help investors' hedge risks at a lower cost but also help different investors make appropriate investment choices based on their own risk preferences. How to reasonably test the effectiveness of option pricing has important practical significance. However, the existing methods to test the effectiveness of the option market mostly focus on posttest methods to verify the validity of the option parity formula (put-call parity) (see Stoll [1]) or the initial option fee and ending income of the spread portfolio model, and whether meet the no-arbitrage relationship put-call parity formula, few on option butterfly portfolio returns strategy method to test the effectiveness of the option market. Therefore, it would be a useful exploration to study how to use the option butterfly portfolio return strategy method to test the effectiveness of the option market.

In this paper, the butterfly portfolio returns have been constructed with the trading data in one minute from April 1, 2019 09:30:00 to May 31, 2019 15:00:00 of all SSE 50ETF European put options with expiration in June and July 2019, for a backtesting on the butterfly portfolio no-arbitrage return at maturity and actual return at maturity, including regression analysis, arbitrage the relationship between opportunities and arbitrage return and option value status, butterfly portfolio exercise price interval, the relationship of underlying asset price volatility rate, the intraday distribution of arbitrage opportunities, and the duration of the day, leading to a conclusion that although the SSE 50ETF option market has not been fully effective.

The butterfly portfolio return strategy is limited in terms of risks and returns, which is only constructed when the volatility of the future spot market is expected to be stable and benefits from the underlying price falling into the middle of the left and right strike prices of the butterfly portfolio. It is an arbitrage return strategy suitable for a market with flat fluctuations. The no-arbitrage return of this strategy is related to the state price density (SPD). Butterfly portfolio return strategy is featured with limited income and closed risk exposure with low risk to achieve stable investment income. Importantly, the core of the option butterfly portfolio return strategy is to seize arbitrage opportunities to obtain profits. This paper will focus on option butterfly portfolio return strategy that generates returns when market volatility is stable.

In this paper, our contribution includes two aspects. Firstly, we extend finding the second derivative of option price of a single-indicator variable of option factor (the combination of the underlying asset price, the exercise price, exercise period, the risk-free rate and implied volatility, etc.) on the basis of nonparametric single-index option pricing model (see Li and Yang [2]). Secondly, we develop an efficient nonparametric estimation deep learning method to estimate the implied SPD function for single-index option pricing model and to present the numerical algorithm we use to estimate the option butterfly portfolio returns, with a comparison of results between parameter estimation method and nonparametric estimation method.

The remaining sections are compiled in the following manner. Section 2 provides a literature review. Section 3 proposes a nonparametric estimation deep learning method for single-index option price model with introducing the classical nonparametric estimation method of European option prices by performing kernel density estimation and local polynomial regression estimation by the classical nonparametric estimation method and the single-index nonparametric estimation method, respectively. Exactly, it is expected to find every minimum estimated mean square error values with these two methods. In Section 4, the empirical analysis of option butterfly portfolio returns based on parameter estimation method is attempted to find the relationship between the MSE of the classical nonparametric method and the MSE of the single-index nonparametric estimation deep learning method. The values of nonparametric estimation deep learning algorithm for option butterfly portfolio returns may generate. A conclusion of this paper is given in Section 5. The fundamental properties for the option pricing, the SPD function and the butterfly portfolio returns, and the calculation program codes are reviewed in the Appendix.

## 2. Literature Review

The time-state preference model proposed by Arrow and Debreu [3] has promoted the development of uncertainty investing theory and introduced Arrow-Debreu securities (i.e., underlying securities) whose prices are determined by the SPD function to define each Arrow-Debreu security that generates a payment at state $x$. The option price can be obtained indirectly by estimating the SPD. The information of SPD not only can be used to derive more than derivatives prices but also to measure the size of financial risks of commercial banks, investment banks, securities companies, and other financial institutions. In order to overcome the shortcomings of traditional VaR risk measurement, Ait-Sahalia and Lo [4] have proposed a new risk measurement method (E-VaR) based on SPD, which has two important characteristics; on the one hand, it contains all relevant economic information, such as investors' risk appetite, asset price dynamics, and market clearing; it can be derived from the preference-based equilibrium model or the measurement on the basis of the Black-Scholes-Merton (BSM) model. Therefore, of risk based on SPD is more attractive than that in the traditional statistical sense. Regarding the estimation of SPD, the method of neural network is used to make nonparametric estimation of option price to estimate SPD by Hutchinson et al. [5]; and the method of binary tree is given to estimate SPD by Rubinstein [6]. The estimation of SPD is obtained by taking the second derivative of the option pricing formula with respect to the strike price on the basis of the BSM model by Ait-Sahalia and Lo [4]. Yang [7] has proposed a new semiparametric estimation method by combining mathematical models and nonparametric estimation methods to estimate the SPD function and has verified that the effect of option price estimation based on the semiparametric estimation method is better than adapted BSM estimation, direct nonparametric estimation, and semiparametric BSM estimation.

As mentioned earlier, the problem of derivatives pricing can be transformed into the problem of estimating SPD function. Breeden and Litzenberger [8] with strong assumptions about the underlying asset have proposed the analytical solution of SPD by the BSM model; that is, if the underlying asset price obeys geometric Brownian motion and the risk-free interest rate is unchanged, SPD obeys logarithmic positive state distribution. However, the underlying asset price is a more complex random process, and the analytical solution cannot be obtained; therefore, the estimation of SPD can only be performed by numerical approximation. Rubinstein [6] and Jackwerth and Rubinstein [9] have minimized the gap between the SPD and the prior distribution by using the method of prior distribution of the SPD. Because of the classical BSM model with too many assumptions, the estimation accuracy of the SPD method is proved to be insufficient by nonparametric estimation method. Therefore, nonparametric estimation methods without any presupposition requirement to estimate the SPD are crucial and necessary. The existing nonparametric estimation method for estimating the SPD function is mainly divided into fully nonparametric Nadaraya-Watson kernel estimation, semiparametric Nadaraya-Watson kernel estimation relying on the BSM formula, and local polynomial estimation methods. However, all nonparametric estimation methods are very dependent on the quality of sample data. If the sample set is sparse in a certain area, the nonparametric point estimation near the area is not effective. Then, it is very meaningful to explore a nonparametric estimation deep learning method to estimate SPD that the performance of point estimation will not be greatly influenced when the sample points are sparse, leading to obtain the nonarbitrage income of option butterfly portfolio returns by estimating SPD.

The estimation method of option butterfly portfolio returns was first proposed by Breeden and Litzenberger [8] who have successfully induced the SPD function by the price of European call option price, and the first derivative of the option price related to the strike price is the distribution function of the asset state price underlying the option price, while the second derivative is SPD function, which has opened the door to estimate option butterfly portfolio returns by nonparametric estimation method. Since then, more and more researches have turned the direction to estimate the SPD function. The performance of parametric estimation method of the BSM formula derivation has been presented with a strong premise, but option price state function actually is hard to meet the assumption of normal distribution form. Based on the analysis of Breeden and Litzenberger [8], one study reported by Ait-Sahalia [10] is mainly about the nonparametric estimation method for the asset SPD function of the underlying option price. On this basis, Kiesel [11] has made a full explanation for the nonparametric estimation of the asset SPD function of the underlying option price under the butterfly arbitrage principle. It has been proved to be arbitrage-free nature of butterfly portfolio return strategy by Carr and Madan [12]. The conditions of arbitrage opportunities in the butterfly portfolio return strategy that are available have been provided by Davis and Hobson [13] with the finite probability space method.

The estimation of option butterfly portfolio return centers on the SPD function of the underlying asset of the option. The method proposed by Breeden and Litzenberger [8] will not be ensured unless the assumptions that state price obeys normal distribution are satisfied. It also assumes that the SPD function is an unknown nonparametric form by Ait-Sahalia [10] and Kiesel [11]. According to the nonparametric estimation theory of Ait-Sahalia [10] and Kiesel [11], the focus of the estimation of the asset SPD function of the underlying option is put on the nonparametric estimation of the option price. It is supposed to first estimate the nonparametric form of the option price, jumping to the nonparametric form of the SPD function after the first-order derivative.

However, previous studies of nonparametric estimation deep learning methods are carried out with lager sample size, while data of delivery option price in a given time (e.g., a trading day) is limited from the number of 20 to 50. It is useful to aggregate the data over time to increase the number of samples for nonparametric pricing methods. For example, option data set for 1 year was adopted by Ait-Sahalia and Lo [14], and two-dimensional rolling model of cross section and time was supported by Fan and Mancini [15] who chose option data set for 3 years. Ludwig [16] has argued that although the option pricing model of aggregating data over time is effective to solve the problem of sample size, it is simple to combine option contracts with different term structures to make nonparametric regression: price contract of different rights under the same term, ignoring the influence of the term structure of option pricing, which is vulnerable to nonstationary and calendar effect.

The founding discovered by Breeden and Litzenberger [8] is that the first derivative of the exercise price of a call option is less than zero (monotonicity constraint), and the second derivative is greater than zero (convexity constraint). The monotonicity and the constraint condition of convexity are called as shape constraints or no-arbitrage constraints, which the pricing model is named as a nonparametric option pricing model no-arbitrage constraints by Ait-Sahalia and Duarte [17]. Compared with the genuine nonparametric option pricing method of Ait-Sahalia and Lo [14], option pricing model based on a nonparametric estimation deep learning without arbitrage constraints is shown as follow aspects. Firstly, it is to ensure that the risk-neutral probability density function is positive value to have arbitrage opportunities. Secondly, the model has no calendar arbitrage effect with no need for scrolling the data set by time (see Ludwig [16]). Lastly, only a small sample of a single term structure is needed without rolling over a larger number of data sets (there are only dozens of option contracts for a single term structure).

Followed by the no-arbitrage constraint pricing model of Ait-Sahalia and Duarte [17], there have been many studies on nonparametric regression under no-arbitrage constraints. For example, the idea of Yatchew and Hardle [18] is to use nonparametric least squares method and Bootstrap method to consider tail constraints under the condition of the call option price to find better the effect of getting the tail constraint. Hardle and Hlavka [19] and Birke and Pilz [20] have further studied the nonparametric estimation method under call option pricing no-arbitrage constraints but only for the different estimation method. Monteiro and Santos [21] have established a nonparametric regression model with both call and put option data, which has been transformed into a quadratic programming model to solve it.

