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Research Article

Adaptation of the Novel Cubic B-Spline Algorithm for Dealing with Conformable Systems of Differential Boundary Value Problems concerning Two Points and Two Fractional Parameters

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Recently, conformable calculus has appeared in many abstract uses in mathematics and several practical applications in engineering and science. In addition, many methods and numerical algorithms have been adapted to it. In this paper, we will demonstrate, use, and construct the cubic B-spline algorithm to deal with conformable systems of differential boundary value problems concerning two points and two fractional parameters in both regular and singular types. Here, several linear and nonlinear examples will be presented, and a model for the Lane-Emden will be one of the applications presented. Indeed, we will show the complete construction of the used spline through the conformable derivative along with the convergence theory, and the error orders together with other results that we will present in detail in the form of tables and graphs using Mathematica software. Through the results we obtained, it became clear to us that the spline approach is effective and fast, and it requires little compulsive and mathematical burden in solving the problems presented. At the end of the article, we presented a summary that contains the most important findings, what we calculated, and some future suggestions.

1. Introduction

At present, in addition to the past tens of years, the applications of FDPs have expanded to include many physical and engineering applications [1–3]. In one place, we find application for them in kinetics energy [4], anomalous diffusion [5], movement of fluids [6], movement of waves [7], electrical engineering [8], and some of the fields of computer science [9], whilst in another place, we see some abstract uses of theories and definitions, which are originally found to organize the mathematical aspect of fractional derivatives in solving several fractional models like cholera outbreak [10] and partial FDPs [11]. From the definition of Riemann, the fractional differential began, and then different definitions appeared, such as Caputo, Fabrizio, and Atangana [12, 13].

Many of the definitions of fractional derivatives have strong features that make them a target in the modeling of many scientific phenomena, and at the same time, they have weaknesses in some characteristics that made some researchers search for a mathematically appropriate definition that is consistent with many of the laws and theorems found in the classical derivative. Therefore, in this paper, we will use the definition of the CD as a new approach to solving BVPs in their regular and singular states by adopting the CBSA to it. Conformable calculus proposed by [14] and theorized by [15] appears in several fields of applied sciences and abstract analysis as stellar mathematical agents to characterize hereditary behaviors with the memory of many substances. The CD has been successfully exercised in diverse physical and engineering application fields (herein, we try to list it briefly so that we do not prolong the reader and do not increase the size of the paper as much as possible) as in the formulation of fuzzy differential problems [16], in Newton mechanics [17], in Burgers' model [18], in population growth model [19], and in traveling wave field [20].

Systems of BVPs which are a mixture of several FDPs subject to given BCs represent very important issues in solving real-world models. Because of this rise, studying numerical and analytical solutions to these systems is an enticing topic for scientists. These kinds of systems are usually difficult to solve analytically, especially for singular, nonlinear, and nonhomogenous cases. To this end, extensive research has been carried out to obtain numerical schemes and various methods as utilized in the literature as follows: n [21], the authors applied the Adomian decomposition scheme; in [22], the authors described the sinc collocation algorithm; and in [23] the authors utilized the fractional Lagrangian approach.

The spline approach is an ongoing research subject in various diverse and pervasive science areas such as numerical analysis, signal processing, and computational physics [24–26]. It is a crucial method for solving-modeling many FDPs like singular BVPs [27], nonfractional Bratu-type BVPs [28], nonfractional LEP [29], and fractional physiology problem [30] (herein, we try to list it briefly so that we do not prolong the reader and do not increase the size of the paper as much as possible). CBS is the most common BS, which Schoenberg coined the expression BS, and it is an abbreviation of the word "basis spline". In computational mathematics, BS is a spline function with the lowest description interval for a given degree of smoothness and domain decomposition.

Here, we will show the complete construction of the used CBSA through the CD along with the convergence theory and other results that we will present in detail in the form of tables and graphs using the Mathematica software. Anyhow, we will solve the following:

(i) Conformable system of FDPs of regular type:

$$\begin{aligned} &T^{\theta_1}\Psi + a_1(\varsigma)T^{\delta_1}\Psi + a_2(\varsigma)\Psi + T^{\theta_2}\Phi + a_3(\varsigma)T^{\delta_2}\Phi \\ &\quad + a_4(\varsigma)\Phi + N_1(\Psi,\Phi) \\ &\quad = \mathscr{F}_1(\varsigma), \\ &T^{\theta_2}\Phi + b_1(\varsigma)T^{\delta_2}\Phi + b_2(\varsigma)\Phi + T^{\theta_1}\Psi + b_3(\varsigma)T^{\delta_1}\Psi \\ &\quad + b_4(\varsigma)\Psi + N_2(\Psi,\Phi) \\ &\quad = \mathscr{F}_2(\varsigma), \end{aligned} \tag{1}$$

concerning the BC

$$\begin{aligned} &\Psi(a) = \alpha_1, \Phi(a) = \alpha_2, \\ &\Psi(b) = \beta_1, \Phi(b) = \beta_2. \end{aligned} \tag{2}$$

(ii) Conformable LEP of singular type as

$$\begin{aligned} &T^{\theta_1}\Psi + \frac{\eta_1}{\varsigma}T^{\delta_1}\Psi + a_2(\varsigma)\Psi + a_4(\varsigma)\Phi + N_1(\Psi,\Phi) = \mathscr{F}_1(\varsigma), \\ &T^{\theta_2}\Phi + \frac{\eta_2}{\varsigma}T^{\delta_2}\Phi + b_2(\varsigma)\Phi + b_4(\varsigma)\Psi + N_2(\Psi,\Phi) = \mathscr{F}_2(\varsigma), \end{aligned} \tag{3}$$

concerning the BC

$$\begin{aligned} &\Psi(0) = \rho_1, \Phi(0) = \varepsilon_1, \\ &\Psi(1) = \rho_2, \Phi(1) = \varepsilon_2. \end{aligned} \tag{4}$$

Herein, $\Psi = \Psi(\varsigma)$, $\Phi = \Phi(\varsigma)$, $0 < \delta_1, \delta_2 \leq 1$, $1 < \theta_1, \theta_2 \leq 2$, $\alpha_{p}, \beta_{p}, \rho_{p}, \varepsilon_{p} \in \mathbb{R}$, $\eta_1, \eta_2 \geq 0$, and $T^{\delta_1}, T^{\delta_2}, T^{\theta_1}, T^{\theta_2}$ stands for CDs of order $\delta_1, \delta_2, \theta_1, \theta_2$, respectively; N_1 and N_2 are nonlinear functions in Ψ, Φ, $\mathscr{F}_1(\varsigma)$, and $\mathscr{F}_2(\varsigma)$; and $a_{q}(\varsigma)$ and $b_{q}(\varsigma)$ with $q = 1.2.3.4$ are continuous functions. Further, the CD of $T^{\delta}\mathscr{H}(\varsigma)$ is expressed as

$$T^{\omega}\mathscr{H}(\varsigma) = \lim_{\xi \longrightarrow 0} \frac{\mathscr{H}^{(\lceil\omega\rceil-1)}\left(\varsigma + \xi\varsigma^{\lceil\omega\rceil-\omega}\right) - \mathscr{H}^{(\lceil\omega\rceil-1)}(\varsigma)}{\xi}, \tag{5}$$

with $\omega \in (n, n+1]$, $\mathscr{H} : [0,\infty) \longrightarrow \mathbb{R}$ be n-differentiable for all $\varsigma > 0$, and $\mathrm{T}^{\delta}\mathscr{H}(\varsigma) = \varsigma^{\lceil\delta\rceil-\delta}\mathscr{H}^{(\lceil\delta\rceil)}(\varsigma)$.

The motivation of our article can be summarized as follows: Often, real solutions to FDPs are not available and cannot be calculated or predicted because most of the problems are of nonlinear or nonhomogeneous type, or their coefficients are variables and not constants. Therefore, dealing with these issues, in this case, requires the utilization of numerical methods and algorithms, and here in our paper, we proposed the CBSA for ease of dealing with it and the ease of writing its computer program and because it is also accurate and does not require combining it with other numerical methods to obtain the required approximation. In addition to its convergence, its error order is guaranteed by the theories and results that we presented in our coming sections.

The basic structure herein was built as next. Section 2 proposes and formulates the CBSA for handling systems of BVPs concerning the CD. Section 3 deals with solving a singular system of conformable LEP by using the CBSA. Section 4 explores and discusses the convergence analysis together with the error order of the utilized CBSA. In Section 5, by using tables and graphs, some treatment examples are examined to offer the accuracy and fineness of the CBSA using Mathematica 11 software. At the end of the article, we presented a summary that contains the most important findings, what we calculated, and some future suggestions.

2. Formulation of the CBSA for Handling Systems of BVPs

In this section, the CBSA is used to construct and obtain approximations of the mentioned systems of conformable FDPs for both regular and singular types. Herein, we will consider two computational cases according to the nature of the shapes functions $N_1(\Psi,\Phi)$ and $N_2(\Psi,\Phi)$.

Assume that $\Pi:\{a=\varsigma_0<\varsigma_1<\cdots<\varsigma_{r-1}<\varsigma_r=b\}$ be a partition of $[a,b]$ with mesh points $\varsigma_k=a+k\hbar$, $k=0,1,\cdots,r$ wherein $\varsigma_0=a$, $\varsigma_r=b$, and $\hbar=(b-a)/r$. By introducing knots $\varsigma_{-2}<\varsigma_{-1}<\varsigma_0$ and $\varsigma_r<\varsigma_{r+1}<\varsigma_{r+2}$, Π becomes

$$\Pi:\{\varsigma_{-2}<\varsigma_{-1}<\varsigma_0=a<\varsigma_1<\cdots<\varsigma_r=b<\varsigma_{r+1}<\varsigma_{r+2}\}. \quad (6)$$

Define $\zeta_3(\Pi)=\{n(\varsigma)\in C^2[a,b]\}$ such that $n(\varsigma)$ is piecewise, 3rd-degree polynomials around Π. Anyhow, the 3rd-degree BSs is

$$B_{k,3}(\varsigma)=\frac{1}{6\hbar^3}\begin{cases}(\varsigma-\varsigma_{k-2})^3, & \varsigma_{k-2}\le\varsigma<\varsigma_{k-1},\\ -3(\varsigma-\varsigma_{k-1})^3+3\hbar(\varsigma-\varsigma_{k-1})^2+3\hbar^2(\varsigma-\varsigma_{k-1})+\hbar^3, & \varsigma_{k-1}\le\varsigma<\varsigma_k,\\ -3(\varsigma_{k+1}-\varsigma)^3+3\hbar(\varsigma_{k+1}-\varsigma)^2+3\hbar^2(\varsigma_{k+1}-\varsigma)+\hbar^3, & \varsigma_k\le\varsigma<\varsigma_{k+1},\\ (\varsigma_{k+2}-\varsigma)^3, & \varsigma_{k+1}\le\varsigma<\varsigma_{k+2},\\ 0, & \text{otherwise}.\end{cases} \quad (7)$$

To solve (1) and (2) together with (3) and (4) numerically, $T^{\delta}B_{k,3}(\varsigma)$ and $T^{\theta}B_{k,3}(\varsigma)$ evaluation is needed, where $0<\delta\le1$ and $1<\theta\le2$. Using the propositions of CD, one has

$$T^{\delta}B_{k,3}(\varsigma)=\frac{\varsigma^{1-\delta}}{2\hbar^3}\begin{cases}(\varsigma-\varsigma_{k-2})^2, & \varsigma_{k-2}\le\varsigma<\varsigma_{k-1},\\ -3(\varsigma-\varsigma_{k-1})^2+2\hbar(\varsigma-\varsigma_{k-1})+\hbar^2, & \varsigma_{k-1}\le\varsigma<\varsigma_k,\\ 3(\varsigma_{k+1}-\varsigma)^2-2\hbar(\varsigma_{k+1}-\varsigma)-\hbar^2, & \varsigma_k\le\varsigma<\varsigma_{k+1},\\ -(\varsigma_{k+2}-\varsigma)^2, & \varsigma_{k+1}\le\varsigma<\varsigma_{k+2},\\ 0, & \text{otherwise}.\end{cases} \quad (8)$$

$$T^{\theta}B_{k,3}(\varsigma)=\frac{\varsigma^{2-\theta}}{\hbar^3}\begin{cases}\varsigma-\varsigma_{k-2}, & \varsigma_{k-2}\le\varsigma<\varsigma_{k-1},\\ \hbar-3(\varsigma-\varsigma_{k-1}), & \varsigma_{k-1}\le\varsigma<\varsigma_k,\\ \hbar-3(\varsigma_{k+1}-\varsigma), & \varsigma_k\le\varsigma<\varsigma_{k+1},\\ (\varsigma_{k+2}-\varsigma), & \varsigma_{k+1}\le\varsigma<\varsigma_{k+2},\\ 0, & \text{otherwise}.\end{cases} \quad (9)$$

To formulate the required approximation using the CBSA, let

$$\widehat{\Psi}(\varsigma)=\sum_{k=-1}^{r+1}\mu_k B_{k,3}(\varsigma),\qquad \widehat{\Phi}(\varsigma)=\sum_{k=-1}^{r+1}\nu_k B_{k,3}(\varsigma), \quad (10)$$

be a cubic BS interpolating function of $\Psi(\varsigma)$ and $\Phi(\varsigma)$, respectively, with knots Π, where μ_k,ν_k are unknown, and $B_{k,3}(\varsigma)$ are the 3rd-degree BS functions which are defined in (7).

