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Research Article

Iterative Positive Solutions to a Coupled Riemann-Liouville Fractional q-Difference System with the Caputo Fractional q-Derivative Boundary Conditions

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This paper is devoted to the existence of positive solutions for a nonlinear coupled Riemann-Liouville fractional q-difference system, with multistrip and multipoint mixed boundary conditions under Caputo fractional q-derivative. We obtain the existence of positive solutions and initial iterative solutions by the monotone iteration technique. Then, we also calculate the error limits of the numerical approximation solution by induction. In the end, two examples are given to illustrate the above research results, and in the second example, some graphs of the iterative solutions are also drawn to give a more intuitive sense of the iterative process.

1. Introduction

Fractional calculus, which originated in 1695 [1], extended the traditional concept of integral calculus to the whole real field [2]. Over the next several hundred years, many mathematicians have studied the properties of fractional calculus. From these studies, it is not difficult to conclude that fractional calculus has some very nice properties and can be applied to various fields, such as random processes [3], non-Newtonian fluid mechanics [4], control theory [5], economics [6], and medical [7]. Especially in the modeling of dynamic problems, fractional differential equations are quite common. Therefore, fractional differential equations derived from fractional calculus have received more and more attention. The existence analysis stability analysis and numerical simulation of solutions have gradually become very important research directions [8–15].

Quantum calculus has been around since the beginning of the twentieth century. It was first studied systematically by Jackson [16, 17]. Readers can refer to the works of Kac and Cheung for further research and applications [18]. Youm used q-calculus to study quantum mechanics in 2000 and came up with the nonexchange theories [19]. Quantum calculus is not only used in quantum mechanics but also plays an important role in economics [20], dynamic system and quantum model [21], heat and wave equation [22], sampling signal analysis theory [23], and so on. Because of its wide application, fractional q-difference equation has entered the field of view of researchers [24–40].

In [25], the following q-difference boundary value problem (BVP) is discussed:

$$\begin{cases} D_q^{\alpha} u(t) + f(t, u(t)) = 0, & t \in [0, 1], \\ u(0) = u(1) = 0, \end{cases} \tag{1}$$

where $0 < q < 1$, and $1 < \alpha \leqslant 2$. By means of the Guo–Krasnoselskii's fixed point theorem, the authors proved the existence of nontrivial solutions to the fractional q-difference equation with Dirichlet boundary condition, which is one of the three well-known and fundamental boundary conditions. Thereafter, more and more researchers began to study the existence and uniqueness of solutions to fractional q-difference boundary value problem.

In 2020, the authors [39] considered the following fractional q-difference system with four-point boundary conditions by using the monotone iterative approach:

$$\begin{cases} D_q^{\alpha}u(t)+f(t,v(t))=0, & t\in(0,1),\\ D_q^{\beta}v(t)+f(t,u(t))=0, & t\in(0,1),\\ u(0)=0, u(1)=\gamma_1 u(\eta_1),\\ v(0)=0, v(1)=\gamma_2 v(\eta_2), \end{cases} \tag{2}$$

where $1<\beta\leqslant\alpha\leqslant 2$, $0<\eta_1,\eta_2<1$, $0<\gamma_1\eta_1^{\alpha-1}<1$, and $0<\gamma_2\eta_2^{\beta-1}<1$. As far as we all know, lots of the papers studied single fractional q-difference equation. This work attracted our attention to fractional q-difference system.

Recently in [29], based on the Leray–Schauder alternative principle and Guo–Krasnoselskii's fixed point theorem, the authors investigated the existence and uniqueness of nonnegative solutions of the nonlinear fractional q-difference boundary value problem:

$$\begin{cases} {}^cD_q^{\sigma}[k](t)+w\left(t,k(t),{}^cD_q^{\zeta}[k](t)\right)=0,\\ k(0)=k''(0)=0,\\ k'(r)=\lambda k''(r), \lambda>0, \end{cases} \tag{3}$$

where $t\in J\coloneqq(0,1)$ and $0<q<1$, and $w:[0,1]\times\mathbb{R}^2\longrightarrow\mathbb{R}$ is a given function with $[0,1], 2<\sigma<3, \zeta,\sigma\in J$. They presented the global convergence of the proposed method with the line searches. In addition, they built some algorithms to assist the proof process. The results of numerical experiments demonstrated the effectiveness of the proposed algorithm.

In [27], the aim is to provide the conclusion of the existence and uniqueness of solutions for the following mixed fractional q-difference boundary value problem:

$$\begin{cases} D_q^{\alpha}u(t)+f(t,u(t))=0, & t\in[0,1],\\ u(0)={}^cD_q^{\beta}u(0)={}^cD_q^{\beta}u(1), \end{cases} \tag{4}$$

where $0<\beta\leqslant 1$, $2<\alpha<2+\beta$, and $D_q^{\alpha},{}^cD_q^{\beta}$ are the Riemann–Liouville fractional q-derivative and Caputo fractional q-derivative of order α and β. Using the Guo–Krasnoselskii's fixed point theorem and the Banach contraction mapping principle as well as the Schaefer's fixed point theorem, the authors obtained the existence and uniqueness of solutions in cone.

In the existing literatures, a large number of papers focus on the subject about equations or systems under the definition of a single type of q-derivative, Riemann-Liouville, or Caputo derivative. In [27], the model of mixed q-derivative is studied for the first time, in which the form of the equation is concise and the boundary conditions are simple. However, as the models in the real world are extremely complex, such simple models cannot describe complex dynamic processes under some conditions. Therefore, we are more interested in the improvement of the model of mixed q-derivative which is more in line with the real needs. In [10], the authors study a system with single derivative, but it is a coupled system with the multistrip and multipoint mixed boundary conditions. We know that the multistrip and multipoint mixed boundary conditions highly summarize the characteristics of boundary conditions in the existing studies and have more research significance. In addition, the coupled derivative system can better depict the interaction between different factors in the real model.

As a consequence, motivated by aforementioned work, we consider that there is no relevant research results about the coupled system under mixed q-derivative with complex boundary conditions, so we fill in this part of the gap. We investigate the following coupled nonlinear fractional q-difference system subject to the multistrip and multipoint mixed boundary conditions, where the Riemann-Liouville fractional q-derivative and Caputo fractional q-derivative are applied to the q-difference system and the boundary conditions separately:

$$\begin{cases} D_q^{\alpha_1}u(t)+f_1(t,u(t),v(t))=0, & t\in(0,1),\\ D_q^{\alpha_2}v(t)+f_2(t,u(t),v(t))=0, & t\in(0,1), \end{cases} \tag{5}$$

subject to the boundary conditions

$$\begin{cases} u(0)=0, v(0)=0,\\ {}^cD_q^{\alpha_1-1}u(1)=\sum_{i=1}^{m}\lambda_{1i}I_q^{\beta_{1i}}v(\xi_i)+\sum_{j=1}^{n}b_{1j}v\left(\eta_j\right),\\ {}^cD_q^{\alpha_2-1}v(1)=\sum_{i=1}^{m}\lambda_{2i}I_q^{\beta_{2i}}u(\xi_i)+\sum_{j=1}^{n}b_{2j}u\left(\eta_j\right), \end{cases} \tag{6}$$

where ${}^cD_q^{\alpha}, D_q^{\alpha}$ are the Caputo fractional q-derivative and Riemann-Liouville fractional q-derivative of order α, respectively, and $I_q^{\beta_{ki}}$ is the Riemann-Liouville fractional q-integral of order β_{ki}, for $k=1,2$ and $i=1,2,\cdots,m$. $1\leqslant\alpha_1,\alpha_2\leqslant 2$; $\lambda_{1i},\lambda_{2i}\geqslant 0$, $\beta_{1i},\beta_{2i}>0$, and $0\leqslant\xi_i\leqslant 1$, for $i=1,2,\cdots,m$; b_{1j}, $b_{2j}\geqslant 0$, and $0\leqslant\eta_j\leqslant 1$, for $j=1,2,\cdots,n$.

At present, there are many theoretical methods to solve the boundary value problem of fractional q-difference equations. The majority use all kinds of the fixed point theorem methods, such as the Guo–Krasnoselskii's fixed point theorem [25–29], the Leray–Schauder alternative principle, and the Banach contraction mapping principle [30–34], and less use monotone iteration techniques [35–39]. All these methods can effectively study the existence of solutions, but the reason why we chose to use the monotone iterative method is that it has more advantages than other methods, which can not only prove the existence of positive solutions but also obtain numerical approximate solutions within certain limits of error.

This article expands from the following aspects: in Section 2, we introduce some fundamental definitions and lemmas. Also, we discuss some crucial results and their proofs. Section 3 contains the main conclusion: the existence results of monotone iterative positive solutions. In addition, two examples are given to prove our result.

2. Preliminaries

Here, we list some basic notion, lemmas, and some auxiliary results for the proof, which will be used in the next section.

For $q \in (0,1)$ and $a \in \mathbb{R}$, define

$$[a]_q = \frac{1-q^a}{1-q}. \tag{7}$$

The q-analogue of the power function is

$$(a-b)_q^{(0)} = 1, \tag{8}$$

$$(a-b)_q^{(k)} = \prod_{i=0}^{k-1}\left(a-bq^i\right), k \in \mathbb{N}, a, b \in \mathbb{R}. \tag{9}$$

Generally, if $\alpha \in \mathbb{R}$, there is

$$(a-b)_q^{(\alpha)} = a^\alpha \prod_{i=0}^{\infty} \frac{a-bq^i}{a-bq^{i+\alpha}}. \tag{10}$$

It is clear that $a^{(\alpha)} = a^\alpha$ for $b=0$ and $0^{(\alpha)} = 0$ for $\alpha \geq 0$.

The q-Gamma function is given by

$$\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}, x \in \mathbb{R} \setminus \{0,-1,-2,\cdots\}, \tag{11}$$

then, we have $\Gamma_q(x+1) = [x]_q \Gamma_q(x)$.

