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Research Article

A Two-Point Boundary Value Problem with Reflection of the Argument

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We consider the following two-point boundary value problems $u''(x)+u(\pi-x)+g(x,u(\pi-x))=h(x)$ in $(0,\pi)$, $u(0)=0=u(\pi)$, and $u''(x)+u(\pi-x)-g(x,u(\pi-x))=-h(x)$ in $(0,\pi)$, $u(0)=0=u(\pi)$, by setting $h\in L^1(0,\pi)$ and $g:(0,\pi)\times R\longrightarrow R$ being a Caratheodory function. When $a,b\in L^1(0,\pi)$, $a(x)\le 3$ for $x\in(0,\pi)$ a.e. with strict inequality on a positive measurable subset of $(0,\pi)$, and $|g(x,u)|\le a(x)|u|+b(x)$ for $x\in(0,\pi)$ a.e. as well as sufficiently large $|u|$, several existence theorems will be obtained, with or without a sign condition.

1. Introduction

Let us study the existence of solutions of the two-point boundary value problems with reflection of the argument:

$$u''(x)+u(\pi-x)+g(x,u(\pi-x))=h(x)\text{ in }(0,\pi),\quad u(0)=0=u(\pi),\tag{1}$$

$$u''(x)+u(\pi-x)-g(x,u(\pi-x))=-h(x)\text{ in }(0,\pi),\quad u(0)=0=u(\pi),\tag{2}$$

by setting $h\in L^1(0,\pi)$ and $g:(0,\pi)\times R\longrightarrow R$ being a Caratheodory function. That means

(i) $g(x,u)$ is continuous in $u\in R$ for $x\in(0,\pi)$ a.e.,

(ii) $g(x,u)$ is measurable in $x\in(0,\pi)$ for all $u\in R$, and

(iii) for each $r>0$, there exists an $a_r\in L^1(0,\pi)$ such that

$$|g(x,u)|\le a_r(x),\tag{3}$$

for $x\in(0,\pi)$ a.e. and for all $|u|\le r$. Concerning the nonlinear growth of g, let us make an assumption (H):

(H) There exists a constant $r_0>0$, for $a,b,c,d\in L^1(0,\pi)$, $a,b\ge 0$ and $a(x)\le 3$ for $x\in(0,\pi)$ a.e. with strict inequality on a positive measurable subset of $(0,\pi)$, such that

(i) for $x\in(0,\pi)$ a.e. and for all $u\ge r_0$

$$c(x)\le g(x,u)\le a(x)|u|+b(x),\tag{4}$$

as well as

(ii) for $x\in(0,\pi)$ a.e. and for all $u\le -r_0$

$$-a(x)|u|-b(x)\le g(x,u)\le d(x)\tag{5}$$

both hold.

On the other hand, either with or without a Landesman-Lazer condition (see (15) below), solvability of the resonance problem

$$u''(x)+u(x)+g(x,u(x))=h(x)\text{ in }(0,\pi),\ u(0)=0=u(\pi)\tag{6}$$

has been extensively studied under the condition that the nonlinearity of $g(x,u)$ is assumed to have either the following:

(i) linear growth in u as $|u| \longrightarrow \infty$ (see [1–11])

(ii) superlinear growth in u in one of the directions $u \longrightarrow \infty$ or $u \longrightarrow -\infty$, as well as may be bounded in the opposite direction (see [12, 13])

Similar study on (1) in addition to a new order term under a different assumption has been done by [14]. The research on (2) has been first studied by [15] when $g(x,u)$ is bounded, while [16] focused on the nonresonance case by allowing $g(x,u)$ to grow linearly in u as $|u| \longrightarrow \infty$. However, research on the boundary value problems (1) and (2) has been studied but not thoroughly enough.

The purpose of this paper is to establish solvability theorems for (1) and (2) when (H) is satisfied. Based on the well-known Leray-Schauder continuation method (see [17, 18]) some new solvability results will be obtained, with or without a sign condition (that is $c = 0 = d$ in (H) with $r_0 = 0$, and $\int_0^\pi h(x)\sin x dx = 0$).