Regarding the nonparametric estimation of the SPD function, local polynomial estimation is a genuine nonparametric estimation method to overcome the boundary effect. Because the least squares method is used to estimate the regression function and the reciprocal in this method, it is easy to calculate the estimated value of the explained variable and its partial derivative. Ait-Sahalia and Duarte [17] have estimated the SPD by using a local polynomial approach, without a proof of the asymptotic nature of the estimator. Based on the research of Ait-Sahalia and Duarte [17], many scholars have discussed the local polynomial method to estimate the SPD. After the discussion of the convergence of nonparametric estimation, Li et al. [22] have obtained the estimator of SPD and the deviation and variance by the method of local polynomial estimation, with the analysis of the convergence of local polynomial estimation and the speed of the convergence of that.

## 3. The Single-Index Nonparametric Estimation Method for the SPD Function

According to the nonparametric estimation method for the SPD function raised by Ait-Sahalia and Lo [14], it is related to option price concerning on the strike price. Actually, the butterfly portfolio return is influenced not only by the SPD function but also by the exercise price and other factors such as volatility. This main idea of this paper is to estimate the option butterfly returns by nonparametric estimation—the local polynomial method. Specially, putting forward a nonparametric estimation method for the SPD function has taken multiple factors into consideration, which is one of the machine learning method.

As shown in Appendix A.1, as to GBM Equation (A.1) for nondividend stocks, it can be extended to the case of dividends as Equation (A.9). If the stock price $S(t)$ satisfies the GBM stochastic differential (Equation (A.9)), the solution to $S = S(t)$ is

$$S(t) = S(0) \exp\left[\sigma(W(t) - W(0)) + \left(\mu - q - \frac{1}{2}\sigma^2\right)t\right]. \quad (1)$$

Therefore

$$S(T) = S(t) \exp\left[\sigma(W(T) - W(t)) + \left(\mu - q - \frac{1}{2}\sigma^2\right)(T - t)\right]. \quad (2)$$

Now defining the drift transformation $\tilde{W}(t)$ of Brownian motion $W(t)$ as

$$\tilde{W}(t) = W(t) + \frac{\mu - r}{\sigma} t, t \le t \le T, \quad (3)$$

so

$$\tilde{W}(T) - \tilde{W}(t) = W(T) - W(t) + \frac{\mu - r}{\sigma}(T - t). \quad (4)$$

Substituting Equation (4) into Equation (2) to get the new expression of $S(T)$ as

$$S(T) = S(t) \exp\left[\sigma\left(\tilde{W}(T) - \tilde{W}(t)\right) + \left(r - q - \frac{1}{2}\sigma^2\right)(T - t)\right], \quad (5)$$

let

$$\begin{aligned} X &= -\frac{W(T) - W(t)}{\sqrt{T - t}}, \\ Y &= -\frac{\tilde{W}(T) - \tilde{W}(t)}{\sqrt{T - t}}, \text{ or} \\ Y &= X - \frac{\mu - r}{\sigma}\sqrt{T - t}, \end{aligned} \quad (6)$$

then

$$S(T) = S(t) \exp\left[-\sigma\sqrt{\tau}y + \left(r - q - \frac{1}{2}\sigma^2\right)\tau\right]. \quad (7)$$

Note that the difference between Equation (7) and Equation (2) is that $\mu$ is replaced by $r$. It is known that under the original probability measure $\mathbb{P}$, the random variable $X$ obeys the standard normal distribution, and its density function is $f_X(x) = (1/\sqrt{2\pi})e^{-(x^2/2)}$. However, the random variable $Y$ in $\mathbb{P}$ does not obey the standard normal distribution. Now, we define the Radon-Nikodym derivative as follows:

$$Z(t, T) = \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = \exp\left[\int_t^T \theta dW(s) - \frac{1}{2}\int_t^T \theta^2 ds\right] = e^{\theta\sqrt{\tau}x - (1/2)\theta^2\tau}, \quad (8)$$

where $\theta = (\mu - r)/\sigma$. Then, we achieve a new probability measure $\widetilde{\mathbb{P}}$, such that

$$\begin{aligned} d\widetilde{\mathbb{P}} &= \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} d\mathbb{P} = e^{\theta\sqrt{\tau}x - (1/2)\theta^2\tau} d\mathbb{P} = \frac{1}{\sqrt{2\pi}} e^{\theta\sqrt{\tau}x - (1/2)\theta^2\tau} e^{-\left(x^2/2\right)} \\ &= \frac{1}{\sqrt{2\pi}} e^{-(x - \theta\sqrt{\tau})^2/2} = \frac{1}{\sqrt{2\pi}} e^{-\left(y^2/2\right)} = f_Y(y) \end{aligned} \quad (9)$$

That is, $y = x - \theta\sqrt{\tau}$, $Y$ obeys the standard normal distribution under $\widetilde{\mathbb{P}}$; that is, the random process $\{\tilde{W}(t), 0 \le t \le T\}$ is a standard Brownian motion under $\widetilde{\mathbb{P}}$. Let $\tilde{\mathbb{E}}(\cdot)$ represent the mathematical expectation about the new measure $\widetilde{\mathbb{P}}$ and $\tilde{\mathbb{E}}_t(\cdot)$ represent the conditional expectation about the new measure $\widetilde{\mathbb{P}}$ up to time $t$, under the condition of known information. Based on Equation (8) and Equation (9), Equation (A.16) (see Appendix A.2) can be written as

$$V(S_t, t) = \tilde{\mathbb{E}}_t\left[e^{-r(T-t)} \max\{S_T - K, 0\}\right]. \quad (10)$$

Furthermore, under the assumption that $y = B_t$ is a random variable with unknown distribution over time with a dividend, that is, $B_t \sim f_Y(B_t)$ represents a random variable, and $f_Y(B_t)$ is specified as probability density function, such as a normal distribution, or a nonparametric probability density function, so the price of the underlying asset of the option is proved to be an equation of the random variable for info set $\mathfrak{F} = (S_t, K, r, q, \tau)$. Changing the time from $\tau = T - t$ to $\tau = t - 0 = t$; that is, considering the time $\tau$ from 0 'to $t$, the state price of the underlying asset is described as $S_0$, and the random variable $B_t$ can be expressed as a state price function $S_t$ as

$$S_t = S_0 \exp\left[\mu(\mathfrak{F}) - \sigma(\mathfrak{F})B_t\right], B_t = -\frac{\ln(S_t/S_0) - \mu(\mathfrak{F})}{\sigma(\mathfrak{F})}, \quad (11)$$

where $\mu(\mathfrak{F}) = (r - q - \sigma^2/2)\tau$ refers to the drift coefficient, and $\sigma(\mathfrak{F}) = \sigma\sqrt{\tau}$ means the diffusion coefficient, while the continuous dividend yield is defined as $q$.

According to the above Formula (11) and Formula (A.18) (see Appendix A.2), the antiderivative of the European call option price with random variable $B_t$ can be deduced as follows:

$$\begin{aligned} c(S_0, \mathfrak{F}) &= e^{-r\tau}\tilde{\mathbb{E}}_0[\max\{S_t - K, 0\}] \\ &= e^{-r\tau}\int_D^{+\infty}\left(S_0 e^{\mu(\mathfrak{F}) - \sigma(\mathfrak{F})B_t} - K\right) f_Y(B_t) dB_t, \end{aligned} \quad (12)$$

where the notation $\tilde{\mathbb{E}}_t$ (here, $t = 0$) for conditional expectations, $D = (\ln (K/S_0) - \mu(\mathfrak{F}))/\sigma(\mathfrak{F})$, which is identified as factor single indicator variable of option price in this paper. On the analysis of Equation (12), the option price is subject to variable transformation to become an antiderivative of random variables $B_t$, and the lower limit of integral turns to be a single-index $D$. Thus, all the influencing factors of the option price are combined to achieve model dimension reduction. At the same time, it is noted that the SPD function $f_X(S_t)$ of the underlying asset of the option has a connection with the probability density function $f_Y(B_t)$ of the random variable as follows:

$$f_X(S_t) = \frac{f_Y(B_t)}{S_0 \sigma(\mathfrak{F}) e^{\mu(\mathfrak{F}) - \sigma(\mathfrak{F}) B_t}}, \tag{13}$$

when the properties of probability density function $f_Y(B_t)$ are the standard normal distribution, namely, $f_Y(B_t) = (1/\sqrt{2\pi})$ $e^{-(B_t^2/2)}$, then

$$\begin{aligned}
c(S_0, \mathfrak{F}) &= e^{-r\tau} \int_D^{+\infty} \left( S_0 e^{\mu(\mathfrak{F}) - \sigma(\mathfrak{F}) B_t} - K \right) f_Y(B_t) dB_t \\
&= e^{-r\tau} \int_D^{+\infty} S_0 e^{\mu(\mathfrak{F}) - \sigma(\mathfrak{F}) B_t} f_Y(B_t) dB_t - e^{-r\tau} K \int_D^{+\infty} f_Y(B_t) dB_t \\
&= e^{-r\tau} \int_D^{+\infty} S_0 e^{\mu(\mathfrak{F}) - \sigma(\mathfrak{F}) B_t} \frac{1}{\sqrt{2\pi}} e^{-(B_t^2/2)} dB_t - e^{-r\tau} K \int_D^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-(B_t^2/2)} dB_t \\
&= e^{-r\tau} S_0 e^{\mu(\mathfrak{F}) + \sigma^2(\mathfrak{F})/2} \int_D^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-(1/2)[B_t + \sigma(\mathfrak{F})]^2} dB_t - e^{-r\tau} K[1 - N(D)] \\
&= e^{-r\tau} S_0 e^{\mu(\mathfrak{F}) + \sigma^2(\mathfrak{F})/2} \int_{D - \sigma(\mathfrak{F})}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-(Z^2/2)} \\
&\quad \cdot dZ - e^{-r\tau} K[1 - N(D)] (Z = B_t + \sigma(\mathfrak{F})) \\
&= e^{-q\tau} S_0 [1 - N(D - \sigma(\mathfrak{F}))] - e^{-r\tau} K[1 - N(D)] \\
&= e^{-q\tau} S_0 N(-D + \sigma(\mathfrak{F})) - e^{-r\tau} K N(-D) \\
&= e^{-q\tau} S_0 N\left( \frac{\ln (S_0/K) + (r - q + \sigma^2/2)\tau}{\sigma\sqrt{\tau}} \right) \\
&\quad - e^{-r\tau} K N\left( \frac{\ln (S_0/K) + (r - q - \sigma^2/2)\tau}{\sigma\sqrt{\tau}} \right) \\
&= e^{-q\tau} S_0 N(d_{10}) - e^{-r\tau} K N(d_{20}) = c_{BSM}(S_0, \tau),
\end{aligned} \tag{14}$$

where $Z = B_t + \sigma(\mathfrak{F})$, $r - q = (\mu(\mathfrak{F}) + \sigma^2/2)/\tau$, $d_{10} = (\ln (S_0/K) + (r - q + \sigma^2/2)\tau)/(\sigma\sqrt{\tau})$ and $d_{20} = (\ln (S_0/K) + (r - q - \sigma^2/2)\tau)/(\sigma\sqrt{\tau}) = d_{10} - \sigma\sqrt{\tau}$. Specifically, in Equation (14), when the volatility $\sigma$ is equal as the classic BSM option pricing formula. When the volatility is supposed to be implied, it appears to be the semiparametric BSM option pricing formula (see Ait-Sahalia and Lo [14]).