For $x, y > 0$, we have

$$B_q(x,y) = \int_0^1 t^{x-1}(1-qt)^{(y-1)} d_q t, \tag{12}$$

especially

$$B_q(x,y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}. \tag{13}$$

The q-derivative of a function f is defined by

$$\left(D_q f\right)(x) = \frac{f(qx)-f(x)}{(q-1)x}, \tag{14}$$

$$\left(D_q f\right)(0) = \lim_{x \longrightarrow 0}\left(D_q f\right)(x), \tag{15}$$

and the q-derivative of higher order by

$$\left(D_q^0 f\right)(x) = f(x), \tag{16}$$

$$\left(D_q^n f\right)(x) = D_q\left(D_q^{n-1} f\right)(x), n \in \mathbb{N}. \tag{17}$$

The q-integral of a function f defined on the interval $[0,b]$ is given by

$$\left(I_q f\right)(x) = \int_0^x f(s) d_q s = x(1-q)\sum_{k=0}^{\infty} f\left(xq^k\right)q^k, x \in [0,b]. \tag{18}$$

If $a \in [0,b]$ and f is defined in the interval $[0,b]$; then, its integral from a to b is defined by

$$\int_a^b f(s) d_q s = \int_0^b f(s) d_q s - \int_0^a f(s) d_q s, \tag{19}$$

and similarly the q-integral of higher order is given by

$$\left(I_q^0 f\right)(x) = f(x), \tag{20}$$

$$\left(I_q^n f\right)(x) = I_q\left(I_q^{n-1} f\right)(x), n \in \mathbb{N}. \tag{21}$$

Definition 1 (see [41]). Let $\alpha \geq 0$ and f be a real function defined on a certain interval $[a,b]$. The Riemann-Liouville fractional q-integral of order α is defined by

$$\left(I_q^0 f\right)(t) = f(t), \tag{22}$$

$$\left(I_q^\alpha f\right)(t) = \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)} f(s) d_q s, \alpha > 0. \tag{23}$$

Definition 2 (see [41]). The fractional q-derivative of the Riemann-Liouville type of order $\alpha \geq 0$ of a continuous and differential function f on the interval $[a,b]$ is given by

$$\left(D_q^0 f\right)(t) = f(t), \tag{24}$$

$$\left(D_q^\alpha f\right)(t) = \left(D_q^l I_q^{l-\alpha} f\right)(t), \alpha > 0, \tag{25}$$

where l is the smallest integer greater than or equal to α.

Definition 3 (see [41]). Let $\alpha \geq 0$, and the Caputo fractional q -derivatives of f be defined by

$$\left({}^c D_q^\alpha f\right)(t) = \left(I_q^{l-\alpha} D_q^l f\right)(t), \tag{26}$$

where l is the smallest integer greater than or equal to α.

Lemma 4 (see [42]). *Let $\alpha, \beta \geq 0$ and $f : [a,b] \longrightarrow \mathbb{R}$ be a continuous function defined on $[a,b]$ and its derivative exist. Then, the following formulas hold:*

$$\left(D_q^\alpha I_q^\alpha f\right)(t) = f(t), \tag{27}$$

$$\left(I_q^\alpha I_q^\beta f\right)(t) = \left(I_q^{\alpha+\beta} f\right)(t). \tag{28}$$

Lemma 5 (see [42]). *Let $\alpha > 0$ and p be a positive integer. Then, the following equality holds:*

$$\left(I_q^\alpha D_q^p f\right)(t) = \left(D_q^p I_q^\alpha f\right)(t) - \sum_{k=0}^{p-1} \frac{t^{\alpha-p+k}}{\Gamma_q(\alpha-p+k+1)} \left(D_q^k f\right)(0). \tag{29}$$

Lemma 6 (see [18, 42]). *Let $\alpha > 0$ and $n = [\alpha] + 1$. Then, we have*

$$\left(I_q^{\alpha c} D_q^\alpha f\right)(t) = f(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \tag{30}$$

where $c_0, c_1, \cdots, c_{n-1}$ are some constants.

For convenience, we denote

$$\begin{cases} l_1 = \dfrac{1}{\Gamma_q(\alpha_2)} \left[\displaystyle\sum_{i=1}^m \frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})} \int_0^{\xi_i} (\xi_i - qs)^{(\beta_{1i}-1)} s^{\alpha_2-1} d_q s + \sum_{j=1}^n b_{1j} \eta_j^{\alpha_2-1}\right], \\ l_2 = \dfrac{1}{\Gamma_q(\alpha_1)} \left[\displaystyle\sum_{i=1}^m \frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})} \int_0^{\xi_i} (\xi_i - qs)^{(\beta_{2i}-1)} s^{\alpha_1-1} d_q s + \sum_{j=1}^n b_{2j} \eta_j^{\alpha_1-1}\right]. \end{cases} \tag{31}$$

In the forthcoming analysis, we always need the following assumptions:

(F_1) $1 \leqslant \alpha_k \leqslant 2, \beta_{ki} > 0$, for $k = 1, 2$ and $i = 1, 2, \cdots, m$

(F_2) $0 \leqslant \eta_j, \xi_i \leqslant 1, \lambda_{1i}, \lambda_{2i} \geqslant 0, b_{1j}, b_{2j} \geqslant 0$, for $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$

(F_3) $1 - l_1 l_2 > 0$, where l_1, l_2 are defined by (12)

(F_4) $f_k : [0, 1] \times [0, +\infty)^2 \longrightarrow [0, +\infty)$ is continuous $(k = 1, 2)$

Subject to BVP (5) and (6), we consider a corresponding linear differential system as follows and establish the expression of the corresponding Green's functions.

Lemma 7. *Assume that (F_1) - (F_3) hold. For $h_1, h_2 \in C(0, 1)$, the fractional differential system*

$$\begin{cases} D_q^{\alpha_1} u(t) + h_1(t) = 0, & t \in (0, 1), \\ D_q^{\alpha_2} v(t) + h_2(t) = 0, & t \in (0, 1), \end{cases} \tag{32}$$

with boundary condition (6) has an integral representation

$$\begin{cases} u(t) = \displaystyle\int_0^1 K_1(t, qs) h_1(s) d_q s + \int_0^1 H_1(t, qs) h_2(s) d_q s, \\ v(t) = \displaystyle\int_0^1 K_2(t, qs) h_2(s) d_q s + \int_0^1 H_2(t, qs) h_1(s) d_q s, \end{cases} \tag{33}$$

where

$$\begin{aligned} K_1(t, qs) &= g_1(t, qs) + \frac{l_1 t^{\alpha_1-1}}{\Gamma_q(\alpha_1)(1 - l_1 l_2)} \\ &\quad \cdot \left[\sum_{i=1}^m \frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})} \int_0^{\xi_i} (\xi_i - q\tau)^{(\beta_{2i}-1)} g_1(\tau, qs) d\tau + \sum_{j=1}^n b_{2j} g_1\left(\eta_j, qs\right)\right], \\ H_1(t, qs) &= \frac{t^{\alpha_1-1}}{\Gamma_q(\alpha_1)(1 - l_1 l_2)} \\ &\quad \cdot \left[\sum_{i=1}^m \frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})} \int_0^{\xi_i} (\xi_i - q\tau)^{(\beta_{1i}-1)} g_2(\tau, qs) d_q \tau + \sum_{j=1}^n b_{1j} g_2\left(\eta_j, qs\right)\right], \end{aligned} \tag{34}$$

$$\begin{aligned} K_2(t, qs) &= g_2(t, qs) + \frac{l_2 t^{\alpha_2-1}}{\Gamma_q(\alpha_2)(1 - l_1 l_2)} \\ &\quad \cdot \left[\sum_{i=1}^m \frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})} \int_0^{\xi_i} (\xi_i - q\tau)^{(\beta_{1i}-1)} g_2(\tau, qs) d\tau + \sum_{j=1}^n b_{1j} g_2\left(\eta_j, qs\right)\right], \\ H_2(t, qs) &= \frac{t^{\alpha_2-1}}{\Gamma_q(\alpha_2)(1 - l_1 l_2)} \\ &\quad \cdot \left[\sum_{i=1}^m \frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})} \int_0^{\xi_i} (\xi_i - q\tau)^{(\beta_{2i}-1)} g_1(\tau, qs) d_q \tau + \sum_{j=1}^n b_{2j} g_1\left(\eta_j, qs\right)\right], \end{aligned} \tag{35}$$

and for $k = 1, 2$,

$$g_k(t, qs) = \frac{1}{\Gamma_q(\alpha_k)} \begin{cases} t^{\alpha_k-1} - (t - qs)^{(\alpha_k-1)}, & 0 \leqslant qs \leqslant t \leqslant 1, \\ t^{\alpha_k-1}, & 0 \leqslant t \leqslant qs \leqslant 1. \end{cases} \tag{36}$$

Proof. From Lemma 5, we can reduce (32) to the following equivalent integral equations:

$$\begin{cases} u(t) = -I_q^{\alpha_1} h_1(t) + c_{11} t^{\alpha_1-1} + c_{12} t^{\alpha_1-2}, \\ v(t) = -I_q^{\alpha_2} h_2(t) + c_{21} t^{\alpha_2-1} + c_{22} t^{\alpha_2-2}, \end{cases} \tag{37}$$

where $c_{11}, c_{12}, c_{21}, c_{22}$ are constants.

From $u(0)=v(0)=0$, we obtain $c_{12}=c_{22}=0$. By using Lemma 6, we get

$$\begin{cases} {}^cD_q^{\alpha_1-1}u(t) = -{}^cD_q^{\alpha_1-1}I_q^{\alpha_1}h_1(t) + c_{11}{}^cD_q^{\alpha_1-1}t^{\alpha_1-1} = -I_qh_1(t) + c_{11}[\alpha_1-1]_qI_q^{2-\alpha_1}t^{\alpha_1-2}, \\ {}^cD_q^{\alpha_2-1}v(t) = -{}^cD_q^{\alpha_2-1}I_q^{\alpha_2}h_2(t) + c_{21}{}^cD_q^{\alpha_2-1}t^{\alpha_2-1} = -I_qh_2(t) + c_{21}[\alpha_2-1]_qI_q^{2-\alpha_2}t^{\alpha_2-2}, \end{cases} \tag{38}$$

Then, from (12), we get

$$\begin{cases} {}^cD_q^{\alpha_1-1}u(1) = -I_qh_1(1) + c_{11}[\alpha_1-1]_qI_q^{2-\alpha_1}1 = -\int_0^1 h_1(s)d_qs + c_{11}\Gamma_q(\alpha_1), \\ {}^cD_q^{\alpha_2-1}v(1) = -I_qh_2(1) + c_{21}[\alpha_2-1]_qI_q^{2-\alpha_2}1 = -\int_0^1 h_2(s)d_qs + c_{21}\Gamma_q(\alpha_2), \end{cases} \tag{39}$$

