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Research Article

Multiple Solutions of a p-th Yamabe Equation on Graph

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Let $G=(V,E)$ be a connected finite graph and Δ_p be the p-Laplacian on G with $p>1$. We consider a perturbed p-th Yamabe equation $-\Delta_p u-\lambda|u|^{p-2}u=h|u|^{\alpha-2}u+\varepsilon f$, where $h,f:V\longrightarrow\mathbb{R}$ are functions with $h,f>0$; $1<p<\alpha$; λ and ε are two positive constants. Using the variational method, we prove that there exists some positive constant ϵ_1 such that for all $\epsilon\in(0,\epsilon_1)$, the above equation has two distinct solutions.

1. Introduction and Main Results

Let $G=(V,E)$ be a finite graph, where V denotes the vertex set and E denotes the edge set. Let $\mu:V\longrightarrow\mathbb{R}^+$ be a finite measure and $w:E\longrightarrow\mathbb{R}^+$ be the weight of an edge. The graph G satisfies the following properties.

(a) For any edge $ij\in E$, $w_{ij}>0$ and $w_{ij}=w_{ji}$ (symmetric)

(b) For any $i\in V$, there are only finite $j\in V$ such that i $j\in E$ (locally finite)

(c) For any $i,j\in V$, there exist finite edges connecting i and j (connected)

(d) There exists a constant $\mu_{\min}>0$ such that $\mu_i\geq\mu_{\min}$ for all $i\in V$ (uniformly positive measure)

(e) The distance d_{ij} of two vertices $i,j\in V$ is defined by the minimal number of edges which connect these two vertices. For a subset Ω of V, the distance d_{ij} is uniformly bounded from above for any $i,j\in\Omega$ (bounded domain)

To do various analysis works, some reasonable assumptions are made about the graph, which results in different prominent features of the graph in different contexts. For example, some similarities and differences in feature between the metric graph and the graph mentioned above can be found. The reader may refer to [1–3] and the references therein for more details.

For any function $u:V\longrightarrow\mathbb{R}$, $p>1$, the p-Laplacian of u is defined as

$$\Delta_p u_i=\frac{1}{\mu_i}\sum_{j\sim i}w_{ij}\left|u_j-u_i\right|^{p-2}\left(u_j-u_i\right), \tag{1}$$

where $j\sim i$ denotes $ij\in E$. Δ_p is a nonlinear operator when $p\neq 2$.

In the case of $p=2$, Grigor'yan et al. used the mountain-pass theorem to establish the existence results for the Yamabe equation [4] and the Schrödinger equation [5] on graphs. They also used a direct method of variation and the method of upper and lower solutions to study the existence of solutions for the Kazdan-Warner equation [6] on graphs. Later, Keller and Schwarz [7] studied the Kazdan-Warner equation on canonically compact graphs. Zhang and Zhao [8] studied the convergence of ground state solutions for a nonlinear Schrödinger equation on graphs. In the case of $p>1$, Ge [9] studied the existence of solutions for the p-th Yamabe equation on graphs. One may refer to [10–16] for more related works.

In this paper, we consider the multiplicity of solutions to a p-th Yamabe equation on a graph. For any function $u:V\longrightarrow\mathbb{R}$, the integral of u over V with respect to the vertex weight μ is defined by

$$\int_V u d\mu = \sum_{i\in V} \mu_i u_i. \tag{2}$$

Set

$$\int_V d\mu = \mathrm{Vol}(G). \tag{3}$$

For any function g defined on the edge set E, the integral of g over E with respect to the edge weight w is defined by

$$\int_E g dw = \sum_{i\sim j} w_{ij} g_{ij}. \tag{4}$$

For any function $u : V \longrightarrow \mathbb{R}$,

$$\int_E |\nabla u|^p dw = \sum_{i\sim j} w_{ij} |u_j - u_i|^p, \tag{5}$$

where $|\nabla u|$ is defined on the edge set E, and $|\nabla u|_{ij} = |u_j - u_i|$ for each edge $i \sim j$.