Therefore, from (7), (8), and (9) the value of $\widehat{\Psi}(\varsigma)$, $T^{\delta_1}\widehat{\Psi}(\varsigma)$, $T^{\theta_1}\widehat{\Psi}(\varsigma)$ and $\widehat{\Phi}(\varsigma)$, $T^{\delta_2}\widehat{\Phi}(\varsigma)$, $T^{\theta_2}\widehat{\Phi}(\varsigma)$ at knot ς_k can be simplified as

$$\begin{aligned}\widehat{\Psi}(\varsigma_k)&=\sum_{k=-1}^{r+1}\mu_k B_{k,3}(\varsigma_k)\\&=\mu_{k-1}B_{k-1,3}(\varsigma_k)+\mu_k B_{k,3}(\varsigma_k)+\mu_{k+1}B_{k+1,3}(\varsigma_k),\\ T^{\delta_1}\widehat{\Psi}(\varsigma_k)&=\sum_{k=-1}^{r+1}\mu_k T^{\delta_1}B_{k,3}(\varsigma_k)\\&=\mu_{k-1}T^{\delta_1}B_{k-1,3}(\varsigma_k)+\mu_k T^{\delta_1}B_{k,3}(\varsigma_k)\\&\quad+\mu_{k+1}T^{\delta_1}B_{k+1,3}(\varsigma_k),\\ T^{\theta_1}\widehat{\Psi}(\varsigma_k)&=\sum_{k=-1}^{r+1}\mu_k T^{\theta_1}B_{k,3}(\varsigma_k)\\&=\mu_{k-1}T^{\theta_1}B_{k-1,3}(\varsigma_k)+\mu_k T^{\theta_1}B_{k,3}(\varsigma_k)\\&\quad+\mu_{k+1}T^{\theta_1}B_{k+1,3}(\varsigma_k),\end{aligned} \quad (11)$$

where B's, $T^{\delta_1}B$'s, and $T^{\theta_1}B$'s are given, respectively, as

$$B_{k-1,3}(\varsigma_k)=\frac{1}{6},\qquad B_{k,3}(\varsigma_k)=\frac{2}{3},\qquad B_{k+1,3}(\varsigma_k)=\frac{1}{6}. \quad (12)$$

$$
\begin{aligned}
T^{\delta_1} B_{\hbar-1,3}(\varsigma_\hbar) &= \frac{1}{2\hbar}\varsigma_\hbar^{1-\delta_1},\\
T^{\delta_1} B_{\hbar,3}(\varsigma_\hbar) &= 0,\\
T^{\delta_1} B_{\hbar+1,3}(\varsigma_\hbar) &= -\frac{1}{2\hbar}\varsigma_\hbar^{1-\delta_1}.
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
T^{\theta_1} B_{\hbar-1,3}(\varsigma_\hbar) &= \frac{1}{\hbar^2}\varsigma_\hbar^{2-\theta_1},\\
T^{\theta_1} B_{\hbar,3}(\varsigma_\hbar) &= -\frac{2}{\hbar^2}\varsigma_\hbar^{2-\theta_1},\\
T^{\theta_1} B_{\hbar+1,3}(\varsigma_\hbar) &= \frac{1}{\hbar^2}\varsigma_\hbar^{2-\theta_1}.
\end{aligned} \tag{14}
$$

Anyhow, one can write

$$
\begin{aligned}
\widehat{\Psi}(\varsigma_\hbar) &= \frac{1}{6}\mu_{\hbar-1} + \frac{2}{3}\mu_\hbar + \frac{1}{6}\mu_{\hbar+1},\\
T^{\delta_1}\widehat{\Psi}(\varsigma_\hbar) &= -\frac{1}{2\hbar}\varsigma_\hbar^{1-\delta_1}\mu_{\hbar-1} + \frac{1}{2\hbar}\varsigma_\hbar^{1-\delta_1}\mu_{\hbar+1},\\
T^{\theta_1}\widehat{\Psi}(\varsigma_\hbar) &= \frac{1}{\hbar^2}\varsigma_\hbar^{2-\theta_1}\mu_{\hbar-1} - \frac{2}{\hbar^2}\varsigma_\hbar^{2-\theta_1}\mu_\hbar + \frac{1}{\hbar^2}\varsigma_\hbar^{2-\theta_1}\mu_{\hbar+1}.
\end{aligned} \tag{15}
$$

Similarly, one can get the following regarding $\widehat{\Phi}$:

$$
\begin{aligned}
\widehat{\Phi}(\varsigma_\hbar) &= \frac{1}{6}\nu_{\hbar-1} + \frac{2}{3}\nu_\hbar + \frac{1}{6}\nu_{\hbar+1},\\
T^{\delta_2}\widehat{\Phi}(\varsigma_\hbar) &= -\frac{1}{2\hbar}\varsigma_\hbar^{1-\delta_2}\nu_{\hbar-1} + \frac{1}{2\hbar}\varsigma_\hbar^{1-\delta_2}\nu_{\hbar+1},\\
T^{\theta_2}\widehat{\Phi}(\varsigma_\hbar) &= \frac{1}{\hbar^2}\varsigma_\hbar^{2-\theta_2}\nu_{\hbar-1} - \frac{2}{\hbar^2}\varsigma_\hbar^{2-\theta_2}\nu_\hbar + \frac{1}{\hbar^2}\varsigma_\hbar^{2-\theta_2}\nu_{\hbar+1}.
\end{aligned} \tag{16}
$$

Firstly, we will theorize the linear conformable BVP systems. In this case, $N_1(\Psi,\Phi) = N_2(\Psi,\Phi) = 0$ in (1). Thus, the approximation solutions (10) and their CDs should satisfy the given differential equation at points $\varsigma = \varsigma_\hbar$ when $\hbar = 1, 2, \cdots, r$. This can be done by substituting (10) with (1). Anyhow, the resulting formulas for $\hbar = 1, 2, \cdots, r$ should be

$$
\begin{cases}
T^{\theta_1}\widehat{\Psi}(\varsigma_\hbar) + a_1(\varsigma_\hbar)T^{\delta_1}\widehat{\Psi}(\varsigma_\hbar) + a_2(\varsigma_\hbar)\widehat{\Psi}(\varsigma_\hbar) + T^{\theta_2}\widehat{\Phi}(\varsigma_\hbar) + a_3(\varsigma_\hbar)T^{\delta_2}\widehat{\Phi}(\varsigma_\hbar) + a_4(\varsigma_\hbar)\widehat{\Phi}(\varsigma_\hbar) = \mathcal{F}_1(\varsigma_\hbar),\\
T^{\theta_2}\widehat{\Phi}(\varsigma_\hbar) + b_1(\varsigma_\hbar)T^{\delta_2}\widehat{\Phi}(\varsigma_\hbar) + b_2(\varsigma_\hbar)\widehat{\Phi}(\varsigma_\hbar) + T^{\theta_1}\widehat{\Psi}(\varsigma_\hbar) + b_3(\varsigma_\hbar)T^{\delta_1}\widehat{\Psi}(\varsigma_\hbar) + b_4(\varsigma_\hbar)\widehat{\Psi}(\varsigma_\hbar) = \mathcal{F}_2(\varsigma_\hbar),
\end{cases} \tag{17}
$$

with the BCs

$$
\begin{aligned}
\widehat{\Psi}(\varsigma_\hbar) &= \alpha_1, \text{ for } \hbar = a,\\
\widehat{\Psi}(\varsigma_\hbar) &= \beta_1, \text{ for } \hbar = b,\\
\widehat{\Phi}(\varsigma_\hbar) &= \alpha_2, \text{ for } \hbar = a,\\
\widehat{\Phi}(\varsigma_\hbar) &= \beta_2, \text{ for } \hbar = b.
\end{aligned} \tag{18}
$$

To proceed more, (15) and (16) are substituted into (17) and (18) and will be resulting in $[G]_{2(r+3)\times 2(r+3)}B = Q$ system of unknowns $\mu_{-1}, \mu_0, \cdots, \mu_{r+1}, \nu_{-1}$, and $\nu_0, \cdots, \nu_{r+1}$ with

$$
\begin{aligned}
B &= [\mu_{-1}, \mu_0, \cdots, \mu_{r+1}, \nu_{-1}, \nu_0, \cdots, \nu_{r+1}]^T,\\
Q &= 6\big[\alpha_1, \hbar^2\mathcal{F}_1(\varsigma_0), \hbar^2\mathcal{F}_1(\varsigma_1), \cdots, \hbar^2\mathcal{F}_1(\varsigma_r), \beta_1, \alpha_2, \hbar^2\mathcal{F}_2\\
&\quad \cdot(\varsigma_0), \hbar^2\mathcal{F}_2(\varsigma_1), \cdots, \hbar^2\mathcal{F}_2(\varsigma_r), \beta_2\big].
\end{aligned} \tag{19}
$$

Herein, $[G]_{2(r+3)\times 2(r+3)}$ and its corresponding elements are provided by

$$
G = \begin{bmatrix} G_1 & \cdots & G_2\\ \vdots & \ddots & \vdots\\ G_3 & \cdots & G_4 \end{bmatrix}. \tag{20}
$$

$$
G_1 = \begin{bmatrix}
1 & 4 & 1 & 0 & \cdots & 0 & 0\\
g_1(\varsigma_0) & p_1(\varsigma_0) & q_1(\varsigma_0) & 0 & \cdots & 0 & 0\\
0 & g_1(\varsigma_1) & p_1(\varsigma_1) & q_1(\varsigma_1) & 0 & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & \cdots & \cdots & 0 & g_1(\varsigma_r) & p_1(\varsigma_r) & q_1(\varsigma_r)\\
0 & \cdots & \cdots & 0 & 1 & 4 & 1
\end{bmatrix}, \tag{21}
$$

$$
G_2 = \begin{bmatrix}
0 & 0 & 0 & 0 & \cdots & 0 & 0\\
g_2(\varsigma_0) & p_2(\varsigma_0) & q_2(\varsigma_0) & 0 & \cdots & 0 & 0\\
0 & g_2(\varsigma_1) & p_2(\varsigma_1) & q_2(\varsigma_1) & 0 & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & \cdots & \cdots & 0 & g_2(\varsigma_r) & p_2(\varsigma_r) & q_2(\varsigma_r)\\
0 & \cdots & \cdots & 0 & 0 & 0 & 0
\end{bmatrix}, \tag{22}
$$

$$G_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ g_3(\varsigma_0) & p_3(\varsigma_0) & q_3(\varsigma_0) & 0 & \cdots & 0 & 0 \\ 0 & g_3(\varsigma_1) & p_3(\varsigma_1) & q_3(\varsigma_1) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & g_3(\varsigma_r) & p_3(\varsigma_r) & q_3(\varsigma_r) \\ 0 & \cdots & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (23)$$

$$G_4 = \begin{bmatrix} 1 & 4 & 1 & 0 & \cdots & 0 & 0 \\ g_4(\varsigma_0) & p_4(\varsigma_0) & q_4(\varsigma_0) & 0 & \cdots & 0 & 0 \\ 0 & g_4(\varsigma_1) & p_4(\varsigma_1) & q_4(\varsigma_1) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & g_4(\varsigma_r) & p_4(\varsigma_r) & q_4(\varsigma_r) \\ 0 & \cdots & \cdots & 0 & 1 & 4 & 1 \end{bmatrix}. \quad (24)$$

