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Research Article

Tent Space Approach of Morrey Spaces and Their Application to Duality and Complex Interpolation

Takahiro Ono

Department of Mathematics, Chuo University, 1-13-27, Kasuga, Bunkyo-ku, Tokyo 112-8551, Japan

Correspondence should be addressed to Takahiro Ono; t.ono.tmu@gmail.com

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The aim in this paper is to establish a new duality property of Morrey spaces and to discover the complex interpolation space between Morrey spaces and Lebesgue spaces. For that purpose, a new space $N^s_{p,q}(\mathbf{R}^n)$ is introduced by using methods of tent spaces. The space generalizes Morrey spaces and Lebesgue spaces. Furthermore, this scale of spaces is amenable to the complex interpolation space.

1. Introduction

The Morrey space $M^s_q(\mathbf{R}^n)$ was introduced by Morrey [1] to investigate the existence and differentiability properties of solutions to the elliptic partial differential equations of second order. Tent spaces were introduced by Coifman et al. [2] to analyze Hardy spaces, and tent spaces have been applied for the theory of parabolic partial differential equations in the previous research. The aim in this paper is to generalize Morrey spaces by applying some properties of tent spaces. To our best knowledge, it seems that tent spaces are not used for the study of function spaces related to Morrey spaces.

As the motivation of this study, let us recall concrete examples of equivalence of homogeneous Triebel-Lizorkin spaces. The homogeneous Triebel-Lizorkin space $\dot{F}^s_{p,q}(\mathbf{R}^n)$ was introduced by Triebel (see [3]). The norm of $\dot{F}^s_{p,q}(\mathbf{R}^n)$ is characterized in terms of tent spaces. The space $\dot{F}^s_{p,q}(\mathbf{R}^n)$ generalizes Lebesgue spaces $L^p(\mathbf{R}^n)(1 < p<\infty)$ and the bounded mean oscillation space $\mathrm{BMO}(\mathbf{R}^n)$. Furthermore, the following properties is known as a particular case:

$$\begin{array}{ccc} L^2(\mathbf{R}^n) & \left[L^2(\mathbf{R}^n), \mathrm{BMO}(\mathbf{R}^n)\right]_{1/2} & \mathrm{BMO}(\mathbf{R}^n) \\ || & || & || \\ \dot{F}^0_{2,2}(\mathbf{R}^n) & \dot{F}^0_{4,2}(\mathbf{R}^n) & \dot{F}^0_{\infty,2}(\mathbf{R}^n) \\ || & || & || \\ \left(\dot{F}^0_{2,2}(\mathbf{R}^n)\right)^* & \left(\dot{F}^0_{4/3,2}(\mathbf{R}^n)\right)^* & \left(\dot{F}^0_{1,2}(\mathbf{R}^n)\right)^*. \end{array} \quad (1)$$

Here, $[X, Y]_\theta$ is the complex interpolation space between X and Y. Furthermore, $(X)^*$ is the dual space of X.

In this paper, the new function space $N^s_{p,q}(\mathbf{R}^n)$ is introduced (see Section 2.2 below). The norm of the space $N^s_{p,q}(\mathbf{R}^n)$ is also defined via tent spaces like homogeneous Triebel-Lizorkin spaces. The motivation of the study of the space $N^s_{p,q}(\mathbf{R}^n)$ is to construct the corresponding chart for Lebesgue spaces and Morrey spaces. The following chart summarizes arguments shown in this paper:

$$\begin{array}{ccc} L^p(\mathbf{R}^n) & \left[L^p(\mathbf{R}^n), M^s_q(\mathbf{R}^n)\right]_\theta & M^s_q(\mathbf{R}^n) \\ || & || & || \\ N^0_{p,p}(\mathbf{R}^n) & N^{\theta s}_{p_\theta,q_\theta}(\mathbf{R}^n) & N^s_{\infty,q}(\mathbf{R}^n) \\ || & || & || \\ \left(N^0_{p',p'}(\mathbf{R}^n)\right)^* & \left(N^{-\theta s}_{p_\theta{}',q_\theta{}'}(\mathbf{R}^n)\right)^* & \left(N^{-s}_{1,q'}(\mathbf{R}^n)\right)^*. \end{array} \quad (2)$$

The explicit expressions of p_θ and q_θ are not given here. We content ourselves with mentioning that there are natural interpolation indices. One of important points in the above chart is that the description of the complex interpolation space between Lebesgue spaces and Morrey spaces is given. A description of the complex interpolation space between two Morrey spaces is known (see Lemma 11). As far as we know, other descriptions of the complex interpolation space between Morrey spaces and other spaces are not known in the recent research.

1.1. Notations. We use the following notations in this paper:

(1) We denote by $L^0(\mathbf{R}^n)$ the set of all measurable functions on $\mathbf{R}^n$

(2) For $1 \le p \le \infty$, the conjugate number p' of p is defined by the number which realizes $1/p + 1/p' = 1$

(3) For $x \in \mathbf{R}^n$ and $r > 0$, $B(x, r)$ is the ball with radius r centered at x

(4) For $f \in L^0(\mathbf{R}^n)$ and $x \in \mathbf{R}^n$, we define the Hardy-Littlewood maximal operator M by

$$Mf(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy. \quad (3)$$

More generally, for $0 < q < \infty$, we define its powered version by

$$M_q(f) := \left(M(|f|^q)\right)^{1/q}. \quad (4)$$

(5) The space $L^1(0,\infty; dt/t)$ consists of measurable functions $\phi : [0,\infty) \longrightarrow [0,\infty)$ satisfying

$$\|\phi\|_{L^1(0,\infty;dt/t)} := \int_0^\infty \phi(t) \frac{dt}{t} < \infty. \quad (5)$$

1.2. Lebesgue Spaces and Morrey Spaces. In this subsection, we shall introduce the definitions of Lebesgue spaces and Morrey spaces. The most basic Banach spaces are Lebesgue spaces $L^p(\mathbf{R}^n)$ defined as follows:

Let $1 \le p \le \infty$. Then, the space $L^p(\mathbf{R}^n)$ is the set of all functions $f \in L^0(\mathbf{R}^n)$ satisfying

$$\|f\|_p := \left(\int_{\mathbf{R}^n} |f(x)|^p \, dx\right)^{1/p} < \infty \quad (6)$$

for $1 \le p < \infty$ and

$$\|f\|_\infty := \operatorname{esssup}_{x \in \mathbf{R}^n} |f(x)| < \infty \quad (7)$$

for $p = \infty$. Next, we recall the definition of Morrey spaces which is the main function spaces in this paper.

Definition 1. Let $1 \le q < \infty$ and $0 \le s \le n/q$. The Morrey space $M_q^s(\mathbf{R}^n)$ is the set of all functions $f \in L^0(\mathbf{R}^n)$ satisfying

$$\|f\|_{M_q^s} := \sup_{(x,r) \in \mathbf{R}^n \times (0,\infty)} r^s \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|^q \, dy\right)^{1/q} < \infty. \quad (8)$$

We note some properties of Morrey spaces.

Remark 2.

(1) Let ω_n be the volume of the unit ball on $\mathbf{R}^n$. Then, the equation

$$\begin{aligned} \|f\|_{M_p^{n/p}} &= \omega_n^{-1/p} \sup_{(x,r) \in \mathbf{R}^n \times (0,\infty)} \left(\int_{B(x,r)} |f(y)|^p \, dy\right)^{1/p} \\ &= \omega_n^{-1/p} \|f\|_p \end{aligned} \quad (9)$$

implies that

$$M_p^{n/p}(\mathbf{R}^n) = L^p(\mathbf{R}^n), \quad (10)$$

with equivalence of norms for $1 \le p < \infty$.

(2) For $1 \le q_2 \le q_1 < \infty$, the embedding

$$M_{q_1}^s(\mathbf{R}^n) \subset M_{q_2}^s(\mathbf{R}^n) \quad (11)$$

follows from Hölder's inequality.

(3) Combining (10) and (11), for any $1 \le q \le n/s$, we have

$$L^{n/s}(\mathbf{R}^n) \subset M_q^s(\mathbf{R}^n). \quad (12)$$

1.3. Dual Spaces. In the study of quasi-Banach spaces, the duality argument plays an important role. We recall how we identify the dual space of function spaces.

Definition 3. For a quasi-Banach space X, we write the dual space of X as X^*. The Banach space X^* is defined as the set of all linear continuous functionals $\ell : X \longrightarrow \mathbf{C}$. We define

$$\|\ell\|_{X^*} := \sup_{\|f\|_X = 1} |\ell(f)|. \quad (13)$$

It is known that a dual space of quasi-Banach space is also a quasi-Banach space with the above norm. We define the following duality relation between quasi-Banach spaces and their duals contained in $L^0(\mathbf{R}^n)$:

Definition 4. For quasi-Banach spaces $X, Y \subset L^0(\mathbf{R}^n)$, we say that $X^* = Y$ if it satisfies the following:

(0) If $f \in X$ and $g \in Y$, then $f \cdot g \in L^1(\mathbf{R}^n)$

(i) For $g \in Y$, we set

$$\ell_g(f) = \int_{\mathbf{R}^n} f(x)g(x)dx \quad (f \in X). \tag{14}$$

Then, we have $\ell_g \in X^*$ and

$$\|\ell_g\|_{X^*} \leq C\|g\|_Y. \tag{15}$$

(ii) For any $\ell \in X^*$, there uniquely exists $g \in Y$ satisfying

$$\ell(f) = \int_{\mathbf{R}^n} f(x)g(x)dx \quad (f \in X), \tag{16}$$

and we have

$$\|g\|_Y \leq C\|\ell\|_{X^*}. \tag{17}$$

Usually, the notion of duality relation is considered for Banach spaces of measurable functions. However, as we will see, the ones for quasi-Banach spaces will be needed since we would like to handle the quasi-Banach space $N^s_{p,q}(\mathbf{R}^n)$ which will be defined in Section 2.2 below. For quasi-Banach spaces X and Y, we say that X is a predual space of Y if it satisfies $X^* = Y$. It is known that a predual space is not always unique. For Lebesgue spaces, we have the following duality property:

Lemma 5. *Let $1 \leq p < \infty$. Then,*

$$(L^p(\mathbf{R}^n))^* = L^{p'}(\mathbf{R}^n). \tag{18}$$

For the proof, see, e.g., Grafakos [4].

In the previous research, some predual spaces of Morrey spaces are known. The following block spaces were found by Long [5].

Definition 6. Let $s \in \mathbf{R}$ and $1 < q < \infty$. We say that a measurable function a is a (q, s)-block in a ball B if a satisfies

(i) supp $a \subset B$,

(ii) $\|a\|_q \leq |B|^{((-1/q')-(s/n))}$

The block space $B^s_q(\mathbf{R}^n)$ is the set of all functions $f \in L^0(\mathbf{R}^n)$ such that

$$\|f\|_{B^s_q} := \inf \left\{ \sum_{j\in\mathbf{N}} |\lambda_j| \,\middle|\, f = \sum_{j\in\mathbf{N}} \lambda_j a_j, \{\lambda_j\}_{j\in\mathbf{N}} \subset \mathbf{C}, \{a_j\}_{j\in\mathbf{N}} : (q,s)-\text{blocks} \right\} < \infty, \tag{19}$$

where the infimum is taken over all decompositions of f. Here, each a_j satisfies (i) and (ii) for B replaced by the ball B_j. Note that B_j can vary according to j.