We shall make use of real Banach spaces $L^p(0,\pi)$, $C[0,\pi]$, $C^1[0,\pi]$, as well as Sobolev spaces $W^{2,1}(0,\pi)$ and $H^1(0,\pi)$ in the following procedure. The norms of $L^p(0,\pi)$, $C[0,\pi]$, $C^1[0,\pi]$, and $H^1(0,\pi)$ are denoted by $\|u\|_{L^p}$, $\|u\|_C$, $\|u\|_{C^1}$, and $\|u\|_{H^1}$, respectively. By saying "a solution of (1)," we mean that $u \in W^{2,1}(0,\pi)$ with $u(0) = 0 = u(\pi)$ and satisfies the differential equation in (1), $x \in (0,\pi)$ a.e.

2. Existence Theorems

For each $v \in W^{2,1}(0,\pi)$ with $v(0) = 0 = v(\pi)$, we write $\bar{v}(x) = (2/\pi\int_0^\pi v(x)\sin t dt)\sin x$ and $\tilde{v} = v - \bar{v}$. To obtain the main results of this paper with reflection of the argument, we need to deduce the following two lemmas which are extensions of [7], Lemma 1.

Lemma 1. *Let a be a nonnegative $L^1(0,\pi)$ -function such that for $x \in (0,\pi)$ a.e., $a(x) \le 3$ with strict inequality on a positive measurable subset of $(0,\pi)$. Then, there exists a constant $K_1 > 0$ such that*

$$\int_0^\pi (\bar{u}(x) - \tilde{u}(x))\left[u''(x) + u(\pi - x) + p(x)u(\pi - x)\right]dx \ge K_1\|\tilde{u}\|_{H^1}^2, \tag{7}$$

whenever $p \in L^1(0,\pi)$ with $0 \le p(x) \le a(x)$ for $x \in (0,\pi)$ a.e., and $u \in W^{2,1}(0,\pi)$ with $u(0) = 0 = u(\pi)$.

Proof. Just as in the proof in [7], Lemma 1, there exists a constant $K_1 > 0$ such that

$$\int_0^\pi \left(\tilde{u}'(x)\right)^2 - (1 + p(x))(\tilde{u}(x))^2 dx \ge K_1\|\tilde{u}\|_{H^1}^2, \tag{8}$$

whenever $p \in L^1(0,\pi)$ with $0 \le p(x) \le a(x)$ for $x \in (0,\pi)$ a.e. and $u \in W^{2,1}(0,\pi)$ with $u(0) = 0 = u(\pi)$. Since $\bar{u}(\pi - x) = \bar{u}(x)$ and $\bar{u}''(x) = -\bar{u}(x)$, we have

$$\begin{aligned}
\int_0^\pi [\bar{u}(x) - \tilde{u}(x)]u(\pi - x)dx &= \int_0^\pi [\bar{u}(x) - \tilde{u}(x)][\bar{u}(\pi - x) + \tilde{u}(\pi - x)]dx \\
&= \int_0^\pi \bar{u}(x)\bar{u}(\pi - x)dx - \int_0^\pi \tilde{u}(x)\tilde{u}(\pi - x)dx \\
&= \int_0^\pi [\bar{u}(x)]^2 dx - \int_0^\pi \tilde{u}(x)\tilde{u}(\pi - x)dx, \\
\int_0^\pi [\bar{u}(x) - \tilde{u}(x)]u''(x)dx &= \int_0^\pi [\bar{u}(x) - \tilde{u}(x)]\left[-\bar{u}(x) + \tilde{u}''(x)\right]dx \\
&= -\int_0^\pi [\bar{u}(x)]^2 dx - \int_0^\pi \tilde{u}(x)\tilde{u}''(x)dx \\
&= -\int_0^\pi [\bar{u}(x)]^2 dx + \int_0^\pi \left[\tilde{u}'(x)\right]^2 dx,
\end{aligned} \tag{9}$$

therefore

$$\begin{aligned}
&\int_0^\pi [\bar{u}(x) - \tilde{u}(x)]\left[u''(x) + u(\pi - x)\right]dx \\
&\quad = \int_0^\pi \left\{\left[\tilde{u}'(x)\right]^2 - \tilde{u}(x)\tilde{u}(\pi - x)\right\}dx.
\end{aligned} \tag{10}$$