Also, the coefficients in the submatrices G_1, G_2, G_3, and G_4 have the form

$$\begin{aligned} g_1(\varsigma_k) &= 6\varsigma_k^{2-\theta_1} - a_1(\varsigma_k)3\hbar\varsigma_k^{1-\delta_1} + \hbar^2 a_2(\varsigma_k), \\ p_1(\varsigma_k) &= -12\varsigma_k^{2-\theta_1} + 4\hbar^2 a_2(\varsigma_k), \\ q_1(\varsigma_k) &= 6\varsigma_k^{2-\theta_1} + a_1(\varsigma_k)3\hbar\varsigma_k^{1-\delta_1} + \hbar^2 a_2(\varsigma_k). \end{aligned} \quad (25)$$

$$\begin{aligned} g_2(\varsigma_k) &= 6\varsigma_k^{2-\theta_2} + a_3(\varsigma_k)3\hbar\varsigma_k^{1-\delta_2} + \hbar^2 a_4(\varsigma_k), \\ p_2(\varsigma_k) &= -12\varsigma_k^{2-\theta_2} + 4\hbar^2 a_4(\varsigma_k), \\ q_2(\varsigma_k) &= 6\varsigma_k^{2-\theta_2} + a_3(\varsigma_k)3\hbar\varsigma_k^{1-\delta_2} + \hbar^2 a_4(\varsigma_k). \end{aligned} \quad (26)$$

$$\begin{aligned} g_3(\varsigma_k) &= 6\varsigma_k^{2-\theta_2} + b_1(\varsigma_k)3\hbar\varsigma_k^{1-\delta_2} + \hbar^2 b_2(\varsigma_k), \\ p_3(\varsigma_k) &= -12\varsigma_k^{2-\theta_2} + 4\hbar^2 b_2(\varsigma_k), \\ q_3(\varsigma_k) &= 6\varsigma_k^{2-\theta_2} + b_1(\varsigma_k)3\hbar\varsigma_k^{1-\delta_2} + \hbar^2 b_2(\varsigma_k). \end{aligned} \quad (27)$$

$$\begin{aligned} g_4(\varsigma_k) &= 6\varsigma_k^{2-\theta_1} + b_3(\varsigma_k)3\hbar\varsigma_k^{2-\delta_1} + \hbar^2 b_4(\varsigma_k), \\ p_4(\varsigma_k) &= -12\varsigma_k^{2-\theta_1} + 4\hbar^2 b_4(\varsigma_k), \\ q_4(\varsigma_k) &= 6\varsigma_k^{2-\theta_1} + b_3(\varsigma_k)3\hbar\varsigma_k^{1-\delta_1} + \hbar^2 b_4(\varsigma_k). \end{aligned} \quad (28)$$

Secondly, we will theorize the nonlinear conformable BVP systems in this case of $N_1(\Psi, \Phi)$ and $N_2(\Psi, \Phi)$ are nonlinear functions of Ψ and Φ differ from zero. Anyhow, the substituting of (10) and its CDs in (1) and (2) at $\varsigma = \varsigma_k$ when $k = 0, 1, \cdots, r$ will gives

$$\begin{aligned} \mathscr{F}_1(\varsigma_k) = {} & \sum_{k=-1}^{r+1} \mu_k \left[T^{\theta_1} B_{k,3}(\varsigma_k) + a_1(\varsigma_k) T^{\delta_1} B_{k,3}(\varsigma_k) + a_2(\varsigma_k) B_{k,3}(\varsigma_k) \right] \\ & + \sum_{k=-1}^{r+1} \nu_k \left[T^{\theta_2} B_{k,3}(\varsigma_k) + a_3(\varsigma_k) T^{\delta_2} B_{k,3}(\varsigma_k) + a_4(\varsigma_k) B_{k,3}(\varsigma_k) \right] \\ & + N_1 \left(\sum_{k=-1}^{r+1} \mu_k B_{k,3}(\varsigma_k), \sum_{k=-1}^{r+1} \nu_k B_{k,3}(\varsigma_k) \right). \end{aligned} \quad (29)$$

$$\begin{aligned} \mathscr{F}_2(\varsigma_k) = {} & \sum_{k=-1}^{r+1} \nu_k \left[T^{\theta_2} B_{k,3}(\varsigma_k) + b_1(\varsigma_k) T^{\delta_2} B_{k,3}(\varsigma_k) + b_2(\varsigma_k) B_{k,3}(\varsigma_k) \right] \\ & + \sum_{k=-1}^{r+1} \mu_k \left[T^{\theta_1} B_{k,3}(\varsigma_k) + b_3(\varsigma_k) T^{\delta_1} B_{k,3}(\varsigma_k) + b_4(\varsigma_k) B_{k,3}(\varsigma_k) \right] \\ & + N_2 \left(\sum_{k=-1}^{r+1} \mu_k B_{k,3}(\varsigma_k), \sum_{k=-1}^{r+1} \nu_k B_{k,3}(\varsigma_k) \right). \end{aligned} \quad (30)$$

subject to the same BCs (18).

Recalling, the BS functions at $\{\varsigma_k\}_{k=0}^{r}$ are determined by substitution (12), (13), and (14) in (29), (30), and (18).

3. The CBSA for Handling Singular Systems of CDs

Now, we will spend the CBSA to build a numerical solution for the singular conformable LEP. We start by overcoming the singularity at $\varsigma = 0$ and then employing our proposed procedure scheme.

To solve the singular LEP in its CD case, we first write (3) in the standard form as

$$\begin{aligned} T^{\theta_1}\Psi(\varsigma) + \frac{\eta_1}{\varsigma} T^{\delta_1}\Psi(\varsigma) + Q_1(\varsigma, \Psi(\varsigma), \Phi(\varsigma)) = 0, \\ T^{\theta_2}\Phi(\varsigma) + \frac{\eta_2}{\varsigma} T^{\delta_2}\Phi(\varsigma) + Q_2(\varsigma, \Psi(\varsigma), \Phi(\varsigma)) = 0, \end{aligned} \quad (31)$$

concerning the BC

$$\begin{aligned} \Psi(0) = \rho_1, \Phi(0) = \varepsilon_1, \\ \Psi(1) = \rho_2, \Phi(1) = \varepsilon_2, \end{aligned} \quad (32)$$

where the set functions Q_1 and Q_2 are given as

$$\begin{aligned} Q_1(\varsigma, \Psi(\varsigma), \Phi(\varsigma)) = a_2(\varsigma)\Psi(\varsigma) + a_4(\varsigma)\Phi(\varsigma) + N_1(\Psi(\varsigma), \Phi(\varsigma)) - \mathscr{F}_1(\varsigma), \\ Q_2(\varsigma, \Psi(\varsigma), \Phi(\varsigma)) = b_2(\varsigma)\Phi(\varsigma) + b_4(\varsigma)\Psi(\varsigma) + N_2(\Psi(\varsigma), \Phi(\varsigma)) - \mathscr{F}_2(\varsigma). \end{aligned} \quad (33)$$

More focused, to take off the singularity $\varsigma = 0$, one can be employing the following next steps:

Multiplying (31) with ς gives

$$\begin{aligned}&\varsigma T^{\theta_1}\Psi(\varsigma)+\eta_1 T^{\delta_1}\Psi(\varsigma)+\varsigma Q_1(\varsigma,\Psi(\varsigma),\Phi(\varsigma))=0,\\&\varsigma T^{\theta_2}\Phi(\varsigma)+\eta_2 T^{\delta_2}\Phi(\varsigma)+\varsigma Q_2(\varsigma,\Psi(\varsigma),\Phi(\varsigma))=0.\end{aligned}\tag{34}$$

(i) Taking the CD of order δ_1 and δ_2, respectively, from both sides of (34), one has

$$\begin{aligned}&T^{\delta_1}\left(\varsigma T^{\theta_1}\Psi(\varsigma)\right)+\eta_1 T^{\delta_1}T^{\delta_1}\Psi(\varsigma)+T^{\delta_1}(\varsigma Q_1(\varsigma,\Psi(\varsigma),\Phi(\varsigma)))=0,\\&T^{\delta_2}\left(\varsigma T^{\theta_2}\Phi(\varsigma)\right)+\eta_2 T^{\delta_2}T^{\delta_2}\Phi(\varsigma)+T^{\delta_2}(\varsigma Q_2(\varsigma,\Psi(\varsigma),\Phi(\varsigma)))=0.\end{aligned}\tag{35}$$

(ii) Using the properties of the CD, one obtains

$$\begin{aligned}&\varsigma^{1-\delta_1}T^{\theta_1}\Psi(\varsigma)+\varsigma T^{\delta_1}T^{\theta_1}\Psi(\varsigma)+\eta_1 T^{\delta_1}T^{\delta_1}\Psi(\varsigma)\\&\quad+\varsigma^{1-\delta_1}Q_1(\varsigma,\Psi(\varsigma),\Phi(\varsigma))+\varsigma T^{\delta_1}Q_1(\varsigma,\Psi(\varsigma),\Phi(\varsigma))=0,\\&\varsigma^{1-\delta_2}T^{\theta_2}\Phi(\varsigma)+\varsigma T^{\delta_2}T^{\theta_2}\Phi(\varsigma)+\eta_2 T^{\delta_2}T^{\delta_2}\Phi(\varsigma)\\&\quad+\varsigma^{1-\delta_2}Q_2(\varsigma,\Psi(\varsigma),\Phi(\varsigma))+\varsigma T^{\delta_2}Q_2(\varsigma,\Psi(\varsigma),\Phi(\varsigma))=0.\end{aligned}\tag{36}$$

(iii) Substituting $\theta_1=\theta_2=2$ and $\delta_1=\delta_2=1$ in (36) at $\varsigma=0$, one gets

$$\begin{aligned}&(\eta_1+1)\Psi''(0)+Q_1(0,\Psi(\varsigma),\Phi(\varsigma))=0,\\&(\eta_2+1)\Phi''(0)+Q_2(0,\Psi(\varsigma),\Phi(\varsigma))=0.\end{aligned}\tag{37}$$

Putting (10) in (31), (32), and (37) at $\varsigma=\varsigma_\hbar$ it follows that

$$\begin{aligned}&T^{\theta_1}\widehat{\Psi}(\varsigma_\hbar)+\frac{\eta_1}{\varsigma_\hbar}T^{\delta_1}\widehat{\Psi}(\varsigma_\hbar)+Q_1\left(\varsigma_\hbar,\widehat{\Psi}(\varsigma_\hbar),\widehat{\Phi}(\varsigma_\hbar)\right)=0,\ \text{for } \hbar=1,\cdots,r,\\&(\eta_1+1)\widehat{\Psi}''(0)+Q_1\left(0,\widehat{\Psi}(\varsigma_\hbar),\widehat{\Phi}(\varsigma_\hbar)\right)=0,\ \text{for } \hbar=0,\\&T^{\theta_2}\widehat{\Phi}(\varsigma_\hbar)+\frac{\eta_2}{\varsigma_\hbar}T^{\delta_2}\widehat{\Phi}(\varsigma_\hbar)+Q_2\left(\varsigma_\hbar,\widehat{\Psi}(\varsigma_\hbar),\widehat{\Phi}(\varsigma_\hbar)\right)=0,\ \text{for } \hbar=1,\cdots,r,\\&(\eta_2+1)\widehat{\Phi}''(0)+Q_2\left(0,\widehat{\Psi}(\varsigma_\hbar),\widehat{\Phi}(\varsigma_\hbar)\right)=0,\ \text{for } \hbar=0,\\&\widehat{\Psi}(\varsigma_0)=\rho_1,\ \text{for } \varsigma_0=0,\\&\widehat{\Psi}(\varsigma_r)=\rho_2,\ \text{for } \varsigma_r=1,\\&\widehat{\Phi}(\varsigma_0)=\varepsilon_1,\ \text{for } \varsigma_0=0,\\&\widehat{\Phi}(\varsigma_r)=\varepsilon_2,\ \text{for } \varsigma_r=1.\end{aligned}\tag{38}$$

This drives a system of $2(r+3)$ equations with the same number of unknowns which can be treated to obtain the vectors $\mu_\hbar$ and $\nu_\hbar$; consequently an approximation of $\Psi(\varsigma)$ and $\Phi(\varsigma)$.