Define

$$\lambda_1 = \inf_{u\not\equiv 0} \frac{\int_E |\nabla u|^p dw + \int_V |u|^p d\mu}{2\int_V |u|^p d\mu}. \tag{6}$$

The well-known Yamabe equation [17, 18]

$$\Delta u + gu = khu^{N-1} \tag{7}$$

derives from the Yamabe problem: Given a compact Riemannian manifold (M, l) of dimension $n \geq 3$, find a metric conformal to l with constant scalar curvature, which is to prove that there is a real number k and a function $u > 0$ satisfying the above Yamabe equation, where g, h are functions on M with $h > 0$ and $N = 2n/(n-2)$.

In [9], Ge studied the following p-th discrete Yamabe equation

$$\Delta_p u + gu^{p-1} = khu^{\alpha-1} \tag{8}$$

on a finite graph G, where g, h are functions with $h > 0$; $\alpha \geq p > 1$. Using a direct method of variation, the author showed that (8) always has a positive solution for some constant k. We consider the following p-th Yamabe equation

$$-\Delta_p u - \lambda |u|^{p-2} u = h|u|^{\alpha-2} u + \epsilon f, \tag{9}$$

where $h, f : V \longrightarrow \mathbb{R}$ are functions with $h, f > 0$; $1 < p < \alpha$; λ and ε are positive constants. Note that, in equation (9), we add a perturbed term ϵf. In order to make our derivation possible, we have to set $g \equiv \lambda$. By using the mountain-pass theorem, which is due to Ambrosetti and Rabinowitz [19], and a direct method of variation, we prove that (9) has two distinct solutions. Now, we can state the theorem as follows.

Theorem 1. *Let $G = (V, E)$ be a finite graph and $h, f : V \longrightarrow \mathbb{R}$ be functions with $h, f > 0$. Assume that $1 < p < \alpha$, $1 < \lambda < \lambda_1$. Then, there exists $\epsilon_1 > 0$ such that for any $\epsilon \in (0, \epsilon_1)$, (9) has two distinct solutions.*

In case $p = 2$, we have the following result.

Corollary 2. *Let $G = (V, E)$ be a finite graph and $h, f : V \longrightarrow \mathbb{R}$ be functions with $h, f > 0$. Assume that $\alpha > 2$, $1 < \lambda < \lambda_1$. Then, there exists $\epsilon_1 > 0$ such that for any $\epsilon \in (0, \epsilon_1)$, the following Yamabe equation*

$$-\Delta u - \lambda u = h|u|^{\alpha-2} u + \epsilon f \tag{10}$$

has two distinct solutions.

The multiplicity of solutions to certain equations on a graph was extensively studied by Grigor'yan et al. [5], Liu and Yang [12], Huang et al. [20], and Liu [21]. More results have been obtained in the Euclidean space; we refer the reader to [22–27] and the references therein.

2. Preliminaries

Define a Sobolev space and a norm on it by

$$W^{1,p}(G) = \left\{ u : V \longrightarrow \mathbb{R} \,\middle|\, \int_E |\nabla u|^p dw + \int_V |u|^p d\mu < +\infty \right\},$$
$$\|u\|_{W^{1,p}(G)} = \left(\int_E |\nabla u|^p dw + \int_V |u|^p d\mu \right)^{1/p}. \tag{11}$$

Since G is a finite graph, then $W^{1,p}(G)$ is exactly the set of all functions on V, a finite dimensional linear space. This implies the following Sobolev embedding.

Lemma 3 (Sobolev embedding theorem, see [4]). *Let $G = (V, E)$ be a finite graph and $p > 1$. Then, $W^{1,p}(G)$ is embedded in $L^q(G)$ for all $1 \leq q \leq +\infty$. In particular, there exists a constant $C_{p,G}$ depending only on p and G such that*

$$\|u\|_{L^q(G)} \leq C_{p,G} \|u\|_{W^{1,p}(G)}, \tag{12}$$

for all $1 \leq q \leq +\infty$ and for all $u \in W^{1,p}(G)$. Moreover, the Sobolev space $W^{1,p}(G)$ is precompact; namely, if $\{u_n\}$ is bounded in $W^{1,p}(G)$, then there exists some $u \in W^{1,p}(G)$ such that, up to a subsequence, $u_n \longrightarrow u$ in $W^{1,p}(G)$.