From the rest of the condition of (6), it can be obtained that

$$\begin{cases} c_{11} = \dfrac{1}{\Gamma_q(\alpha_1)}\left[\sum_{i=1}^m \lambda_{1i}I_q^{\beta_{1i}}v(\xi_i) + \sum_{j=1}^n b_{1j}v\left(\eta_j\right) + \int_0^1 h_1(s)d_qs\right], \\ c_{21} = \dfrac{1}{\Gamma_q(\alpha_2)}\left[\sum_{i=1}^m \lambda_{2i}I_q^{\beta_{2i}}u(\xi_i) + \sum_{j=1}^n b_{2j}u\left(\eta_j\right) + \int_0^1 h_2(s)d_qs\right], \end{cases} \tag{40}$$

Further, we can reduce (37) to

$$\begin{cases} u(t) = \dfrac{t^{\alpha_1-1}}{\Gamma(\alpha_1)}\left[\sum_{i=1}^m \dfrac{\lambda_{1i}}{\Gamma_q(\beta_{1i})}\int_0^{\xi_i}(\xi_i-qs)^{(\beta_{1i}-1)}v(s)d_qs + \sum_{j=1}^n b_{1j}v\left(\eta_j\right)\right] + \int_0^1 g_1(t,qs)h_1(s)d_qs, \\ v(t) = \dfrac{t^{\alpha_2-1}}{\Gamma(\alpha_2)}\left[\sum_{i=1}^m \dfrac{\lambda_{2i}}{\Gamma_q(\beta_{2i})}\int_0^{\xi_i}(\xi_i-qs)^{(\beta_{2i}-1)}u(s)d_qs + \sum_{j=1}^n b_{2j}u\left(\eta_j\right)\right] + \int_0^1 g_2(t,qs)h_2(s)d_qs, \end{cases} \tag{41}$$

where $g_k(t,qs)(k=1,2)$ is introduced by (36). Then, we can get

$$\begin{aligned} &\sum_{i=1}^m \frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})}\int_0^{\xi_i}(\xi_i-qs)^{(\beta_{1i}-1)}v(s)d_qs + \sum_{j=1}^n b_{1j}v\left(\eta_j\right) \\ &\quad = \frac{1}{\Gamma_q(\alpha_2)}\left[\sum_{i=1}^m \frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})}\int_0^{\xi_i}(\xi_i-qs)^{(\beta_{1i}-1)}s^{\alpha_2-1}d_qs + \sum_{j=1}^n b_{1j}\eta_j^{\alpha_2-1}\right] \\ &\qquad \cdot \left[\sum_{i=1}^m \frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})}\int_0^{\xi_i}(\xi_i-qs)^{(\beta_{2i}-1)}u(s)d_qs + \sum_{j=1}^n b_{2j}u\left(\eta_j\right)\right] \\ &\qquad + \sum_{i=1}^m \frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{1i}-1)}\int_0^1 g_2(\tau,qs)h_2(s)d_qsd_q\tau \\ &\qquad + \sum_{j=1}^n b_{1j}\int_0^1 g_2\left(\eta_j,qs\right)h_2(s)d_qs, \end{aligned} \tag{42}$$

$$\begin{aligned} &\sum_{i=1}^m \frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})}\int_0^{\xi_i}(\xi_i-qs)^{(\beta_{2i}-1)}u(s)d_qs + \sum_{j=1}^n b_{2j}u\left(\eta_j\right) \\ &\quad = \frac{1}{\Gamma_q(\alpha_1)}\left[\sum_{i=1}^m \frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})}\int_0^{\xi_i}(\xi_i-qs)^{(\beta_{2i}-1)}s^{\alpha_1-1}d_qs + \sum_{j=1}^n b_{2j}\eta_j^{\alpha_1-1}\right] \\ &\qquad \cdot \left[\sum_{i=1}^m \frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})}\int_0^{\xi_i}(\xi_i-qs)^{(\beta_{1i}-1)}v(s)d_qs + \sum_{j=1}^n b_{1j}v\left(\eta_j\right)\right] \\ &\qquad + \sum_{i=1}^m \frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{2i}-1)}\int_0^1 g_1(\tau,qs)h_1(s)d_qsd_q\tau \\ &\qquad + \sum_{j=1}^n b_{2j}\int_0^1 g_1\left(\eta_j,qs\right)h_1(s)d_qs. \end{aligned} \tag{43}$$

Combining (42) and (43), it can be seen that

$$\begin{aligned} &\sum_{i=1}^m \frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})}\int_0^{\xi_i}(\xi_i-qs)^{(\beta_{1i}-1)}v(s)d_qs + \sum_{j=1}^n b_{1j}v\left(\eta_j\right) \\ &\quad = \frac{1}{1-l_1l_2}\left[l_1\left(\sum_{i=1}^m \frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{2i}-1)}\int_0^1 g_1(\tau,qs)h_1(s)d_qsd_q\tau\right.\right. \\ &\qquad \left. + \sum_{j=1}^n b_{2j}\int_0^1 g_1\left(\eta_j,qs\right)h_1(s)d_qs\right) + \sum_{i=1}^m \frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{1i}-1)} \\ &\qquad \left. \times \int_0^1 g_2(\tau,qs)h_2(s)d_qsd_q\tau + \sum_{j=1}^n b_{1j}\int_0^1 g_2\left(\eta_j,qs\right)h_2(s)d_qs\right], \end{aligned}$$

$$\times\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})}\int_0^{\xi_i}(\xi_i-qs)^{(\beta_{2i}-1)}u(s)d_qs+\sum_{j=1}^{n}b_{2j}u\left(\eta_j\right)$$
$$=\frac{1}{1-l_1l_2}\left[l_2\left(\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{1i}-1)}\int_0^1 g_2(\tau,qs)h_2(s)d_qsd_q\tau\right.\right.$$
$$+\sum_{j=1}^{n}b_{1j}\int_0^1 g_2\left(\eta_j,qs\right)h_2(s)d_qs\right)+\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{2i}-1)}$$
$$\times\int_0^1 g_1(\tau,qs)h_1(s)d_qsd_q\tau+\sum_{j=1}^{n}b_{2j}\int_0^1 g_1\left(\eta_j,qs\right)h_1(s)d_qs], \tag{44}$$

where $l_k(k=1,2)$ is defined by (31). From (41) and (44), we have

$$u(t)=\int_0^1 g_1(t,qs)h_1(s)d_qs+\frac{t^{\alpha_1-1}}{\Gamma_q(\alpha1)(1-l_1l_2)}$$
$$\cdot\left[\int_0^1 l_1\left(\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{2i}-1)}g_1(\tau,qs)d_q\tau\right.\right.$$
$$\left.+\sum_{j=1}^{n}b_{2j}g_1\left(\eta_j,qs\right)\right)h_1(s)d_qs+\int_0^1$$
$$\cdot\left(\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{1i}-1)}g_2(\tau,qs)d_q\tau\right.$$
$$\left.+\sum_{j=1}^{n}b_{1j}g_2\left(\eta_j,qs\right)\right)h_2(s)d_qs]=\int_0^1 K_1(t,qs)h_1(s)d_qs$$
$$+\int_0^1 H_1(t,qs)h_2(s)d_qs, \tag{45}$$

where $K_1(t,qs)$ and $H_1(t,qs)$ are introduced by (34). Similarly, we also have

$$v(t)=\int_0^1 g_2(t,qs)h_2(s)d_qs+\frac{t^{\alpha_2-1}}{\Gamma_q(\alpha_2)(1-l_1l_2)}$$
$$\cdot\left[\int_0^1 l_2\left(\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{1i}-1)}g_2(\tau,qs)d_q\tau\right.\right.$$
$$\left.+\sum_{j=1}^{n}b_{1j}g_2\left(\eta_j,qs\right)\right)h_2(s)d_qs+\int_0^1$$
$$\cdot\left(\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{2i}-1)}g_1(\tau,qs)d_q\tau\right.$$
$$\left.+\sum_{j=1}^{n}b_{2j}g_1\left(\eta_j,qs\right)\right)h_1(s)d_qs]=\int_0^1 K_2(t,qs)h_2(s)d_qs$$
$$+\int_0^1 H_2(t,qs)h_1(s)d_qs, \tag{46}$$

where $K_2(t,qs)$ and $H_2(t,qs)$ are also given by (35).

This completes the proof of the lemma. □

Lemma 8. *Assume that (F_1) holds. Then, the functions $g_k(t,s)$, for $k=1,2$, defined by (35) have the following properties:*

$$0\leqslant g_k(t,qs)\leqslant\frac{1}{\Gamma_q(\alpha_k)}t^{\alpha_k-1},\text{for } t,qs\in[0,1]. \tag{47}$$

Proof. For $0\leqslant qs\leqslant t\leqslant 1$, we have

$$g_k(t,qs)=\frac{1}{\Gamma_q(\alpha_k)}\left[t^{\alpha_k-1}-(t-qs)^{(\alpha_k-1)}\right]\geqslant 0, \tag{48}$$

$$g_k(t,qs)=\frac{1}{\Gamma_q(\alpha_k)}\left[t^{\alpha_k-1}-(t-qs)^{(\alpha_k-1)}\right]\leqslant\frac{1}{\Gamma_q(\alpha_k)}t^{\alpha_k-1}. \tag{49}$$

For $0\leqslant t\leqslant qs\leqslant 1$, we have

$$0\leqslant g_k(t,qs)=\frac{1}{\Gamma_q(\alpha_k)}t^{\alpha_k-1}. \tag{50}$$

This completes the proof of the lemma. □

For convenience, we denote

$$\varrho_1=\frac{1}{\Gamma_q(\alpha_1)}\left[1+\frac{l_1}{\Gamma_q(\alpha_1)(1-l_1l_2)}\cdot\left(\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{2i}-1)}\tau^{\alpha_1-1}d_q\tau+\sum_{j=1}^{n}b_{2j}\eta_j^{\alpha_1-1}\right)\right], \tag{51}$$

$$\varrho_2=\frac{1}{\Gamma_q(\alpha_2)}\left[1+\frac{l_2}{\Gamma_q(\alpha_2)(1-l_1l_2)}\cdot\left(\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{1i}-1)}\tau^{\alpha_2-1}d_q\tau+\sum_{j=1}^{n}b_{1j}\eta_j^{\alpha_2-1}\right)\right], \tag{52}$$

$$\rho_1=\frac{1}{\Gamma_q(\alpha_1)\Gamma_q(\alpha_2)(1-l_1l_2)}\cdot\left[\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{1i}-1)}\tau^{\alpha_2-1}d_q\tau+\sum_{j=1}^{n}b_{1j}\eta_j^{\alpha_2-1}\right], \tag{53}$$

$$\rho_2=\frac{1}{\Gamma_q(\alpha_1)\Gamma_q(\alpha_2)(1-l_1l_2)}\cdot\left[\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})}\int_0^{\xi_i}(\xi_i-q\tau)^{(\beta_{2i}-1)}\tau^{\alpha1-1}d_q\tau+\sum_{j=1}^{n}b_{2j}\eta_j^{\alpha_1-1}\right]. \tag{54}$$

Lemma 9. *Assume that (F_1) - (F_3) hold. Then, for $(t,s)\in[0,1]\times[0,1]$, the functions $K_i(t,s)$ and $H_i(t,s)$, for $i=1,2$,* *defined by (33) and (34) satisfy the following results:*

$$0 \leqslant K_1(t, qs) \leqslant \varrho_1 t^{\alpha_1 - 1}, \quad 0 \leqslant K_2(t, qs) \leqslant \varrho_2 t^{\alpha_2 - 1}, \tag{55}$$

$$0 \leqslant H_1(t, qs) \leqslant \rho_1 t^{\alpha_1 - 1}, \quad 0 \leqslant H_2(t, qs) \leqslant \rho_2 t^{\alpha_2 - 1}. \tag{56}$$

Proof.