The dual space of $B^{-s}_{q'}(\mathbf{R}^n)$ is the Morrey space $M^s_q(\mathbf{R}^n)$.

Lemma 7 (see [5]). *Let $1 < q < \infty$ and $0 < s \leq n/q$. Then,*

$$\left(B^{-s}_{q'}(\mathbf{R}^n)\right)^* = M^s_q(\mathbf{R}^n). \tag{20}$$

We refer to Adams and Xiao [6] and Gogatishvili and Mustafayev [7] for other representations of predual spaces of Morrey spaces.

1.4. Complex Interpolation Spaces. The theory of complex interpolation spaces plays an important role in operator theory. Our second aim is to obtain the complex interpolation between Morrey spaces and Lebesgue spaces. For this purpose, we prepare the following.

For quasi-Banach function spaces X, Y, let $X + Y$ be the sum space such that

$$X + Y := \{f = f_1 + f_2 : f_1 \in X, f_2 \in Y\}. \tag{21}$$

We define $\mathscr{F}(X, Y)$ be the set of all mappings $f : \mathbf{C} \longrightarrow X + Y$ which are analytic in $S := \{z \in \mathbf{C} : 0 < \mathrm{Re}(z) < 1\}$, continuous on $\bar{S} := \{z \in \mathbf{C} : 0 \leq \mathrm{Re}(z) \leq 1\}$ and satisfy

(i) $\{f(z) : z \in S\} \subset X + Y$

(ii) $\{f(it) : t \in \mathbf{R}\} \subset X$, $\lim_{t'\longrightarrow t} \|f(it') - f(it)\|_X = 0$, for each $t \in \mathbf{R}$ and $\sup_{t\in\mathbf{R}} \|f(it)\|_X < \infty$

(iii) $\{f(1+it) : t \in \mathbf{R}\} \subset Y$, $\lim_{t'\longrightarrow t} \|f(1+it') - f(1+it)\|_Y = 0$, for each $t \in \mathbf{R}$ and $\sup_{t\in\mathbf{R}} \|f(1+it)\|_Y < \infty$

Furthermore, introducing the norm of $\mathscr{F}(X, Y)$ by

$$\|f\|_{\mathscr{F}(X,Y)} := \max \left\{ \sup_{t\in\mathbf{R}} \|f(it)\|_X, \sup_{t\in\mathbf{R}} \|f(1+it)\|_Y \right\}, \tag{22}$$

we define the complex interpolation space $[X, Y]_\theta$ between X and Y with respect to $0 \leq \theta \leq 1$, by

$$[X, Y]_\theta := \{x \in X + Y : x = f(\theta), f \in \mathscr{F}(X, Y)\} \tag{23}$$

equipped with norm

$$\|x\|_{[X,Y]_\theta} := \inf \left\{ \|f\|_{\mathscr{F}(X,Y)} \middle| f \in \mathscr{F}(X, Y), x = f(\theta) \right\}. \tag{24}$$

We can interpolate the operator norm of bounded linear operator, which is the main thrust of investigating interpolation spaces.

Lemma 8 (Bergh and Löfström [8]). *For pairs of Banach spaces (X_1, X_2) and (Y_1, Y_2), assume that a linear operator T is bounded from X_j to $Y_j (j = 1, 2)$. Then, T is also bounded from $[X_1, X_2]_\theta$ to $[Y_1, Y_2]_\theta$ and satisfies*

$$\|T\|_{[X_1,X_2]_\theta \longrightarrow [Y_1,Y_2]_\theta} \le \|T\|_{X_1\longrightarrow Y_1}^{1-\theta} \|T\|_{X_2\longrightarrow Y_2}^{\theta}, \tag{25}$$

for any $0 < \theta < 1$. Here, we define the operator norm by

$$\|T\|_{X\longrightarrow Y} \coloneqq \sup_{\|f\|_X=1} \|Tf\|_Y, \tag{26}$$

for an operator from a Banach space X to a Banach space Y.

We shall give another characterization of complex interpolation spaces by using Carderón product spaces defined as follows:

Definition 9. For quasi-Banach spaces $X, Y \subset L^0(\mathbf{R}^n)$, and $0 \le \theta \le 1$, we define

$$X^{1-\theta}Y^\theta \coloneqq \left\{ f \in X + Y : \|f\|_{X^{1-\theta}Y^\theta} < \infty \right\}, \tag{27}$$

where

$$\|f\|_{X^{1-\theta}Y^\theta} = \inf \left\{ \|f_1\|_X^{1-\theta} \|f_2\|_Y^\theta : |f(x)| \le |f_1(x)|^{1-\theta} |f_2(x)|^\theta \text{ a.e.} \right\}. \tag{28}$$

The space $X^{1-\theta}Y^\theta$ is called the Carderón product space between X and Y with respect to θ.

Below, we recall the relationship between complex interpolation spaces and Calderón product spaces. We say that a quasi-Banach space X is a quasi-Banach function lattice if it satisfies

$$|g(x)| \le |f(x)| \text{ a.e.for } g \in L^0(\mathbf{R}^n) \Longrightarrow g \in X \text{ with } \|g\|_X \le \|f\|_X, \tag{29}$$

for any $f \in X$. We say that a quasi-Banach space X satisfies the Fatou property if

$$0 \le f_k \in X \text{ and } \sup_{k\in\mathbf{N}} \|f_k\|_X < \infty, \text{with } f_k \longrightarrow f \in L^0(\mathbf{R}^n) \text{ a.e.} \Longrightarrow f \in X \quad \text{and} \quad \|f\|_X = \lim_{k\longrightarrow\infty} \|f_k\|_X, \tag{30}$$

for $\{f_k\}_{k\in\mathbf{N}} \subset X$. Furthermore, we say that a quasi-Banach space X is r-convex for $0 < r < \infty$ if it satisfies

$$\left\| \left(\sum_{j=1}^k |f_j|^r \right)^{1/r} \right\|_X \le \left(\sum_{j=1}^k \|f_j\|_X^r \right)^{1/r}, \tag{31}$$

for $\{f_j\}_{j=1}^k \subset X$. In Kalton and Mitrea [9], it was shown that, if two quasi-Banach function lattices X_1 and X_2 satisfy the Fatou property, the r_i-convexity $(i = 1, 2)$, and if either of X_1 or X_2 is separable, then

$$[X_1, X_2]_\theta = X_1^{1-\theta} X_2^\theta, \tag{32}$$

for any $0 < \theta < 1$.

We recall two classical formulas on complex interpolation for Lebesgue spaces and for Morrey spaces.

Lemma 10 (see [4]). *Let $1 \le p_1, p_2 \le \infty$ and $1 < \theta < 1$. Then,*

$$[L^{p_1}(\mathbf{R}^n), L^{p_2}(\mathbf{R}^n)]_\theta = L^{p_\theta}(\mathbf{R}^n), \tag{33}$$

for $1/p_\theta = (1-\theta)/p_1 + \theta/p_2$.

The complex interpolation of Morrey spaces is more complicated than that of Lebesgue spaces.

Lemma 11 (Hakim and Sawano [10]). *Let $0 < \theta < 1$, $1 \le q_i < \infty$, and $0 < s_i \le n/p_i (i = 1, 2)$. If $s_1 q_1 = s_2 q_2$, then*

$$\left[M_{q_1}^{s_1}(\mathbf{R}^n), M_{q_2}^{s_2}(\mathbf{R}^n) \right]_\theta = \left\{ f \in \bar{M}_{q_\theta}^{s_\theta}(\mathbf{R}^n) : \lim_{a\longrightarrow 0^+} \left\| \chi_{\{|f|<a\}} f \right\|_{M_q^s} = 0 \right\}, \tag{34}$$

for $1/q_\theta = (1-\theta)/q_1 + \theta/q_2$ and $s_\theta = (1-\theta)s_1 + \theta s_2$. Here, $\bar{M}_{q_\theta}^{s_\theta}(\mathbf{R}^n)$ denotes the closure with respect to $M_{q_\theta}^{s_\theta}(\mathbf{R}^n)$ of the set of all essentially bounded functions in $M_{q_\theta}^{s_\theta}(\mathbf{R}^n)$.

The organization of the remain part is as follows: In Section 2, we introduce tent spaces and our new space $N_{p,q}^s(\mathbf{R}^n)$. In Section 3, we shall state the main theorems. Further, in Section 4, some properties and the proofs of the main theorems are given and in Section 5, we present their application.

2. Tent Spaces and New Spaces

In this section, we recall the definition and some properties of tent spaces, and after that, we introduce the new space $N_{p,q}^s(\mathbf{R}^n)$. We organize this section as follows: In Section 2.1, we present the definition and investigate some properties of tent spaces. In Section 2.2, we give the definition of the space $N_{p,q}^s(\mathbf{R}^n)$.

2.1. Tent Spaces. In this section, we recall the definition and some properties of tent spaces. Tent spaces were initially introduced by Coifman et al. [2]. In the previous research, tent spaces have been applied for study of boundedness of Calderón-Zygmund operators (see Section 5 for the definition) and the theory of parabolic differential equations (see David and Journé [11] and Koch and Tataru [12]). We write $\mathbf{R}_+^{n+1} := \mathbf{R}^n \times (0,\infty)$.

Definition 12. Let $1 \le p \le \infty$, $1 \le q < \infty$, and $s \in \mathbf{R}$. The tent space $T_{p,q}^s(\mathbf{R}_+^{n+1})$ is the set of all measurable functions $F : \mathbf{R}_+^{n+1} \longrightarrow \mathbf{C}$ satisfying

$$\|F\|_{T_{p,q}^s} := \left\| \left(\int_0^\infty \int_{B(x,t)} t^{sq} |F(y,t)|^q \frac{dydt}{t^{n+1}} \right)^{1/q} \right\|_{L_x^p(\mathbf{R}^n)} < \infty, \tag{35}$$

for $p < \infty$, and

$$\|F\|_{T_{\infty,q}^s} := \sup_{(x,r)\in\mathbf{R}^n\times(0,\infty)} \left(\frac{1}{|B|} \int_0^r \int_{|x-y|<r-t} t^{sq} |F(y,t)|^q \frac{dydt}{t} \right)^{1/q} < \infty, \tag{36}$$

for $p = \infty$.

It is easy to show that the triangle inequality for $\|*\|_{T_{p,q}^s}$ holds. Moreover, we note that by applying Fubini's lemma, we have

$$\|F\|_{T_{p,p}^0} \sim \left(\int_{\mathbf{R}_+^{n+1}} |F(x,t)|^p \frac{dxdt}{t} \right)^{1/p}, \tag{37}$$

for $1 \le p < \infty$ and $F \in L^0(\mathbf{R}_+^{n+1})$.