Furthermore, since $\int_0^\pi (\tilde{u}(\pi - x))^2 dx = \int_0^\pi (\tilde{u}(x))^2 dx$ and $\int_0^\pi (\tilde{u}'(\pi - x))^2 dx = \int_0^\pi (\tilde{u}'(x))^2 dx$, we have $\int_0^\pi (u'(x))^2 - (u(x))^2 dx = \int_0^\pi (u'(\pi - x))^2 - (u(\pi - x))^2 dx$, and

$$\begin{aligned}
&\int_0^\pi (\bar{u}(x) - \tilde{u}(x))\left[u''(x) + u(\pi - x) + p(x)u(\pi - x)\right]dx \\
&= \int_0^\pi \left(\tilde{u}'(x)\right)^2 - \tilde{u}(x)\tilde{u}(\pi - x)dx \\
&\quad + \int_0^\pi [\bar{u}(x) - \tilde{u}(x)]p(x)u(\pi - x)dx \\
&\ge \int_0^\pi \left(\tilde{u}'(x)\right)^2 - \frac{1}{2}\left[(\tilde{u}(x))^2 + (\tilde{u}(\pi - x))^2\right]dx \\
&\quad + \int_0^\pi [\bar{u}(x) - \tilde{u}(x)]p(x)u(\pi - x)dx \\
&= \int_0^\pi \left(\tilde{u}'(x)\right)^2 - \frac{1}{2}\left[(\tilde{u}(x))^2 + (\tilde{u}(\pi - x))^2\right]dx \\
&\quad + \int_0^\pi [\bar{u}(x) - \tilde{u}(x)]p(x)u(\pi - x)dx \\
&= \int_0^\pi \left(\tilde{u}'(x)\right)^2 - \frac{1}{2}\left[(\tilde{u}(x))^2 + (\tilde{u}(\pi - x))^2\right]dx \\
&\quad - \frac{1}{2}\int_0^\pi p(x)\left[(\tilde{u}(x))^2 + (\tilde{u}(\pi - x))^2\right]dx \\
&\quad + \frac{1}{2}\int_0^\pi p(x)\left[(u(\pi - x) - \tilde{u}(x))^2 + (\bar{u}(x))^2\right]dx \\
&\ge K_1\|\tilde{u}\|_{H^1}^2.
\end{aligned} \tag{11}$$

Just as in the proof of Lemma 1, we can apply the equality

$$\begin{aligned}&\int_0^\pi [\bar{u}(x)-\tilde{u}(x)]\left[u''(x)+u(x)\right]dx\\&=\int_0^\pi\left[\tilde{u}'(x)\right]^2dx-\int_0^\pi[\tilde{u}(x)]^2dx\\&=\int_0^\pi\left[\tilde{u}'(x)\right]^2dx-\frac{1}{2}\int_0^\pi\left\{[\tilde{u}(x)]^2+[\tilde{u}(\pi-x)]^2\right\}dx\end{aligned}\tag{12}$$

to obtain the next lemma when $u(\pi-x)$ is replaced by $u(x)$. □

Lemma 2. *Let a be a nonnegative $L^1(0,\pi)$-function such that for $x\in(0,\pi)$ a.e., $a(x)\le 3$ with strict inequality on a positive measurable subset of $(0,\pi)$. Then, there exists a constant $K_1>0$ such that*

$$\int_0^\pi -u(x)\left(u''(x)+u(\pi-x)-p(x)u(\pi-x)\right)dx\ge K_1\|\tilde{u}\|_{H^1}^2,\tag{13}$$

whenever $p\in L^1(0,\pi)$ with $0\le p(x)\le a(x)$ for $x\in(0,\pi)$ a. e., and $u\in W^{2,1}(0,\pi)$ with $u(0)=0=u(\pi)$.

Proof.

$$\begin{aligned}&\int_0^\pi -u(x)\left(u''(x)+u(\pi-x)-p(x)u(\pi-x)\right)dx\\&=\int_0^\pi\left(u'(x)\right)^2-u(x)u(\pi-x)+p(x)u(\pi-x))u(x)dx\\&\ge\int_0^\pi\left(u'(x)\right)^2dx-\frac{1}{2}\int_0^\pi\left[(u(x))^2+(u(\pi-x))^2\right]dx\\&\quad-\frac{1}{2}\int_0^\pi p(x)\left[(\tilde{u}(x))^2+(\tilde{u}(\pi-x))^2\right]dx\\&\quad+\frac{1}{2}\int_0^\pi p(x)\left[(u(\pi-x)+\tilde{u}(x))^2+(\bar{u}(x))^2\right]dx\\&\ge K_1\|\tilde{u}\|_{H^1}^2,\end{aligned}\tag{14}$$

for some constant $K_1>0$ independent of $p\in L^1(0,\pi)$ with $0\le p(x)\le a(x)$ for $x\in(0,\pi)$ a.e., and $u\in W^{2,1}(0,\pi)$ with $u(0)=0=u(\pi)$. □

Theorem 3. *Let $g:(0,\pi)\times R\longrightarrow R$ be a Caratheodory function satisfying (H). Then for each $h\in L^1(0,\pi)$, the problem (1) has a solution u, provided that*

$$\int_0^\pi g_-(x)\sin x dx<\int_0^\pi h(x)\sin x dx<\int_0^\pi g_+(x)\sin x dx\tag{15}$$

holds, where $g_+(x)=\liminf_{u\longrightarrow\infty}g(x,u)$ and $g_-(x)=\limsup_{u\longrightarrow-\infty}g(x,u)$.