(1) According to (F_3), Lemma 8, and the definition of $K_i(t, qs)(k = 1, 2)$, we obtain

$$\begin{aligned} 0 \leqslant K_1(t, qs) &= g_1(t, qs) + \frac{l_1 t^{\alpha_1 - 1}}{\Gamma_q(\alpha_1)(1 - l_1 l_2)} \\ &\cdot \left[\sum_{i=1}^{m} \frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})} \int_0^{\xi_i} (\xi_i - q\tau)^{(\beta_{2i} - 1)} g_1(\tau, qs) d_q\tau + \sum_{j=1}^{n} b_{2j} g_1(\eta_j, qs)\right] \\ &\leqslant \frac{t^{\alpha_1 - 1}}{\Gamma_q(\alpha_1)} + \frac{l_1 t^{\alpha_1 - 1}}{\Gamma_q(\alpha_1)(1 - l_1 l_2)} \\ &\cdot \left[\sum_{i=1}^{m} \frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})\Gamma_q(\alpha_1)} \int_0^{\xi_i} (\xi_i - q\tau)^{(\beta_{2i} - 1)} \tau^{\alpha_1 - 1} d_q\tau + \frac{1}{\Gamma_q(\alpha_1)} \sum_{j=1}^{n} b_{2j} \eta_j^{\alpha_1 - 1}\right] \\ &= \frac{t^{\alpha_1 - 1}}{\Gamma_q(\alpha_1)} \left[1 + \frac{l_1}{\Gamma_q(\alpha_1)(1 - l_1 l_2)}\right. \\ &\left.\cdot \left(\sum_{i=1}^{m} \frac{\lambda_{2i}}{\Gamma_q(\beta_{2i})} \int_0^{\xi_i} (\xi_i - q\tau)^{(\beta_{2i} - 1)} \tau^{\alpha_1 - 1} d_q\tau + \sum_{j=1}^{n} b_{2j} \eta_j^{\alpha_1 - 1}\right)\right] = \varrho_1 t^{\alpha_1 - 1}, \end{aligned} \tag{57}$$

where ϱ_1 is defined by (51). Similarly, we get

$$0 \leqslant K_2(t, qs) \leqslant \varrho_2 t^{\alpha_2 - 1}, \tag{58}$$

where ϱ_2 is defined by (52).

(2) According to (F_3), Lemma 8, and the definition of $H_k(t, s)(k = 1, 2)$, we can also obtain

$$\begin{aligned} 0 \leqslant H_1(t, qs) &= \frac{t^{\alpha_1 - 1}}{\Gamma_q(\alpha_1)(1 - l_1 l_2)} \\ &\cdot \left[\sum_{i=1}^{m} \frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})} \int_0^{\xi_i} (\xi_i - q\tau)^{(\beta_{1i} - 1)} g_2(\tau, qs) d_q\tau + \sum_{j=1}^{n} b_{1j} g_2(\eta_j, qs)\right] \\ &\leqslant \frac{t^{\alpha_1 - 1}}{\Gamma_q(\alpha_1)(1 - l_1 l_2)} \left[\sum_{i=1}^{m} \frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})\Gamma_q(\alpha_2)} \int_0^{\xi_i} (\xi_i - q\tau)^{(\beta_{1i} - 1)} \tau^{\alpha_2 - 1} d_q\tau\right. \\ &\left.+ \frac{1}{\Gamma_q(\alpha_2)} \sum_{j=1}^{n} b_{1j} \eta_j^{\alpha_2 - 1}\right] = \frac{t^{\alpha_1 - 1}}{\Gamma_q(\alpha_1)\Gamma_q(\alpha_2)(1 - l_1 l_2)} \\ &\cdot \left[\sum_{i=1}^{m} \frac{\lambda_{1i}}{\Gamma_q(\beta_{1i})} \int_0^{\xi_i} (\xi_i - q\tau)^{(\beta_{1i} - 1)} \tau^{\alpha_2 - 1} d_q\tau + \sum_{j=1}^{n} b_{1j} \eta_j^{\alpha_2 - 1}\right] = \rho_1 t^{\alpha_1 - 1}, \end{aligned} \tag{59}$$

where ρ_1 is defined by (53). Analogously, we get

$$0 \leqslant H_2(t, s) \leqslant \rho_2 t^{\alpha_2 - 1}, \tag{60}$$

where ρ_2 is defined by (54).

This completes the proof of the lemma. □

Let $X = \{u | u \in C[0, 1]\}$ be a Banach space endowed with the norm

$$\|u\|_X = \max_{t \in [0,1]} |u(t)|. \tag{61}$$

Also, let $Y = \{v | v \in C[0, 1]\}$ be a Banach space endowed with the norm

$$\|v\|_Y = \max_{t \in [0,1]} |v(t)|. \tag{62}$$

Then, we introduce the product space $(X \times Y, \|(u, v)\|)$ endowed with the norm $\|(u, v)\| = \max \{\|u\|_X, \|v\|_Y\}$ and define a partial order over the product space

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \geqslant \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}, \tag{63}$$

while $u_1(t) \geqslant u_2(t)$, $v_1(t) \geqslant v_2(t)$, and $t \in [0, 1]$. And we can know that $(X \times Y, \|(u, v)\|)$ is also a Banach space.

Further, we define a cone $P \subset X \times Y$ by $P = \{(u, v) \in X \times Y : u(t) \geqslant 0, v(t) \geqslant 0, t \in [0, 1]\}$. For all $(u, v) \in P$, in view of Lemma 7 and (F_4), let $T : P \longrightarrow P$ be the operator defined by

$$T(u, v)(t) = \begin{pmatrix} T_1(u, v)(t) \\ T_2(u, v)(t) \end{pmatrix}, \tag{64}$$

where

$$T_1(u, v)(t) = \int_0^1 K_1(t, qs) f_1(u, v)(s) d_qs + \int_0^1 H_1(t, qs) f_2(u, v)(s) d_qs, \tag{65}$$

$$T_2(u, v)(t) = \int_0^1 K_2(t, qs) f_2(u, v)(s) d_qs + \int_0^1 H_2(t, qs) f_1(u, v)(s) d_qs, \tag{66}$$

and for convenience, we set

$$\begin{cases} f_1(u, v)(s) \triangleq f_1(s, u(s), v(s)), \\ f_2(u, v)(s) \triangleq f_2(s, u(s), v(s)). \end{cases} \tag{67}$$

Lemma 10. *The operator $T : P \longrightarrow P$ is completely continuous.*

Proof. There are three steps to complete the whole proof.

(a) The operator $T : P \longrightarrow P$ is continuous

By the continuity of the functions $K_1(t, qs), K_2(t, qs), H_1(t, qs), H_2(t, qs), f_1$, and f_2, the operator T is continuous.

(b) The operator $T : P \longrightarrow P$ is uniformly bounded

Let Ω be any bounded subset of P. There exists a positive constant M satisfying the inequality

$$\max_{t\in[0,1]}\{f_1(u,v)(t), f_2(u,v)(t)\} \leqslant M, \forall (u,v)\in\Omega. \tag{68}$$

For any $(u,v)\in\Omega$, combining Lemma 9 with (65) and (68), we get

$$\begin{aligned}\|T_1(u,v)\| &= \max_{t\in[0,1]}\left|\int_0^1 K_1(t,qs)f_1(u,v)(s)d_qs + \int_0^1 H_1(t,qs)f_2(u,v)(s)d_qs\right| \\ &\leqslant \varrho_1 t^{\alpha_1-1}\int_0^1 |f_1(u,v)(s)|d_qs + \rho_1 t^{\alpha_1-1}\int_0^1 |f_2(u,v)(s)|d_qs \\ &\leqslant M\varrho_1 t^{\alpha_1-1} + M\rho_1 t^{\alpha_1-1} \leqslant M(\varrho_1+\rho_1).\end{aligned} \tag{69}$$

So, for $(u,v)\in\Omega$, T_1 is uniformly bounded. Furthermore, we get T_2 is also uniformly bounded. Thus, it follows from the above inequalities that the operator T is uniformly bounded.

(c) The operator $T: P \longrightarrow P$ is equicontinous

For any $(u,v)\in\Omega$ and $t_1, t_2\in[0,1]$, in view of Lemma 9, (65), and (68), we infer that

$$\begin{aligned}&|T_1(u,v)(t_2) - T_1(u,v)(t_1)| \\ &\leqslant \left|\int_0^1 K_1(t_2,qs)f_1(u,v)(s)d_qs - \int_0^1 K_1(t_1,qs)f_1(u,v)(s)d_qs\right| \\ &\quad + \left|\int_0^1 H_1(t_2,qs)f_2(u,v)(s)d_qs - \int_0^1 H_1(t_1,qs)f_2(u,v)(s)d_qs\right| \\ &\leqslant M\left[\int_0^1 |K_1(t_2,qs)-K_1(t_1,qs)|d_qs + \int_0^1 |H_1(t_2,qs)-H_1(t_1,qs)|d_qs\right] \\ &\leqslant M\left[\int_0^1 \varrho_1\left|t_2^{\alpha_1-1}-t_1^{\alpha_1-1}\right|d_qs + \int_0^1 \rho_1\left|t_2^{\alpha_1-1}-t_1^{\alpha_1-1}\right|d_qs\right] \\ &= M(\varrho_1+\rho_1)\left|t_2^{\alpha_1-1}-t_1^{\alpha_1-1}\right| \\ &\leqslant M(\varrho_1+\rho_1)(\alpha_1-1)|t_2-t_1| \longrightarrow 0 \, as \, t_2\longrightarrow t_1.\end{aligned} \tag{70}$$

Therefore, T_1 is equicontinous for all $(u,v)\in\Omega$. Similarly, we can get T_2 is equicontinous for all $(u,v)\in\Omega$. As a consequence, the operator T is equicontinous for all $(u,v)\in\Omega$.