The functional related to (9) is

$$J_\epsilon(u) = \frac{1}{p}\int_E |\nabla u|^p dw - \frac{\lambda}{p}\int_V |u|^p d\mu - \frac{1}{\alpha}\int_V h|u|^\alpha d\mu - \epsilon \int_V f u d\mu. \tag{13}$$

The existence of solutions of (9) is transformed into finding the critical points of J_ϵ.

Let $(X, \|\cdot\|)$ be a Banach space; we say that $J : X \longrightarrow \mathbb{R}$ satisfies the $(PS)_c$ condition for some real number c if, for any sequence of functions $u_n : X \longrightarrow \mathbb{R}$ such that $J(u_n) \longrightarrow c$ and $J'(u_n) \longrightarrow 0$ for all $\phi \in W^{1,p}(G)$ as $n \longrightarrow +\infty$, there holds up to a subsequence $u_n \longrightarrow u$ in X. To prove Theorem 1, we need the following mountain-pass theorem.

Theorem 4 (mountain-pass theorem, see [19]). *Let $(X, \|\cdot\|)$ be a Banach space, $J \in C^1(X, \mathbb{R})$, $e \in X$, and $r > 0$ be such that $\|e\| > r$ and*

$$b := \inf_{\|u\|=r} J(u) > J(0) \geq J(e). \tag{14}$$

If J satisfies the $(PS)_c$ condition with $c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t))$, where

$$\Gamma := \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e\}, \tag{15}$$

then c is a critical value of J.

3. Proof of the Main Results

Lemma 5. *There exist positive constants r_ϵ and δ_ϵ such that $J_\epsilon(u) \geq \delta_\epsilon$ for all $u \in W^{1,p}(G)$ with $r_\epsilon/p \leq \|u\|_{W^{1,p}(G)} \leq r_\epsilon$ if $0 < \epsilon < \epsilon_1$ for a sufficiently small ϵ_1.*

Proof. Let $f_M = \max_{i \in V} f_i > 0$. For $p > 1$, by the Hölder inequality, we have

$$\begin{aligned} \int_V f u \, d\mu &\leq f_M \left(\int_V 1^{p/(p-1)} d\mu \right)^{(p-1)/p} \left(\int_V |u|^p d\mu \right)^{1/p} \\ &\leq f_M \mathrm{Vol}(G)^{(p-1)/p} \|u\|_{W^{1,p}(G)} \\ &= C_{f,p,G} \|u\|_{W^{1,p}(G)}, \end{aligned} \tag{16}$$

where $C_{f,p,G} > 0$ is a constant depending on f, p, G.

Let $h_M = \max_{i \in V} h_i > 0$. By Lemma 3, there exists some constant $C_{p,G}$ depending on p and G such that

$$\int_V h|u|^\alpha d\mu \leq h_M \|u\|^\alpha_{L^\alpha(G)} \leq C^\alpha_{p,G} h_M \|u\|^\alpha_{W^{1,p}(G)}. \tag{17}$$

From (16) and (17), and noting that $1 < \lambda < \lambda_1$, we have

$$\begin{aligned} J_\epsilon(u) &\geq \frac{1}{p} \int_E |\nabla u|^p dw - \frac{1}{p} \left(\frac{\lambda}{\lambda_1} \|u\|^p_{W^{1,p}(G)} - \int_V |u|^p d\mu \right) \\ &\quad - \frac{C^\alpha_{p,G} h_M}{\alpha} \|u\|^\alpha_{W^{1,p}(G)} - \epsilon C_{f,p,G} \|u\|_{W^{1,p}(G)} \\ &\geq \|u\|_{W^{1,p}(G)} \left(\frac{\tau}{p} \|u\|^{p-1}_{W^{1,p}(G)} - \frac{C^\alpha_{p,G} h_M}{\alpha} \|u\|^{\alpha-1}_{W^{1,p}(G)} - \epsilon C_{f,p,G} \right), \end{aligned} \tag{18}$$

where $\tau = (\lambda_1 - \lambda)/\lambda_1$. Let $r_\epsilon = \epsilon^{1/p}$, then $(1/p)\epsilon^{1/p} \leq \|u\|_{W^{1,p}(G)} \leq \epsilon^{1/p}$. For $1 < p < \alpha$, we have

$$\lim_{\epsilon \longrightarrow 0^+} \frac{(\tau/p^p)\epsilon^{(p-1)/p} - \left(C^\alpha_{p,G} h_M/\alpha \right) \epsilon^{(\alpha-1)/p} - \epsilon C_{f,p,G}}{(\tau/p^p)\epsilon^{(p-1)/p}} = 1. \tag{19}$$