The following is the known duality theorem for tent spaces based on Definition 4:

Lemma 13 (Huang [13]). *Let $1 \le p < \infty$, $1 < q < \infty$, and $s \in \mathbf{R}$. If $F \in T_{p,q}^s(\mathbf{R}_+^{n+1})$ and $G \in T_{p',q'}^{-s}(\mathbf{R}_+^{n+1})$, then $F \cdot G$ is integrable with respect to the measure $dxdt/t$ and*

$$\left(T_{p,q}^s(\mathbf{R}_+^{n+1}) \right)^* = T_{p',q'}^{-s}(\mathbf{R}_+^{n+1}) \tag{38}$$

via the coupling

$$\langle F, G \rangle = \int_{\mathbf{R}_+^{n+1}} F(x,t) G(x,t) \frac{dxdt}{t}. \tag{39}$$

Furthermore, the following inequalities hold:

$$\left| \int_{\mathbf{R}_+^{n+1}} F(x,t) G(x,t) \frac{dxdt}{t} \right| \le C \|F\|_{T_{p,q}^s} \|G\|_{T_{p',q'}^{-s}}, \tag{40}$$

for $1 < p < \infty$, and

$$\left| \int_{\mathbf{R}_+^{n+1}} F(x,t) G(x,t) \frac{dxdt}{t} \right| \le C \|F\|_{T_{1,q}^{-s}} \|G\|_{T_{\infty,q'}^s}, \tag{41}$$

for $p = 1$. Moreover, the inequality

$$\|F \cdot G\|_{T_{p_0,q_0}^{s_0}} \le C \|F\|_{T_{p_1,q_1}^{s_1}} \|G\|_{T_{p_2,q_2}^{s_2}} \tag{42}$$

holds for $s_0 = s_1 + s_2 \in \mathbf{R}$, $0 \le 1/p_0 = 1/p_1 + 1/p_2 \le 1$, and $0 < 1/q_0 + 1/q_1 + 1/q_2 < 1$ with $1 \le p_1, p_2 \le \infty$, and $1 < q_1, q_2 < \infty$.

We note that Lemma 13 is a particular case of Theorem 4.3 in [13] where $\beta_0 = 0$, $\beta = s$, and $r = q$.

We have also the Sobolev embedding theorem for tent spaces as follows:

Lemma 14 (Amenta [14]). *Let $1 < q < \infty$. For $1 < p_1, p_2 \le \infty$ and $s_1 \le s_2$, if $s_1 + n/p_1 = s_2 + n/p_2$, one has*

$$T_{p_1,q}^{s_1}(\mathbf{R}_+^{n+1}) \subset T_{p_2,q}^{s_2}(\mathbf{R}_+^{n+1}), \tag{43}$$

with a norm inequality

$$\|F\|_{T_{p_2,q_2}^{s_2}} \le C \|F\|_{T_{p_1,q_2}^{s_1}}, \tag{44}$$

for any $F \in T_{p_1,p_2}^{s_1}(\mathbf{R}_+^{n+1})$.

The following lemma states that the Calderón product for tent spaces can be calculated.

Lemma 15 (see [13]). *Let $1 \le q_1, q_2 < \infty$, $1 \le p_1, p_2 \le \infty$(not both$\infty$), and $s \in \mathbf{R}$. Then,*

$$\left(T_{p_1,q_1}^{s_1}(\mathbf{R}_+^{n+1}) \right)^{1-\theta} \left(T_{p_2,q_2}^{s_2}(\mathbf{R}_+^{n+1}) \right)^{\theta} = T_{p_0,q_0}^{s_0}(\mathbf{R}_+^{n+1}), \tag{45}$$

where

$$s_0 = (1-\theta)s_1 + \theta s_2, \ \frac{1}{p_0} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \quad \text{and} \quad \frac{1}{q_0} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}. \tag{46}$$

2.2. The Definition of $N_{p,q}^s$-Spaces. In this section, we introduce new function spaces called $N_{p,q}^s$-spaces by using tent spaces defined in Section 2.1. First, we define the $N_{p,q}^s$-norms with respect to $\phi \in L^1(0,\infty;dt/t)$.

Definition 16. Let $\phi \in L^1(0,\infty;dt/t)$. For $1 \le p \le \infty$, $1 \le q < \infty$, and $s \le n/q$, we define

$$\begin{aligned} \|f\|_{N_{p,q}^{s,\phi}} &:= \left\| \phi(t)^{1/q} f(x) \right\|_{T_{p,q}^s} \\ &= \left\| \left(\int_0^\infty \int_{B(x,t)} t^{sq} \phi(t) |f(y)|^q \frac{dydt}{t^{n+1}} \right)^{1/q} \right\|_{L_x^p(\mathbf{R}^n)}, \end{aligned} \tag{47}$$

for $1 \le p < \infty$, and

$$\|f\|_{N^{s,\phi}_{\infty,q}} := \left\|\phi(t)^{1/q} f(x)\right\|_{T^{s}_{\infty,q}} = \sup_{(x,r)\in\mathbf{R}^n\times(0,\infty)} \left(\frac{1}{|B|}\int_0^r\int_{|x-y|<r-t} t^{sq}\phi(t)|f(y)|^q \frac{dydt}{t}\right)^{1/q}, \quad (48)$$

for $p = \infty$.

Using the above norm, we define the $N^{s}_{p,q}$-norm for $s \ge 0$.

Definition 17. For $1 \le q < \infty$, $q \le p \le \infty$, and $0 \le s \le n/q - n/p$, we define

$$N^{s}_{p,q}(\mathbf{R}^n) := \left\{ f \in L^0(\mathbf{R}^n):\ \|f\|_{N^{s,\phi}_{p,q}} < \infty \text{for any } \phi \in L^1\left(0,\infty;\frac{dt}{t}\right)\right\}, \quad (49)$$

where

$$\|f\|_{N^{s}_{p,q}} = \sup_{\|\phi\|_{L^1(0,\infty;dt/t)}=1} \|f\|_{N^{s,\phi}_{p,q}}. \quad (50)$$

Remark that for $s \ge 0$, the definition of the $N^{s}_{p,q}$-norm demanded $q \le p$. The way of defining the $N^{s}_{p,q}$-norm for $p \le q$ is different from the one for $q \le p$.

Definition 18. Let $1 \le p \le q \le \infty$, $n/q - n/p \le -s \le 0$, and $\phi \in L^1(0,\infty,dt/t)$. We say that the function $f_t(x)$ is an $(N^{-s}_{p,q}, \phi)$-atom if $f_t(x) \ge 0$ for all $x \in \mathbb{R}^n$ and $t \in (0,\infty)$ and

$$\left\|\phi(t)^{1/q} f_t(x)\right\|_{T^{-s}_{p,q}} \le 1. \quad (51)$$

The space $N^{-s}_{p,q}(\mathbb{R}^n)$ is the set of all $f \in L^0(\mathbb{R}^n)$ such that there exists $\lambda > 0$ and an $(N^{-s}_{p,q}, \phi)$-atom f_t for which

$$|f(x)| \le \lambda \int_0^\infty \phi(t) f_t(x) \frac{dt}{t}, \quad (52)$$

for almost all $x \in \mathbb{R}^n$. Let $\lambda(\phi, f_t)$ be the infimum of such λ, and let

$$\|f\|_{N^{-s}_{p,q}} := \inf \lambda(\phi, f_t), \quad (53)$$

where the infimum is taken over all functions ϕ which satisfy $\|\phi\|_{L^1(0,\infty,dt/t)} = 1$ and all $(N^{-s}_{p,q}, \phi)$-atoms f_t.

Remark that for $s \le 0$, the definition of the $N^{s}_{p,q}$-norm demanded $p \le q$. For the case $p = q$, we defined the $N^{0}_{p,p}$-norm in two ways. However, those are essentially equivalent (see Lemma 24 in Section 4.1 below). The definition of the norm $\|f\|_{N^{-s}_{p,q}}$ is complicated; therefore, we introduce $\tilde{N}^{-s}_{p,q}$-"norms" for convenience.

Definition 19. For $1 \le p \le q < \infty$ and $n/q - n/p < -s \le 0$, we define

$$\|f\|_{\tilde{N}^{-s}_{p,q}} := \inf_{\|\phi\|_{L^1(0,\infty;dt/t)}=1} \|f\|_{N^{-s,\phi}_{p,q}}, \quad (54)$$

for measurable function f. The space $\tilde{N}^{-s}_{p,q}(\mathbf{R}^n)$ is the set given by

$$\tilde{N}^{-s}_{p,q}(\mathbf{R}^n) := \left\{ f \in L^0(\mathbf{R}^n):\ \|f\|_{\tilde{N}^{-s}_{p,q}} < \infty \right\}. \quad (55)$$

Then, we include the following:

Lemma 20. *Assume that the parameters p, q, s satisfy the conditions in Definition 18. Then, one has* $\tilde{N}^{-s}_{p,q}(\mathbf{R}^n) \subset N^{-s}_{p,q}(\mathbf{R}^n)$, *with the inequality*

$$\|f\|_{N^{-s}_{p,q}} \le \|f\|_{\tilde{N}^{-s}_{p,q}}, \quad (56)$$

for $f \in \tilde{N}^{-s}_{p,q}(\mathbf{R}^n)$.

Proof. If $f = 0$, then the conclusion is clear. So, assume otherwise. Let $f \in \tilde{N}^{-s}_{p,q}(\mathbf{R}^n)$. Then, for any $\varepsilon > 0$, there exists $\phi \in L^1(0,\infty;dt/t)$ satisfying $\|\phi\|_{L^1(0,\infty;dt/t)} = 1$ such that

$$\|f\|_{N^{-s,\phi}_{p,q}} \le \|f\|_{\tilde{N}^{-s}_{p,q}} + \varepsilon. \quad (57)$$

Subsequently, from $\|\phi\|_{L^1(0,\infty;dt/t)} = 1$, we have

$$|f(x)| = \int_0^\infty \phi(t)|f(x)|\frac{dt}{t} = \|f\|_{N^{-s,\phi}_{p,q}} \int_0^\infty \phi(t)\frac{|f(x)|}{\|f\|_{N^{-s,\phi}_{p,q}}}\frac{dt}{t}. \quad (58)$$

Then, trivially,

$$\left\|\frac{f}{\|f\|_{N^{-s,\phi}_{p,q}}}\right\|_{N^{-s,\phi}_{p,q}} = 1. \quad (59)$$

Thus, we obtain

$$\|f\|_{N^{-s}_{p,q}} \le \|f\|_{N^{-s,\phi}_{p,q}} \le \|f\|_{\tilde{N}^{-s}_{p,q}} + \varepsilon, \quad (60)$$

for any $\varepsilon > 0$, which concludes the proof. □

It is unclear that the $\tilde{N}^{-s}_{p,q}$-"norms" are quasinorms because they do not satisfy the quasitriangle inequality $\|f+g\|_{\tilde{N}^{-s}_{p,q}} \le C(\|f\|_{\tilde{N}^{-s}_{p,q}} + \|g\|_{\tilde{N}^{-s}_{p,q}})$. Meanwhile, we claim that the $N^{s}_{p,q}$-norms do satisfy the quasitriangle inequality. For $s \ge 0$, it is evident from Definition 17. Thus, we consider the case where $s \le 0$. Let $f, g \in N^{-s}_{p,q}(\mathbf{R}^n)$. Then, for any

$\varepsilon_1, \varepsilon_2 > 0$, there exists $\phi_1, \phi_2 \in L^1(0,\infty;dt/t)$ satisfying $\|\phi_1\|_{L^1(0,\infty;dt/t)} = \|\phi_2\|_{L^1(0,\infty;dt/t)} = 1$, and we have

$$|f(x)| \le \left(\|f\|_{N^{-s}_{p,q}} + \varepsilon_1\right)\int_0^\infty \phi_1(t) f_t(x)\frac{dt}{t},$$
$$|g(x)| \le \left(\|g\|_{N^{-s}_{p,q}} + \varepsilon_2\right)\int_0^\infty \phi_2(t) g_t(x)\frac{dt}{t}, \tag{61}$$

for some $(N^{-s}_{p,q}, \phi_1)$-atom f_t and $(N^{-s}_{p,q}, \phi_2)$-atom g_t. Note that

$$\|\phi_1(t)^{1/q} f_t\|_{T^{-s}_{p,q}}, \|\phi_2(t)^{1/q} g_t\|_{N^{-s,\phi_2}_{p,q}} \le 1. \tag{62}$$