Proof. Given a fixed $\alpha\in R$, $0<\alpha<3$. Consider the boundary value problems

$$\begin{aligned}u''(x)+u(\pi-x)+(1-t)\alpha u(\pi-x)+tg(x,u(\pi-x))&=th(x)\text{ in }(0,\pi),\\u(0)=0=u(\pi),&\end{aligned}\tag{16}$$

for $0\le t\le 1$, which becomes the original problem when $t=1$. Since $0<\alpha<3$, (16) has only a trivial solution when $t=0$ by Lemma 1. To apply the Leray-Schauder continuation method, it suffices to show first that solutions to (16) for $0<t<1$ have a priori bound in $H^1(0,\pi)$. To this end, let $\theta:R\longrightarrow R$ be a continuous function such that $0\le\theta\le 1$, $\theta(u)=0$ for $|u|\le r_0$, and $\theta(u)=1$ for $|u|\ge 2r_0$. Define $e(x)=\max\{a_{r_0}(x),b(x),|c(x)|,|d(x)|\}$,

$$g_1(x,u)=\begin{cases}\min\{g(x,u)+e(x),a(x)u\}\theta(u),&\text{if }u\ge 0,\\\max\{g(x,u)-e(x),a(x)u\}\theta(u),&\text{if }u\le 0,\end{cases}\tag{17}$$

and $g_2(x,u)=g(x,u)-g_1(x,u)$. Then, $g_1,g_2:(0,\pi)\times R\longrightarrow R$ are Caratheodory functions, such that for $x\in(0,\pi)$ a. e. and $u\in R$, $u\ne 0$,

$$0\le\frac{g_1(x,u)}{u}\le a(x),\tag{18}$$

and for $x\in(0,\pi)$ a.e. and $u\in R$

$$|g_2(x,u)|\le e(x).\tag{19}$$

If u is a possible solution to (16) for some $0<t<1$, then by using (18), (19), and Lemma 1, we have

$$\begin{aligned}0&=\int_0^\pi(\bar{u}(x)-\tilde{u}(x))\left[u''(x)+u(\pi-x)\right.\\&\quad\left.+(1-t)\alpha u(\pi-x)+tg(x,u(\pi-x))-th(x)\right]dx\\&\ge K_1\|\tilde{u}\|_{H^1}^2-(\|e\|_{L^1}+\|h\|_{L^1})(|\bar{u}|+\|\tilde{u}\|_C)\\&\ge K_1\|\tilde{u}\|_{H^1}^2-C_1(|\bar{u}|+\|\tilde{u}\|_{H^1}),\end{aligned}\tag{20}$$

which implies that

$$\|\tilde{u}\|_{H^1}^2\le\frac{C_1}{K_1}(|\bar{u}|+\|\tilde{u}\|_{H^1}),\tag{21}$$

for some constant $C_1>0$ independent of u. It remains to show that solutions of (16) for $0<t<1$ have an a priori bound in $H^1(0,\pi)$. We will show this by contradiction. Suppose that there exists a sequence $\{u_n\}$ and a corresponding sequence $\{t_n\}$ in $(0,1)$ such that u_n is a solution of (16) with $t=t_n$ and $\|u_n\|_{H^1}\ge n$ for all n. Let $v_n=u_n/\|u_n\|_{H^1}$, then $\|v_n\|_{H^1}=1$ for all $n\in N$, and by (21), we have $\|\tilde{v}_n\|_{H^1}\longrightarrow 0$ as $n\longrightarrow\infty$. Since $\|v_n\|_{H^1}=1$ and $\|\bar{v}_n\|_{H^1}\le\|v_n\|_{H^1}+\|\tilde{v}_n\|_{H^1}$ for all $n\in N$, we may assume without loss of generality that $\{v_n\}$ converges to v in $H^1(0,\pi)$ with $v(x)=\beta\sin x$ for some $\beta\ne 0$, and $\{v''_n\}$