Thus, there exists some sufficiently small ϵ_1 such that $0 < \epsilon < \epsilon_1$ and

$$\frac{\tau}{p^p} \epsilon^{(p-1)/p} - \frac{C^\alpha_{p,G} h_M}{\alpha} \epsilon^{(\alpha-1)/p} - \epsilon C_{f,p,G} \geq \frac{\tau}{2p^{p+1}} \epsilon^{(p-1)/p}. \tag{20}$$

Let $\delta_\epsilon = \tau\epsilon/2p^{p+2}$; we get $J_\epsilon(u) \geq \delta_\epsilon$ for $0 < \epsilon < \epsilon_1$. □

Lemma 6. *J_ϵ satisfies the $(PS)_c$ condition for any real number c.*

Proof. For any $c \in \mathbb{R}$, take $\{u_n\} \subset W^{1,p}(G)$ such that $J_\epsilon(u_n) \longrightarrow c$ and $J'_\epsilon(u_n)(\phi) \longrightarrow 0$ for all $\phi \in W^{1,p}(G)$ as $n \longrightarrow +\infty$. Namely,

$$\begin{aligned} &\frac{1}{p} \int_E |\nabla u_n|^p dw - \frac{\lambda}{p} \int_V |u_n|^p d\mu - \frac{1}{\alpha} \int_V h|u_n|^\alpha d\mu - \epsilon \int_V f u_n d\mu \\ &\quad = c + o_n(1), \end{aligned} \tag{21}$$

$$\begin{aligned} &\left| \int_V \left(-\Delta_p u_n - \lambda |u_n|^{p-2} u_n - h|u_n|^{\alpha-2} u_n - \epsilon f \right) \phi \, d\mu \right| \\ &\quad = o_n(1) \|\phi\|_{W^{1,p}(G)}, \end{aligned} \tag{22}$$

for all $\phi \in W^{1,p}(G)$.

Taking $\{u_n\}$ as the test function ϕ in (22), we have

$$\begin{aligned} &\int_E |\nabla u_n|^p dw - \lambda \int_V |u_n|^p d\mu - \int_V h|u_n|^\alpha d\mu - \epsilon \int_V f u_n d\mu \\ &\quad = o_n(1) \|u_n\|_{W^{1,p}(G)}. \end{aligned} \tag{23}$$

From (21) and (23), we obtain that

$$\begin{aligned} \frac{\alpha - p}{p\alpha} \int_E |\nabla u_n|^p dw &= \frac{\lambda(\alpha - p)}{p\alpha} \int_V |u_n|^p d\mu + \frac{\epsilon(\alpha - 1)}{\alpha} \int_V f u_n d\mu \\ &\quad + c + o_n(1) \|u_n\|_{W^{1,p}(G)} + o_n(1) \\ &\leq \frac{\alpha - p}{p\alpha} \left(\frac{\lambda}{\lambda_1} \|u_n\|^p_{W^{1,p}(G)} - \int_V |u_n|^p d\mu \right) \\ &\quad + \frac{\epsilon(\alpha - 1)}{\alpha} C_{f,p,G} \|u_n\|_{W^{1,p}(G)} + c + o_n(1), \end{aligned} \tag{24}$$

which implies that

$$\frac{\tau(\alpha - p)}{p\alpha} \|u_n\|^p_{W^{1,p}(G)} \leq \frac{\epsilon(\alpha - 1)}{\alpha} C_{f,p,G} \|u_n\|_{W^{1,p}(G)} + c + o_n(1). \tag{25}$$

Suppose that $\{u_n\}$ is unbounded in $W^{1,p}(G)$. For $1 < p < \alpha$, we have

$$\frac{\tau(\alpha-p)}{p\alpha}\|u_n\|^p_{W^{1,p}(G)} - \frac{\epsilon(\alpha-1)}{\alpha}C_{f,p,G}\|u_n\|_{W^{1,p}(G)} - c + o_n(1) \longrightarrow +\infty, \tag{26}$$

as $n \longrightarrow +\infty$, which contradicts (25). Hence, $\{u_n\}$ is bounded in $W^{1,p}(G)$.