4. Error and Convergence Analysis

Herein, to guarantee the behavior of the approximate CBSA solutions, we utilized two main analyses: the first one concerning error analysis and the second one concerning convergence analysis.

Using the CBSA approximations (15) and (16), the following relations can be established:

$$\begin{aligned}&\frac{\hbar}{6}\left[\left(\frac{1}{6}\right)\widehat{\Psi}'(\varsigma_{\hbar-1})+\left(\frac{2}{3}\right)\widehat{\Psi}'(\varsigma_\hbar)+\left(\frac{1}{6}\right)\widehat{\Psi}'(\varsigma_{\hbar+1})\right]\\&\quad=\frac{1}{2}\varsigma_\hbar^{1-\delta_1}\left[\widehat{\Psi}(\varsigma_{\hbar+1})+\widehat{\Psi}(\varsigma_{\hbar-1})\right],\end{aligned}\tag{39}$$

$$\begin{aligned}\hbar^2T^{\theta_1}\widehat{\Psi}(\varsigma_\hbar)=\varsigma_\hbar^{2-\theta_1}&\left[6\left(\widehat{\Psi}(\varsigma_{\hbar+1})+\widehat{\Psi}(\varsigma_\hbar)\right)\right.\\&\left.-2\hbar\left(2\widehat{\Psi}'(\varsigma_\hbar)+\widehat{\Psi}'(\varsigma_{\hbar+1})\right)\right].\end{aligned}\tag{40}$$

In notation for the operator $E^\epsilon(\widehat{\Psi}(\varsigma_\hbar))=\widehat{\Psi}(\varsigma_{\hbar+\epsilon})$ with $\epsilon\in\mathbb{Z}$, we can write (39) and (40) as

$$\frac{\hbar}{6}\left[\left(\frac{1}{6}\right)E^{-1}+\left(\frac{2}{3}\right)I+\left(\frac{1}{6}\right)E\right]\widehat{\Psi}'(\varsigma_\hbar)=\frac{1}{2}\varsigma_\hbar^{1-\delta_1}\left[E+E^{-1}\right]\Psi(\varsigma_\hbar),\tag{41}$$

$$\hbar^2T^{\theta_1}\widehat{\Psi}(\varsigma_\hbar)=\varsigma_\hbar^{2-\theta_1}\left[6(E+\mathrm{I})\Psi(\varsigma_\hbar)-2\hbar(2\mathrm{I}+E)\Psi'(\varsigma_\hbar)\right].\tag{42}$$

Moreover, if $\Lambda=d/d\varsigma$, we have got

$$\begin{aligned}E\left(\widehat{\Psi}(\varsigma_\hbar)\right)&=\widehat{\Psi}(\varsigma_\hbar+\hbar)=\sum_{\hbar=0}^{\infty}\frac{\hbar^\hbar\widehat{\Psi}^{(\hbar)}(\varsigma_\hbar)}{\hbar!}\\&=\sum_{\hbar=0}^{\infty}\frac{(\hbar\Lambda)^\hbar}{\hbar!}\widehat{\Psi}(\varsigma_\hbar)=e^{(\hbar\Lambda)}\widehat{\Psi}(\varsigma_\hbar).\end{aligned}\tag{43}$$

It implies that $E=e^{\hbar\Lambda}$. Similarly, we have $E^{-1}=e^{-\hbar\Lambda}$ and we can get

$$\begin{aligned}&E+E^{-1}=2\left(1+\frac{(\hbar\Lambda)^2}{2!}+\frac{(\hbar\Lambda)^4}{4!}+\frac{(\hbar\Lambda)^6}{6!}+\cdots\right),\\&E-E^{-1}=2\left(\hbar\Lambda+\frac{(\hbar\Lambda)^3}{3!}+\frac{(\hbar\Lambda)^5}{5!}+\frac{(\hbar\Lambda)^7}{7!}+\cdots\right).\end{aligned}\tag{44}$$

Thus, (39) can be represented serially as

$$\left[1+\frac{1}{3}\left(\frac{(\hbar\Lambda)^2}{2!}+\frac{(\hbar\Lambda)^4}{4!}+\frac{(\hbar\Lambda)^6}{6!}+\cdots\right)\right]\widehat{\Psi}'(\varsigma_\hbar) = \varsigma_\hbar^{1-\delta_1}\left(\Lambda+\frac{\hbar^2\Lambda^3}{3!}+\frac{\hbar^4\Lambda^5}{5!}+\frac{\hbar^6\Lambda^7}{7!}+\cdots\right)\Psi(\varsigma_\hbar), \tag{45}$$

$$\widehat{\Psi}'(\varsigma_\hbar) = \varsigma_\hbar^{1-\delta_1}\left(\Lambda+\frac{\hbar^2\Lambda^3}{3!}+\frac{\hbar^4\Lambda^5}{5!}+\frac{\hbar^6\Lambda^7}{7!}+\cdots\right)\cdot\left[1+\left(\frac{(\hbar\Lambda)^2}{6}+\frac{(\hbar\Lambda)^4}{72}+\frac{(\hbar\Lambda)^6}{2160}+\cdots\right)\right]^{-1}\Psi(\varsigma_\hbar), \tag{46}$$

$$\begin{aligned}\widehat{\Psi}'(\varsigma_\hbar) &= \varsigma_\hbar^{1-\delta_1}\left(\Lambda+\frac{\hbar^2\Lambda^3}{3!}+\frac{\hbar^4\Lambda^5}{5!}+\frac{\hbar^6\Lambda^7}{7!}+\cdots\right)\\ &\quad\cdot\left[1-\left(\frac{(\hbar\Lambda)^2}{6}+\frac{(\hbar\Lambda)^4}{72}+\frac{(\hbar\Lambda)^6}{2160}+\cdots\right)+\left(\frac{(\hbar\Lambda)^2}{6}+\frac{(\hbar\Lambda)^4}{72}+\cdots\right)^2+\cdots\right]\Psi(\varsigma_\hbar)\\ &= \varsigma_\hbar^{1-\delta_1}\left(\Lambda+\frac{\hbar^2\Lambda^3}{3!}+\frac{\hbar^4\Lambda^5}{5!}+\frac{\hbar^6\Lambda^7}{7!}+\cdots\right)\cdot\left(1-\frac{(\hbar\Lambda)^2}{6}+\frac{(\hbar\Lambda)^4}{72}-\frac{(\hbar\Lambda)^6}{2160}+\cdots\right)\Psi(\varsigma_\hbar)\\ &= \varsigma_\hbar^{1-\delta_1}\left(\Lambda-\frac{\hbar^4\Lambda^5}{180}+\frac{\hbar^6\Lambda^7}{1512}-\cdots\right)\Psi(\varsigma_\hbar).\end{aligned} \tag{47}$$

Hence, after ranking, one can write

$$\begin{aligned}T^{\delta_1}\widehat{\Psi}(\varsigma_\hbar) &= \varsigma_\hbar^{1-\delta_1}\Psi'(\varsigma_\hbar)-\frac{\hbar^4}{180}\varsigma_\hbar^{1-\delta_1}\Psi^{(5)}(\varsigma_\hbar)+\cdots\\ &= T^{\delta_1}\Psi(\varsigma_\hbar)-\frac{\hbar^4}{180}\varsigma_\hbar^{1-\delta_1}\Psi^{(5)}(\varsigma_\hbar)+\cdots.\end{aligned} \tag{48}$$

By applying the same technique as (40), we may extract

$$\begin{aligned}T^{\theta_1}\widehat{\Psi}(\varsigma_\hbar) &= \varsigma_\hbar^{2-\theta_1}\Psi''(\varsigma_\hbar)-\frac{\hbar^2}{12}\varsigma_\hbar^{2-\theta_1}\Psi^{(4)}(\varsigma_\hbar)+\frac{\hbar^4}{360}\varsigma_\hbar^{2-\theta_1}\Psi^{(6)}(\varsigma_\hbar)+\cdots\\ &= T^{\theta_1}\Psi(\varsigma_\hbar)-\frac{\hbar^2}{12}\varsigma_\hbar^{2-\theta_1}\Psi^{(4)}(\varsigma_\hbar)+\frac{\hbar^4}{360}\varsigma_\hbar^{2-\theta_1}\Psi^{(6)}(\varsigma_\hbar)+\cdots.\end{aligned} \tag{49}$$

Let us now describe the expression $e_1(\varsigma)=\Psi(\varsigma)-\widehat{\Psi}(\varsigma)$ for error. Using (48) and (49) in $e(\varsigma_\hbar)$ expansion of the Taylor series, one gets

$$\begin{aligned}e_1(\varsigma_\hbar+\hbar) &= e(\varsigma_\hbar)+\hbar e'(\varsigma_\hbar)+\frac{\hbar^2}{2!}e''(\varsigma_\hbar)+\cdots\\ &= \left(\Psi(\varsigma_\hbar)-\widehat{\Psi}(\varsigma_\hbar)\right)+\hbar\left(\Psi'(\varsigma_\hbar)-\widehat{\Psi}'(\varsigma_\hbar)\right)+\frac{\hbar^2}{2!}\left(\Psi''(\varsigma_\hbar)-\widehat{\Psi}''(\varsigma_\hbar)\right)+\cdots\\ &= \left(\Psi(\varsigma_\hbar)-\widehat{\Psi}(\varsigma_\hbar)\right)+\hbar\varsigma_\hbar^{\delta_1-1}\left(T^{\delta_1}\widehat{\Psi}(\varsigma_\hbar)-T^{\delta_1}\Psi(\varsigma_\hbar)\right)+\frac{\hbar^2}{2!}\varsigma_\hbar^{\theta_1-2}\left(T^{\theta_1}\widehat{\Psi}(\varsigma_\hbar)-T^{\theta_1}\Psi(\varsigma_\hbar)\right)+\cdots.\end{aligned} \tag{50}$$

Hence,

$$e_1(\varsigma_\hbar+\hbar) = -\frac{\hbar^4}{24}\Psi^{(4)}(\varsigma_\hbar)+\frac{\hbar^5}{180}\Psi^{(5)}(\varsigma_\hbar)+\frac{\hbar^6}{720}\Psi^{(6)}(\varsigma_\hbar)+\cdots. \tag{51}$$

Similarly, we have

$$e_2(\varsigma_\hbar+\hbar) = -\frac{\hbar^4}{24}\Phi^{(4)}(\varsigma_\hbar)+\frac{\hbar^5}{180}\Phi^{(5)}(\varsigma_\hbar)+\frac{\hbar^6}{720}\Phi^{(6)}(\varsigma_\hbar)+\cdots. \tag{52}$$

As a score, it is obvious that our CBSA approximation is $O(\hbar^4)$ accurate.

In the convergence approach, we will prove the convergence of the CBSA for Dirichlet BC. Let $\Psi(\varsigma)$ and $\Phi(\varsigma)$ be the exact solutions of (1) and (2). Also, let $\widehat{\Psi}(\varsigma)$ and $\widehat{\Phi}(\varsigma)$ in (10) be the cubic BS approximations to $\Psi(\varsigma)$ and $\Phi(\varsigma)$, respectively. Due to round-off errors in computations, we will assume that $\mathcal{S}_1(\varsigma)=\sum_{\hbar=-1}^{r+1}\widehat{\mu}_\hbar B_{\hbar,3}(\varsigma)$ and $S_2(\varsigma)=\sum_{\hbar=-1}^{r+1}\widehat{\nu}_\hbar B_{\hbar,3}(\varsigma)$ be the computed BS approximations to $\widehat{\Psi}(\varsigma)$ and $\widehat{\Phi}(\varsigma)$, respectively, where $\widehat{\mu}_\hbar=(\widehat{\mu}_{-1},\widehat{\mu}_0,\cdots,\widehat{\mu}_{r+1})$ and $\widehat{\nu}_\hbar=(\widehat{\nu}_{-1},\widehat{\nu}_0,\cdots,\widehat{\nu}_{r+1})$.