From the above steps, we get the operator $T: P\longrightarrow P$ is completely continuous. The proof is completed. □

3. Existence Results of Monotone Iterative Positive Solutions

Theorem 11. *Assume that* (F_1) - (F_4) *hold. Let* A_i, L_i, M_i, $(i=1,2)$, l *and* $\gamma<1/2$ *be positive constants satisfying*

$$\gamma = \sup\{\varrho_1 L_1+\rho_1 M_1, \varrho_1 L_2+\rho_1 M_2, \varrho_2 M_1+\rho_2 L_1, \varrho_2 M_2+\rho_2 L_2\}, \tag{71}$$

$$l = \max\{A_1\varrho_1 + A_2\rho_1, A_2\varrho_2 + A_1\rho_2\}, \tag{72}$$

(S_1) For $t\in[0,1]$, $f_j(t,x_1,x_2)$ is increasing in $x_i\in[0,l]$ $(i=1,2)$ for $j=1,2$

(S_2) $\max_{0\leqslant t\leqslant 1} f_i(t,l,l)\leqslant A_i$, $f_i(t,0,0)\equiv 0$, $0\leqslant t\leqslant 1$ for $i=1,2$

(S_3) For any $x_i, y_i\in C[0,1]$ with $x_i\leqslant y_i (i=1,2)$, there exist nonnegative constants L_i and $M_i (i=1,2)$ such that

$$\begin{aligned}&0\leqslant f_1(t,y_1,y_2) - f_1(t,x_1,x_2)\leqslant L_1(y_1-x_1)+L_2(y_2-x_2), t\in[0,1], \\ &0\leqslant f_2(t,y_1,y_2) - f_2(t,x_1,x_2)\leqslant M_1(y_1-x_1)+M_2(y_2-x_2), t\in[0,1].\end{aligned} \tag{73}$$

Then BVP (5) and (6) has iterative positive solutions (u^*,v^*) and (w^*,z^*) satisfying $0\leqslant\|(u^*,v^*)\|\leqslant l$, which $\lim_{n\longrightarrow\infty}(u_n,v_n)=(u^*,v^*)$,

$$(u_n,v_n) = T(u_{n-1},v_{n-1}) = \begin{pmatrix} T_1(u_{n-1},v_{n-1}) \\ T_2(u_{n-1},v_{n-1})\end{pmatrix}, n=1,2,\cdots, \tag{74}$$

$$\begin{pmatrix} u_0(t) \\ v_0(t)\end{pmatrix} = \begin{pmatrix} l t^{\alpha_1-1} \\ l t^{\alpha_2-1}\end{pmatrix}; \tag{75}$$

$0\leqslant\|(w^*,z^*)\|_{X\times Y}\leqslant l$, which $\lim_{n\longrightarrow\infty}(w_n,z_n)=(w^*,z^*)$,

$$(w_n,z_n) = T(w_{n-1},z_{n-1}) = \begin{pmatrix} T_1(w_{n-1},z_{n-1}) \\ T_2(w_{n-1},z_{n-1})\end{pmatrix}, n=1,2,\cdots, \tag{76}$$

$$\begin{pmatrix} w_0(t) \\ z_0(t)\end{pmatrix} = \begin{pmatrix} 0 \\ 0\end{pmatrix}. \tag{77}$$

Moreover, the error estimate of the solution is given by

$$\|u_n-u^*\|\leqslant\frac{(2\gamma)^n}{2(1-2\gamma)}(\|u_0-u_1\|+\|v_0-v_1\|), \tag{78}$$

$$\|v_n-v^*\|\leqslant\frac{(2\gamma)^n}{2(1-2\gamma)}(\|u_0-u_1\|+\|v_0-v_1\|), \tag{79}$$

$$\|w^*-w_n\|\leqslant\frac{(2\gamma)^n}{2(1-2\gamma)}(\|w_1-w_0\|+\|z_1-z_0\|), \tag{80}$$

$$\|z^*-z_n\|\leqslant\frac{(2\gamma)^n}{2(1-2\gamma)}(\|w_1-w_0\|+\|z_1-z_0\|). \tag{81}$$

Proof. Denote $P_l=\{(u,v)\in X\times Y: \|(u,v)\|_{X\times Y}\leqslant l\}$, where l is introduced by (72). In the following, we first prove that $T: P_l\longrightarrow P_l$. For any $(u,v)\in P_l$, then for $t\in[0,1]$, we have

$$0\leqslant u(t)\leqslant\|u\|\leqslant l, 0\leqslant v(t)\leqslant\|v\|\leqslant l. \tag{82}$$

So, for $t\in[0,1]$ and $i=1,2$, by $(S1)$ and $(S2)$, we get

$$0\leqslant f_i(t,u(t),v(t))\leqslant\max_{0\leqslant t\leqslant 1} f_i(t,l,l)\leqslant A_i. \tag{83}$$

Consequently, for $t \in [0,1]$, in view of Lemma 9 and (83), we have

$$\begin{aligned}\|T_1(u,v)\| &= \max_{0\le t\le 1}\left|\int_0^1 K_1(t,qs)f_1(u,v)(s)d_qs + \int_0^1 H_1(t,qs)f_2(u,v)(s)d_qs\right| \\ &\le \max_{0\le t\le 1}\left[\varrho_1 t^{\alpha_1-1}\int_0^1 |f_1(u,v)(s)|d_qs + \rho_1 t^{\alpha_1-1}\int_0^1 |f_2(u,v)(s)|d_qs\right] \\ &\le \max_{0\le t\le 1}\left[(A_1\varrho_1 + A_2\rho_1)t^{\alpha_1-1}\right] \le A_1\varrho_1 + A_2\rho_1 \le l,\end{aligned} \tag{84}$$

$$\begin{aligned}\|T_2(u,v)\| &= \max_{0\le t\le 1}\left|\int_0^1 K_2(t,qs)f_2(u,v)(s)d_qs + \int_0^1 H_2(t,qs)f_1(u,v)(s)d_qs\right| \\ &\le \max_{0\le t\le 1}\left[\varrho_2 t^{\alpha_2-1}\int_0^1 |f_2(u,v)(s)|d_qs + \rho_2 t^{\alpha_2-1}\int_0^1 |f_1(u,v)(s)|d_qs\right] \\ &\le \max_{0\le t\le 1}\left[(A_2\varrho_2 + A_1\rho_2)t^{\alpha_2-1}\right] \le A_2\varrho_2 + A_1\rho_2 \le l.\end{aligned} \tag{85}$$

As a result, we obtain

$$\|T(u,v)\| \le l, \tag{86}$$

and thus $T : P_l \longrightarrow P_l$.

According to (75) and (77), it is obvious that (u_0, v_0), $(w_0, z_0) \in P_l$. By using the completely continuous operator T, we define the sequences $\{(u_n, v_n)\}$ and $\{(w_n, z_n)\}$ as $(u_n, v_n) = T(u_{n-1}, v_{n-1})$, $(w_n, z_n) = T(w_{n-1}, z_{n-1})$, for $n = 1, 2, \cdots$. Since $T : P_l \longrightarrow P_l$, we get that $(u_n, v_n), (w_n, z_n) \in P_l$, for $n = 1, 2, \cdots$.

For $t \in [0,1]$, by the definition of the iterative scheme, we have

$$\begin{aligned}T_1(u_0,v_0)(t) &= \int_0^1 K_1(t,qs)f_1(u_0,v_0)(s)d_qs + \int_0^1 H_1(t,qs)f_2(u_0,v_0)(s)d_qs \\ &\le \int_0^1 \varrho_1 t^{\alpha_1-1}f_1(u_0,v_0)(s)d_qs + \int_0^1 \rho_1 t^{\alpha_1-1}f_2(u_0,v_0)(s)d_qs \\ &\le (A_1\varrho_1 + A_2\rho_1)t^{\alpha_1-1} \le lt^{\alpha_1-1} = u_0(t).\end{aligned} \tag{87}$$

Similarly,

$$\begin{aligned}T_2(u_0,v_0)(t) &= \int_0^1 K_2(t,qs)f_2(u_0,v_0)(s)d_qs \\ &\quad + \int_0^1 H_2(t,qs)f_1(u_0,v_0)(s)d_qs \le (A_2\varrho_2 + A_1\rho_2)t^{\alpha_2-1} \\ &\le lt^{\alpha_2-1} = v_0(t).\end{aligned} \tag{88}$$

From (87) and (88), we can get

$$\begin{pmatrix} u_1(t) \\ v_1(t) \end{pmatrix} = \begin{pmatrix} T_1(u_0,v_0)(t) \\ T_2(u_0,v_0)(t) \end{pmatrix} \le \begin{pmatrix} u_0(t) \\ v_0(t) \end{pmatrix}. \tag{89}$$

Thus, for $t \ge 1$, from (89)and ($S1$), we do the second iteration

$$\begin{pmatrix} u_2(t) \\ v_2(t) \end{pmatrix} = \begin{pmatrix} T_1(u_1,v_1)(t) \\ T_2(u_1,v_1)(t) \end{pmatrix} \le \begin{pmatrix} T_1(u_0,v_0)(t) \\ T_2(u_0,v_0)(t) \end{pmatrix} = \begin{pmatrix} u_1(t) \\ v_1(t) \end{pmatrix}. \tag{90}$$

By induction, for $n = 0, 1, 2, \cdots$, we get

$$\begin{pmatrix} u_{n+1}(t) \\ v_{n+1}(t) \end{pmatrix} \le \begin{pmatrix} u_n(t) \\ v_n(t) \end{pmatrix}, \text{for } 0 \le t \le 1. \tag{91}$$

So we assert that $\lim_{n\longrightarrow\infty}(u_n, v_n) = (u^*, v^*)$ and $T(u^*, v^*) = (u^*, v^*)$, since T is completely continuous and $(u_{n+1}, v_{n+1}) = T(u_n, v_n)$.