Similarly, it is easy to derive the pricing formula of European put options with dividends. When the volatility $\sigma$ in the formula is the historical volatility, the formula is the classic BSM option pricing formula; when the volatility $\sigma$ changes to the implied volatility, the formula turns to be the semiparametric BSM option pricing formula. The put option price formula of the single-index model can be deduced as the following (Formula (15)):

$$\begin{aligned}
p(S_0, \mathfrak{F}) &= e^{-r\tau} \int_{-\infty}^{D} \left( K - S_0 e^{\mu(\mathfrak{F}) - \sigma(\mathfrak{F}) B_t} \right) f_Y(B_t) dB_t \\
&= e^{-r\tau} K \int_{-\infty}^{D} f_Y(B_t) dB_t - e^{-r\tau} \int_{-\infty}^{D} S_0 e^{\mu(\mathfrak{F}) - \sigma(\mathfrak{F}) B_t} f_Y(B_t) dB_t \\
&= e^{-r\tau} K \int_{-\infty}^{D} \frac{1}{\sqrt{2\pi}} e^{-(B_t^2/2)} dB_t - e^{-r\tau} \int_{-\infty}^{D} S_0 e^{\mu(\mathfrak{F}) - \sigma(\mathfrak{F}) B_t} \frac{1}{\sqrt{2\pi}} e^{-(B_t^2/2)} dB_t \\
&= e^{-r\tau} K N(D) - e^{-r\tau} S_0 e^{\mu(\mathfrak{F}) + \sigma^2(Z)/2} \int_{-\infty}^{D} \frac{1}{\sqrt{2\pi}} e^{-(1/2)[B_t + \sigma(\mathfrak{F})]^2} dB_t \\
&= e^{-r\tau} K N(D) - e^{-r\tau} S_0 e^{\mu(\mathfrak{F}) + \sigma^2(\mathfrak{F})/2} \int_{-\infty}^{D - \sigma(\mathfrak{F})} \frac{1}{\sqrt{2\pi}} e^{-(Z^2/2)} dZ (Z = B_t + \sigma(\mathfrak{F})) \\
&= e^{-r\tau} K N(D) - e^{-q\tau} S_0 N(D - \sigma(\mathfrak{F})) \\
&= e^{-r\tau} K N\left( -\frac{\ln (S_0/K) + (r - q - \sigma^2/2)\tau}{\sigma\sqrt{\tau}} \right) \\
&\quad - e^{-q\tau} S_0 N\left( -\frac{\ln (S_0/K) + (r - q + \sigma^2/2)\tau}{\sigma\sqrt{\tau}} \right) \\
&= e^{-r\tau} K N(-d_{20}) - e^{-q\tau} S_0 N(-d_{10}) = p_{BSM}(S_0, \tau).
\end{aligned} \tag{15}$$

Based on the existing research of the single-index nonparametric option pricing model proposed by Li and Yang [2], the five influencing factors of option price $\mathfrak{F} = (S, K, r, q, \tau)$ are transformed into integral function of random variables $B_t$ by the process of variable transformation, leading to the regression equation of put option price to single-index variable $D$:

$$\begin{aligned}
p(S_0, \mathfrak{F}) &= e^{-r\tau} \int_{-\infty}^{D} \left( K - S_0 e^{\mu(\mathfrak{F}) - \sigma(\mathfrak{F}) B_t} \right) f_Y(B_t) dB_t \\
&= e^{-r\tau} \int_{-\infty}^{D} K\sigma(\mathfrak{F}) F_Y(B_t) dB_t = e^{-r\tau} K\sigma(\mathfrak{F}) \int_{-\infty}^{D} F_Y(B_t) dB_t.
\end{aligned} \tag{16}$$

The above formula can also be written as

$$p(S_0, \mathfrak{F}) \frac{e^{r\tau}}{K\sigma(\mathfrak{F})} = \int_{-\infty}^{D} F_Y(B_t) dB_t. \tag{17}$$

We can see that Equation (17) provides us with a method for estimating the SPD function. $F_Y(B_t)$ is the distribution function of the random variable $B_t$. The first derivative of $D$ is the probability density function of the random variable $B_t$. When Formula (17) is used to estimate the option price, the integral part of Formula (17) can be regarded as a whole, such as the function $G(D)$ about $D$ to estimate the whole; that is, the above Formula (17) can be expressed as

$$Y_i = G(X_i) + \varepsilon_i, \tag{18}$$

where $Y_i = p(S_0, \mathfrak{F}_i)(e^{r\tau}/K\sigma(\mathfrak{F}_i))$, $X_i = D_i = \ln (K_i/S_0) - \mu(\mathfrak{F})_i/\sigma(\mathfrak{F})_i$. Therefore, in order to obtain the SPD function, it is only necessary to obtain the second-order partial derivative of the estimated function $G(D)$ with respect to the single-index $D$, such as the following formula.

$$\left. \frac{\partial^2 G(D)}{\partial D^2} \right|_{D = B_t} = f_Y(B_t). \tag{19}$$

Table 1: The historical dividend situation of the SSE 50ETF.

| | | | | | | |
|---|---|---|---|---|---|---|
| Years | 2019 | 2018 | 2017 | 2016 | 2014 | 2013 |
| Equity registration date | 29/11/2019 | 30/11/2018 | 27/11/2017 | 28/11/2016 | 14/11/2014 | 14/11/2013 |
| Dividends per share (RMB) | 0.047 | 0.049 | 0.054 | 0.053 | 0.043 | 0.053 |
| Years | 2012 | 2012 | 2010 | 2008 | 2006 | 2006 |
| Equity registration date | 12/11/2012 | 15/5/2012 | 15/11/2010 | 18/11/2008 | 15/11/2006 | 18/5/2006 |
| Dividends per share (RMB) | 0.037 | 0.011 | 0.026 | 0.060 | 0.037 | 0.024 |

## 4. Empirical Analysis to Nonparametric Estimation Deep Learning Method for Butterfly Portfolio Models

Since the focus of this paper is put on the stock options of the SSE 50ETF, the underlying asset of which contains dividends, it depends on the BSM formula with dividends. Underlying assets are selected from January 4, 2016 to July 24, 2018 while the options come from the closing price of the day at each strike price for each contract month. The information of SSE 50ETF option contract terms is recently associated with four kinds of contracts on the market such as July, August, September, and December, and all of contracts expire on the fourth Wednesday of each month while July contract is about to expire. All of the data are source from strike price options in circulation in the market from July 3, 2018 to July 24, 2018 for contracts in August and December as well as June 1, 2018 to July 24, 2018 for contracts in September.

In this paper, the risk-free interest rate is replaced by the 20-day average of SHIBOR interest rate on July 24, 2018. More precisely, there are 21 working days left to expire for the August contracts, and SHIBOR2W is used as the risk-free interest rate. However, SHIBOR1M is selected for the September contracts which have 46 days left. For the December contract, which has 111 days to remain, we use SHIBOR3M as the risk-free rate. In addition, the dividend has been regarded as a constant $q_i$ that stands for dividend per share at $i$ times. Fortunately, the data of dividend per share of each time since the establishment of China Shanghai 50ETF Fund on December 30, 2004 are available. As for the dividend yield, the historical average dividend rate under continuous compound interest is more effective to calculate.

In order to determine the dividend rate $q$, this paper has inquired the historical dividend situation of SSE 50ETF Fund, which has paid out 12 dividends by 2019, as shown in Table 1.

In this paper, the historical average dividend rate under continuous compounding is used to represent the dividend rate $q$. Details are below:

$$q = \ln\left(1 + \frac{1/n\sum_{i=1}^{n} q_i}{S_0}\right). \tag{20}$$

Among them, $q_i$ is the dividend of each dividend at the $i$-th dividend, and $S_0 = 2.938$ refers to the closing price of the SSE 50ETF on the day of equity registration in 2019. Therefore, the calculation result of $q$ obtained by Formula (20) is $q = 0.0139$.

According to the above assumption about risk-free interest rate and dividend yield, the historical annual volatility $\sigma_{\text{year}}$ of the underlying SSE 50ETF is specified as the volatility $\sigma_{\text{day}}$ as follows:

$$\begin{aligned} R_t &= \ln\left(\frac{S_t}{S_{t-1}}\right), \\ \mu &= \frac{1}{T}\sum_{t=1}^{T} R_t, \\ \sigma_{\text{day}}^2 &= \frac{1}{T-1}\sum_{t-1}^{T} (R_t - \mu)^2, \\ \sigma_{\text{year}} &= \sigma_{\text{day}} \cdot \sqrt{252}, \end{aligned} \tag{21}$$

where $S_t$ means the closing price of the SSE 50ETF on $t$ day, and $R_t$ refers to the daily yield, while $T$ stands for the sample length of SSE 50ETF. The calculated annual historical volatility is $\sigma_{\text{year}} = 0.20657$.