To estimate the errors $\|(\Psi(\varsigma),\Phi(\varsigma))-(\widehat{\Psi}(\varsigma),\widehat{\Phi}(\varsigma))\|_\infty$ we must estimate $\|(\Psi(\varsigma),\Phi(\varsigma))-(\mathcal{S}_1(\varsigma),\mathcal{S}_1(\varsigma))\|_\infty$ and $\|(\mathcal{S}_1(\varsigma),\mathcal{S}_1(\varsigma))-(\widehat{\Psi}(\varsigma),\widehat{\Phi}(\varsigma))\|_\infty$, wherein $\|.\|$ represents the ∞-norm.

Firstly, we will consider the linear cases as follows:

$$\begin{aligned}L_1\left(\widehat{\Psi},\widehat{\Phi}\right) &= \mathcal{F}_1(\varsigma),\\ L_2\left(\widehat{\Psi},\widehat{\Phi}\right) &= \mathcal{F}_2(\varsigma),\end{aligned} \tag{53}$$

with the BCs (2) will lead to the linear system $GB=Q$.

$$\begin{aligned}L_1(\mathcal{S}_1,\mathcal{S}_2) &= \mathcal{F}_1^*(\varsigma),\\ L_2(\mathcal{S}_1,\mathcal{S}_2) &= \mathcal{F}_2^*(\varsigma),\end{aligned} \tag{54}$$

with the BCs (2) will lead to the linear system $GB^*=Q^*$.

Then it follows that $G(B-B^*)=(Q-Q^*)$, where

$$B-B^*=[(\widehat{\mu}_{-1}-\mu_{-1}),\cdots,(\widehat{\mu}_{r+1}-\mu_{r+1}),(\widehat{\nu}_{-1}-\nu_{-1}),\cdots,(\widehat{\nu}_{r+1}-\nu_{r+1})]^T, \tag{55}$$

$$\begin{aligned} Q-Q^* = 6\hbar^2[0,(\mathcal{F}_1(\varsigma_0)-\mathcal{F}_1^*(\varsigma_0)),\cdots,(\mathcal{F}_1(\varsigma_r) \\ -\mathcal{F}_1^*(\varsigma_r)),0,0,(\mathcal{F}_2(\varsigma_0) \\ -\mathcal{F}_2^*(\varsigma_0)),\cdots,(\mathcal{F}_2(\varsigma_r)-\mathcal{F}_2^*(\varsigma_r)),0]. \end{aligned} \tag{56}$$

Theorem 1. *Suppose that $\Psi(\varsigma),\Phi(\varsigma)\in C^5[a,b]$ and $\Pi:\{a=\varsigma_0<\varsigma_1<\cdots<\varsigma_{r-1}<\varsigma_r=b\}$ be the equally spaced partition of $[a,b]$ with step size $\hbar$. If $\mathcal{S}$ is the cubic BS function that interpolates the values of the function u at the knots $\varsigma_0,\cdots,\varsigma_r\in\Pi$, then there exist constants γ_q which do not depend on $\hbar$ such that for $\varsigma\in[a,b]$ with $b>a\geq 0$, we have*

$$\begin{aligned} &\|(\Psi(\varsigma),\Phi(\varsigma))-(\mathcal{S}_1(\varsigma),\mathcal{S}_1(\varsigma))\|_\infty\leq\gamma_1\hbar^4, \\ &\left\|T^\delta(\Psi(\varsigma),\Phi(\varsigma))-T^\delta(\mathcal{S}_1(\varsigma),\mathcal{S}_1(\varsigma))\right\|_\infty\leq\gamma_2\hbar^4, 0<\delta\leq 1, \\ &\left\|T^\theta(\Psi(\varsigma),\Phi(\varsigma))-T^\theta(\mathcal{S}_1(\varsigma),\mathcal{S}_1(\varsigma))\right\|_\infty\leq\gamma_3\hbar^2, 1<\theta\leq 2. \end{aligned} \tag{57}$$

Proof. Using the prior results, one can find

$$\begin{aligned} |\mathcal{F}_1(\varsigma_k)-\mathcal{F}_1^*(\varsigma_k)| &= \left|\mathrm{L}_1\left(\widehat{\Psi},\widehat{\Phi}\right)-\mathrm{L}_1(\mathcal{S}_1,\mathcal{S}_2)\right| \\ &\leq \left|T^{\theta_1}\widehat{\Psi}(\varsigma_k)-T^{\theta_1}\mathcal{S}_1(\varsigma_k)\right| \\ &\quad+|a_1(\varsigma_k)|\left|T^{\delta_1}\widehat{\Psi}(\varsigma_k)-T^{\delta_1}\mathcal{S}_1(\varsigma_k)\right| \\ &\quad+|a_2(\varsigma_k)|\left|\widehat{\Psi}(\varsigma_k)-\mathcal{S}_1(\varsigma_k)\right| \\ &\quad+\left|T^{\theta_2}\widehat{\Phi}(\varsigma_k)-T^{\theta_2}\mathcal{S}_2(\varsigma_k)\right| \\ &\quad+|a_3(\varsigma_k)|\left|T^{\delta_2}\widehat{\Phi}(\varsigma_k)-T^{\delta_2}\mathcal{S}_2(\varsigma_k)\right| \\ &\quad+|a_4(\varsigma_k)|\left|\widehat{\Phi}(\varsigma_k)-\mathcal{S}_2(\varsigma_k)\right|. \end{aligned} \tag{58}$$

Again, one can write

$$\begin{aligned} |\mathcal{F}_1(\varsigma_k)-\mathcal{F}_1^*(\varsigma_k)| \leq \gamma_3\hbar^2+\|a_1(\varsigma_k)\|_\infty\gamma_2\hbar^4 \\ +\|a_2(\varsigma_k)\|_\infty\gamma_1\hbar^4+\gamma_3'\hbar^2 \\ +\|a_3(\varsigma_k)\|_\infty\gamma_2'\hbar^4+\|a_4(\varsigma_k)\|_\infty\gamma_1'\hbar^4. \end{aligned} \tag{59}$$

Since $\hbar\ll 1$, one has

$$\|\mathcal{F}_1(\varsigma_k)-\mathcal{F}_1^*(\varsigma_k)\|\leq M\hbar^2, \tag{60}$$

$$\|\mathcal{F}_2(\varsigma_k)-\mathcal{F}_2^*(\varsigma_k)\|\leq M_1\hbar^2. \tag{61}$$

From (60) and (61), we can find

$$\|Q-Q^*\|_\infty\leq 6\hbar^4 M. \tag{62}$$

The matrix G is monotone and thus nonsingular [31]. Hence, we can write

$$(B-B^*)=G^{-1}(Q-Q^*). \tag{63}$$

Now, we determine row sums $\mathcal{S}_{-1},\mathcal{S}_0,\cdots,\mathcal{S}_{2(r+2)}$ of the matrix G as follows:

$$\begin{aligned} &\mathcal{S}_{-1}=6, \\ &\mathcal{S}_p=\sum_{q=0}^{r}a_{pq}=6\hbar^2a_2(\varsigma_k)+6\hbar^2a_4(\varsigma_k), p=0,\cdots,r+1, \\ &\mathcal{S}_{r+1}=6, \\ &\mathcal{S}_{r+2}=6, \\ &\mathcal{S}_p'=\sum_{q=r+3}^{2r+3}a_{pq}=6\hbar^2b_2(\varsigma_k)+6\hbar^2b_4(\varsigma_k), p=r+3,\cdots,2r+3, \\ &\mathcal{S}_{2r+4}=6. \end{aligned} \tag{64}$$

Thus, if a_{kq} indicates the $(k,q)^{\text{th}}$ element of the matrix G, then we can write

$$\mathcal{S}_k=\sum_{q=-1}^{2r+4}a_{kq}, k=-1,\cdots,2r+4. \tag{65}$$

Let a_{pk}^{-1} indicates the $(k,q)^{\text{th}}$ element of G^{-1}. Then, the matrix norms are defined as

$$\|G^{-1}\|=\max_{-1\leq p\leq 2r+4}\sum_{k=-1}^{2r+4}|a_{pk}^{-1}|. \tag{66}$$

So, we have

$$I=G^{-1}G=\sum_{k=-1}^{2r+4}a_{pk}^{-1}a_{kq}, p=-1,\cdots,2r+4, q=-1,\cdots,2r+4, \tag{67}$$

and $\|G^{-1}G\|=1$ which gives also

$$\begin{aligned} &\sum_{q=-1}^{2r+4}\sum_{k=-1}^{2r+4}a_{pk}^{-1}a_{kq}=1, p=-1,\cdots,2r+4, \\ &\sum_{k=-1}^{2r+4}a_{pk}^{-1}\left(\sum_{q=-1}^{2r+4}a_{kq}\right)=1, p=-1,\cdots,2r+4, \\ &\sum_{k=-1}^{2r+4}a_{pk}^{-1}\mathcal{S}_k=1, p=-1,\cdots,2r+4. \end{aligned} \tag{68}$$

Let $\mathcal{S}^*_{\hbar} = \min \mathcal{S}_{\hbar}$. Then we get

$$\sum_{\hbar=-1}^{2r+4} a_{p\hbar}^{-1} \leq \frac{1}{\mathcal{S}^*_{\hbar}}, \tag{69}$$

where $\mathcal{S}^*_{\hbar} = 6\hbar^2 \min\left(a_2(\varsigma_{\hbar}) + a_4(\varsigma_{\hbar}), b_2(\varsigma_{\hbar}) + b_4(\varsigma_{\hbar})\right) = 6\hbar^2 M$. Thus

$$\|B - B^*\| = \left\|G^{-1}\right\| \|(Q - Q^*\| \leq \hbar^2 \widehat{M}. \tag{70}$$

Using the definition of cubic BS basis functions in (7), one can obtain that

$$\sum_{\hbar=-1}^{r+1} \left|B_{\hbar,3}(\varsigma)\right| \leq \frac{5}{3}, a \leq \varsigma \leq b, \tag{71}$$

$$\begin{aligned} &\left\|\left(\mathcal{S}_1(\varsigma) - \widehat{\Psi}(\varsigma), \mathcal{S}_2(\varsigma) - \widehat{\Phi}(\varsigma)\right)\right\|_{\infty} \\ &= \left\|\left(\sum_{\hbar=-1}^{r+1} \widehat{\mu}_{\hbar} B_{\hbar,3}(\varsigma) - \sum_{\hbar=-1}^{r+1} \mu_{\hbar} B_{\hbar,3}(\varsigma), \sum_{\hbar=-1}^{r+1} \widehat{\nu}_{\hbar} B_{\hbar,3}(\varsigma) - \sum_{\hbar=-1}^{r+1} \nu_{\hbar} B_{\hbar,3}(\varsigma)\right)\right\| \\ &= \|(\widehat{\mu}_{\hbar} - \mu_{\hbar}, \widehat{\nu}_{\hbar} - \nu_{\hbar})\| \sum_{\hbar=-1}^{r+1} \left|B_{\hbar,3}(\varsigma)\right| \leq \frac{5}{3}\hbar^2 \widehat{M}. \end{aligned} \tag{72}$$

Hence,

$$\begin{aligned} &\left\|(\mathcal{S}_1(\varsigma), \mathcal{S}_1(\varsigma)) - \left(\widehat{\Psi}(\varsigma), \widehat{\Phi}(\varsigma)\right)\right\|_{\infty} \\ &= \left\|\left(\mathcal{S}_1(\varsigma) - \widehat{\Psi}(\varsigma), \mathcal{S}_2(\varsigma) - \widehat{\Phi}(\varsigma)\right)\right\|_{\infty} \\ &\leq \frac{5}{3}\hbar^2 \widehat{M}. \end{aligned} \tag{73}$$

Thus, $\|(\Psi(\varsigma), \Phi(\varsigma)) - (\mathcal{S}_1(\varsigma), \mathcal{S}_1(\varsigma))\|_{\infty} \leq \gamma_1 \hbar^4$ and $\|(\Psi(\varsigma), \Phi(\varsigma)) - (\widehat{\Psi}(\varsigma), \widehat{\Phi}(\varsigma))\|_{\infty} \leq w\hbar^2$. □

5. Application and Numerical Simulation

To highlight the importance and strength of what we presented in terms of analysis and mathematical construction concerning the CBSA, we need to discuss several practical examples, and this is what we will present in this special part.