For the sequence $\{(w_n, z_n)\}_{n=1}^{\infty}$, we apply a similar argument. For $t \in [0,1]$, we have

$$\begin{aligned}\begin{pmatrix} w_1(t) \\ z_1(t) \end{pmatrix} &= \begin{pmatrix} T_1(w_0,z_0)(t) \\ T_2(w_0,z_0)(t) \end{pmatrix} \\ &= \begin{pmatrix} \int_0^1 K_1(t,qs)f_1(w_0,z_0)(s)ds + \int_0^1 H_1(t,qs)f_2(w_0,z_0)(s)ds, \\ \int_0^1 K_2(t,qs)f_2(w_0,z_0)(s)ds + \int_0^1 H_2(t,qs)f_1(w_0,z_0)(s)ds \end{pmatrix} \\ &\ge \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} w_0(t) \\ z_0(t) \end{pmatrix}.\end{aligned} \tag{92}$$

To sum up, for $0 \le t \le 1$, we have

$$\begin{pmatrix} w_2(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} T_1(w_1,z_1)(t) \\ T_2(w_1,z_1)(t) \end{pmatrix} \ge \begin{pmatrix} T_1(w_0,z_0)(t) \\ T_2(w_0,z_0)(t) \end{pmatrix} = \begin{pmatrix} w_1(t) \\ z_1(t) \end{pmatrix}. \tag{93}$$

Analogously, for $n = 1, 2, \cdots$, we get

$$\begin{pmatrix} w_{n+1}(t) \\ z_{n+1}(t) \end{pmatrix} \ge \begin{pmatrix} w_n(t) \\ z_n(t) \end{pmatrix}, \text{for } 0 \le t \le 1. \tag{94}$$

Similarly, we can also assert that $\lim_{n\longrightarrow\infty}(w_n, z_n) = (w^*, z^*)$ and $T(w^*, z^*) = (w^*, z^*)$, since T is completely continuous and $(w_{n+1}, z_{n+1}) = T(w_n, z_n)$.

Consequently, there exist (u^*, v^*) and (w^*, z^*) in P_l which are nonnegative extremal solutions of BVP (5) and (6). Since $f_k(t,0,0) \equiv 0, \forall t \in [0,1]$, for $k = 1, 2$; then, zero is not a solution of problem (5). It is obvious that $(u^*, v^*)(t) > 0$, $(w^*, z^*)(t) > 0$.

It follows from Lemma 9 and $(S3)$ that

$$\begin{aligned}
\|u_n-u_{n+1}\| &= \|T_1(u_{n-1},v_{n-1})-T_1(u_n,v_n)\| \\
&= \max_{0\leqslant t\leqslant 1}\left|\int_0^1 K_1(t,qs)f_1(u_{n-1},v_{n-1})(s)d_qs \right.\\
&\quad +\int_0^1 H_1(t,qs)f_2(u_{n-1},v_{n-1})(s)d_qs \\
&\quad -\int_0^1 K_1(t,qs)f_1(u_n,v_n)(s)d_qs \\
&\quad \left.-\int_0^1 H_1(t,qs)f_2(u_n,v_n)(s)d_qs\right| \\
&\leqslant \max_{0\leqslant t\leqslant 1}\left(\varrho_1 t^{\alpha_1-1}\int_0^1|f_1(u_{n-1},v_{n-1})(s)-f_1(u_n,v_n)(s)|d_qs\right.\\
&\quad \left.+\rho_1 t^{\alpha 1-1}\int_0^1|f_2(u_{n-1},v_{n-1})(s)-f_2(u_n,v_n)(s)|d_qs\right) \\
&\leqslant \varrho_1\int_0^1(L_1|u_{n-1}-u_n|+L_2|v_{n-1}-v_n|)d_qs+\rho_1 \\
&\quad \cdot\int_0^1(M_1|u_{n-1}-u_n|+M_2|v_{n-1}-v_n|)d_qs \\
&\leqslant \varrho_1(L_1\|u_{n-1}-u_n\|+L_2\|v_{n-1}-v_n\|) \\
&\quad +\rho_1(M_1\|u_{n-1}-u_n\|+M_2\|v_{n-1}-v_n\|) \\
&= (\varrho_1L_1+\rho_1M_1)\|u_{n-1}-u_n\|+(\varrho_1L_2+\rho_1M_2)\|v_{n-1}-v_n\| \\
&\leqslant \gamma(\|u_{n-1}-u_n\|+\|v_{n-1}-v_n\|).
\end{aligned} \tag{95}$$

Homoplastically, we have

$$\begin{aligned}
\|v_n-v_{n+1}\| &\leqslant \gamma(\|u_{n-1}-u_n\|+\|v_{n-1}-v_n\|), \\
\|w_{n+1}-w_n\| &\leqslant \gamma(\|w_n-w_{n-1}\|+\|z_n-z_{n-1}\|), \\
\|z_{n+1}-z_n\| &\leqslant \gamma(\|w_n-w_{n-1}\|+\|z_n-z_{n-1}\|).
\end{aligned} \tag{96}$$

Combining (95) and (96), we get

$$\begin{aligned}
&\|u_1-u_2\|\leqslant\gamma(\|u_0-u_1\|+\|v_0-v_1\|), \\
&\|v_1-v_2\|\leqslant\gamma(\|u_0-u_1\|+\|v_0-v_1\|)\,; \\
&\|u_2-u_3\|\leqslant\gamma(\|u_1-u_2\|+\|v_1-v_2\|)\leqslant\gamma\cdot 2\gamma(\|u_0-u_1\|+\|v_0-v_1\|), \\
&\|v_2-v_3\|\leqslant\gamma(\|u_1-u_2\|+\|v_1-v_2\|)\leqslant\gamma\cdot 2\gamma(\|u_0-u_1\|+\|v_0-v_1\|)\,; \\
&\cdots\cdots
\end{aligned} \tag{97}$$

Then, we obtain

$$\begin{aligned}
\|u_n-u_{n+1}\| &\leqslant \frac{(2\gamma)^n}{2}(\|u_0-u_1\|+\|v_0-v_1\|), \\
\|v_n-v_{n+1}\| &\leqslant \frac{(2\gamma)^n}{2}(\|u_0-u_1\|+\|v_0-v_1\|).
\end{aligned} \tag{98}$$

Therefore, for $n,p\in\mathbb{Z}^+$, it comes that

$$\begin{aligned}
\|u_n-u_{n+p}\| &\leqslant \|u_n-u_{n+1}\|+\|u_{n+1}-u_{n+2}\|+\cdots+\|u_{n+p-1}-u_{n+p}\| \\
&\leqslant \left(\frac{(2\gamma)^n}{2}+\frac{(2\gamma)^{n+1}}{2}+\cdots+\frac{(2\gamma)^{n+p-1}}{2}\right) \\
&\quad \cdot(\|u_0-u_1\|+\|v_0-v_1\|)=\frac{(2\gamma)^n}{2} \\
&\quad \cdot\left((2\gamma)^0+(2\gamma)^1+\cdots+(2\gamma)^{p-1}\right)\cdot(\|u_0-u_1\|+\|v_0-v_1\|) \\
&= \frac{(2\gamma)^n}{2}\cdot\frac{1-(2\gamma)^p}{1-2\gamma}\cdot(\|u_0-u_1\|+\|v_0-v_1\|).
\end{aligned} \tag{99}$$

For $0<\gamma<1/2$, we can obtain $\|u_n-u^*\|\leqslant((2\gamma)^n/2(1-2\gamma))\cdot(\|u_0-u_1\|+\|v_0-v_1\|)$ as $p\longrightarrow\infty$, and similarly, $\|u_n-u^*\|\leqslant((2\gamma)^n/2(1-2\gamma))\cdot(\|u_0-u_1\|+\|v_0-v_1\|)$.

In the same manner, we can see that $\|w^*-w_n\|\leqslant((2\gamma)^n/2(1-2\gamma))\cdot(\|w_1-w_0\|+\|z_1-z_0\|)$, and $\|z^*-z_n\|\leqslant((2\gamma)^n/2(1-2\gamma))\cdot(\|w_1-w_0\|+\|z_1-z_0\|)$.

The proof is completed. □

When the system degenerates into the following form and the boundary conditions remain unchanged, we still have the same conclusion:

$$\begin{cases} D_q^{\alpha_1}u(t)+f_1(t,u(t))=0, & t\in(0,1), \\ D_q^{\alpha_2}v(t)+f_2(t,v(t))=0, & t\in(0,1), \end{cases} \tag{100}$$

Corollary 12. *Assume that (F_1) - (F_4) hold. Let $A_i(i=1,2)$, L,M,l and $\gamma<1/2$ be positive constants satisfying*

$$\gamma=\sup\{\varrho_1L,\rho_1M,\varrho_2M,\rho_2L\}, \tag{101}$$

$$l=\max\{A_1\varrho_1+A_2\rho_1,A_2\varrho_2+A_1\rho_2\}, \tag{102}$$

(S_1) For $t\in[0,1]$, $f_j(t,x)$ is increasing in $x\in[0,l]$ for $j=1,2$

$(S_2)\max_{0\leqslant t\leqslant 1}f_i(t,l)\leqslant A_i$, $f_i(t,0)\equiv 0$, $0\leqslant t\leqslant 1$ for $i=1,2$

(S_3) For any $x,y\in C[0,1]$ with $x\leqslant y$, there exist nonnegative constants L and M such that

$$0\leqslant f_1(t,y)-f_1(t,x)\leqslant L(y-x),\ t\in[0,1], \tag{103}$$

$$0\leqslant f_2(t,y)-f_2(t,x)\leqslant M(y-x),\ t\in[0,1]. \tag{104}$$

Then, BVP (100) and (6) has iterative positive solutions (u^*,v^*) and (w^*,z^*) satisfying the same conclusion as Theorem 11.

The absence of coupling in the BVP (100) and (6) results in a reduction of the value of the parameter γ, so the overall iteration accuracy is greatly improved. By comparing the error estimates and the plot of the solutions in the latter Example 3 with (139), we can verify that the BVP (100) and (6) has fewer iterations but higher accuracy.