*4.1. The Classical Nonparametric Estimation Method for Option Price.* From Equation (A.27) (see Appendix A.3), the SPD function is easy to be deduced as long as the option price function is valid. The parametric form results are obtained from the direct derivation of the classical BSM formula. Much more previous literatures have discussed the limitations of the parametric form method. For example, the calculation of the option price formula has been provided by Ait-Sahalia and Lo [14] depending on the nonparametric kernel estimation method, which estimate the nonparametric estimator $\widehat{p}(\cdot)$ of the put option price very intuitively with the financial market data to obtain the second-order partial differential $\partial^2\widehat{p}/\partial K^2$. Under appropriate regular conditions, when $\widehat{p}(\cdot)$ converges to the real's put option price $p(\cdot)$ according to the probability, $A$ also probabilistically converges to $\partial^2 p/\partial K^2$ proportional to the SPD. When with a call option price market data set $\{p_i\}$, accompanying with the feature sets $\{\mathfrak{F}_i = (S_i, K_i, r_{t_i,\tau_i}, q_{t_i,\tau_i}, \tau_i)\}$, the Nadaraya-Watson kernel estimator of the put option price function is estimated as

$$\widehat{p}(\cdot) = \mathbb{E}(p|\mathfrak{F}) = \frac{\sum_{i=1}^{n} \mathbb{K}_h(\mathfrak{F} - \mathfrak{F}_i) p_i}{\sum_{i=1}^{n} \mathbb{K}_h(\mathfrak{F} - \mathfrak{F}_i)}, \tag{22}$$

where $\mathbb{K}_h(x_i - x_0) = \mathbb{K}((x_i - x_0)/h)$ with bandwidth $h$, since Formula (22) involves 5 variables, and in the case of a limited number of samples data, the accuracy of the put option price function estimation will decrease with the increase of the number of variables.

Ait-Sahalia and Lo [14] used two methods to reduce the number of variables. First, it is assumed that the option pricing formula is not a function of the underlying asset price, risk-free interest rate, and dividend rate but depends on the future price of these variables $F_t = S_t e^{(r-q)\tau}$. Under this assumption, the number of regressors is reduced from 5 to 4. The put option pricing formula is reformulated as the following:

$$\hat{p}(F_{t,\tau}, K, r_{t,\tau}, \tau) = \frac{\sum_{i=1}^{n}\mathbb{K}_{h_F}(F_{t,\tau} - F_{t_i,\tau_i})\mathbb{K}_{h_K}(K - K_i)\mathbb{K}_{h_\tau}(\tau - \tau_i)\mathbb{K}_{h_r}(r - r_{t_i,\tau_i})p_i}{\sum_{i=1}^{n}\mathbb{K}_{h_F}(F_{t,\tau} - F_{t_i,\tau_i})\mathbb{K}_{h_K}(K - K_i)\mathbb{K}_{h_\tau}(\tau - \tau_i)\mathbb{K}_{h_r}(r - r_{t_i,\tau_i})}. \tag{23}$$

The second method is semiparametric methods. Here, the option pricing formula is still given by the BSM model, but the volatility $\sigma$ is based on the result of a nonparametric estimation $\hat{\sigma}(F_{t,\tau}, K, r_{t,\tau}, \tau)$, that is

$$\hat{p}(F_{t,\tau}, K, r_{t,\tau}, \tau) = p_{BSM}(F_{t,\tau}, K, r_{t,\tau}, \tau, \hat{\sigma}(F_{t,\tau}, K, r_{t,\tau}, \tau)). \tag{24}$$

Then, the kernel estimator of volatility $\hat{\sigma}(F_{t,\tau}, K, r_{t,\tau}, \tau)$ can be written as

$$\hat{\sigma}^2(F_{t,\tau}, K, r_{t,\tau}, \tau) = \mathbb{E}(\sigma^2 | F_{t,\tau}, K, r_{t,\tau}, \tau) = \frac{\sum_{i=1}^{n}\mathbb{K}_{h_F}(F_{t,\tau} - F_{t_i,\tau_i})\mathbb{K}_{h_K}(K - K_i)\mathbb{K}_{h_\tau}(\tau - \tau_i)\mathbb{K}_{h_r}(r - r_{t_i,\tau_i})\sigma_i^2}{\sum_{i=1}^{n}\mathbb{K}_{h_F}(F_{t,\tau} - F_{t_i,\tau_i})\mathbb{K}_{h_K}(K - K_i)\mathbb{K}_{h_\tau}(\tau - \tau_i)\mathbb{K}_{h_r}(r - r_{t_i,\tau_i})}. \tag{25}$$

Suppose that $X_1, X_2, \cdots, X_n$ are independent and identically distributed samples extraded through the future price from a one-dimensional population $X$, and the probability density functions $f(x), x \in R$ of $X$ are unknown, then, the kernel density of $f(x)$ is estimated as

$$\hat{f}(x) = \frac{1}{nh}\sum_{i=1}^{n}\mathbb{K}_h(x - X_i), \tag{26}$$

where $\hat{f}(x)$ is the estimation of the probability density function and $n$ means the number of samples. Here, $h$ is the bandwidth, and $\mathbb{K}_h(x)$ is represented as the kernel function.

Two problems in using kernel density estimation are the choice of kernel function and the choice of bandwidth. Firstly, taking the one-dimensional case as an example, there are six commonly used kernel functions. Generally speaking, when the amount of data is large enough, the choice of the kernel function is not important. According to theoretical calculations, the kernel density estimation is very similar by using different kernel functions, which means that kernel density estimation is not sensitive to the choice of kernel function. Relatively speaking, kernel density estimation is more sensitive to the choice of bandwidth with a total of 9901 pieces of data screened for the empirical study in this paper, and the sample size should be classified into a sufficient category. Therefore, it is more appropriate to select the Gaussian kernel function with the bandwidth $h$, which is $\mathbb{K}_h(x) = (1/\sqrt{2\pi})e^{-x^2/(2h^2)}$. Secondly, in theory, the bandwidth should decrease as the sample size increases, when $n \longrightarrow \infty$, $h \longrightarrow 0$. Based on Formula (26), the bandwidth $h$ controls the degree of smoothness. If the bandwidth $h$ is smaller, the influence of randomness will increase, and the kernel density function $f(x)$ will become an irregular shape. The important features of kernel density may be concealed, causing the estimated value of the kernel density function to fluctuate greatly and resulting in overfitting; and if the bandwidth $h$ is larger, the sample information will be averaged by $(x - X_i)/h$, and the participation of each sample point will be reduced. The estimated result will be very smooth and accompanied by a large deviation. So under a given sample, the choice of bandwidth is crucial. Therefore, the cross-validated (CV) method (see Li and Racine [23]) is applied to obtain the empirically optimal bandwidth $h$ in the nonparametric regression model as follows:

$$h_{\text{opt}} = \left(\frac{4\hat{\sigma}^5}{3n}\right)^{1/5} = 1.06\hat{\sigma}n^{-1/5}. \tag{27}$$

Because the purpose of this paper is mainly to use a nonparametric estimation deep learning method to estimate the butterfly portfolio strategy return, the use of the multivariate Nadaray-Watson estimation method with a Gaussian kernel function or bandwidth matrix follows the general form in nonparametric estimation. If the heteroscedasticity is considered, the bandwidth $h$ can be selected by using a nonparametric estimation model with variable bandwidth, but the focus of this paper is to estimate the butterfly portfolio returns, so there is no special requirement for the selection of the method.

Therefore, this paper uses the usual training mean squared error (MSE) method to select the bandwidth $h$, which assigned values in turn from 0.1 to 1 with an interval of 0.1 to select the bandwidth that minimizes MSE. More details are showing in

$$\mathbb{K}_h(x - X_i) = \frac{1}{\sqrt{2\pi}}e^{-(1/2)((x-X_i)/h)^2}, \tag{28}$$

$\text{MSE} = 1/n\sum_{i=1}^{n}(p_i - \hat{p}_i)^2$.

As shown in Table 2, when the bandwidth is selected as $h_{\text{opt}} = 0.1$, the MSE results of the kernel density nonparametric estimation for the SSE 50ETF put option price are the smallest. That is, the nonparametric estimation method can take the test see to the smallest MSE of option price estimation, e.g., MSE = 0.0005965. After determining the optimal bandwidth $h_{\text{opt}} = 0.1$, the data set is divided into two equal parts by random sampling, which are used as training set and test set, respectively. Next, it is designed to observe

Table 2: The MSE results of the kernel density nonparametric estimation for put option prices.

| | | | | | |
|---|---|---|---|---|---|
| Bandwidth | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
| MSE | 0.0005965 | 0.0021889 | 0.0039184 | 0.0051961 | 0.0061221 |
| Bandwidth | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
| MSE | 0.0068078 | 0.0073195 | 0.0077041 | 0.0079962 | 0.0082210 |

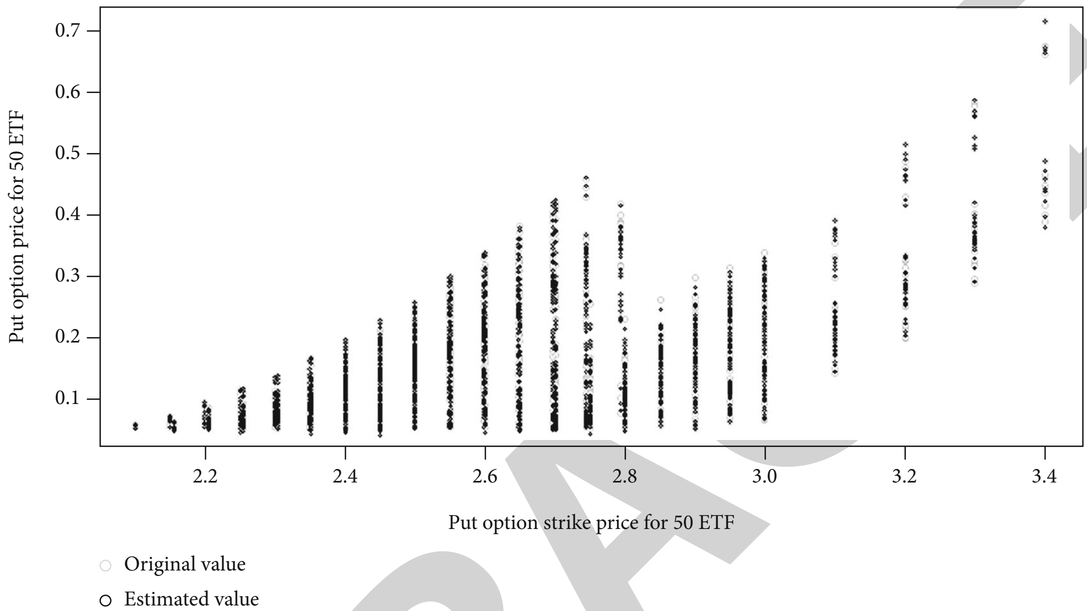


Figure 1: The values of kernel density nonparametric estimation for SSE 50ETF put option prices.