Hither, $\widehat{\Psi}(\varsigma_{\hbar}), \widehat{\Phi}(\varsigma_{\hbar})$ will approximate $\Psi(\varsigma_{\hbar}), \Phi(\varsigma_{\hbar})$, respectively. Indeed, $\mathcal{A}_{\Psi}(\varsigma_{\hbar}) = |\Psi - \widehat{\Psi}|(\varsigma_{\hbar})$ and $\mathcal{A}_{\Phi}(\varsigma_{\hbar}) = |\Phi - \widehat{\Phi}|(\varsigma_{\hbar})$ denote the absolute errors, whilst $\mathcal{R}_{\Psi}(\varsigma_{\hbar}) = |\Psi - \widehat{\Psi}||\Psi|^{-1}(\varsigma_{\hbar})$ and $\mathcal{R}_{\Phi}(\varsigma_{\hbar}) = |\Phi - \widehat{\Phi}||\Phi|^{-1}(\varsigma_{\hbar})$ denote the relative errors.

Example 2. We test the following conformable linear system:

$$\begin{aligned} &T^{4/3}\Psi(\varsigma) - 3\varsigma^3 T^{1/2}\Psi(\varsigma) + T^{5/4}\Phi(\varsigma) \\ &\quad + T^{1/3}\Phi(\varsigma) + (1+\varsigma)\Phi(\varsigma) = \mathcal{F}_1(\varsigma), \\ &T^{5/4}\Phi(\varsigma) + \cosh(\varsigma) T^{1/3}\Phi(\varsigma) + \frac{\varsigma^3}{\varsigma^2(1-\varsigma)+1}\Phi(\varsigma) + T^{4/3}\Psi(\varsigma) \\ &\quad + T^{1/2}\Psi(\varsigma) + \left(2\varsigma^2 - 3\varsigma\right)\Psi(\varsigma) = \mathcal{F}_2(\varsigma), \end{aligned} \tag{74}$$

$$\begin{aligned} \mathcal{F}_1(\varsigma) &= -2\varsigma^{2/3} - 2\varsigma^{3/4} - 2\varsigma^{5/3} + 6\varsigma^{7/4} \\ &\quad - \varsigma^2 + 3\varsigma^{8/3} - 3\varsigma^{7/2} + \varsigma^4 + 6\varsigma^{9/2}, \\ \mathcal{F}_2(\varsigma) &= (1-2\varsigma)\sqrt{\varsigma} - \varsigma^{2/3} + 2\varsigma^{3/4}(-1+3\varsigma) \\ &\quad + \varsigma^3\left(3 - 5\varsigma + 2\varsigma^2\right) - \frac{(-1+\varsigma)\varsigma^4}{1+\varsigma^2-\varsigma^3} \\ &\quad + \varsigma^{5/3}(-2+3\varsigma)\cosh(\varsigma), \end{aligned} \tag{75}$$

concerning the BCs

$$\begin{aligned} &\Psi(0.5) = 0.25, \Phi(0.5) = -0.125, \\ &\Psi(1) = \Phi(1) = 0. \end{aligned} \tag{76}$$

Herein, the exact solutions are

$$\begin{aligned} &\Psi(\varsigma) = \varsigma(1-\varsigma), \\ &\Phi(\varsigma) = \varsigma^2(\varsigma - 1). \end{aligned} \tag{77}$$

Concerning Example 2 and using CBSA, the related numerical solutions for $r = 10$ are displayed in Table 1. Additionally, the graphics of $\mathcal{A}_{\Psi}(\varsigma_{\hbar})$ and $\mathcal{A}_{\Phi}(\varsigma_{\hbar})$ for $r = 10$ are given in Figure 1. Hither, it can be observed from the figure and table that the result data is sufficient accuracy and are firmly connected.

Example 3. We test the following conformable nonlinear system:

$$\begin{aligned} &T^{3/2}\Psi(\varsigma) + T^{1/4}\Psi(\varsigma) + T^{3/2}\Psi(\varsigma)\Psi(\varsigma) \\ &\quad + \exp(\Phi(\varsigma)) + \cos(\varsigma)\Phi(\varsigma) = \mathcal{F}_1(\varsigma), \\ &T^{3/2}\Phi(\varsigma) + T^{1/4}\Phi(\varsigma) + T^{1/2}\Psi(\varsigma) T^{1/4}\Phi(\varsigma) \\ &\quad + \ln(\Psi(\varsigma)) + 2\varsigma\Phi(\varsigma) = \mathcal{F}_2(\varsigma), \end{aligned} \tag{78}$$

$$\begin{aligned} \mathcal{F}_1(\varsigma) &= e^{-2\varsigma^2}\left[e^{2\varsigma^2}\left(\left(1 + \varsigma^2 + \cos(\varsigma)\ln\left(1+\varsigma^2\right)\right)\right)\right. \\ &\quad + 2\sqrt{\varsigma}\left(-1 + 2\varsigma^2\right) + 2\sqrt{\varsigma}e^{\varsigma^2}\left(-1 - \varsigma^{5/4} + 2\varsigma^2\right], \\ \mathcal{F}_2(\varsigma) &= \frac{1}{1+\varsigma^2}\left(-\frac{2\sqrt{\varsigma}\left(1-\varsigma^2\right)}{1+\varsigma^2} + 2\varsigma^{7/4} - 4e^{\varsigma^2}\varsigma^{13/4}\right) \\ &\quad + e^{-2\varsigma^2} + 2\varsigma\ln\left(1+\varsigma^2\right), \end{aligned} \tag{79}$$

TABLE 1: Solutions result with $\mathscr{r} = 10$ in Example 2.

$\varsigma_{\mathscr{k}}$	$\Psi(\varsigma_{\mathscr{k}})$	$\widehat{\Psi}(\varsigma_{\mathscr{k}})$	$\mathscr{A}_{\Psi}(\varsigma_{\mathscr{k}})$	$\Phi(\varsigma_{\mathscr{k}})$	$\widehat{\Phi}(\varsigma_{\mathscr{k}})$	$\mathscr{A}_{\Phi}(\varsigma_{\mathscr{k}})$
0.50	0.2500	0.2500	0	-0.125000	-0.125000	0
0.55	0.2475	0.2475	0	-0.136125	-0.136125	1.1102×10^{-16}
0.60	0.2400	0.2400	1.9429×10^{-16}	-0.144000	-0.144000	2.4980×10^{-16}
0.65	0.2275	0.2275	2.7756×10^{-17}	-0.147875	-0.147875	1.3045×10^{-15}
0.70	0.2100	0.2100	9.4369×10^{-16}	-0.147000	-0.147000	3.0531×10^{-16}
0.75	0.1875	0.1875	1.1102×10^{-16}	-0.140625	-0.140625	1.1657×10^{-15}
0.80	0.1600	0.1600	9.4369×10^{-16}	-0.128000	-0.128000	8.3267×10^{-17}
0.85	0.1275	0.1275	2.7756×10^{-17}	-0.108375	-0.108375	1.3739×10^{-15}
0.90	0.0900	0.0900	1.1935×10^{-15}	-0.081000	-0.081000	2.3592×10^{-16}
0.95	0.0475	0.0475	1.2490×10^{-16}	-0.045125	-0.045125	1.2212×10^{-15}
1.00	0	0	0	0	0	0

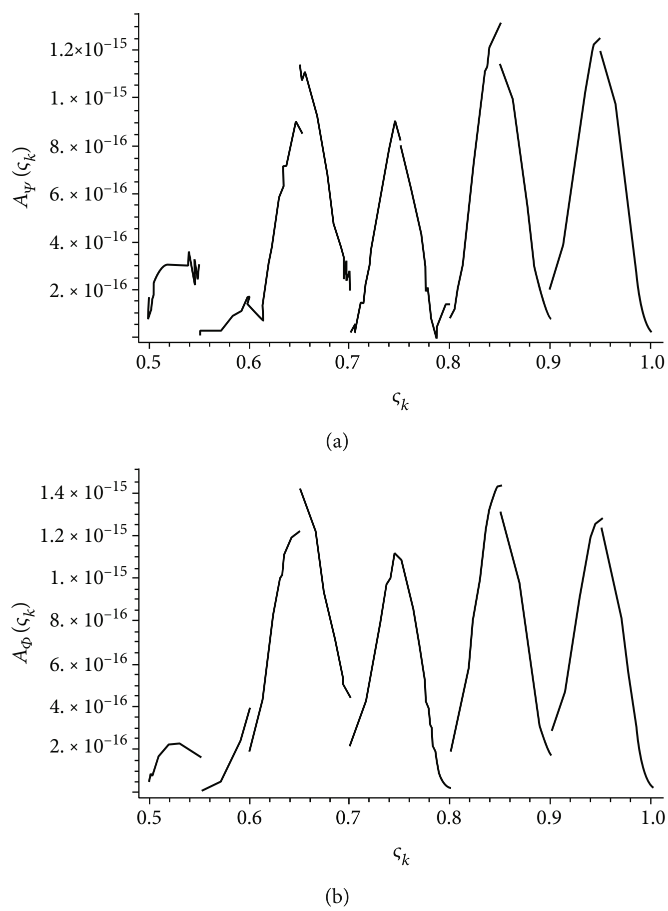


FIGURE 1: Graphical results with $\mathscr{r} = 10$ in Example 2: (a) $\mathscr{A}_{\Psi}(\varsigma_{\mathscr{k}})$ and (b) $\mathscr{A}_{\Phi}(\varsigma_{\mathscr{k}})$.

concerning the BCs

$$\Psi(1) = e^{-1}, \Phi(1) = \ln(2), \quad \Psi(2) = e^{-4}, \Phi(2) = \ln(5). \tag{80}$$

Herein, the exact solutions are

$$\Psi(\varsigma) = e^{-\varsigma^2}, \quad \Phi(\varsigma) = \ln\left(1 + \varsigma^2\right). \tag{81}$$

Table 2: Solutions result of $\mathscr{A}_{\Psi}(\varsigma_{k})$ with $r=\{10,20,40,80,160\}$ in Example 3.

ς_{k}	$r=10$	$r=20$	$r=40$	$r=80$	$r=160$
1	0	0	0	0	0
1.1	1.41417×10^{-4}	3.53949×10^{-5}	8.85131×10^{-6}	2.21299×10^{-6}	5.53257×10^{-7}
1.2	2.17765×10^{-4}	5.44737×10^{-5}	1.36205×10^{-5}	3.40526×10^{-6}	8.51323×10^{-7}
1.3	2.41486×10^{-4}	6.03663×10^{-5}	1.50913×10^{-5}	3.77281×10^{-6}	9.43202×10^{-7}
1.4	2.27054×10^{-4}	5.67117×10^{-5}	1.41747×10^{-5}	3.54349×10^{-6}	8.85859×10^{-7}
1.5	1.88993×10^{-4}	4.71574×10^{-5}	1.17837×10^{-5}	2.94558×10^{-6}	7.36374×10^{-7}
1.6	1.40251×10^{-4}	3.49512×10^{-5}	8.73088×10^{-6}	2.18229×10^{-6}	5.45546×10^{-7}
1.7	9.11547×10^{-5}	2.26784×10^{-5}	5.66277×10^{-6}	1.41527×10^{-6}	3.53789×10^{-7}
1.8	4.89733×10^{-5}	1.21551×10^{-5}	3.03330×10^{-6}	7.57986×10^{-7}	1.89475×10^{-7}
1.9	1.80044×10^{-5}	4.45136×10^{-6}	1.10975×10^{-6}	2.77246×10^{-7}	6.92995×10^{-8}
2	0	0	0	0	0

Concerning Example 3 and to show the compatibility between $(\Psi(\varsigma_{k}),\Phi(\varsigma_{k}))$ and $(\widehat{\Phi}(\varsigma_{k}),\widehat{\Psi}(\varsigma_{k}))$, the values of $\mathscr{A}_{\Psi}(\varsigma_{k})$ and $\mathscr{A}_{\Phi}(\varsigma_{k})$ are summarized in Tables 2 and 3, respectively, for $r=\{10,20,40,80,160\}$. Additionally, the graphics of $\mathscr{A}_{\Psi}(\varsigma_{k})$ and $\mathscr{A}_{\Phi}(\varsigma_{k})$ for $r=160$ are given in Figure 2. Whilst, the graphics of $(\Psi(\varsigma_{k}),\widehat{\Psi}(\varsigma_{k}))$ and $(\Phi(\varsigma_{k}),\widehat{\Phi}(\varsigma_{k}))$ for $r\in\{20,40\}$ are given in Figure 3. Again, it can be observed from the figure and table that the result data is sufficient accuracy and firmly connected.