If we put $\phi_0(t) = (\phi_1(t) + \phi_2(t))/2$, then $\|\phi_0\|_{L^1(0,\infty;dt/t)} = 1$, and we obtain

$$|f(x) + g(x)| \le 2^{1/q'}\left(\|f\|_{N^{-s}_{p,q}} + \|g\|_{N^{-s}_{p,q}} + \varepsilon_3\right)\int_0^\infty \phi_0(t)\frac{1}{2^{1/q'}} \cdot \left(\frac{\phi_1(t)}{\phi_0(t)} f_t(x) + \frac{\phi_2(t)}{\phi_0(t)} g_t(x)\right)\frac{dt}{t}, \tag{63}$$

where we put $\varepsilon_3 = \varepsilon_1 + \varepsilon_2$. The triangle inequality in tent spaces implies that

$$\left\|\phi_0(t)^{1/q}\frac{1}{2^{1/q'}}\left(\frac{\phi_1(t)}{\phi_0(t)} f_t + \frac{\phi_2(t)}{\phi_0(t)} g_t\right)\right\|_{T^{-s}_{p,q}} \le \left\|\phi_0(t)^{1/q}\frac{1}{2^{1/q'}}\frac{\phi_1(t)}{\phi_0(t)} f_t\right\|_{T^{-s}_{p,q}} + \left\|\phi_0(t)^{1/q}\frac{1}{2^{1/q'}}\frac{\phi_2(t)}{\phi_0(t)} g_t\right\|_{T^{-s}_{p,q}}. \tag{64}$$

Then, from (62), we obtain

$$\left\|\phi_0(t)^{1/q}\frac{1}{2^{1/q'}}\frac{\phi_1(t)}{\phi_0(t)} f_t\right\|_{T^{-s}_{p,q}} = \frac{1}{2}\left\|\frac{\phi_1(t)}{(\phi_1(t) + \phi_2(t))^{1/q'}} f_t\right\|_{T^{-s}_{p,q}} \le \frac{1}{2}. \tag{65}$$

Similarly, we have

$$\left\|\phi_0(t)^{1/q}\frac{1}{2^{1/q'}}\frac{\phi_2(t)}{\phi_0(t)} g_t\right\|_{T^{-s}_{p,q}} \le \frac{1}{2}. \tag{66}$$

Thus, the function $((\phi_1/\phi_0)(t) f_t + (\phi_2/\phi_0)(t) g_t)/2^{1/q'}$ is an $(N^{-s}_{p,q}, \phi_0)$-atom so that we get

$$\|f + g\|_{N^{-s}_{p,q}} \le 2^{1/q'}\left(\|f\|_{N^{-s}_{p,q}} + \|g\|_{N^{-s}_{p,q}} + \varepsilon_3\right), \tag{67}$$

for any $\varepsilon_3 > 0$. This shows that the norm $\|*\|_{N^{-s}_{p,q}}$ satisfies the quasitriangle inequality.

3. Main Theorems

In this section, we introduce our main theorems. The first result is the duality theorem of $N^s_{p,q}$-spaces. This derives the duality theorem on Morrey spaces by combining with Proposition 31 in Section 4.1 below.

Theorem 21. *Let $1 < p \le \infty$, $1 \le q < \infty$, and $0 \le s < n/q - n/p$. Then, one has $(N^{-s}_{p',q'}(\mathbf{R}^n))^* = N^s_{p,q}(\mathbf{R}^n)$ in the sense of Definition 4.*

(0) Let $f \in N^{-s}_{p',q'}(\mathbf{R}^n)$ and $g \in N^s_{p,q}(\mathbf{R}^n)$. Then,$\cdot g \in L^1(\mathbf{R}^n)$

(i) For $g \in N^s_{p,q}(\mathbf{R}^n)$, set

$$\ell_g(f) = \int_{\mathbf{R}^n} f(x) g(x)\, dx. \tag{68}$$

Then, $\ell_g \in (N^{-s}_{p',q'}(\mathbf{R}^n))^*$ and $\|\ell_g\|_{(N^{-s}_{p',q'})^*} \le C\|g\|_{N^s_{p,q}}$.

(ii) For any $\ell \in (N^{-s}_{p',q'}(\mathbf{R}^n))^*$, there uniquely exists $g \in N^s_{p,q}(\mathbf{R}^n)$ satisfying $\ell = \ell_g$, and one has $\|g\|_{N^s_{p,q}} \le C\|\ell\|_{(N^{-s}_{p',q'})^*}$.

Letting $p = \infty$ and combining Theorem 21 with Proposition 31 in Section 4.1 below, we obtain the following corollary:

Corollary 22. *Let $1 < q < \infty$ and $0 \le s < n/q$. Then,*

$$\left(N^{-s}_{1,q'}(\mathbf{R}^n)\right)^* = N^s_{\infty,q}(\mathbf{R}^n) = M^s_q(\mathbf{R}^n). \tag{69}$$

For homogeneous Triebel-Lizorkin spaces $\dot{F}^s_{p,q}(\mathbf{R}^n)$ and the bounded mean oscillation space $BMO(\mathbf{R}^n)$, it is known that the duality $(\dot{F}^0_{1,2}(\mathbf{R}^n))^* = \mathrm{BMO}(\mathbf{R}^n)$ holds (see Sawano [15] for their definitions and the precise statement of the duality). Furthermore, it is generalized to the duality $(\dot{F}^s_{p,q}(\mathbf{R}^n))^* = \dot{F}^{-s}_{p',q'}(\mathbf{R}^n)$ (see Frazier and Jawerth [16]). For Morrey spaces, Theorem 21 realizes the above generalization of $\dot{F}^s_{p,q}(\mathbf{R}^n)$. There are several predual of Morrey spaces. The block spaces described in Section 1.3 were first introduced in [5]. Subsequently, other predual spaces were found (see [6, 7]). The space $N^{-s}_{1,q'}(\mathbf{R}^n)$ is the new one we found.

The second theorem concerns a complex interpolation between Lebesgue spaces and Morrey spaces.

Theorem 23. *Let $1 \le p, q < \infty$, $0 \le s \le n/q$, and $0 \le \theta \le 1$. Then,*

$$\left[L^p(\mathbf{R}^n), M^s_q(\mathbf{R}^n)\right]_\theta = N^{\theta s}_{p_\theta,q_\theta}(\mathbf{R}^n), \tag{70}$$

where $p_\theta = p/(1-\theta)$ and $1/q_\theta = (1-\theta)/p + \theta/q$.

We must assume $s \le n/q - n/p$ in the definition of $N^s_{p,q}$-norms when $s \ge 0$. The above θs, p_θ, and q_θ satisfy the condition. In fact, from $s \le n/q$, we have

$$\theta s \le \frac{\theta n}{q} = \frac{(1-\theta)n}{p} + \frac{\theta n}{q} - \frac{(1-\theta)n}{p} = \frac{n}{q_\theta} - \frac{n}{p_\theta}. \quad (71)$$

Theorem 23 implies that $N^s_{p,q}$-spaces with $s \ge 0$ can be characterized by complex interpolation spaces between some Lebesgue space and some Morrey space.

4. Proofs

In this section, we prove some lemmas of $N^s_{p,q}$-spaces, and after that, we prove the main theorems introduced in Section 3. We organize this section as follows: In Section 4.1, we investigate some properties of $N^s_{p,q}$-spaces. In Section 4.2, we prove Theorem 21. In Section 4.3, we prove Theorem 23.

4.1. Properties. In this section, we prove some properties of $N^s_{p,q}$-spaces. At first, we prove that the space $N^s_{p,q}(\mathbf{R}^n)$ generalizes Lebesgue spaces and Morrey spaces. This relation is similar to the fact that homogeneous Triebel-Lizorkin space $\dot{F}^s_{p,q}(\mathbf{R}^n)$ generalizes $L^p(\mathbf{R}^n)$ and $\mathrm{BMO}(\mathbf{R}^n)$. The coincidence with Lebesgue spaces is the following. Here, we must keep in mind that we defined $\|*\|_{N^0_{p,p}}$ in two ways, which are $N^s_{p,p}$-norms for $s \ge 0$ and for $s \le 0$. We will show that Lemma 24 below holds in both senses.

Lemma 24. *For $1 \le p < \infty$, we have $L^p(\mathbf{R}^n) = N^0_{p,p}(\mathbf{R}^n)$, and the following equivalence holds:*

$$\|f\|_{N^0_{p,p}} \sim \|f\|_p, \quad (72)$$

for any $f \in L^0(\mathbf{R}^n)$.

Proof. Let $f \in L^0(\mathbf{R}^n)$. First, for $N^s_{p,q}$-norms for $s \ge 0$, (37) implies that

$$\begin{aligned}\|f\|_{N^0_{p,p}} &= \sup_{\|\phi\|_{L^1(0,\infty;dt/t)}=1} \left\|\phi^{1/p} f\right\|_{T^0_{p,p}} \sim \sup_{\|\phi\|_{L^1(0,\infty;dt/t)}=1} \\ &\quad \cdot \left(\int_{\mathbf{R}^n}\int_0^\infty \phi(t)|f(x)|^p \frac{dt\,dx}{t}\right)^{1/p} \\ &= \|f\|_p.\end{aligned} \quad (73)$$

Conversely, for $N^s_{p,p}$-norms for $s \le 0$, Lemma 20 and (37) imply that

$$\|f\|_{N^0_{p,p}} \le \|f\|_{\tilde{N}^0_{p,p}} = \inf_{\|\phi\|_{L^1(0,\infty;dt/t)}=1} \left\|\phi^{1/p} f\right\|_{T^0_{p,p}} \sim \|f\|_p. \quad (74)$$

Meanwhile, if we assume $f \in N^0_{p,p}(\mathbf{R}^n)$ in the sense of Definition 18, then for any $\varepsilon > 0$, there exists $\phi \in L^1(0,\infty;dt/t)$ satisfying $\|\phi\|_{L^1(0,\infty;dt/t)} = 1$ and $(N^0_{p,p}, \phi)$-atom f_t such that

$$|f(x)| \le \left(\|f\|_{N^0_{p,p}} + \varepsilon\right)\int_0^\infty \phi(t) f_t(x)\frac{dt}{t}. \quad (75)$$

Thus, by Hölder's inequality for t, we have

$$\begin{aligned}\|f\|_p &\le \left(\|f\|_{N^0_{p,p}} + \varepsilon\right)\left\|\int_0^\infty \phi(t) f_t \frac{dt}{t}\right\|_p \\ &\le C\left(\|f\|_{N^0_{p,p}} + \varepsilon\right)\left(\int_0^\infty \phi(t)\frac{dt}{t}\right)^{1/p'} \left\|\phi^{1/p} f_t\right\|_{T^0_{p,p}} \\ &\le \|f\|_{N^0_{p,p}} + \varepsilon,\end{aligned} \quad (76)$$

for any $\varepsilon > 0$. It concludes the proof in both senses. □

The scale $N^s_{p,q}(\mathbf{R}^n)$ is monotone in $1 \le q < \infty$ as Lemma 25 shows, which is similar to homogeneous Triebel-Lizorkin spaces $\dot{F}^s_{p,q}(\mathbf{R}^n)$ for $1 \le q < \infty$.