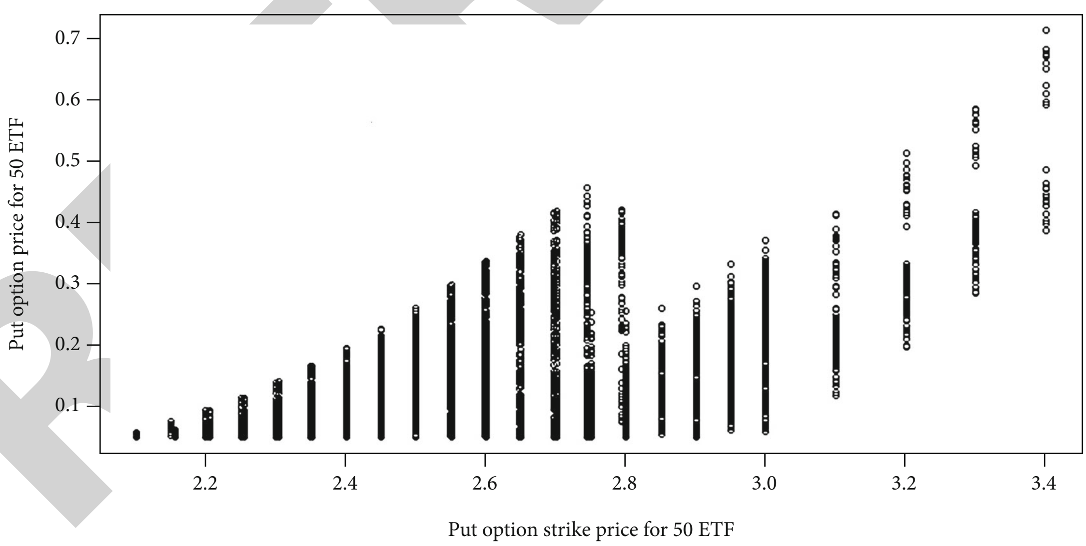


Figure 2: The relationship of all SSE 50ETF put option prices and corresponding strike prices.

the degree of deviation of the option price estimated according to Formula (23) from the original price series, as shown in Figure 1.

In order to observe the difference between the original values and the estimated values more clearly, it is necessary to plot the relationship between all SSE 50ETF European put option prices with expiration time equal to 27 days on 2019-05-30, as well as the original value and their corresponding strike prices during the sample period as shown in Figure 2.

Based on the estimated option price, it is easy to derive the strike price. In practice, an approximate solution to the SPD can be obtained according to the finite difference between the market-observed strike price and the discrete

Table 3: The provisions of the SSE's execution price and exercise price spacing.

| | | | | |
|---|---|---|---|---|
| Exercise price (RMB) | $K \leq 3$ | $3 < K \leq 5$ | $5 < K \leq 10$ | $10 < K \leq 20$ |
| Exercise price spacing (RMB) | 0.05 | 0.1 | 0.25 | 0.5 |
| Exercise price (RMB) | $20 < K \leq 50$ | $50 < K \leq 100$ | $K > 100$ | |
| Exercise price spacing (RMB) | 1.0 | 2.5 | 5.0 | |

option price. Assuming that there are $N$ different strike prices with expiration time $\tau$, $K_1$ represents the lowest strike price, and $K_n$ represents the highest strike price. Here, three options with sequential exercise prices are $K_{n-1}$, $K_n$, and $K_{n+1}$. Generally speaking, $K_n - K_{n-1}$ and $K_{n+1} - K_n$ are not necessarily equal. The state price distribution function value $F(K_n)$ centered on $K_n$ can be estimated by the following formula (29), namely

$$F(K_n) \approx e^{r\tau} \frac{p_{n+1} - p_{n-1}}{K_{n+1} - K_{n-1}}. \tag{29}$$

Therefore, the estimation formula of the SDP function is

$$f(K_n) \approx e^{r\tau} \frac{p_{n+1} - 2p_n + p_{n-1}}{(K_n - K_{n-1})^2}. \tag{30}$$

Formula (30) involves the exercise price interval of different options. It is noted that the provisions of SSE are shown in Table 3.

More importantly, the curve concerning the second derivative of the strike price is the SPD. It is found that the slope of the curve raises along with the increasing of the execution price, which is in line with the conclusion of Equation (A.9) (see Appendix A.1).

The idea of the Nadaraya-Watson kernel density estimation is that the option price at a given point $\mathfrak{F}$ is obtained by the weighted average of the option price observations in the neighborhood of point $\mathfrak{F}$. When there are fewer points on both sides of the characteristic variable $\mathfrak{F}$, the results show a relatively large errors by the method, especially near boundary points, where there are no observations on one side of the boundary point. Therefore, the local polynomial method is useful to estimate the SPD.

*4.2. The Nonparametric Estimation Deep Learning Method for Single-Index Option Price.* Ait-Sahalia and Duarte [17] have estimated the SPD using a local polynomial estimation approach, without the shortcomings of Nadaraya-Watson kernel density estimation method, which overcomes the problem of large deviations on boundary points. The local polynomial estimation has the same order of magnitude on boundary points and interior points. At the same time, by the least squares method, the regression function and derivative can be estimated, without taking partial derivatives of the nonparametric estimators of the function.

The basic idea of local polynomial estimation is to let the value range of the independent variable $D$ be $\mathfrak{D}$, for $\forall D_0 \in \mathfrak{D}$, and to select a certain neighborhood of $D_0$. Those observation values of the dependent variable correspond to the observations of the independent variable, which is fitted in some way in this neighborhood, and the value of the curve obtained by this local fitting at $D_0$ is used as the estimated value $\widehat{G}(D_0)$ of the regression function $G(\cdot)$ at $G_0$. Supposing the model as follows

$$Y = G(D) + \varepsilon, E(\varepsilon) = 0, \mathrm{Var}(\varepsilon) = \sigma^2 < \infty. \tag{31}$$

Let $G^{(k)}(D)$ be the $k$ derivative of the regression function in the model (31), and then, $G(D) = G^{(0)}(D)$. Assuming that $G(D)$ has $p+1$ order derivative, for $\forall D_0 \in \mathfrak{D}$, $G(D)$'s Taylor-expanded in the $\varepsilon$-neighborhood of $D = D_0$ as

$$G(D) \approx G(D_0) + G'(D_0)(D - D_0) + \frac{G''(D_0)}{2!}(D - D_0)^2 + \cdots + \frac{G^{(p)}(D_0)}{p!}(D - D_0)^p. \tag{32}$$

Set $G^{(k)}(D)/k! = \beta_k(D)$, the above Equation (32) can be rewritten as

$$\begin{aligned} G(D) &\approx \beta_0(D_0) + \beta_1(D_0)(D - D_0) \\ &\quad + \beta_2(D_0)(D - D_0)^2 + \cdots + \beta_p(D_0)(D - D_0)^p \\ &= \sum_{j=1}^{p} \beta_j(D_0)(D - D_0)^j. \end{aligned} \tag{33}$$

In order to draw the SPD function curve derived from the single-index model, after obtaining the estimation of the function $G(D)$, according to Formula (31), the second-order derivative of $G(D)$ with respect to the single-index $D$ is the SPD function, which is effective to estimate this second derivative by the local polynomial estimation method, and the parameter $\beta_j(D_0)$ is selected to minimize Formula (34) as

$$\min_{\beta_0, \cdots, \beta_p} \sum_{i=1}^{n} \left[ G(D_i) - \sum_{j=0}^{p} \beta_j(D_0)(D_i - D_0)^j \right]^2 \mathbb{K}_h(D_i - D_0), \tag{34}$$

where $h$ is the bandwidth that controls the local neighborhood, $\mathbb{K}_h(D_i - D_0) = \mathbb{K}(D_i - D_0/h)$, and $\mathbb{K}(\cdot)$ represents the kernel function. The kernel function $\mathbb{K}(\cdot)$ and bandwidth $h$ are still selected by the Gaussian kernel and the method of Formula (27), respectively. Since the inversion matrix part in the local polynomial estimation solution process is usually a singular matrix after substituting the actual data, the inversion matrix

Table 4: MSE of $G(D)$'s local polynomial estimation.

| | | | | | |
|---|---|---|---|---|---|
| Bandwidth | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
| MSE | 0.0033341 | 0.0033006 | 0.0032892 | 0.0032945 | 0.0033008 |
| Bandwidth | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
| MSE | 0.0033142 | 0.0033707 | 0.0034453 | 0.0036354 | 0.0057721 |

cannot be solved normally. Therefore, the Moore-Penrose generalized inverse matrix solution method is also used as an alternative method in the actual calculation. Let

$$X(D_0)=\begin{pmatrix}1 & D_1-D_0 & (D_1-D_0)^2\\ 1 & D_2-D_0 & (D_2-D_0)^2\\ \vdots & \vdots & \vdots\\ 1 & D_n-D_0 & (D_n-D_0)^2\end{pmatrix},$$
$$Y=\begin{pmatrix}G(D_1)\\ G(D_2)\\ \vdots\\ G(D_n)\end{pmatrix},$$
$$\beta(D_0)=\begin{pmatrix}\beta_0(D_0)\\ \beta_1(D_0)\\ \vdots\\ \beta_p(D_0)\end{pmatrix},$$
$$W(D_0)=\mathrm{diag}\,(\mathbb{K}_h(D_1-D_0),\cdots,\mathbb{K}_h(D_n-D_0)). \tag{35}$$

Therefore, the estimated value of $\beta(D_0)$ can be obtained by the weighted least squares method as follows:

$$\widehat{\beta}(D_0)=\left(X^TWX\right)^{-1}X^TWY. \tag{36}$$

In particular, when $p=0$, we can get

$$\widehat{\beta}_0(D_0)=\widehat{G}(D_0)=\frac{\sum_{i=1}^n\mathbb{K}_h(D_i-D_0)Y_i}{\sum_{i=1}^n\mathbb{K}_h(D_i-D_0)}, \tag{37}$$

where $Y_i=G(D_i)$, Formula (37) changes form as the same as Formula (22). Formula (37) is the Nadaraya-Watson kernel estimation of $G(D_0)$; that is, the Nadaraya-Watson estimation is a zero-order local polynomial estimation. At the same time, the estimated value of the second-order partial derivative of the estimated function $G(D)$ with respect to a single-index $D$ is

$$2\widehat{\beta}_2(D_0)=\left.\frac{\partial^2G(D)}{\partial D^2}\right|_{D=D_0}, \tag{38}$$

which derivative $\widehat{\beta}_2(D_0)$ is the SPD function in the new sense proposed in this paper.