Taking a function $u^* \in W^{1,p}(G)$ with $u^* \equiv 0$ and passing to the limit $t \longrightarrow +\infty$, we have

$$\begin{aligned} J_\epsilon(tu^*) = {} & \frac{t^p}{p}\int_E |\nabla u^*|^p dw - \frac{\lambda t^p}{p}\int_V |u^*|^p d\mu \\ & - \frac{t^\alpha}{\alpha}\int_V h|u^*|^\alpha d\mu - t\epsilon\int_V f u^* d\mu \longrightarrow -\infty. \end{aligned} \tag{27}$$

It is obvious that $J_\epsilon \in C^1(W^{1,p}(G), \|\cdot\|)$, $J_\epsilon(0) = 0$; $J_\epsilon(u) \geq \delta_\epsilon > 0$ with $\|u\|_{W^{1,p}(G)} = r_\epsilon/p$; $J_\epsilon(\tilde{u}) < 0$ for some $\tilde{u}$ with $\|\tilde{u}\|_{W^{1,p}(G)} > r_\epsilon/p$. Moreover, J_ϵ satisfies the $(PS)_c$ condition with $\bar{c} = \min_{\gamma\in\Gamma}\max_{t\in[0,1]} J_\epsilon(\gamma(t))$, where

$$\Gamma = \left\{\gamma \in C\left([0,1], W^{1,p}(G)\right) : \gamma(0) = 0, \gamma(1) = \tilde{u}\right\}, \tag{28}$$

and $\bar{c}$ is a critical value of $J_\epsilon(u)$. Thus, there exists a solution $\bar{u}$ in $W^{1,p}(G)$ such that $J_\epsilon(\bar{u}) = \bar{c} \geq \delta_\epsilon > 0$. □

Next, we prove that there exists another solution $\widehat{u}$ such that $J_\epsilon(\widehat{u}) = \widehat{c} < 0$, where $\widehat{c}$ is another critical value of $J_\epsilon(u)$.

Lemma 7. *There exist some ρ and $u \in W^{1,p}(G)$ with $\|u\|_{W^{1,p}(G)} = 1$ such that $J_\epsilon(tu) < 0$ if $0 < t < \rho$.*

Proof. Consider the equation

$$-\Delta_p u - \lambda|u|^{p-2}u = f, \tag{29}$$

in $W^{1,p}(G)$. Define the functional

$$J_f(u) = \frac{1}{p}\int_E |\nabla u|^p dw - \frac{\lambda}{p}\int_V |u|^p d\mu - \int_V f u d\mu. \tag{30}$$

Note that

$$\begin{aligned} J_f(u) & \geq \frac{1}{p}\|u\|^p_{W^{1,p}(G)} - \frac{\lambda}{p\lambda_1}\|u\|^p_{W^{1,p}(G)} - \frac{\eta}{p}\int_V |u|^p d\mu - C_{p,q,\eta,f} \\ & \geq \frac{\tau-\eta}{p}\|u\|^p_{W^{1,p}(G)} - C_{p,q,\eta,f}, \end{aligned} \tag{31}$$

where $1/p + 1/q = 1$; $\eta > 0$ is a sufficiently small constant; $C_{p,q,\eta,f}$ is a constant depending on p, q, η, f; and we use Young's inequality in the proof of the first inequality. Hence, J_f has a lower bound in $W^{1,p}(G)$ for a sufficiently small η. Let $m_f = \inf_{u\in W^{1,p}(G)} J_f(u)$ and taking a sequence $\{u_n\}$ satisfies $J_f(u_n) \longrightarrow m_f$ as $n \longrightarrow +\infty$. Moreover, $\{u_n\}$ is bounded in $W^{1,p}(G)$. By Lemma 3, there exists some $u_0 \in W^{1,p}(G)$ up to a subsequence $u_n \longrightarrow u_0$ in $W^{1,p}(G)$. Then,

$$J_f(u_0) = \lim_{n\longrightarrow+\infty} J_f(u_n) = m_f, \tag{32}$$

and u_0 is a solution of (29). It follows that

$$\int_E |\nabla u_0|^p dw - \lambda\int_V |u_0|^p d\mu = \int_V f u_0 d\mu \geq \tau\|u_0\|_{W^{1,p}(G)} > 0. \tag{33}$$