4. Examples

Example 1. For $t \in [0,1]$, consider the following fractional differential system:

$$\begin{cases} D_{0.5}^{1.9}u(t) + 0.09t^2 + \sin^2\frac{t}{5}u(t) + \sin\frac{t}{5}v(t) = 0, \\ D_{0.5}^{1.8}v(t) + 0.11t + \sin\frac{t}{10}u(t) + \sin^2\frac{t}{10}v(t) = 0, \end{cases} \tag{105}$$

with the coupled integral and discrete mixed boundary conditions:

$$\begin{cases} u(0) = 0, \quad {}^cD_{0.5}^{0.9}u(1) = \sum_{i=1}^{2}\lambda_{1i}I_{0.5}^{\beta_{1i}}v(\xi_i) + \sum_{j=1}^{2}b_{1j}v(\eta_j), \\ v(1) = 0, \quad {}^cD_{0.5}^{0.8}v(1) = \sum_{i=1}^{2}\lambda_{2i}I_{0.5}^{\beta_{2i}}u(\xi_i) + \sum_{j=1}^{2}b_{2j}u(\eta_j). \end{cases} \tag{106}$$

In this model, we set

$$\lambda_{11} = 0.25, \lambda_{21} = 0.2, \beta_{11} = 1.5, \beta_{21} = 1.4, \xi_1 = 0.25, b_{11} = 0.33, b_{21} = 0.17, \eta_1 = 0.33, \tag{107}$$

$$\lambda_{12} = 0.5, \lambda_{22} = 0.1, \beta_{12} = 2.5, \beta_{22} = 2.4, \xi_2 = 0.75, b_{12} = 0.67, b_{22} = 0.83, \eta_2 = 0.67. \tag{108}$$

It is obvious that $(F1)$ and $(F2)$ hold. By calculation, we get

$$l_1 < 0.753252, l_2 < 0.681115, \tag{109}$$

$$\varrho_1 < 2.106456, \varrho_2 < 2.152105, \tag{110}$$

$$\rho_1 < 1.586692, \rho_2 < 1.465830. \tag{111}$$

Setting $A_1 = 0.5, A_2 = 0.3$, we get $l = 1.529236$. And for any $u_i, v_i \in C[0,1]$ with $u_i \leqslant v_i (i = 1, 2)$, we have

$$0 \leqslant f_1(t, v_1, v_2) - f_1(t, u_1, u_2) \leqslant \sin^2\frac{1}{5}(u_2 - u_1) + \sin\frac{1}{5}(v_2 - v_1), t \in [0,1] \tag{112}$$

$$0 \leqslant f_2(t, v_1, v_2) - f_2(t, u_1, u_2) \leqslant \sin\frac{1}{10}(u_2 - u_1) + \sin^2\frac{1}{10}(v_2 - v_1), t \in [0,1], \tag{113}$$

so $L_1 = 0.039470, L_2 = 0.198669, M_1 = 0.099833, M_2 = 0.009967$. Thus, it can be obtained that $\gamma < 1/2$. Then, all the hypotheses of Theorem 11 are satisfied. BVP (105) and (106) have monotone positive solutions (u^*, v^*) and (w^*, z^*), which can be approximated by the following iterative sequences:

$$\begin{pmatrix} u_{n+1}(t) \\ v_{n+1}(t) \end{pmatrix} = \begin{pmatrix} T_1(u_n, v_n)(t) \\ T_2(u_n, v_n)(t) \end{pmatrix} = \begin{pmatrix} \int_0^1 K_1(t, qs)f_1(u_n, v_n)(s)d_qs + \int_0^1 H_1(t, qs)f_2(u_n, v_n)(s)d_qs \\ \int_0^1 K_2(t, qs)f_2(u_n, v_n)(s)d_qs + \int_0^1 H_2(t, qs)f_1(u_n, v_n)(s)d_qs \end{pmatrix}, \tag{114}$$

$$\begin{pmatrix} w_{n+1}(t) \\ z_{n+1}(t) \end{pmatrix} = \begin{pmatrix} \int_0^1 K_1(t, qs)f_1(w_n, z_n)(s)ds + \int_0^1 H_1(t, qs)f_2(w_n, z_n)(s)ds \\ \int_0^1 K_2(t, qs)f_2(w_n, z_n)(s)ds + \int_0^1 H_2(t, qs)f_1(w_n, z_n)(s)ds \end{pmatrix}, \tag{115}$$

and the initial values are

$$\begin{pmatrix} u_0(t) \\ v_0(t) \end{pmatrix} = \begin{pmatrix} lt^{\alpha_1 - 1} \\ lt^{\alpha_2 - 1} \end{pmatrix} = \begin{pmatrix} 1.529236t^{0.9} \\ 1.529236t^{0.8} \end{pmatrix}, \tag{116}$$

$$\begin{pmatrix} w_0(t) \\ z_0(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{117}$$

Example 2. For $t \in [0,1]$, consider the following fractional differential system:

$$\begin{cases} D_{0.5}^{2}u(t) + 0.09t^2 + 0.03tu(t) + 0.14t^2v(t) = 0, \\ D_{0.5}^{2}v(t) + 0.11t + 0.07t^2u(t) + 0.01tv(t) = 0, \end{cases} \tag{118}$$

with the coupled integral and discrete mixed boundary

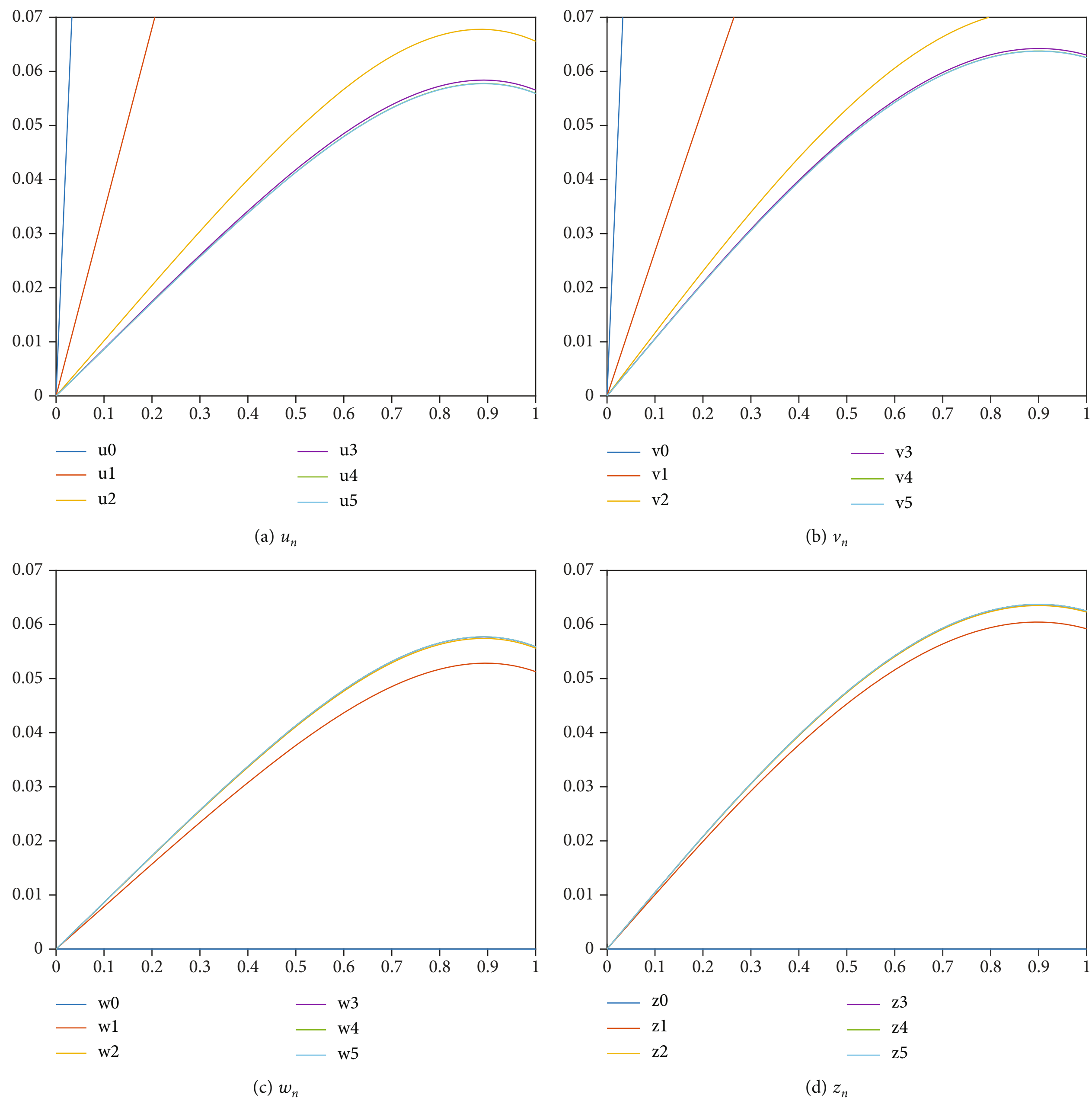


FIGURE 1: A plot of the solutions of Example 2.

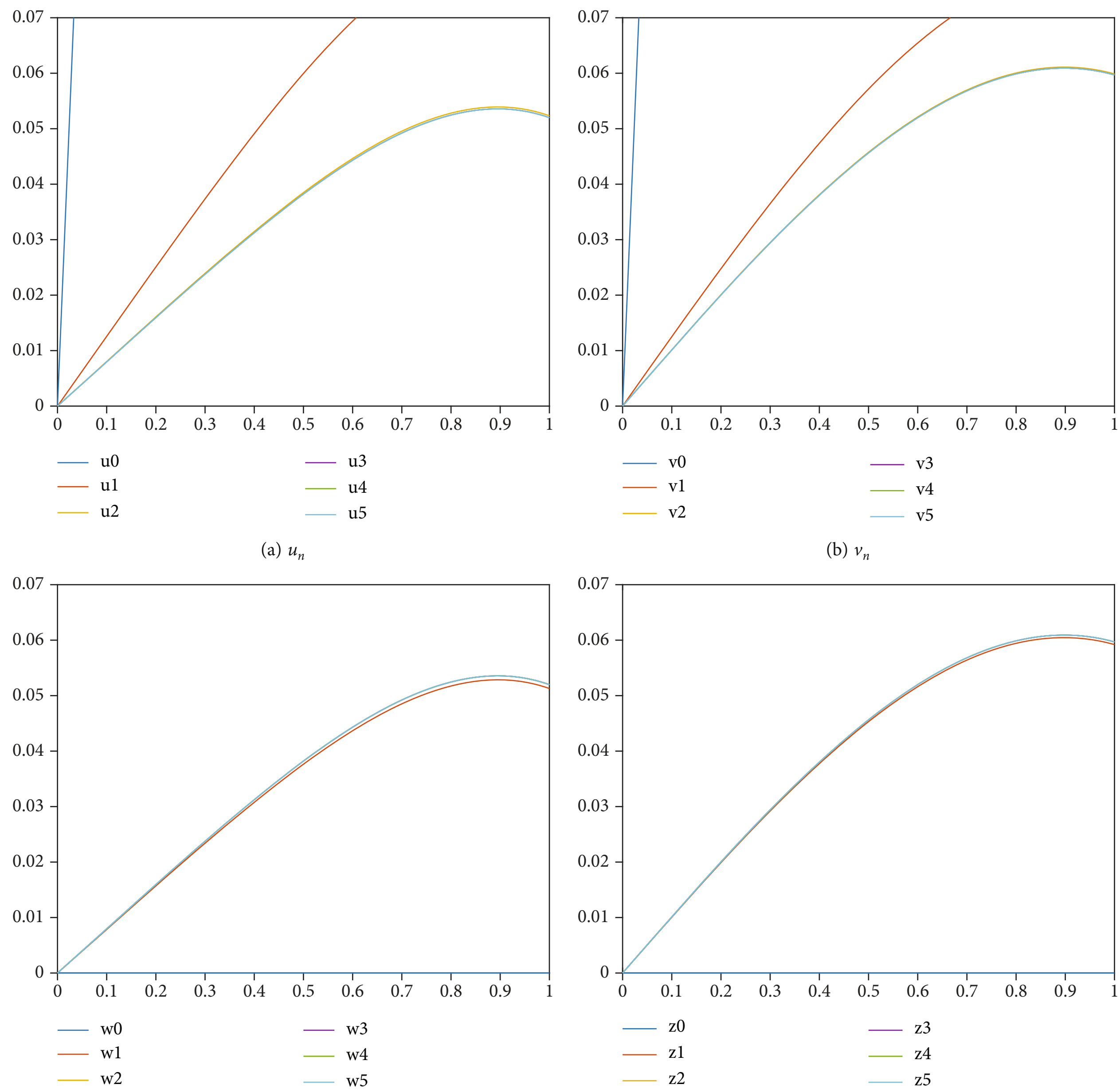
conditions:

$$\begin{cases} u(0)=0, & {}^{c}D_{0.5}^{1}u(1)=\sum_{i=1}^{2}\lambda_{1i}I_{0.5}^{\beta_{1i}}v(\xi_i)+\sum_{j=1}^{2}b_{1j}v\left(\eta_j\right), \\ v(1)=0, & {}^{c}D_{0.5}^{1}v(1)=\sum_{i=1}^{2}\lambda_{2i}I_{0.5}^{\beta_{2i}}u(\xi_i)+\sum_{j=1}^{2}b_{2j}u\left(\eta_j\right). \end{cases} \tag{119}$$

In this model, we set

$$\lambda_{11}=0.25,\quad \lambda_{21}=0.2,\quad \beta_{11}=1,\quad \beta_{21}=1,\quad \xi_1=0.25,\quad b_{11}=0.33,\quad b_{21}=0.17,\quad \eta_1=0.33, \tag{120}$$

$$\lambda_{12}=0.75,\quad \lambda_{22}=0.8,\quad \beta_{12}=1,\quad \beta_{22}=1,\quad \xi_2=0.75,\quad b_{12}=0.67,\quad b_{22}=0.83,\quad \eta_2=0.67. \tag{121}$$



(a) u_n (b) v_n (c) w_n (d) z_n

Figure 2: A plot of the solutions of Example 3.

It is obvious that $(F1)$ and $(F2)$ hold. By calculation, we get

$$l_1 = 0.77655, \quad l_2 = 0.84345, \tag{122}$$

$$\varrho_1 = 2.8983919, \quad \varrho_2 = 2.8983919, \tag{123}$$

$$\rho_1 = 2.2507462, \quad \rho_2 = 2.4446487. \tag{124}$$

Setting $A_1 = 0.5, A_2 = 0.3$, we get $l = 2.1244199$. And for any $u_i, v_i \in C[0,1]$ with $u_i \leqslant v_i (i = 1, 2)$, we have

$$0 \leqslant f_1(t, v_1, v_2) - f_1(t, u_1, u_2) \leqslant 0.03t(u_2 - u_1) + 0.14t^2(v_2 - v_1), t \in [0, 1], \tag{125}$$

$$0 \leqslant f_2(t, v_1, v_2) - f_2(t, u_1, u_2) \leqslant 0.07t^2(u_2 - u_1) + 0.01t(v_2 - v_1), t \in [0, 1], \tag{126}$$

so $L_1 = 0.03, L_2 = 0.14, M_1 = 0.07, M_2 = 0.01$. Thus, it can be obtained that $\gamma = 0.4282824 < 1/2$. Then, all the hypotheses of Theorem 11 are satisfied. BVP (118) and (119) have monotone positive solutions (u^*, v^*) and (w^*, z^*), which can be approximated by the following iterative sequences:

$$\begin{pmatrix} u_{n+1}(t) \\ v_{n+1}(t) \end{pmatrix} = \begin{pmatrix} T_1(u_n, v_n)(t) \\ T_2(u_n, v_n)(t) \end{pmatrix} = \begin{pmatrix} \int_0^1 K_1(t, qs) f_1(u_n, v_n)(s) d_q s + \int_0^1 H_1(t, qs) f_2(u_n, v_n)(s) d_q s \\ \int_0^1 K_2(t, qs) f_2(u_n, v_n)(s) d_q s + \int_0^1 H_2(t, qs) f_1(u_n, v_n)(s) d_q s \end{pmatrix}, \quad (127)$$

$$\begin{pmatrix} w_{n+1}(t) \\ z_{n+1}(t) \end{pmatrix} = \begin{pmatrix} \int_0^1 K_1(t, qs) f_1(w_n, z_n)(s) ds + \int_0^1 H_1(t, qs) f_2(w_n, z_n)(s) ds \\ \int_0^1 K_2(t, qs) f_2(w_n, z_n)(s) ds + \int_0^1 H_2(t, qs) f_1(w_n, z_n)(s) ds \end{pmatrix}, \quad (128)$$

and the initial values are

$$\begin{pmatrix} u_0(t) \\ v_0(t) \end{pmatrix} = \begin{pmatrix} lt^{\alpha_1 - 1} \\ lt^{\alpha_2 - 1} \end{pmatrix} = \begin{pmatrix} 2.1244199t \\ 2.1244199t \end{pmatrix}, \quad (129)$$

$$\begin{pmatrix} w_0(t) \\ z_0(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (130)$$

From Figure 1(a), we can see that the images of the two functions u_4 and u_5 almost coincide, indicating that when the fifth iteration occurs, u_5 is already very close to the extremal solution u^*. So we approximate that

$$u^*(t) \approx u_5(t). \quad (131)$$

Analogously, from Figures 1(b)–1(d), we can also get that

$$v^*(t) \approx v_5(t), \quad (132)$$

$$w^*(t) \approx w_5(t), \quad (133)$$

$$z^*(t) \approx z_5(t). \quad (134)$$

The error estimates are as follows

$$\|u_5(t) - u^*(t)\| \leqslant \frac{(2\gamma)^5}{2(1-2\gamma)} (\|u_0(t) - u_1(t)\| + \|v_0(t) - v_1(t)\|) < 6.2023006, \quad (135)$$

$$\|v_5(t) - v^*(t)\| \leqslant \frac{(2\gamma)^5}{2(1-2\gamma)} (\|u_0(t) - u_1(t)\| + \|v_0(t) - v_1(t)\|) < 6.2023006, \quad (136)$$

$$\|w^*(t) - w_5(t)\| \leqslant \frac{(2\gamma)^5}{2(1-2\gamma)} (\|w_1(t) - w_0(t)\| + \|z_1(t) - z_0(t)\|) < 0.1821279, \quad (137)$$

$$\|z^*(t) - z_5(t)\| \leqslant \frac{(2\gamma)^5}{2(1-2\gamma)} (\|w_1(t) - w_0(t)\| + \|z_1(t) - z_0(t)\|) < 0.1821279. \quad (138)$$

Example 3. For $t \in [0, 1]$, consider the following fractional differential system:

$$\begin{cases} D_{0.5}^2 u(t) + 0.09t^2 + 0.03tu(t) = 0 \\ D_{0.5}^2 v(t) + 0.11t + 0.01tv(t) = 0, \end{cases} \quad (139)$$

with the coupled integral and discrete mixed boundary conditions (119). Then, we can also get (122) by calculation. Setting $A_1 = 0.5, A_2 = 0.3$, we get $l = 2.1244199$. And for any $u_i, v_i \in C[0, 1]$ with $u \leqslant v$, we have

$$0 \leqslant f_1(t, v) - f_1(t, u) \leqslant 0.03t(v - u), t \in [0, 1] \quad (140)$$

$$0 \leqslant f_2(t, v) - f_2(t, u) \leqslant 0.01t(v - u), t \in [0, 1], \quad (141)$$

so $L = 0.03, M = 0.01$. Thus, it can be obtained that $\gamma = 0.0869518 < 1/2$. Then, all the hypotheses of Corollary 12 are satisfied. BVP (139) and (119) have monotone positive solutions (u^*, v^*) and (w^*, z^*), which are the same conclusion as (127)–(130) in Example 2.

From Figure 2, we can also get that

$$u^*(t) \approx u_3(t) \quad (142)$$

$$v^*(t) \approx v_3(t), \quad (143)$$

$$w^*(t) \approx w_3(t), \quad (144)$$

$$z^*(t) \approx z_3(t). \quad (145)$$

The error estimates are as follows

$$\|u_3(t) - u^*(t)\| \leq \frac{(2\gamma)^3}{2(1-2\gamma)} (\|u_0(t) - u_1(t)\| + \|v_0(t) - v_1(t)\|) < 0.013027938, \tag{146}$$

$$\|v_3(t) - v^*(t)\| \leq \frac{(2\gamma)^3}{2(1-2\gamma)} (\|u_0(t) - u_1(t)\| + \|v_0(t) - v_1(t)\|) < 0.013027938, \tag{147}$$

$$\|w^*(t) - w_3(t)\| \leq \frac{(2\gamma)^3}{2(1-2\gamma)} (\|w_1(t) - w_0(t)\| + \|z_1(t) - z_0(t)\|) < 0.0003606834, \tag{148}$$

$$\|z^*(t) - z_3(t)\| \leq \frac{(2\gamma)^3}{2(1-2\gamma)} (\|w_1(t) - w_0(t)\| + \|z_1(t) - z_0(t)\|) < 0.0003606834. \tag{149}$$

Data Availability

No underlying data was collected or produced in this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed equally and significantly to writing this article. All the authors read and approved the final manuscript.

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