The distance between $D_i$ and $D_0$ is designed to measure the weight of $D_i$ when the estimation the density is $D_0$.

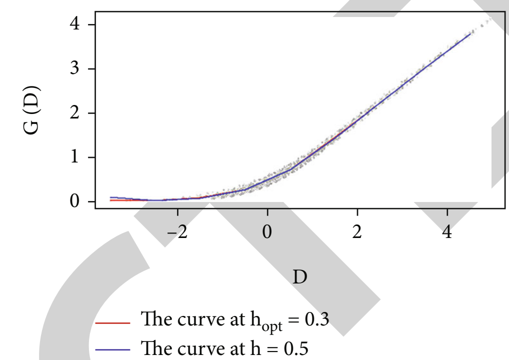


Figure 3: The curve of local polynomial fitting for the $G(D)$ function.

Besides, $h$ has defined the size of $D_0$ in the process of estimation as the bandwidth. Here, $Y$ for $G(D)$ the same as $X$ for $D$. Similarly, the Gaussian kernel function is also applied. Specifically, the CV method is formulated to select the bandwidth of minimum MSE on the test set, with assigning 10 numbers from 0.1 to 1 with an interval of 0.1 as shown in Table 4.

As shown in Table 4, it can be seen that the selection of optimal bandwidth by using the CV method that minimizes MSE on the test set is $h_{opt}=0.3$ (the optimal curve of local polynomial fitting for the $G(D)$ function represented by the red curve). The curve of local polynomial fitting for the $G(D)$ function can be seen in Figure 3. For comparison, the curve at $h=0.5$ is included in Figure 3, in which real data are successful to be shown in the gray scattered points, and the blue curve is identified as the suboptimal curve of local polynomial fitting for the $G(D)$ function.

The put option price on the test set can be estimated by using the previously estimated Formula (19) which can be calculated after estimating $G(D)$. It can be seen that the comparison of all put option strike prices and the estimated value of the put option prices with their true values during the sample period has been presented in Figure 4.

By comparison, it is clear to see that the results estimated by the method in this section are more approximate to the actual data than the method provided in previous Section 4.1. It is evident that the best MSE of the put option price calculated by the nonparametric estimation method for single-index put option prices is 0.0001599, which is more superior than that calculated by the common nonparametric estimation method for put option prices as 0.0005965.

From Figure 4, the estimated value of the SPD function curve by using the single-index model is distributed around the value of Formula (38). Although the center of these values is not as close to the curve of Formula (38) as in the

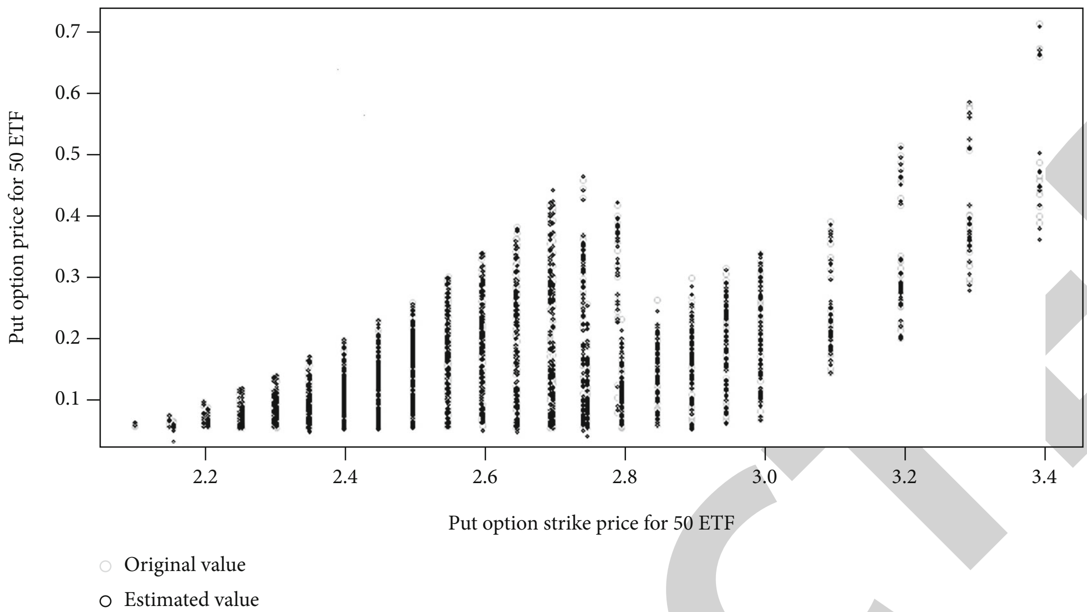


FIGURE 4: Original and estimated values of single-index model for SSE 50ETF put option price.

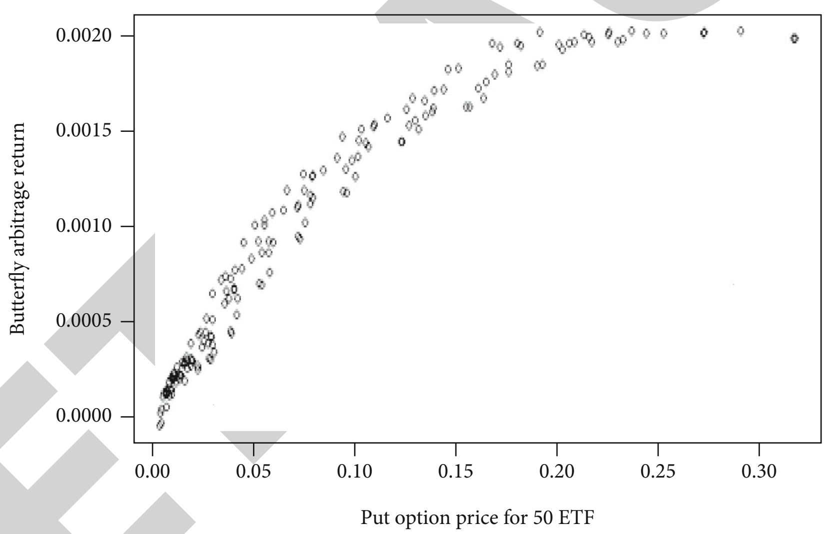


FIGURE 5: Put option contract price—butterfly portfolio returns of SSE 50ETF option in August.

previous Section 4.1, the curve estimated by the single-index model is smoother. The strike price near the sample boundary point of 3.4 RMB does not happen to wild volatility and breakpoint problems like the estimated value in the previous Section 4.1, which is especially important to obtain as much information as possible about the put option prices when the sample is limited or some points are missing. In practice, estimating the SPD can be as accurate as possible by combining models, such as using a nonparametric local polynomial estimation method at the nonboundary points, or a nonparametric single-index model method near the boundary points. Therefore, such a combined model has certain requirements on the sample size. Under small samples are methods such as biasing local polynomial estimation greatly. Similarly, the SPD function estimated by the single-index model estimation method may also show the characteristic that the SPD value decreases with the increase of the expiration time.

*4.3. The Nonparametric Estimation Deep Learning Method for Butterfly Portfolio Returns.* With the proof of optimality, it has access to the single-index option price nonparametric estimation method provided in this paper to estimate the possible butterfly portfolio returns. After the estimation of $G(D)$, it is known that the first-order derivative of $D$ has the property of SPD function, but Formula (13) is supposed to be valid only when it is satisfied with the condition of $\varepsilon \longrightarrow 0$. As a matter of fact, the equation is established only when the strike price lies in the minimum interval in the real market.

Indeed, the value of the derivative required in the Formula (13) can be found by numerical solution. Furthermore,

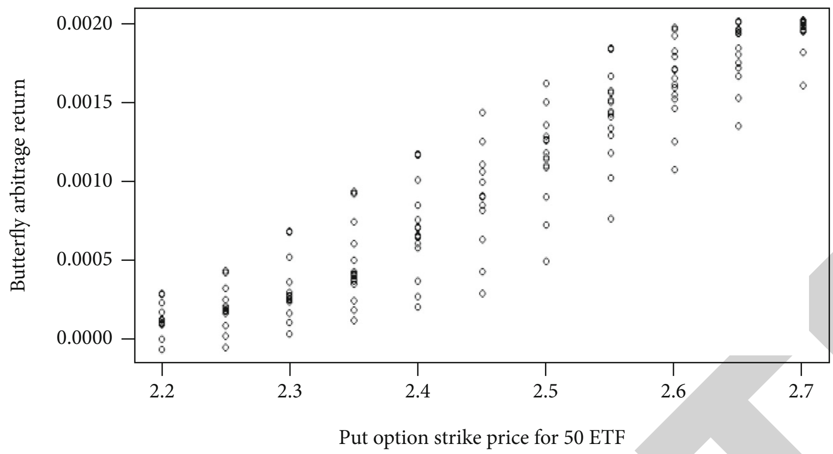


Figure 6: Put option strike price—butterfly portfolio returns of SSE 50ETF option in August.

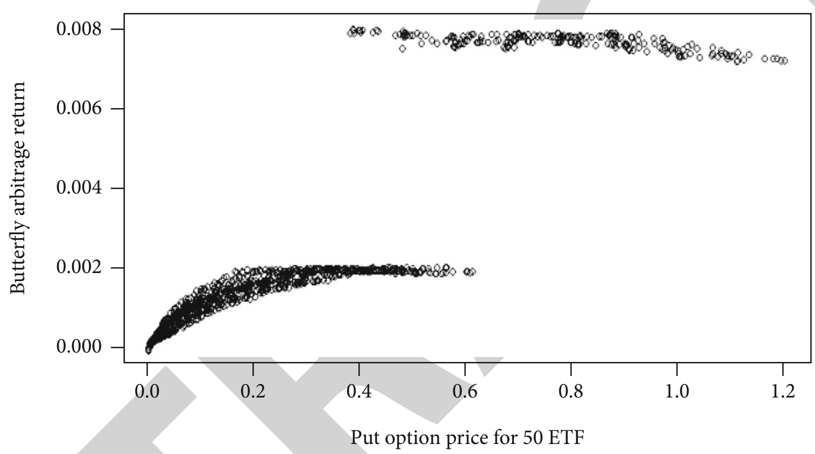


Figure 7: The chart of put option contract price—butterfly portfolio returns of SSE 50ETF option.