Lemma 25. *For any $1 \le p \le \infty$, $1 \le q_1 \le q_2 < \infty$, and $s \in \mathbf{R}$, one has $N^s_{p,q_2}(\mathbf{R}^n) \subset N^s_{p,q_1}(\mathbf{R}^n)$, and*

$$\|f\|_{N^s_{p,q_1}} \le C\|f\|_{N^s_{p,q_2}}, \quad (77)$$

for any $f \in N^s_{p,q_2}(\mathbf{R}^n)$.

Proof. If suffices to show that $\|f\|_{N^{s,\phi}_{p,q_1}} \le C\|f\|_{N^{s,\phi}_{p,q_2}}$ for any $\phi \in L^1(0,\infty;dt/t)$ satisfying $\|\phi\|_{L^1(0,\infty;dt/t)} = 1$, it is easy to show for $p = \infty$ using Hölder's inequality. So, we assume that $1 \le p < \infty$. Setting $1/q_1 = 1/q_2 + 1/q_3$ and using Hölder's inequality twice, we learn

$$\begin{aligned}\|f\|_{N^{s,\phi}_{p,q_1}} &= \left\|\left(\int_0^\infty\int_{B(x,t)} t^{sq_1}\phi(t)|f(y)|^{q_1}\frac{dy\,dt}{t^{n+1}}\right)^{1/q_1}\right\|_{L^p_x} \\ &\le C\left\|\left(\int_0^\infty \phi(t)\left(\int_{B(x,t)} t^{sq_2}|f(y)|^{q_2}\frac{dy}{t^n}\right)^{q_1/q_2}\frac{dt}{t}\right)^{1/q_1}\right\|_{L^p_x} \\ &\le C\left\|\left(\int_0^\infty \phi(t)\frac{dt}{t}\right)^{1/q_3}\left(\int_0^\infty\int_{B(x,t)} t^{sq_2}\phi(t)|f(y)|^{q_2}\frac{dy\,dt}{t^{n+1}}\right)^{1/q_2}\right\|_{L^p_x} \\ &= C\|f\|_{N^{s,\phi}_{p,q_2}},\end{aligned} \quad (78)$$

from $\|\phi\|_{L^1(0,\infty;dt/t)} = 1$. □

The following is the Sobolev embedding in $N^s_{p,q}$-spaces:

Lemma 26. *Let $1 < q < \infty$. For $1 < p_1, p_2 \le \infty$, and $0 \le s_1 \le s_2$ (or $s_1 \le s_2 \le 0$), if $s_1 + n/p_1 = s_2 + n/p_2$, then one has $N^{s_1}_{p_1,q}(\mathbf{R}^n) \subset N^{s_2}_{p_2,q}(\mathbf{R}^n)$, and*

$$\|f\|_{N^{s_2}_{p_2,q}} \le C\|f\|_{N^{s_1}_{p_1,q}}, \tag{79}$$

for any $f \in N^{s_1}_{p_1,q}(\mathbf{R}^n)$.

Lemma 26 is a direct consequence of Lemma 14. We omit its proof. Lemmas 24 and 26 imply that $N^{-s}_{p,q}(\mathbf{R}^n) \subset L^q(\mathbf{R}^n)$ for $-s + n/p = n/q$. The following lemma claims that the converse embedding locally holds.

Lemma 27. *Let $1 \le p \le q < \infty$, $n/q - n/p < -s \le 0$, and $B = B(x_0, r) \subset \mathbf{R}^n$ be a ball. For $f \in L^q(\mathbf{R}^n)$ with* supp $f \subset B$, *one has*

$$\|f\|_{\tilde{N}^{-s}_{p,q}} \le Cr^{-s+n/p-n/q}\|f\|_q. \tag{80}$$

In particular, the following inequality holds:

$$\|f\|_{\tilde{N}^{-s}_{1,q}} \le Cr^{\left(-s+\left(n/q'\right)\right)}\|f\|_q, \tag{81}$$

for $0 \le s < n/q'$.

Before proving Lemma 27, we prepare the definition and some properties of Lorentz spaces.

Definition 28. Let $0 < p < \infty$ and $0 < q \le \infty$. For $f \in L^0(\mathbf{R}^n)$, we define the decreasing rearrangement function $f^* : [0,\infty) \longrightarrow [0,\infty]$ of f as

$$f^*(t) := \inf\left\{\alpha \in (0,\infty) : d_f(\alpha) \le t\right\}, \tag{82}$$

where d_f is the distribution function of f, given by

$$d_f(\alpha) = |\{x \in \mathbf{R}^n : |f(x)| > \alpha\}| (\alpha > 0). \tag{83}$$

Here, $\inf\varnothing$ stands for ∞. Let $f \in L^0(\mathbf{R}^n)$. If $0 < q < \infty$, then define

$$\|f\|_{L^{p,q}} := \left(\int_0^\infty \left(t^{1/p} f^*(t)\right)^q \frac{dt}{t}\right)^{1/q}. \tag{84}$$

If $q = \infty$, then define

$$\|f\|_{L^{p,q}} := \sup_{t>0} t^{1/p} f^*(t). \tag{85}$$

We define the Lorentz space $L^{p,q}(\mathbf{R}^n)$ as the set of all functions $f \in L^0(\mathbf{R}^n)$ for which $\|f\|_{L^{p,q}} < \infty$.

We note some properties of Lorentz spaces.

Remark 29.

(i) Let $1 < p_1, p_2, p_3 < \infty$ satisfy $1/p_3 = 1/p_1 + 1/p_2$ and $1 \le q \le \infty$. Then, for $f \in L^{p_1,q}(\mathbf{R}^n)$ and $g \in L^{p_2,q'}(\mathbf{R}^n)$, we have

$$\|fg\|_{L^{p_3}} \le \|f\|_{L^{p_1,q}}\|g\|_{L^{p_2,q'}}. \tag{86}$$

(ii) For $0 < q < \infty$ and $f \in L^q(\mathbf{R}^n)$, we have

$$\left\|M_q(f)\right\|_{L^{q,\infty}} \le C\|f\|_q, \tag{87}$$

where M_q is the q-powered Hardy–Littlewood maximal operator (see Section 1.1).

(iii) For any $0 < p < \infty$ and ball $B = B(x_0, r) \subset \mathbf{R}^n$, we have

$$\|\chi_B\|_{L^{p,1}} \sim |B|^{n/p}. \tag{88}$$

See, e.g., [4] for the above properties.

All preparations to prove Lemma 27 were given.

Proof of Lemma 30. For a fixed $\alpha > sq$, let $\phi(t) = c\chi_{[0,r)}(t)t^\alpha/r^\alpha$, where $c > 0$ is a normalization parameter satisfying $\|\phi\|_{L^1(0,\infty;dt/t)} = 1$. Since supp $f \subset B(x_0, r)$, for $y \in$ supp f, we get $|y - x_0| < r$. Fix $x \in \mathbf{R}^n$ and $0 < t < r$ with supp $f \cap B(x, t) \ne \varnothing$. Then, for $y \in$ supp $f \cap B(x, t)$, we have

$$|x - x_0| \le |x - y| + |y - x_0| < t + r < 2r. \tag{89}$$

Keeping in mind that in Remark 29, we have

$$\begin{aligned}
\|f\|_{\tilde{N}^{-s}_{p,q}} &\le \left\|\left(\int_0^\infty \int_{B(x,t)} t^{-sq}\phi(t)|f(y)|^q \frac{dydt}{t^{n+1}}\right)^{1/q}\right\|_{L^p_x(\mathbf{R}^n)} \\
&\le C\left\|\left(\int_0^r \int_{B(x,t)} \frac{t^{\alpha-sq}}{r^\alpha}|f(y)|^q \frac{dydt}{t^{n+1}}\right)^{1/q}\right\|_{L^p_x(B(x_0,2r))} \\
&\le C\left\|r^{-s}M_q(f)\right\|_{L^p(B(x_0,2r))} \\
&\le Cr^{-s}\left\|M_q(f)\right\|_{L^{q,\infty}}\left\|\chi_{B(x_0,2r)}\right\|_{L^{p_0,1}} \\
&\le Cr^{-s+n/p-n/q}\|f\|_q,
\end{aligned} \tag{90}$$

where $1/p = 1/q + 1/p_0$. □

The following proposition shows that $N^s_{p,q}$-spaces generalize Morrey spaces.

Proposition 31. *For $1 \le q < \infty$ and $0 < s < n/q$, one has $N^s_{\infty,q}(\mathbf{R}^n) = M^s_q(\mathbf{R}^n)$ with the norm equivalence*

$$\|f\|_{N^s_{\infty,q}} \sim \|f\|_{M^s_q}, \quad (91)$$

for any $f \in M^s_q(\mathbf{R}^n)$.

Proof. We can easily show that $\|f\|_{N^s_{\infty,q}} \le \|f\|_{M^s_q}$ for $f \in M^s_q(\mathbf{R}^n)$:

$$\begin{aligned}\|f\|_{N^s_{\infty,q}} &= \sup_{\|\phi\|_{L^1(0,\infty;dt/t)}=1} \sup_{(x,r)\in\mathbf{R}^n\times(0,\infty)} \cdot \left(\frac{1}{|B|}\int_0^r\int_{|x-y|<r-t} t^{sq}\phi(t)|f(y)|^q \frac{dydt}{t}\right)^{1/q} \\ &\le \sup_{\|\phi\|_{L^1(0,\infty;dt/t)}=1} \sup_{(x,r)\in\mathbf{R}^n\times(0,\infty)} \cdot \left(\frac{r^{sq}}{|B|}\int_0^\infty\int_{|x-y|<r} \phi(t)|f(y)|^q \frac{dydt}{t}\right)^{1/q} \\ &= \|f\|_{M^s_q}.\end{aligned} \quad (92)$$

Conversely, we assume $f \in N^s_{\infty,q}(\mathbf{R}^n)$. For $B = B(x_0, r) \subset \mathbf{R}^n$, we have

$$\|f\|_{L^q(B)} \sim \sup_{\|g\|_{L^{q'}(B)}=1} \left|\int_B f(x)g(x)dx\right|. \quad (93)$$

Then, Lemma 27 implies $g\chi_B \in L^{q'}(\mathbf{R}^n) \subset N^{-s}_{1,q'}(\mathbf{R}^n)$ for any $g \in L^{q'}(B)$. Moreover, for any $\varepsilon > 0$, there exists $\phi \in L^1(0,\infty;dt/t)$ satisfying $\|\phi\|_{L^1(0,\infty;dt/t)} = 1$ such that $\|\phi^{1/q'} g\chi_B\|_{T^{-s}_{1,q'}} \le \|g\chi_B\|_{\tilde{N}^{-s}_{1,q'}} + \varepsilon$. Thus, Hölder's inequality and Lemmas 13 and 27 imply that

$$\begin{aligned}&\sup_{\|g\|_{L^{q'}(B)}=1} \left|\int_B f(x)g(x)dx\right| \\ &= \sup_{\|g\|_{L^{q'}(B)}=1} \left|\int_0^\infty\int_{\mathbf{R}^n} \phi(t)f(x)g\chi_B(x)\frac{dxdt}{t}\right| \\ &\le C \sup_{\|g\|_{L^{q'}(B)}=1} \|\phi^{1/q}f\|_{T^s_{\infty,q}} \left\|\phi^{1/q'} g\chi_B\right\|_{T^{-s}_{1,q'}} \\ &\le C \sup_{\|g\|_{L^{q'}(B)}=1} \|f\|_{N^s_{\infty,q}} \left(\|g\chi_B\|_{\tilde{N}^{-s}_{1,q'}} + \varepsilon\right) \\ &\le C \sup_{\|g\|_{L^{q'}(B)}=1} \|f\|_{N^s_{\infty,q}} \left(r^{-s+(n/q)}\|g\chi_B\|_{q'} + \varepsilon\right) \\ &= C\left(r^{-s+(n/q)}\|f\|_{N^s_{\infty,q}} + \|f\|_{N^s_{\infty,q}}\varepsilon\right),\end{aligned} \quad (94)$$

for any $\varepsilon > 0$. Consequently, the arbitrariness of $\varepsilon > 0$ implies that

$$r^{s-(n/q)}\|f\|_{L^q(B)} \le C\|f\|_{N^s_{\infty,q}}, \quad (95)$$

for any ball $B = B(x_0, r)$, and it concludes the proof. □

Lemma 25 shows that the $N^s_{p,q}$-spaces are vested. We combine Lemmas 25 and 26 and Proposition 31.