is pointwise bounded by an $L^1(0,\pi)$-function independent of $n\in N$. In particular, $\{v_n'\}$ converges uniformly on $[0,\pi]$, which implies that $\{v_n\}$ converges to v in $C^1[0,\pi]$. Now, let us consider only the case $\beta>0$, for the case $\beta<0$ can be treated similarly. Using the elementary inequality

$$\left|\frac{w(x)}{\sin x}\right| \le \frac{\pi}{2}\left\|w'\right\|_C, \tag{22}$$

for all $x\in[0,\pi]$ as well as $w\in C^1[0,\pi]$ with $w(0)=0=w(\pi)$, and the fact that $\{\tilde{v}_n\}$ converges to 0 uniformly on $[0,\pi]$, we have $v_n(x)\ge(\beta/2)\sin x$ on $[0,\pi]$ for sufficiently large n. Multiplying each side of (16) by $\sin x$, and integrating them over $[0,\pi]$ when $u=u_n$ and $t=t_n$, we find

$$\begin{aligned} t_n\int_0^\pi g(x,u_n(\pi-x))\sin x dx &< (1-t_n)\alpha\int_0^\pi u_n(\pi-x)\sin x dx \\ &\quad + t_n\int_0^\pi g(x,u_n(\pi-x))\sin x dx = t_n\int_0^\pi h(x)\sin x dx, \end{aligned} \tag{23}$$

for sufficiently large n. It follows from (3) and (H) that $g(x,u_n(\pi-x))$ is bounded below by a function in $L^1(0,\pi)$ independent of $n\in N$. By applying Fatou's lemma to the inequality $\int_0^\pi g(x,u_n(\pi-x))\sin x dx \le \int_0^\pi h(x)\sin x dx$, we find $\int_0^\pi g_+(x)\sin x dx\le\int_0^\pi h(x)\sin x dx$, which contradicts the second inequality in (15). Therefore, the theorem is proven. □

Theorem 4. *Let $g:(0,\pi)\times R\longrightarrow R$ be a Caratheodory function satisfying (H) with $g(x,u)u\ge 0$ for $x\in(0,\pi)$ a.e. and for all $u\in R$. Then for each $h\in L^1(0,\pi)$, the problem (1) has a solution u, provided that $\int_0^\pi h(x)\sin x dx=0$.*

By modifying the proof of Theorem 3 slightly, we obtain the next solvability theorem, where the nonlinearity of g satisfies the following condition:

(F) There exist constants $r_0^{\sim}\ge 0, 0\le\gamma,\delta\le 1$ and $c^{\sim}$, $d^{\sim}\in L^1(0,\pi)$ such that for $x\in(0,\pi)$ a.e. and for all $u\ge r_0^{\sim}$

$$g(x,u)u\ge\tilde{c}(x)|u|^{1-\gamma}. \tag{24}$$

Also for $x\in(0,\pi)$ a.e. and for all $u\le -r_0^{\sim}$

$$g(x,u)u\ge\tilde{d}(x)|u|^{1-\delta}. \tag{25}$$

At the same time, condition (15) may be replaced by the inequality:

$$\int_0^\pi g_-^\delta(x)\sin^{1-\delta}x dx<\int_0^\pi h(x)\sin x dx=0<\int_0^\pi g_+^\gamma(x)\sin^{1-\gamma}x dx, \tag{26}$$

where $g_-^\delta(x)=limsup_{u\longrightarrow-\infty}g(x,u)|u|^\delta$ *and* $g_+^\gamma(x)=\liminf_{u\longrightarrow\infty}g(x,u)|u|^\gamma$.

Theorem 5. *Let $g:(0,\pi)\times R\longrightarrow R$ be a Caratheodory function satisfying (F) and (H). Then for each $h\in L^1(0,\pi)$, problem (1) has a solution u, provided that (26) holds.*