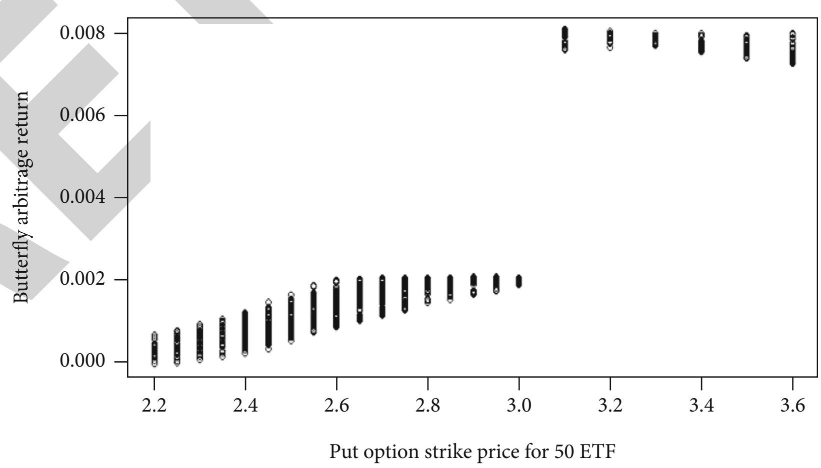


Figure 8: The chart of put option strike price—butterfly portfolio returns of SSE 50ETF option.

it is necessary to work out the result of $\Delta G(D)$) within the minimum change of $D$, such as $\Delta D = 0.00001$ so that the slope with a tiny move approximates to derivative value. Finally, substituting them into the formula to obtain parameter estimations of possible return from a short or long butterfly portfolio at minimum, the strike price interval of $\mathfrak{F} = (S, K, r, q, \tau)$. After estimating the optimal function $G($ $D)$, it is easy to obtain the possible butterfly portfolio returns at each point $\mathfrak{F}$ in the sample period, taking the August contract as an example.

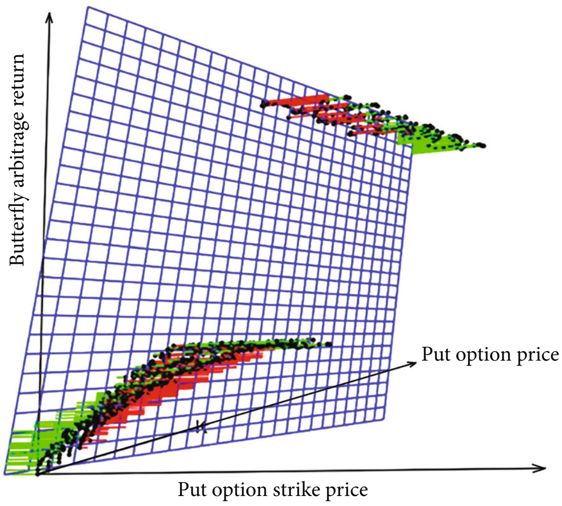


Figure 9: Three-dimensional graph of butterfly portfolio returns—strike prices—option prices.

Generally, the risk-free interest rate, volatility, and dividend yield are considered to be constant in the analysis. From Figure 5, it can be observed that the butterfly portfolio returns raise along with the increase of option price, taking the change of expiration time and the difference of strike price into consideration. On account of the change of expiration time and option price, it can be seen that there is a growth trend in butterfly portfolio returns with the growth of the strike price from Figure 6. Therefore, all of option contracts in August, September, and December in the market are summarized as follows.

As shown in Figures 7 and 8, it is feasible to extend the analysis period from the August contracts to the full-contract month put options listed in the market. The jump gap in the figure is attributed to the change of the strike price interval as specified in Table 4. In addition, this paper also draws three-dimensional graphs of nonparametric estimation for butterfly portfolio returns, strike prices, and option prices for butterfly portfolio construction at all points $\mathfrak{F}$ in the selected sample period (see Figure 9).

For more discussion on the issue of put option butterfly portfolio returns, the corresponding put option price is calculated based on the bullish and bearish parity relation. The nonparametric estimation deep learning method in this paper can be used for the same research.

## 5. Conclusions

In this paper, more emphases have been put on the nonparametric estimation method of option price, and the practical application has been added to obtain the possible butterfly portfolio return at the minimum strike price interval. Firstly, the evidence of classical definition of SPD and the origin of butterfly portfolio return formula have been reviewed. Secondly, the nonparametric estimation model of single-index option price is proposed to deduce another form of SPD function calculated by European put option price, which provides a new model to calculate the important index of financial risk measurement. As a result, the classic nonparametric estimation method of European option price is supposed to arrive at the minimum MSE values with approach of kernel density estimation, while the single-index nonparametric method of that to the minimum MSE values is required with local polynomial regression estimation. In comparison, it is reasonable to explain that the MSE of single-index method is smaller than classical nonparametric estimation method by the comparison, which illustrates that the method proposed has the superiority. Finally, the single-index nonparametric model is performed to estimate a new form of the SPD function, making it possible to obtain parameter estimations of possible butterfly portfolio returns at the minimum of the strike price interval, which has provided a powerful reference for investors taking butterfly positions in the options market. In future, the advanced method to estimate the option-implied state price density (SPD) will be proposed, such as sieve method which is one of the seminonparametric models.

## Appendix

## A. Fundamental Properties for Option Pricing, SPD and Butterfly Portfolio

*A.1. Black-Scholes-Merton Option Pricing.* Assuming that the change in the stock price $S(t)$ obeys the generalized Wiener process $W(t)$, or the stock price respects geometric Brownian movement (GBM), that is, with constant drift rate and variance rate, the model is derived as the following; the stochastic differential equation (SDE) shows

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t), \quad \text{(A.1)}$$

where $S(t)$ denotes the stock price, $\sigma$ means the stock price volatility, and $\mu$ is the stock return expectation. Assuming that the stock price $S(t) = S$ respects the Itô process, the assumption about $V = V(S, t)$ is the derivative price related to $S$, and the variable $V$ is a function of $S$ and $t$ by Itô's lemma

$$dV = \left(\frac{\partial V}{\partial S}\mu S + \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial V}{\partial S}\sigma S dW(t). \quad \text{(A.2)}$$

It is appropriate to choose the combination to remove uncertainty, such as taking a short position in a derivative and regarding a $\partial V/\partial S$ amount of stock as a combination. After defining $\Pi$ as the value of the combination, then

$$\Pi = V - \frac{\partial V}{\partial S}S, \qquad d\Pi = dV - \frac{\partial V}{\partial S}dS. \quad \text{(A.3)}$$

Substituting Equation (A.1) and Equation (A.2) into Equation (A.3), the equation in discrete form is

$$d\Pi = \left( \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt. \tag{A.4}$$

It is found that the $W(t)$ term in the equation has been eliminated, so the investment portfolio must be risk-free within $dt$ time, and at the same time, it can be discovered that an instantaneous rate of return is equal to the market risk-free rate that can be obtained; therefore

$$\frac{d\Pi}{\Pi} = rdt, r\Pi dt = \left( \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt, \tag{A.5}$$

where $r$ is the risk-free interest rate. Substituting Equations (A.3) and (A.4) into Equation (A.5), we can get

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0. \tag{A.6}$$

Equation (A.6) is the Black-Scholes-Merton (BSM) equation.

When $t = T$, the boundary conditions for European call option are $C(S, T) = \max(S_T - K, 0)$, where $K$ is the option strike price. When $S_T = S(T) = 0$, the call option yield is 0, so when $S(t) = S = 0$, there is $C(0, t) = 0$. When $S \longrightarrow \infty$, the value of the call option becomes the value of the stock, that is, $C(S, t) \sim S$. The boundary conditions for European put option are $P(S, T) = \max(K - S_T, 0)$, if $S \equiv 0$, then, the terminal payoff of the put option is $K$, assuming the interest rate $r$ is constant, and the boundary condition for $S = 0$ is $P(0, t) = Ke^{-r(T-t)}$. When $S \longrightarrow \infty$, the put option cannot be exercised, so $P(S, t) \longrightarrow 0$. Substituting the boundary conditions into the BSM (Equation (A.6)) to solve the pricing formulas for European call option price and European put option price, then

$$C(S, t) = SN(d_1) - Ke^{-r(T-t)}N(d_2), \tag{A.7}$$

$$P(S, t) = Ke^{-r(T-t)}N(-d_2) - SN(-d_1), \tag{A.8}$$

where $d_1 = (\ln(S/K) + (r + \sigma^2/2)(T-t))/(\sigma\sqrt{T-t})$ and $d_2 = (\ln(S/K) + (r - \sigma^2/2)(T-t))/(\sigma\sqrt{T-t}) = d_1 - \sigma\sqrt{T-t}$.

When the underlying common stock pays dividends, $D(S, t)$ is denoted as the dividends paid by one stock per unit time, and the expected rate of return in the stochastic differential Equation (A.1) that the stock price obeys goes to become $\mu - D(S, t)/S$, and then, the value of the option with dividends is defined as $H(S, t)$, and when $r, \sigma$ are constants, $D(S, t) = qS$, on account of further proof in the book by Hull [24], the stock price satisfies the stochastic differential equation.

$$dS(t) = (\mu - q)S(t)dt + \sigma S(t)dW(t). \tag{A.9}$$

The corresponding option value equation is

$$\frac{\partial H}{\partial t} + (r - q)S\frac{\partial H}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 H}{\partial S^2} - rH = 0. \tag{A.10}$$

In Formula (A.10), making variable substitution $\tilde{S} = Se^{-q(T-t)}$, we can get

$$\frac{\partial H}{\partial t} + r\tilde{S}\frac{\partial H}{\partial \tilde{S}} + \frac{1}{2}\sigma^2 \tilde{S}^2 \frac{\partial^2 H}{\partial \tilde{S}^2} - rH = 0. \tag{A.11}$$

The terminal condition for a call option with a dividend payment remains as $H(S, T) = \max(S_T - K, 0)$, when $S = 0$, there still is $H(0, t) = 0$. When $S \longrightarrow \infty$, the value of the call option becomes the value of the stock; that is, $H(S, t) \sim Se^{-q(T-t)}$, we can get

$$H(S, t) = C\left(\tilde{S}, t\right) = e^{-q(T-t)}SN(d_{10}) - Ke^{-r(T-t)}N(d_{20}), \tag{A.12}$$

where $d_{10} = (\ln(S/K) + (r - q + \sigma^2/2)(T-t))/(\sigma\sqrt{T-t})$ and $d_{20} = (\ln(S/K) + (r - q - \sigma^2/2)(T-t))/(\sigma\sqrt{T-t}) = d_{10} - \sigma\sqrt{T-t}$. It also shows that stock holders have dividend income, while option holders have no dividend income.