Example 4. We test the following linear conformable LEP system:

$$T^{4/3}\Psi(\varsigma)+\frac{2}{\varsigma}T^{1/2}\Psi(\varsigma)+\varsigma\Phi(\varsigma)+e^{\varsigma}\Psi(\varsigma)=\mathscr{F}_1(\varsigma),$$
$$T^{4/3}\Phi(\varsigma)+\frac{1}{\varsigma}T^{1/2}\Phi(\varsigma)+2\sin(\varsigma)\Phi(\varsigma)+2\varsigma\Psi(\varsigma)=\mathscr{F}_2(\varsigma), \quad (82)$$

$$\mathscr{F}_1(\varsigma)=e^{\varsigma}+\varsigma-\varsigma^2+\varsigma^3+2\pi\left(1+2\varsigma^{1/6}\right)\cos(\pi\varsigma)+\left(4\sqrt{\varsigma}+2\varsigma^{2/3}+\varsigma^2e^{\varsigma}-\pi^2\varsigma^{8/3}\right)\sin(\pi\varsigma),$$
$$\mathscr{F}_2(\varsigma)=2\varsigma^{5/6}-\varsigma^{-1/4}(1+2\varsigma)+2\left(1-\varsigma+\varsigma^2\right)\sin(\varsigma)+2\varsigma\left(1+\varsigma^2\right)\sin(\pi\varsigma), \quad (83)$$

concerning the BCs

$$\Psi(0)=\Phi(0)=1,\quad \Psi(1)=\Phi(1)=1. \quad (84)$$

Herein, the exact solutions are

$$\Psi(\varsigma)=\varsigma^2\sin(\pi\varsigma)+1,\quad \Phi(\varsigma)=\varsigma^2-\varsigma+1. \quad (85)$$

Concerning Example 4 and to show the compatibility between $(\Psi(\varsigma_{k}),\Phi(\varsigma_{k}))$ and $(\widehat{\Phi}(\varsigma_{k}),\widehat{\Psi}(\varsigma_{k}))$, the values of $(\mathscr{A}_{\Psi}(\varsigma_{k}),\mathscr{R}_{\Psi}(\varsigma_{k}))$ and $(\mathscr{A}_{\Phi}(\varsigma_{k}),\mathscr{R}_{\Phi}(\varsigma_{k}))$ are summarized together in Table 4 for $r=60$. Whilst the graphics of $(\mathscr{A}_{\Psi}(\varsigma_{k}),\mathscr{R}_{\Psi}(\varsigma_{k}))$ and $(\mathscr{A}_{\Phi}(\varsigma_{k}),\mathscr{R}_{\Phi}(\varsigma_{k}))$ for $r=60$ are given in Figure 4. Indeed, the graphics of $(\Psi(\varsigma_{k}),\widehat{\Psi}(\varsigma_{k}))$ and $(\Phi(\varsigma_{k}),\widehat{\Phi}(\varsigma_{k}))$ for $r\in\{60,80\}$ are given in Figure 5. Hither, it can be observed from the figure and table that the result data is sufficient accuracy and are firmly connected.

Example 5. We test the following nonlinear singular LEP system:

$$T^{5/4}\Psi(\varsigma)+\frac{2}{\varsigma}T^{1/5}\Psi(\varsigma)+\sinh(\varsigma)\Phi^2(\varsigma)+\frac{\Psi(\varsigma)}{\Psi^2(\varsigma)+1}=\mathscr{F}_1(\varsigma),$$
$$T^{5/4}\Phi(\varsigma)-\frac{1}{\varsigma}T^{1/5}\Phi(\varsigma)+2\cos(\Psi(\varsigma))=\mathscr{F}_2(\varsigma), \quad (86)$$

$$\mathscr{F}_1(\varsigma)=\varsigma^{3/4}(-3+6\varsigma)+\varsigma^{-1/5}\left(1-6\varsigma+6\varsigma^2\right)+\frac{2+\varsigma-3\varsigma^2+2\varsigma^3}{2+0.5\left(2+\varsigma-3\varsigma^2+2\varsigma^3\right)^2}+\cos^2(\varsigma)\sinh(\varsigma),$$
$$\mathscr{F}_2(\varsigma)=-\varsigma^{3/4}-\cos\left(0.5\left(2+\varsigma-3\varsigma^2+2\varsigma^3\right)\right)+\varsigma^{-1/5}\sin(\varsigma), \quad (87)$$

concerning the BCs

$$\Psi(0)=\Phi(0)=1,\quad \Psi(1)=1,\Phi(1)=\cos(1). \quad (88)$$

Herein, the exact solutions are

$$\Psi(\varsigma)=\varsigma^3-1.5\varsigma^2+0.5\varsigma+1,\quad \Phi(\varsigma)=\cos(\varsigma). \quad (89)$$

Table 3: Solutions result of $\mathscr{A}_{\Phi}(\varsigma_{k})$ with $r = \{10,20,40,80,160\}$ in Example 3.

ς_{k}	$r = 10$	$r = 20$	$r = 40$	$r = 80$	$r = 160$
1	0	0	0	0	0
1.1	3.76804×10^{-7}	1.68330×10^{-8}	6.23761×10^{-10}	4.57952×10^{-10}	1.33368×10^{-16}
1.2	3.49244×10^{-6}	1.00246×10^{-6}	2.58702×10^{-7}	6.51818×10^{-8}	1.63269×10^{-8}
1.3	9.15243×10^{-6}	2.44919×10^{-6}	6.22382×10^{-7}	1.56226×10^{-7}	3.90960×10^{-8}
1.4	1.47071×10^{-5}	3.85284×10^{-6}	9.74249×10^{-7}	2.44253×10^{-7}	6.11065×10^{-8}
1.5	1.86288×10^{-5}	4.83271×10^{-6}	1.21920×10^{-6}	3.05490×10^{-7}	7.64158×10^{-8}
1.6	1.97846×10^{-5}	5.10561×10^{-6}	1.28643×10^{-6}	3.22237×10^{-7}	8.05984×10^{-8}
1.7	1.76475×10^{-5}	4.54048×10^{-6}	1.14322×10^{-6}	2.86312×10^{-7}	7.16099×10^{-8}
1.8	1.25707×10^{-5}	3.22885×10^{-6}	8.12645×10^{-7}	2.03502×10^{-7}	5.08967×10^{-8}
1.9	5.94475×10^{-6}	1.52547×10^{-6}	3.83849×10^{-7}	9.61176×10^{-8}	2.40391×10^{-8}
2	0	0	0	0	0

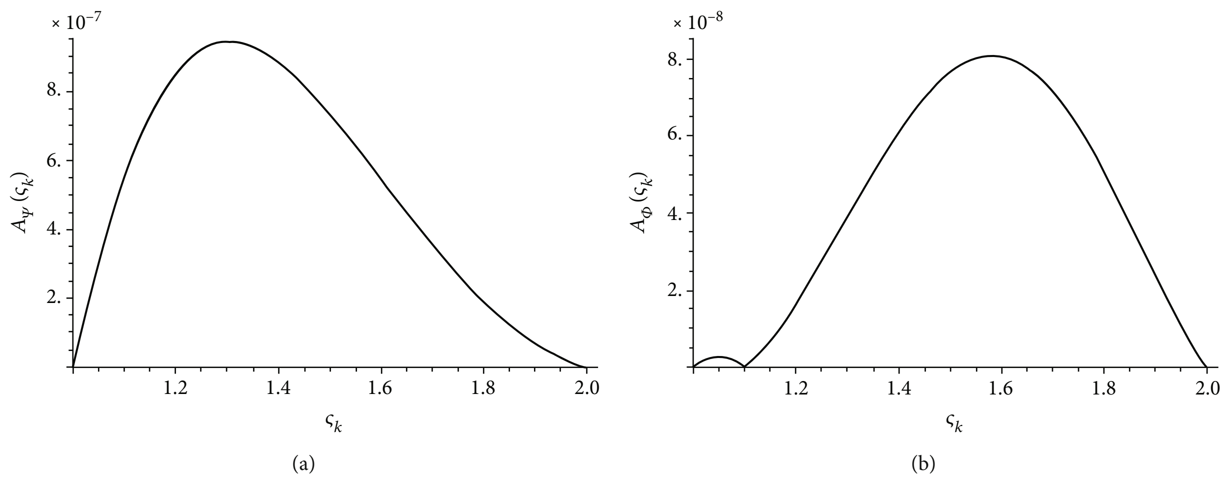


Figure 2: Graphical results with $r = 160$ in Example 3: (a) $\mathscr{A}_{\Psi}(\varsigma_{k})$ and (b) $\mathscr{A}_{\Phi}(\varsigma_{k})$.

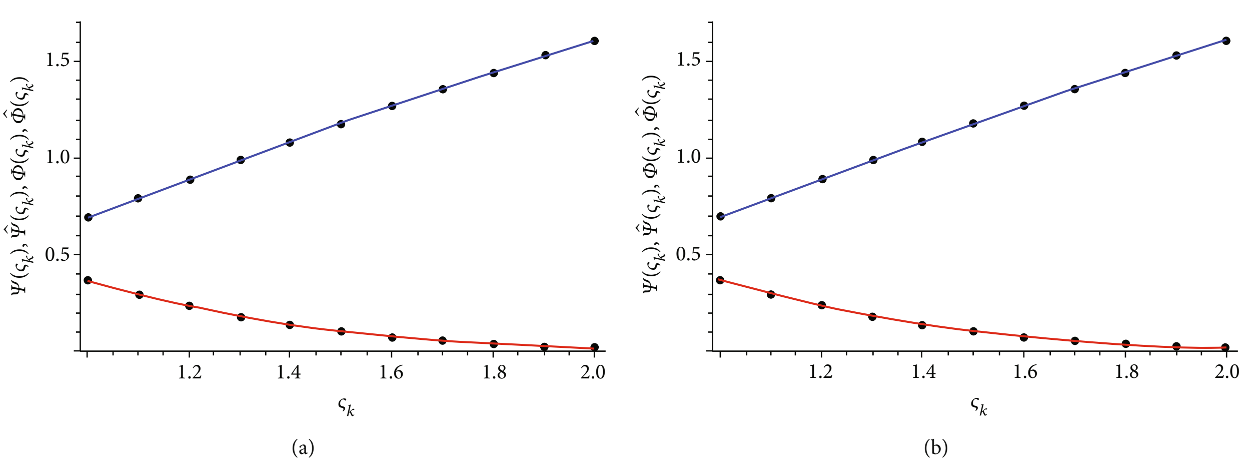
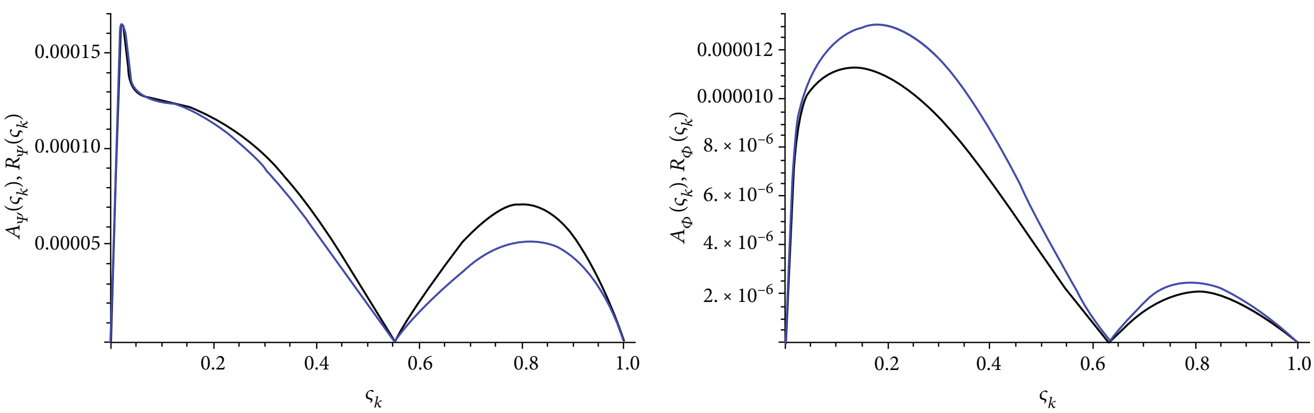


Figure 3: Graphical results of $(\Psi(\varsigma_{k}), \widehat{\Psi}(\varsigma_{k}))$ and $(\Phi(\varsigma_{k}), \widehat{\Phi}(\varsigma_{k}))$ in Example 3 as red: Ψ, blue: Φ, black stars: $(\widehat{\Psi}, \widehat{\Phi})$: (a) $r = 10$ and (b) $r = 40$.