Proof. In the process of showing Theorem 3, (15) is used only to see the contradiction in the end. Thus, we may follow exactly the same process as in the proof of Theorem 3, to the point where $\beta>0$ on $(0,\pi)$ is considered and (23) holds. By (F) and (22), we find that

$$\begin{aligned} g(x,u_n(\pi-x))\|u_n\|_{H^1}^\gamma &= g(x,u_n(\pi-x))|u_n(\pi-x)|^\gamma|v_n(\pi-x)|^{-\gamma} \\ &\ge -\left|\tilde{c}(x)\right||v_n(\pi-x)|^{-\gamma} \\ &\ge -\left|\tilde{c}(x)\right|\left[\frac{\beta}{2}\sin(\pi-x)\right]^{-\gamma} \\ &= -\left|\tilde{c}(x)\right|\left(\frac{\beta}{2}\sin x\right)^{-\gamma}, \end{aligned} \tag{27}$$

for $x\in(0,\pi)$ a.e. with $|u_n(\pi-x)|\ge r_0^{\sim}$ and for all $n\ge n_0$. By (3) and (22), we also have

$$\begin{aligned} g(x,u_n(\pi-x))\|u_n\|_{H^1}^\gamma &= g(x,u_n(\pi-x))|u_n(\pi-x)|^\gamma|v_n(\pi-x)|^{-\gamma} \\ &\ge -\left|a_{\tilde{r}_0}(x)\right|\tilde{r}_0^{\gamma}|v_n(\pi-x)|^{-\gamma} \\ &\ge -\left|a_{\tilde{r}_0}(x)\right|\tilde{r}_0^{\gamma}\left[\frac{\beta}{2}\sin(\pi-x)\right]^{-\gamma} \\ &= -\left|a_{\tilde{r}_0}(x)\right|\tilde{r}_0^{\gamma}\left(\frac{\beta}{2}\sin x\right)^{-\gamma}, \end{aligned} \tag{28}$$

for $x\in(0,\pi)$ a.e. with $|u_n(\pi-x)|\le r_0^{\sim}$ and for all $n\ge n_0$. Combining (27) and (28), we find that $g(x,u_n(\pi-x))\|u_n\|_{H^1}^\gamma\sin x$ is bounded below by a function in $L^1(0,\pi)$ independent of $n\ge n_0$. By applying Fatou's lemma to the left hand side of the following inequality

$$\|u_n\|_{H^1}^\gamma\int_0^\pi g(x,u_n(\pi-x))\sin x dx<\|u_n\|_{H^1}^\gamma\int_0^\pi h(x)\sin x dx=0, \tag{29}$$

we find

$$\beta^{-\gamma}\int_0^\pi g_+^\gamma(x)\sin^{1-\gamma}x dx\le 0, \tag{30}$$

or equivalently

$$\int_0^\pi g_+^\gamma(x)\sin^{1-\gamma}x dx\le 0, \tag{31}$$

which contradicts the second inequality in (26). Thus, we have proven the theorem. □

Again, by modifying the proofs of Theorem 3, Theorem 4, and Theorem 5 slightly, we may use Lemma 2 to obtain the following solvability results for (2).

Theorem 6. *Let $g : (0, \pi) \times R \longrightarrow R$ be a Caratheodory function satisfying (H). Then for each $h \in L^1(0, \pi)$, the problem (2) has a solution u, provided that (15) holds.*

Theorem 7. *Let $g : (0, \pi) \times R \longrightarrow R$ be a Caratheodory function satisfying (H) with $g(x, u)u \geq 0$ for $x \in (0, \pi)$ a.e. and for all $u \in R$. Then for each $h \in L^1(0, \pi)$, the problem (2) has a solution u, provided that $\int_0^{\pi} h(x) \sin x dx = 0$.*

Theorem 8. *Let $g : (0, \pi) \times R \longrightarrow R$ be a Caratheodory function satisfying (F) and (H). Then for each $h \in L^1(0, \pi)$, the problem (2) has a solution u, provided that (26) holds.*

Remark 9. The conclusions of Theorem 5 and Theorem 8 both remain true if a_r, $c^{\sim}$, $d^{\sim} \in L^{\infty}(0, \pi)$, but to use the condition $0 \leq \gamma, \delta < 2$ instead of $0 \leq \gamma, \delta \leq 1$ in (F).

Also by applying Lemma 2, the next result may be obtained when $u(\pi - x)$ is replaced by $u(x)$.

Remark 10. Our results remain valid when problems (1) and (2) are replaced by

$$u''(x) + u(x) + g(x, u(\pi - x)) = h(x) \text{ in } (0, \pi), u(0) = 0 = u(\pi),$$
$$u''(x) + u(x) - g(x, u(\pi - x)) = -h(x) \text{ in } (0, \pi), u(0) = 0 = u(\pi), \tag{32}$$

respectively.

Data Availability

Data will be made available upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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