Now, we consider the derivative of $J_\epsilon(tu_0)$:

$$\begin{aligned} \frac{d}{dt}J_\epsilon(tu_0) = {} & t^{p-1}\int_E |\nabla u_0|^p dw - \lambda t^{p-1}\int_V |u_0|^p d\mu \\ & - t^{\alpha-1}\int_V h|u_0|^\alpha d\mu - \epsilon\int_V f u_0 d\mu. \end{aligned} \tag{34}$$

By (33), we get

$$\left.\frac{d}{dt}\right|_{t=0} J_\epsilon(tu_0) < 0. \tag{35}$$

Let $u = u_0/\|u_0\|_{W^{1,p}(G)}$, and we finish the proof. □

Now, we prove that there exists another solution $\widehat{u} \in W^{1,p}(G)$ with $\|\widehat{u}\|_{W^{1,p}(G)} < r_\epsilon/p$ such that

$$J_\epsilon(\widehat{u}) = \widehat{c} = \inf_{\|u\|_{W^{1,p}(G)} \leq r_\epsilon} J_\epsilon(u) < 0, \tag{36}$$

for $0 < \epsilon < \epsilon_1$, where $r_\epsilon = \epsilon^{1/p}$. By Lemma 5, we know that $J_\epsilon(u)$ has a lower bound on $B_{r_\epsilon} = \{u \in W^{1,p}(G) : \|u\|_{W^{1,p}(G)} \leq r_\epsilon\}$. By Lemma 7, we get that $\inf_{\|u\|_{W^{1,p}(G)}\leq r_\epsilon} J_\epsilon(u) = \widehat{c} < 0$.

Take the sequence $\{u_n\} \subset W^{1,p}(G)$ with $\|u_n\|_{W^{1,p}(G)} \leq r_\epsilon$ such that $J_\epsilon(u_n) \longrightarrow \widehat{c}$ as $n \longrightarrow +\infty$. Since $\{u_n\}$ is bounded in $W^{1,p}(G)$, by Lemma 3, there exists some $\widehat{u} \in W^{1,p}(G)$ up to a subsequence $u_n \longrightarrow \widehat{u}$ in $W^{1,p}(G)$. Moreover,

$$\begin{aligned} & \lim_{n\longrightarrow+\infty}\|u_n\|_{W^{1,p}(G)} = \|\widehat{u}\|_{W^{1,p}(G)}, \\ & \lim_{n\longrightarrow+\infty}\int_V |u_n|^p d\mu = \int_V |\widehat{u}|^p d\mu, \\ & \lim_{n\longrightarrow+\infty}\epsilon\int_V f u_n d\mu = \epsilon\int_V f\widehat{u} d\mu, \\ & \lim_{n\longrightarrow+\infty}\int_V h|u_n|^\alpha d\mu = \int_V h|\widehat{u}|^\alpha d\mu. \end{aligned} \tag{37}$$

Then,

$$J_{\epsilon}(\widehat{u}) = \lim_{n \longrightarrow +\infty} J_{\epsilon}(u_n) = \widehat{c} < 0, \tag{38}$$

and $\widehat{u}$ is the minimizer of $J_{\epsilon}(u)$ on $B_{r_{\epsilon}}$. Lemma 5 implies that $\|\widehat{u}\|_{W^{1,p}(G)} < r_{\epsilon}/p$. Calculating the Euler–Lagrange equation of $J_{\epsilon}(\widehat{u})$ for $\phi \in W^{1,p}(G)$, we get that

$$0 = \frac{d}{dt}\bigg|_{t=0} J_{\epsilon}(\widehat{u} + t\phi) = \int_V \left(-\Delta_p \widehat{u} - \lambda |\widehat{u}|^{p-2}\widehat{u} - h|\widehat{u}|^{\alpha-2}\widehat{u} - \epsilon f\right)\phi d\mu. \tag{39}$$

Hence,

$$-\Delta_p \widehat{u} - \lambda |\widehat{u}|^{p-2}\widehat{u} = h|\widehat{u}|^{\alpha-2}\widehat{u} + \epsilon f. \tag{40}$$

Thus, $\widehat{u}$ is a solution of (9). This ends the proof of Theorem 1.

Data Availability

Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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