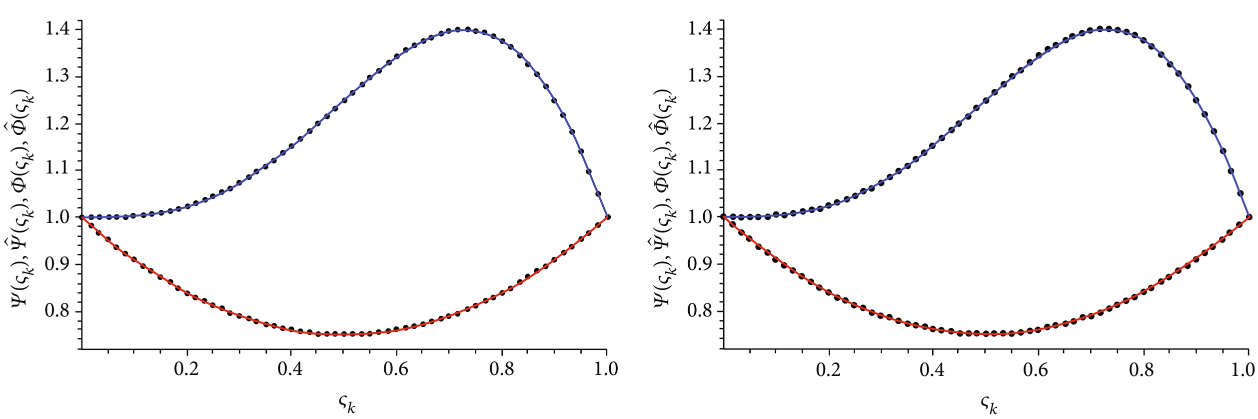
Table 4: Solutions result with $r = 60$ in Example 4.

ς_{k}	$\widehat{\Psi}(\varsigma_{k})$	$\mathscr{A}_{\Psi}(\varsigma_{k})$	$\mathscr{R}_{\Psi}(\varsigma_{k})$	$\widehat{\Phi}(\varsigma_{k})$	$\mathscr{A}_{\Phi}(\varsigma_{k})$	$\mathscr{R}_{\Phi}(\varsigma_{k})$
0	0	0	0	0	0	0
0.1	1.00322	1.257708×10^{-4}	1.253833×10^{-4}	0.910011	1.123178×10^{-5}	1.234262×10^{-5}
0.2	1.02363	1.163980×10^{-4}	1.137241×10^{-4}	0.840011	1.091802×10^{-5}	1.299764×10^{-5}
0.3	1.07291	9.657157×10^{-5}	9.001727×10^{-5}	0.790009	9.231926×10^{-6}	1.168598×10^{-5}
0.4	1.15223	6.463408×10^{-5}	5.609774×10^{-5}	0.760007	6.638852×10^{-6}	8.735332×10^{-6}
0.5	1.25002	2.332187×10^{-5}	1.865750×10^{-5}	0.750004	3.618730×10^{-6}	4.824973×10^{-6}
0.6	1.34238	2.035114×10^{-5}	1.516049×10^{-5}	0.760001	7.687164×10^{-7}	1.011469×10^{-6}
0.7	1.39636	5.596469×10^{-5}	4.007731×10^{-5}	0.789999	1.289339×10^{-6}	1.632074×10^{-6}
0.8	1.37611	7.143946×10^{-5}	5.191132×10^{-5}	0.839998	2.083354×10^{-6}	2.480184×10^{-6}
0.9	1.25025	5.543125×10^{-5}	4.433423×10^{-5}	1.25025	1.502923×10^{-6}	1.651564×10^{-6}
1	1	0	0	1	0	0



(a) (b)

Figure 4: Graphical results with $r = 60$ in Example 4 as black: $\mathscr{A}$ and blue: $\mathscr{R}$: (a) $(\mathscr{A}_{\Psi}(\varsigma_{k}), \mathscr{R}_{\Psi}(\varsigma_{k}))$ and (b) $(\mathscr{A}_{\Phi}(\varsigma_{k}), \mathscr{R}_{\Phi}(\varsigma_{k}))$.



(a) (b)

Figure 5: Graphical results of $(\Psi(\varsigma_{k}), \widehat{\Psi}(\varsigma_{k}))$ and $(\Phi(\varsigma_{k}), \widehat{\Phi}(\varsigma_{k}))$ in Example 4 as blue: Ψ, red: Φ, and black stars: $(\widehat{\Psi}, \widehat{\Phi})$: (a) $r = 60$, and (b) $r = 80$.

Table 5: Solutions result with $r = 60$ in Example 5.

ς_{k}	$\widehat{\Psi}(\varsigma_{k})$	$\mathscr{A}_{\Psi}(\varsigma_{k})$	$\mathscr{R}_{\Psi}(\varsigma_{k})$	$\widehat{\Phi}(\varsigma_{k})$	$\mathscr{A}_{\Phi}(\varsigma_{k})$	$\mathscr{R}_{\Phi}(\varsigma_{k})$
0	1	0	0	1	0	0
0.1	1.036	4.687946×10^{-7}	4.525045×10^{-7}	0.995004	4.007344×10^{-7}	4.027465×10^{-7}
0.2	1.048	4.781665×10^{-7}	4.562657×10^{-7}	0.980066	1.038737×10^{-6}	1.059864×10^{-7}
0.3	1.042	4.717943×10^{-7}	4.527776×10^{-7}	0.955335	1.677880×10^{-6}	1.756324×10^{-6}
0.4	1.024	4.499979×10^{-7}	4.394512×10^{-7}	0.921059	2.198900×10^{-6}	2.387355×10^{-6}
0.5	1.000	4.099687×10^{-7}	4.099687×10^{-7}	0.877580	2.526306×10^{-6}	2.878711×10^{-6}
0.6	0.976	3.505588×10^{-7}	3.591791×10^{-7}	0.825333	2.609380×10^{-6}	3.161598×10^{-6}
0.7	0.958	2.734979×10^{-7}	2.854884×10^{-7}	0.764840	2.414309×10^{-6}	3.156611×10^{-6}
0.8	0.952	1.837305×10^{-7}	1.929942×10^{-7}	0.696705	1.920084×10^{-6}	2.755944×10^{-6}
0.9	0.964	8.918082×10^{-8}	9.251122×10^{-8}	0.621609	1.116007×10^{-6}	1.795349×10^{-7}
1	1	0	0	0.540302	0	0

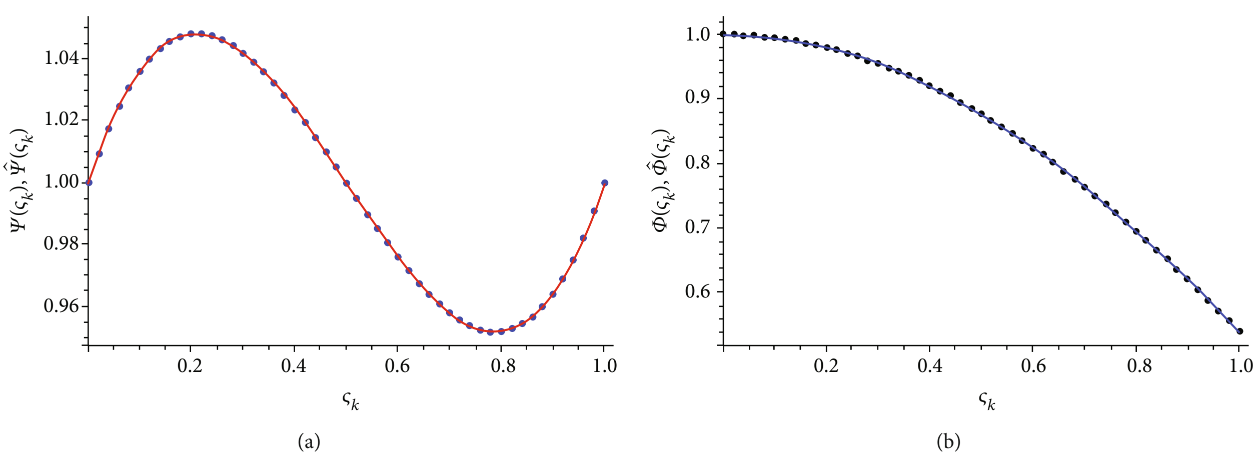


Figure 6: Graphical results with $r = 60$ in Example 5 as red: Ψ, blue: Φ, and black stars: $(\widehat{\Psi}, \widehat{\Phi})$: (a) $(\Psi(\varsigma_{k}), \widehat{\Psi}(\varsigma_{k}))$ and (b) $(\Phi(\varsigma_{k}), \widehat{\Phi}(\varsigma_{k}))$.

Concerning Example 5 and to show the compatibility between $(\Psi(\varsigma_{k}), \Phi(\varsigma_{k}))$ and $(\widehat{\Phi}(\varsigma_{k}), \widehat{\Psi}(\varsigma_{k}))$, the values of $(\mathscr{A}_{\Psi}(\varsigma_{k}), \mathscr{R}_{\Psi}(\varsigma_{k}))$ and $(\mathscr{A}_{\Phi}(\varsigma_{k}), \mathscr{R}_{\Phi}(\varsigma_{k}))$ are summarized together in Table 5 for $r = 60$. Indeed, the graphics of $(\Psi(\varsigma_{k}), \widehat{\Psi}(\varsigma_{k}))$ and $(\Phi(\varsigma_{k}), \widehat{\Phi}(\varsigma_{k}))$ for $r = 60$ are given in Figure 6. Hither, it can be observed from the figure and table that the result data is sufficient accuracy and are firmly connected.

6. Summary and Future Suggestions

Throughout this study, the CBSA is used to get soft and fineness approximations of BVPs for conformable systems concerning two points and two fractional parameters in both regular and singular types. Several linear and nonlinear examples will be examined, and a model for the Lane-Emden will be one of the applications presented. The complete construction of the used spline through the CD along with the convergence theory, and the error orders together with other results are utilized in detail in the form of tables and graphs using Mathematica 11 software. From the reported results, it can be concluded that CBSA is a very effective scheme that obtains numerical approximations to conformable systems of BVPs. The main characteristics noted here are that the spline approach is effective and fast, and it requires little compulsive and mathematical burden in solving the problems presented. In the coming work, we will apply the CBSA to solve the Lotka-Volterra model despite CD.

Abbreviations

CD: Conformable derivative
BVP: Boundary value problem
CBSA: Cubic B-spline algorithm
FDP: Fractional differential problem
BS: B-spline
LEP: Lane-Emden problem
BC: Boundary condition.

Data Availability

No datasets are associated with this manuscript. The datasets used for generating the plots and results during the current study can be directly obtained from the numerical simulation of the related mathematical equations in the manuscript.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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