Remark 32. Let $1 \le q < p \le \infty$ and $0 \le s \le n/q - n/p$. Then,

$$N^s_{p,q}(\mathbf{R}^n) \subset N^{s+(n/p)}_{\infty,q}(\mathbf{R}^n) = M^{s+(n/p)}_q(\mathbf{R}^n). \quad (96)$$

Here, we used Lemma 26 and Proposition 31. Furthermore, we obtain the following:

Remark 33. Let $1 \le p \le q < \infty$ and $n/q - n/p < -s \le 0$. Then,

$$N^{-s}_{p,q}(\mathbf{R}^n) \subset N^0_{p_0,q}(\mathbf{R}^n) \subset L^{p_0}(\mathbf{R}^n), \quad (97)$$

where $p_0 = (1/p - s/n)^{-1} < q$. Here, we used Lemmas 26 and 25.

4.2. Proof of Theorem 21. In this section, we prove Theorem 21.

Proof of Theorem 34 (0), (i). Let $f \in N^{-s}_{p',q'}(\mathbf{R}^n)$ and $g \in N^s_{p,q}(\mathbf{R}^n)$. Then, for any $\varepsilon > 0$, there exists $\phi \in L^1(0,\infty;dt/t)$ satisfying $\|\phi\|_{L^1(0,\infty;dt/t)} = 1$ and $(N^{-s}_{p',q'}, \phi)$-atom f_t such that

$$|f(x)| \le \left(\|f\|_{N^{-s}_{p',q'}} + \varepsilon\right)\int_0^\infty \phi(t)f_t(x)\frac{dt}{t}. \quad (98)$$

□

Thus, (40) implies that

$$\begin{aligned}&\int_{\mathbf{R}^n} |f(x)||g(x)|\, dx \\ &\le \left(\|f\|_{N^{-s}_{p',q'}} + \varepsilon\right)\int_0^\infty\int_{\mathbf{R}^n} \phi(t)f_t(x)|g(x)|\frac{dxdt}{t} \\ &\le C\left(\|f\|_{N^{-s}_{p',q'}} + \varepsilon\right)\left\|\phi^{1/q'}f_t\right\|_{T^{-s}_{p',q'}} \|\phi^{1/q}g\|_{T^s_{p,q}} \\ &\le \left(\|f\|_{N^{-s}_{p',q'}} + \varepsilon\right)\|g\|_{N^s_{p,q}},\end{aligned} \quad (99)$$

for any $\varepsilon > 0$. Thus, we have

$$\int_{\mathbf{R}^n} |f(x)||g(x)|\, dx \le C\|f\|_{N^{-s}_{p',q'}} \|g\|_{N^s_{p,q}}, \quad (100)$$

and it concludes the proof of Theorem 34 (0). From (100), we obtain

$$\|\ell_g\|_{\left(N^{-s}_{p',q'}\right)^*} = \sup_{\|f\|_{N^{-s}_{p',q'}}=1} \left|\int_{\mathbf{R}^n} f(x)g(x)dx\right| \le C\|g\|_{N^s_{p,q}}. \quad (101)$$

It shows Theorem 21 (i).

Proof of Theorem 34 (ii). We assume that ℓ is a bounded linear functional of $N^{-s}_{p',q'}(\mathbf{R}^n)$. Fix a ball $B = B(x_0, r)$. If f is supported on $B(x_0, r)$, then

$$\|f\|_{N^{-s}_{p',q'}} \le Cr^{-s+((n/q)-(n/p))}\left\|f\chi_{B(x_0,r)}\right\|_{q'} \quad (102)$$

from Lemma 27. Hence, ℓ induces a bounded linear functional on $L^{q'}(B(x_0, r))$ and acts with $g^B \in L^q(B(x_0, r))$. By taking $B_j = B(0, j)$ $(j \in \mathbf{N})$, we have $g^{B_j} = g^{B_{j+1}}$ on B_j, and thus we obtain a unique function g on $\mathbf{R}^n$ that is locally in $L^q(\mathbf{R}^n)$, such that $\ell(f) = \int_{\mathbf{R}^n} f(x)g(x)dx$ when $f \in N^{-s}_{p',q'}(\mathbf{R}^n)$ is supported on some ball. We can extend ℓ as a global functional over $\mathbf{R}^n$ using the following lemma:

Lemma 35. *Let $1 \le p \le q < \infty$, and $n/q - n/p < -s \le 0$. For $f \in N^{-s}_{p,q}(\mathbf{R}^n)$, let $f_j = f\chi_{B_j}$, where $B_j = B(0, j)$ $(j \in \mathbf{N})$. Then,*

$$\lim_{j\to\infty}\left\|f - f_j\right\|_{N^{-s}_{p,q}} = 0. \quad (103)$$

Proof. Because $f \in N^{-s}_{p,q}(\mathbf{R}^n)$, for any $\varepsilon_1 > 0$, there exists $\phi \in L^1(0,\infty;dt/t)$ satisfying $\|\phi\|_{L^1(0,\infty;dt/t)} = 1$ and an $(N^{-s}_{p,q}, \phi)$-atom f_t such that

$$\left|f - f_j\right| = \left|f\chi_{(B_j)^c}\right| \le \left(\|f\|_{N^{-s}_{p,q}} + \varepsilon_1\right)\int_0^\infty \phi(t)f_t\chi_{(B_j)^c}\frac{dt}{t}. \quad (104)$$

From the dominated convergence theorem, for any $\varepsilon_2 > 0$, there exists $N \in \mathbf{N}$ such that

$$\left\|\phi^{1/q}f_t\chi_{(B_j)^c}\right\|_{T^{-s}_{p,q}} < \varepsilon_2, \quad (105)$$

for $j \ge N$. Thus, the function $f_t\chi_{(B_j)^c}/\varepsilon_2$ is an $(N^{-s}_{p,q}, \phi)$-atom, and we have

$$\left\|f - f_j\right\|_{N^{-s}_{p,q}} \le \left(\|f\|_{N^{-s}_{p,q}} + \varepsilon_1\right)\cdot\varepsilon_2, \quad (106)$$

for $j \ge N$. This concludes the proof. □

To prove Theorem 21 (ii), it suffices to prove $\|g\|_{N^s_{p,q}} \le C\|\ell\|_{(N^{-s}_{p',q'})^*}$. At first, we prove $\|g\|_{N^s_{\infty,q}} \le C\|\ell\|_{(N^{-s}_{1,q'})^*}$, which is the case when $p = \infty$. Let $B = B(x_0, r)$. From Lemma 27, we have

$$\begin{aligned}\left(\int_{B(x_0,r)}|g(x)|^q\,dx\right)^{1/q} &\sim \sup_{\|f\|_{L^{q'}(B)}=1}\left|\int_B f(x)g(x)dx\right| \\ &\le \sup_{\|f\|_{L^{q'}(B)}=1}\|\ell\|_{\left(N^{-s}_{1,q'}\right)^*}\|f\chi_B\|_{N^{-s}_{1,q'}} \\ &\le Cr^{-s+(n/q)}\|\ell\|_{\left(N^{-s}_{1,q'}\right)^*},\end{aligned} \quad (107)$$

for any $x_0 \in \mathbf{R}^n$ and $r > 0$. Thus, we have $\|g\|_{N^s_{\infty,q}} \le C\|\ell\|_{(N^{-s}_{1,q'})^*}$ by Proposition 31. Next, we prove Theorem 21 (ii) for the case when $1 < p < \infty$. To prove $\|g\|_{N^s_{p,q}} \le C\|\ell\|_{(N^{-s}_{p',q'})^*}$ for $1 < p < \infty$, we present the following lemma.

Lemma 36. *Let $1 < q \le p < \infty$ and $n/q' - n/p' \le -s \le 0$. For $\phi \in L^1(0,\infty;dt/t)$ satisfying $\|\phi\|_{L^1(0,\infty;dt/t)} = 1$, define*

$$\pi^{s,\phi}_q(F)(x) = \int_0^\infty \phi(t)^{1/q}t^sF(x,t)\frac{dt}{t}. \quad (108)$$

Then,

$$\left\|\pi^{s,\phi}_q(F)\right\|_{N^{-s}_{p',q'}} \le \|F\|_{T^0_{p',q'}}, \quad (109)$$

for $F \in T^0_{p',q'}(\mathbf{R}^{n+1}_+)$.

Proof. From the definition of $\pi^{s,\phi}_q(F)(x)$, we get

$$\left|\pi^{s,\phi}_q(F)(x)\right| \le \|F\|_{T^0_{p',q'}}\int_0^\infty \phi(t)\frac{1}{\|F\|_{T^0_{p',q'}}}\phi(t)^{-1/q'}t^s|F(x,t)|\frac{dt}{t}. \quad (110)$$

Note that

$$\left\|\phi(t)^{1/q'}\frac{1}{\|F\|_{T_{p',q'}}}\phi(t)^{-1/q'}t^s|F(x,t)|\right\|_{T^{-s}_{p',q'}} = 1. \quad (111)$$

The function $(1/\|F\|_{T_{p',q'}})\phi(t)^{-1/q'}t^s|F(x,t)|$ is an $(N^{-s}_{p',q'}, \phi)$-atom, and we obtain

$$\left\|\pi^{s,\phi}_q(F)\right\|_{N^{-s}_{p',q'}} \le \|F\|_{T^0_{p',q'}}. \quad (112)$$

□

We now prove that $\|g\|_{N^s_{p,q}} \le C\|\ell\|_{(N^{-s}_{p',q'})^*}$ for $1<p<\infty$. For any $\phi \in L^1(0,\infty;dt/t)$ satisfying $\|\phi\|_{L^1(0,\infty;dt/t)}=1$, by applying Lemma 13, (40), and Lemma 36, we have

$$\begin{aligned}\|g\|_{N^{s,\phi}_{p,q}} &= \|t^s\phi^{1/q}g\|_{T^0_{p,q}} \sim \sup_{\|F\|_{T^0_{p',q'}}=1}\left|\int_0^\infty\int_{\mathbf{R}^n} t^s\phi(t)^{1/q}g(x)F(x,t)\frac{dxdt}{t}\right| \\ &= \sup_{\|F\|_{T^0_{p',q'}}=1}\left|\int_{\mathbf{R}^n} g(x)\pi^{s,\phi}_q(F)(x)dx\right| \\ &\le \sup_{\|F\|_{T^0_{p',q'}}=1}\|\ell\|_{\left(N^{-s}_{1,q'}\right)^*}\left\|\pi^{s,\phi}_q(F)\right\|_{N^{-s}_{p',q'}} \\ &\le C\sup_{\|F\|_{T^0_{p',q'}}=1}\|\ell\|_{\left(N^{-s}_{1,q'}\right)^*}\|F\|_{T^0_{p',q'}} \\ &= C\|\ell\|_{\left(N^{-s}_{1,q'}\right)^*}.\end{aligned} \tag{113}$$

Taking the supremum over $\phi\in L^1(0,\infty;dt/t)$ satisfying $\|\phi\|_{L^1(0,\infty;dt/t)}=1$, we obtain the conclusion.