*A.2. State Price Density (SPD) Function.* Because of the difficulty in solving the BSM differential equation, the most commonly used derivative pricing method is the equivalent martingale measure. The solution of BSM option pricing (Equation (A.6)) has probabilistic expression as

$$V(S, t) = e^{-r(T-t)} \int_0^\infty \max\{S_T - K, 0\} f(S_T, T; S, t) dS_T, \tag{A.13}$$

where $f(S_T, T; S, t)$ is the risk-neutral transfer density function or SPD function of the stock taking the value of $S_T$ at time $T$ under the condition of $S$. Equation (A.13) means that the price of the option at time $t$ is the discount of its value at time $T$ under the risk-neutral probability. Defining $\max\{S_T - K, 0\} = \Phi(S_T)$ and differentiating Formula (A.13), we can get

$$\begin{aligned} \frac{\partial V}{\partial t} &= rV + e^{-r(T-t)} \int_0^\infty \Phi(S_T) \frac{\partial f}{\partial t} dS_T, \\ \frac{\partial V}{\partial S} &= e^{-r(T-t)} \int_0^\infty \Phi(S_T) \frac{\partial f}{\partial S} dS_T, \\ \frac{\partial^2 V}{\partial S^2} &= e^{-r(T-t)} \int_0^\infty \Phi(S_T) \frac{\partial^2 f}{\partial S^2} dS_T. \end{aligned} \tag{A.14}$$

Since the geometric Brown motion $S$ satisfies Equation (A.1), its transfer density function $f(S_T, T; S, t)$ satisfies the Kolmogorov equation

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = 0. \tag{A.15}$$

In particular, for Equation (A.6), with boundary conditions $V(0, t) = 0$ and $V(S, t) < S$ and a terminal payoff of

$V(S,T)=\Phi(S)$, the value of an undetermined interest at time $t$ is

$$V(S,t)=e^{-r(T-t)}\int_0^{\infty}\Phi(S_T)f(S_T,T;S,t)dS_T=e^{-r(T-t)}\tilde{\mathbb{E}}_t(\Phi(S_T)), \tag{A.16}$$

where the notation $\tilde{\mathbb{E}}_t$ for conditional expectations is explained in Section 4.

Under the assumption that the market is complete, assuming that there is a market in which bonds and stocks can be traded freely, and the risk-free interest on bonds is fixed as $r$, that is, when there is a unique risk-neutral probability measure in the market, at any given time $t$, for a derivative security that generates a payment on the maturity date, its price can be presented with the risk-neutral pricing formula provided by Harrison and Kreps [25] and Pliska [26] for a derivative security with a payment of $K=\Phi(S_T)$ at maturity $T$. When the risk-free interest rate $r$ in the above formula is constant, $f(S_T,T;S,t)$ also refers to the SPD function, which summarizes all the required information in the course of pricing derivative securities. The SPD function $f(S_T,T;S,t)$ is the solution of the backward Kolmogorov Equation (A.15), which is derived from the European call option by Breeden and Litzenberger [8]:

$$f(S_T,T;S_t,t)=\frac{1}{S_t\sqrt{2\pi\sigma^2(T-t)}}\cdot\exp\left[-\frac{\left[(r-q-\sigma^2/2)(T-t)-\ln S_T+\ln S_t\right]^2}{2\sigma^2(T-t)}\right]. \tag{A.17}$$

If noting $S(t)=S=S_t$, $\tau=T-t$, and substituting Formula (A.17) into Formula (A.16), we can get

$$V(S_t,t)=e^{-r\tau}\int_0^{\infty}\max\{S_T-K,0\}\frac{1}{S_t\sqrt{2\pi\sigma^2\tau}}\cdot\exp\left[-\frac{\left[(r-q-\sigma^2/2)\tau-\ln S_T+\ln S_t\right]^2}{2\sigma^2\tau}\right]dS_T, \tag{A.18}$$

making $y=((r-q-\sigma^2/2)\tau-\ln S_T+\ln S_t)/(\sigma\sqrt{\tau})$, and then, Formula (A.18) becomes

$$V(S_t,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-r\tau}\max\left\{S_te^{-\sigma\sqrt{\tau}y+(r-q-\sigma^2/2)\tau}-K,0\right\}e^{-(y^2/2)}dy. \tag{A.19}$$

For a no-dividend European call option, the process of the first-order derivative of Equation (A.16) of the strike price is designed to obtain the SPD function as follows:

$$\frac{\partial c(S_t,\tau)}{\partial K}=\frac{\partial\left(e^{-r\tau}\int_K^{\infty}(S_T-K)f(S_T,T;S_t,t)dS_T\right)}{\partial K}=-e^{-r\tau}\int_K^{\infty}f(S_T,T;S_t,t)dS_T. \tag{A.20}$$

Since the density function is definitely greater than 0 in the value range, corresponding to the European call option, when $S_T>K$, then $f(S_T,T;S_t,t)>0$

$$\left.\frac{\partial^2 c(S_t,\tau)}{\partial K^2}\right|_{K=x}=e^{-r\tau}f(x,S_t,\tau)>0. \tag{A.21}$$

To be more specific, the call option price $c(S_t,\tau)$ is a convex function of the strike price $K$, and there is

$$c(S_t,\lambda K_1+(1-\lambda)K_3,\tau)<\lambda c(S_t,K_1,\tau)+(1-\lambda)c(S_t,K_3,\tau),\forall K_1<K_3,0<\lambda<1. \tag{A.22}$$

Let

$$\lambda K_1+(1-\lambda)K_3=K_2\longrightarrow\lambda=\frac{K_3-K_2}{K_3-K_1}, \tag{A.23}$$

Then

$$c(S_t,K_2,\tau)<\lambda c(S_t,K_1,\tau)+(1-\lambda)c(S_t,K_3,\tau),\forall K_1<K_2<K_3,0<\lambda<1. \tag{A.24}$$

Theoretically, the price of European call options without arbitrage is supposed to conform with the above equation.

*A.3. Butterfly Portfolio Model.* It is intuitive to construct an option butterfly portfolio returns to the explained Equation (A.17). Here, we can see that a call option butterfly portfolio returns are designed to buy a call option with a strike price of $K-\varepsilon$ and $K+\varepsilon$, respectively, with the same expiration date, and sell two calls with a strike price of $K$. Then, the gain and loss of the butterfly portfolio on the option premium are shown as

$$2c(S_t,K,\tau)-c(S_t,K-\varepsilon,\tau)-c(S_t,K+\varepsilon,\tau). \tag{A.25}$$

The profit of call option butterfly portfolio returns is shown in Figure 10.

When $\varepsilon\longrightarrow 0$, the return function form of the butterfly portfolio is similar to Dirac $\delta$ function ($\delta(x)=0, x\neq 0$ and $\int_{-\infty}^{+\infty}\delta(x)dx=1$). More precisely, only allowing the underlying stock price to be $K$, in this case, the return is shown as the area $\varepsilon^2$ of the shaded part in Figure 11.

However in other cases, it is virtually believed to be 0. Geometrically, regarding the combination at this time as a single security, it turns out to be $\varepsilon^2 f(x)e^{-r\tau}$ at $(t)$ the moment, and $f(x)$ serves as the SPD function. In the principle of no arbitrage, theoretically, the profit and loss for option of constructing butterfly portfolio returns are

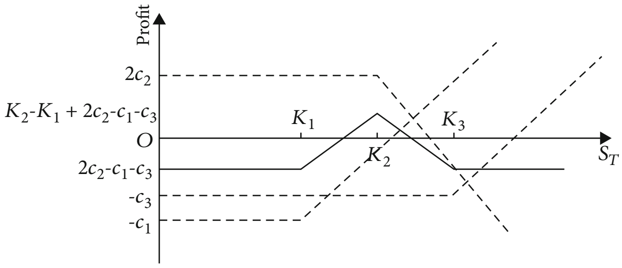


FIGURE 10: The profit of call option butterfly portfolio returns.

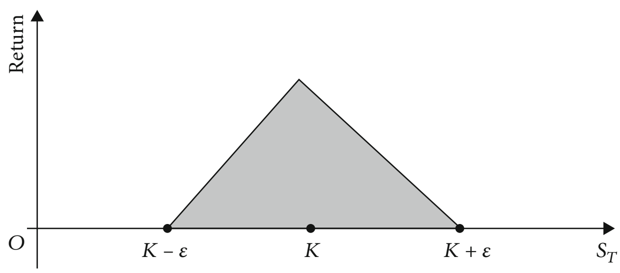


FIGURE 11: The butterfly portfolio returns.

possible to be equal to the return from the portfolio, so when $\varepsilon \longrightarrow 0$ :

$$c(S_t, K-\varepsilon, \tau) + c(S_t, K+\varepsilon, \tau) - 2c(S_t, K, \tau) = \varepsilon^2 f(K, S_t, \tau) e^{-r\tau}. \tag{A.26}$$

Substituting Formula (A.21) into the above Formula (A.26), then

$$\frac{c(S_t, K-\varepsilon, \tau) + c(S_t, K+\varepsilon, \tau) - 2c(S_t, K, \tau)}{\varepsilon^2} = \frac{\partial^2 c(S_t, K, \tau)}{\partial K^2}. \tag{A.27}$$

Equation (A.27) has express the intuitive interpretation of option butterfly portfolio returns of Equation (A.21). It is acknowledged that the option butterfly portfolio returns are capable of being calculated by SPD function. However, it has been assumed that the SPD function is normal density function with the normal distribution in most existing researches. Actually, it is not. Therefore, the nonparametric estimation method comes into being.

## Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

## Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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