4.3. Proof of Theorem 23. In this section, we prove Theorem 23. Notice that $L^p(\mathbf{R}^n)(1\le p<\infty)$ and $M^s_q(\mathbf{R}^n)(0\le s\le n/q)$ are Banach function lattices, satisfy the Fatou property, and are 1-convex. Furthermore, $L^p(\mathbf{R}^n)(1\le p<\infty)$ is separable. Thus, thanks to (32), it suffices to demonstrate the following:

Theorem 37. *Let $1\le p,q<\infty$, $0\le s\le n/q$, and $0\le\theta\le 1$. Then,*

$$(L^p(\mathbf{R}^n))^{1-\theta}\left(M^s_q(\mathbf{R}^n)\right)^{\theta} = N^{\theta s}_{p/(1-\theta),q_\theta}(\mathbf{R}^n) \tag{114}$$

where $1/q_\theta=(1-\theta)/p+\theta/q$.

Before proving Theorem 37, we check the consistency of Theorem 37 for the critical cases $s=0$ and $s=n/q$.

Lemma 38. *Let $1\le p<\infty$. Then,*

$$\|f\|_{N^0_{p,q}} \sim \|f\|_p, \tag{115}$$

for any $1\le q\le p$ and $f\in L^0(\mathbf{R}^n)$.

Proof. From Lemmas 24 and 25, we have

$$\|f\|_{N^0_{p,q}} \le \|f\|_p, \tag{116}$$

for any $q\le p$. Conversely, we set

$$\phi_\varepsilon(t) = \frac{\varepsilon^{n+\delta}}{t^{n+\delta}}\chi_{[\varepsilon,\infty)}(t), \tag{117}$$

for some fixed $\delta>0$. We also set

$$c_1 := \int_1^\infty \frac{1}{t^{n+\delta}}\frac{dt}{t} \quad\text{and}\quad c_2 := \int_1^\infty \frac{1}{t^\delta}\frac{dt}{t}. \tag{118}$$

Then, by Lebesgue differentiation theorem and $c_1=\|\phi_\varepsilon\|_{L^1(0,\infty;/dt/t)}$, we have

$$\begin{aligned}\|f\|_p &= \left\|\liminf_{\varepsilon\longrightarrow 0}\left(\frac{1}{\varepsilon^n}\int_{B(x,\varepsilon)}|f(y)|^q\,dy\right)^{1/q}\right\|_{L^p_x} \\ &= \frac{1}{c_1}\left\|\liminf_{\varepsilon\longrightarrow 0}\left(\int_0^\infty\int_{B(x,\varepsilon)}\phi_\varepsilon(t)\frac{1}{\varepsilon^n}|f(y)|^q\,\frac{dydt}{t}\right)^{1/q}\right\|_{L^p_x} \\ &\le \frac{1}{c_1}\liminf_{\varepsilon\longrightarrow 0}\left\|\left(\int_0^\infty\int_{B(x,t)}\frac{\varepsilon^\delta}{t^\delta}\chi_{[\varepsilon,\infty)}(t)|f(y)|^q\,\frac{dydt}{t^{n+1}}\right)^{1/q}\right\|_{L^p_x} \\ &\le \frac{c_2}{c_1}\|f\|_{N^0_{p,q}}.\end{aligned} \tag{119}$$

So, we are done. □

Letting $s=0$ in Theorem 37, we see that the property $M^0_q(\mathbf{R}^n)=L^\infty(\mathbf{R}^n)$ for any $1\le q<\infty$ yields

$$(L^p(\mathbf{R}^n))^{1-\theta}(L^\infty(\mathbf{R}^n))^\theta = N^0_{p/(1-\theta),q_\theta}(\mathbf{R}^n). \tag{120}$$

From Lemma 38, we have $N^0_{p/1-\theta,q_\theta}(\mathbf{R}^n)=L^{p/1-\theta}(\mathbf{R}^n)$, and it does not contradict Lemma 10. Furthermore, we have the following.

Proposition 39. *If $s=n/q-n/p$, one has*

$$\|f\|_{N^s_{p,q}} \sim \|f\|_q, \tag{121}$$

for any $f\in L^q(\mathbf{R}^n)$.

Proof. From Proposition 31 and Lemma 26, we have

$$\|f\|_q = \|f\|_{M^{n/q}_q} \sim \|f\|_{N^{n/q}_{\infty,q}} \le C\|f\|_{N^{((n/q)-(n/p))}_{p,q}}. \tag{122}$$

Next, Lemmas 24 and 26 imply that

$$\|f\|_{N^{((n/q)-(n/p))}_{p,q}} \le C\|f\|_{N^0_{q,q}} = \|f\|_q. \tag{123}$$

□

Setting $s=n/q$ in Theorem 37, we learn that $M^{n/q}_q(\mathbf{R}^n)=L^q(\mathbf{R}^n)$ implies that

$$(L^p(\mathbf{R}^n))^{1-\theta}(L^q(\mathbf{R}^n))^\theta = N^{\theta(n/q)}_{p/(1-\theta),q_\theta}(\mathbf{R}^n). \tag{124}$$

From $\theta n/q = n/q_\theta - (1-\theta)n/p$ and Proposition 39, we obtain $N^{\theta(n/q)}_{p/(1-\theta),q_\theta}(\mathbf{R}^n) = L^{q_\theta}(\mathbf{R}^n)$, and it does not contradict Lemma 10.

We now prove Theorem 37. First, we prove that $(L^p(\mathbf{R}^n))^{1-\theta}(M^s_q(\mathbf{R}^n))^\theta \subset N^{\theta s}_{p/(1-\theta),q_\theta}(\mathbf{R}^n)$. Let $f \in (L^p(\mathbf{R}^n))^{1-\theta}(M^s_q(\mathbf{R}^n))^\theta$. Then, for any $\varepsilon > 0$, there exists $f_1 \in L^p(\mathbf{R}^n)$ and $f_2 \in M^s_q(\mathbf{R}^n)$ such that $|f(x)| \le |f_1(x)|^{1-\theta}|f_2(x)|^\theta$ and

$$\|f_1\|^{1-\theta}_{L^p}\|f_2\|^\theta_{M^s_q} \le \|f\|_{(L^p)^{1-\theta}(M^s_q)^\theta} + \varepsilon. \quad (125)$$

It should be noted that $\|f^\theta\|_{T^s_{p,q}} = \|f\|^\theta_{T^{s/\theta}_{\theta p,\theta q}}$ for any $\theta > 0$. From (42), for any $\phi \in L^1(0,\infty; dt/t)$ satisfying $\|\phi\|_{L^1(0,\infty;dt/t)} = 1$, we have

$$\begin{aligned}
&\left\|\phi(t)^{1/q_\theta} f(x)\right\|_{T^{\theta s}_{p/(1-\theta),q_\theta}} \\
&\le \left\|\phi(t)^{1/q_\theta} f_1(x)^{1-\theta} f_2(x)^\theta\right\|_{T^{\theta s}_{p/(1-\theta),q_0}} \\
&\le C\left\|\phi(t)^{(1-\theta)/p} f_1(x)^{1-\theta}\right\|_{T^0_{p/(1-\theta),p/(1-\theta)}} \left\|\phi(t)^{\theta/q} f_2(x)^\theta\right\|_{T^{\theta s}_{\infty,q/\theta}} \\
&\le C\|f_1\|^{1-\theta}_p \|f_2\|^\theta_{M^s_q} \\
&\le C\left(\|f\|_{(L^p)^{1-\theta}(M^s_q)^\theta} + \varepsilon\right),
\end{aligned} \quad (126)$$

for any $\varepsilon > 0$. Thus, we obtain $\|f\|_{N^{\theta s}_{p/(1-\theta),q_\theta}} \le C\|f\|_{(L^p)^{1-\theta}(M^s_q)^\theta}$, which concludes the proof of $(L^p(\mathbf{R}^n))^{1-\theta}(M^s_q(\mathbf{R}^n))^\theta \subset N^{\theta s}_{p/1-\theta,q_\theta}(\mathbf{R}^n)$.

Next, we must prove $N^{\theta s}_{p/(1-\theta),q_\theta}(\mathbf{R}^n) \subset (L^p(\mathbf{R}^n))^{1-\theta}(M^s_q(\mathbf{R}^n))^\theta$. Let $\phi_0(t) = c\,t^\alpha \chi_{[0,1)}(t)$ for a sufficiently large $\alpha > 0$, where $c > 0$ denotes the number that satisfies $\|\phi_0\|_{L^1(0,\infty;dt/t)} = 1$. We assume that $f \in N^{\theta s}_{p/(1-\theta),q_\theta}(\mathbf{R}^n)$. Then, we have $\phi_0(t)^{1/q_\theta} f(x) \in T^{\theta s}_{p/(1-\theta),q_\theta}(\mathbf{R}^{n+1}_+)$. Thus, Lemma 15 implies that for any $\varepsilon > 0$, there exists functions $F(x,t) \in T^0_{p,p}(\mathbf{R}^{n+1}_+)$ and $G(x,t) \in T^s_{\infty,q}(\mathbf{R}^{n+1}_+)$, such that

$$\phi_0(t)^{1/q_\theta}|f(x)| \le |F(x,t)|^{1-\theta}|G(x,t)|^\theta, \quad (127)$$

$$\|F\|^{1-\theta}_{T^0_{p,p}}\|G\|^\theta_{T^s_{\infty,q}} \le \left\|\phi_0(t)^{1/q_\theta} f(x)\right\|_{T^{\theta s}_{p/(1-\theta),q_\theta}} + \varepsilon. \quad (128)$$

From (127), we obtain

$$\phi_0(t)|f(x)| \le \phi_0(t)^{1/q'_\theta}|F(x,t)|^{1-\theta}|G(x,t)|^\theta. \quad (129)$$

Since $\|\phi_0\|_{L^1(0,\infty;dt/t)} = 1$ and $1/q_\theta = (1-\theta)/p' + \theta/q'$, integrating both sides of the above inequality against the measure dt/t and using Hölder's inequality, we obtain

$$\begin{aligned}
|f(x)| &\le \int_0^\infty \phi_0(t)^{1/q'_\theta}|F(x,t)|^{1-\theta}|G(x,t)|^\theta \frac{dt}{t} \\
&\le \left(\int_0^\infty \phi_0(t)^{1/p'}|F(x,t)|\frac{dt}{t}\right)^{1-\theta}\left(\int_0^\infty \phi_0(t)^{1/q'}|G(x,t)|\frac{dt}{t}\right)^\theta \\
&=: |f_1(x)|^{1-\theta}|f_2(x)|^\theta.
\end{aligned} \quad (130)$$

Using $\|\phi_0\|_{L^1(0,\infty;dt/t)} = 1$ and Hölder's inequality again, we obtain

$$\|f_1\|_p \le \left\|\left(\int_0^\infty \phi_0(t)\frac{dt}{t}\right)^{1/p'}\left(\int_0^\infty |F(x,t)|\frac{dt}{t}\right)^{1/p}\right\|_p \sim \|F\|_{T^0_{p,p}}, \quad (131)$$

where we used (37) in the last equality. Next, we estimate f_2. Using (41), we have

$$\begin{aligned}
\|f_2\|_{L^q(B(x_0,1))} &\sim \sup_{\|g\|_{L^{q'}(B(x_0,1))}=1}\left|\int_0^\infty\int_{\mathbf{R}^n}\phi_0(t)^{1/q'}|G(x,t)|\,g(x)\chi_{B(x_0,1)}\frac{dx\,dt}{t}\right| \\
&\le C\sup_{\|g\|_{L^{q'}(B(x_0,1)}=1}\|G\|_{T^s_{\infty,q}}\left\|\phi_0(t)^{1/q'}g\chi_{B(x_0,1)}(x)\right\|_{T^{-s}_{1,q'}},
\end{aligned} \quad (132)$$

for any $x_0 \in \mathbf{R}^n$. Using the argument in the proof of Lemma 27, we obtain

$$\left\|\phi_0(t)^{1/q'}g\chi_{B(x_0,1)}(x)\right\|_{T^{-s}_{1,q'}} \le C\|g\|_{L^{q'}(B(x_0,1))}, \quad (133)$$

which implies that

$$\|f_2\|_{L^q(B(x_0,1))} \le C\|G\|_{T^s_{\infty,q}}, \quad (134)$$

for any $x_0 \in \mathbf{R}^n$. By combining (131), (134), and (128), we obtain that for any $\varepsilon > 0$, there exists functions $f_1 \in L^p(\mathbf{R}^n)$ and $f_2 \in L^q_{loc}(\mathbf{R}^n)$, such that $|f(x)| \le |f_1(x)|^{1-\theta}|f_2(x)|^\theta$ and

$$\begin{aligned}
\|f_1\|^{1-\theta}_p\left(\sup_{x_0\in\mathbf{R}^n}\|f_2\|_{L^q(B(x_0,1))}\right)^\theta &\le C\|F\|^{1-\theta}_{T^0_{p,p}}\|G\|^\theta_{T^s_{\infty,q}} \\
&\le \left\|\phi_0^{1/q_\theta}f\right\|_{T^{\theta s}_{p/(1-\theta),q_\theta}} + \varepsilon,
\end{aligned} \quad (135)$$

for any $f \in N^{\theta s}_{p/(1-\theta),q_\theta}(\mathbf{R}^n)$. Thus, we obtain

$$\|f_1\|_p^{1-\theta}\left(\sup_{x_0\in\mathbf{R}^n}\|f_2\|_{L^q(B(x_0,1))}\right)^\theta \le C\|f\|_{N^{\theta s}_{p/(1-\theta),q_\theta}}. \tag{136}$$

To conclude, we investigate the dilation property.

Lemma 40. *Let $1 \le p \le \infty$, $1 \le q < \infty$, and $0 \le s \le n/q - n/p$. Define $f_r(x) = f(x/r)$ for $r > 0$ and $f \in L^0(\boldsymbol{R}^n)$. Then,*

$$\|f_r\|_{N^s_{p,q}} = r^{s+(n/p)}\|f\|_{N^s_{p,q}}, \tag{137}$$

for any $f \in N^s_{p,q}(\mathbf{R}^n)$.

Proof. Let $f \in N^s_{p,q}(\mathbf{R}^n)$ and $\phi \in L^1(0,\infty;dt/t)$ satisfying $\|\phi\|_{L^1(0,\infty;dt/t)} = 1$. Considering that $\phi(r\,\cdot) \in L^1(0,\infty;dt/t)$ with $\|\phi(r\,\cdot)\|_{L^1(0,\infty;dt/t)} = 1$ for any $r > 0$, we have

$$\begin{aligned}\|f_r\|_{N^{s,\phi}_{p,q}} &= \left\|\left(\int_0^\infty\int_{B(x,t)} t^{sq}\phi(t)\left|f\left(\frac{y}{r}\right)\right|^q \frac{dydt}{t^{n+1}}\right)^{1/q}\right\|_{L^p_x}\\ &= \left\|\left(\int_0^\infty\int_{B(x,t)} r^{sq}\left(\frac{t}{r}\right)^{sq}\phi\left(r\frac{t}{r}\right)\left|f\left(\frac{y}{r}\right)\right|^q \frac{dydt}{t^{n+1}}\right)^{1/q}\right\|_{L^p_x}\\ &= r^{(s+(n/p))}\left\|\left(\int_0^\infty\int_{B(x,t)} t^{sq}\phi(rt)|f(y)|^q \frac{dydt}{t^{n+1}}\right)^{1/q}\right\|_{L^p_x}\\ &= r^{(s+(n/p))}\|f\|_{N^{s,\phi(r\cdot)}_{p,q}}.\end{aligned} \tag{138}$$

Thus, we obtain $\|f_r\|_{N^s_{p,q}} = r^{(s+(n/p))}\|f\|_{N^s_{p,q}}$. □

Now, all preparations to prove $N^{\theta s}_{p/(1-\theta),q_\theta}(\mathbf{R}^n) \subset (L^p(\mathbf{R}^n))^{1-\theta}(M^s_q(\mathbf{R}^n))^\theta$ have been provided. From Lemma 40, we have $f \in N^{\theta s}_{p/(1-\theta),q_\theta}(\mathbf{R}^n)$ if and only if $f_r \in N^{\theta s}_{p/(1-\theta),q_\theta}(\mathbf{R}^n)$ for any $r > 0$. We fix an arbitrary $r > 0$. Then, from (136) and Lemma 40, we obtain

$$\begin{aligned}&\|(f_1)_r\|_p^{1-\theta}\left\{\sup_{x_0\in\mathbf{R}^n} r^s\left(\frac{1}{r^n}\int_{B(x_0,r)}|(f_2)_r(y)|^q\,dy\right)^{1/q}\right\}^\theta\\ &= r^{\theta s+(n(1-\theta)/p)}\|f_1\|_p\left(\sup_{x_0\in\mathbf{R}^n}\|f_2\|_{L^q(B(x_0,1))}\right)^\theta\\ &\le Cr^{\theta s+(n(1-\theta)/p)}\|f\|_{N^{\theta s}_{p/(1-\theta),q_\theta}}\\ &= \|f_r\|_{N^{\theta s}_{p/(1-\theta),q_\theta}}.\end{aligned} \tag{139}$$

Note that

$$|f(x)| \le |f_1(x)|^{1-\theta}|f_2(x)|^\theta \Leftrightarrow |f_r(x)| \le \left|(f_1)_r(x)\right|^{1-\theta}\left|(f_2)_r(x)\right|^\theta, \tag{140}$$

for any $r > 0$. By replacing $f_r = g$ for any $g \in N^{\theta s}_{p/(1-\theta),q_\theta}(\mathbf{R}^n)$, there exist meaurable functions g_1, g_2 such that

$$\begin{aligned}&|g(x)| \le |g_1(x)|^{1-\theta}|g_2(x)|^\theta,\\ &\|g_1\|_p^{1-\theta}\left\{\sup_{x_0\in\mathbf{R}^n} r^s\left(\frac{1}{r^n}\int_{B(x_0,r)}|g_2(y)|^q\,dy\right)^{1/q}\right\}^\theta\\ &\quad\le C\|g\|_{N^{\theta s}_{p/(1-\theta),q_\theta}}.\end{aligned} \tag{141}$$

Because $r > 0$ is arbitrary, we have the conclusion.

5. An Application of Complex Interpolation Spaces

As basic properties of function spaces, we will prove the boundedness of Calderón–Zygmund operators. Calderón–Zygmund operators are defined by the following:

For a bounded linear operator $T : L^2(\mathbf{R}^n) \longrightarrow L^2(\mathbf{R}^n)$, we say that T is a Calderón–Zygmund operator if there exists a kernel $k(x, y)$, and it satisfies the following conditions:

(i) $Tf(x) = \int_{\mathbf{R}^n} k(x,y)f(y)\,dy$ for $x \notin \operatorname{supp} f$

(ii) $|k(x,y)| \le C(1/|x-y|^n)$ for any $x, y \in \mathbf{R}^n$

(iii) $|k(x,y) - k(z,y)| + |k(y,x) - k(y,z)| \le C(|z-y|/|x-y|^{n+1})$ whenever $|z-y| < (1/2)|x-y|$

In fact, these operators are bounded on Lebesgue spaces $L^p(\mathbf{R}^n)$ $(1 < p < \infty)$ (see Stein [17]) and Morrey spaces $M^s_q(\mathbf{R}^n)$ $(1 < q < \infty,\ 0 < s < n/q)$ (see F. Chiarenza and M. Frasca [18]). Using Theorem 37, we obtain the boundedness in $N^s_{p,q}$-spaces.

Theorem 41. *For $1 < p \le \infty$, $1 < q < \infty$, and $0 \le s \le n/q - n/p$, any Calderón–Zygmund operator T are bounded on $N^s_{p,q}(\boldsymbol{R}^n)$.*

Proof. For the case where $s = 0$, it immediately follows by the boundedness of T in Lebesgue spaces and Lemma 38. Furthermore, for the case where $p = \infty$, we use the boundedness in Morrey spaces and Proposition 31. We prove the other cases. Combining Lemma 8 with Theorem 23 and the boundedness of T in $L^p(\mathbf{R}^n)$ and $M^s_q(\mathbf{R}^n)$, we see

$$\|T\|_{N^{\theta s}_{p/(1-\theta),q_\theta}\longrightarrow N^{\theta s}_{p/(1-\theta),q_\theta}} \le C\|T\|^{1-\theta}_{L^p\longrightarrow L^p}\|T\|^\theta_{M^s_q\longrightarrow M^s_q}, \tag{142}$$

for any $0 < \theta < 1$, $1 < p, q < \infty$, and $0 < s < n/q$. Thus, we obtain $\|T\|_{N^s_{p,q}\longrightarrow N^s_{p,q}} < \infty$ for any $1 < p \le \infty$, $1 < q < \infty$, and $0 \le s \le n/q - n/p$. □

Data Availability

Data is not applicable in this manuscript. I introduce some previous researches I referred. A. Amenta, Tent spaces over metric measure spaces under doubling and related assumptions, Operator Theory in Harmonic and Non-commutative Analysis Vol. 240 (2014), 1-29. Y. Huang, Weighted tent spaces with Whitney averages: factorizations, interpolation, and duality, Math. Zeitschrift\textbf {24} (2016), 913-933. R. R. Coifman, Y. Meyer, E. M. Stein, Some new function spaces and their applications to harmonic analysis, J. Funct. Anal. Vol. 62 No. 2 (1985), 304-335.

Conflicts of Interest

The author declares no conflicts of interest.

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