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Research Article

On Multiple Positive Solutions for Singular Fractional Boundary Value Problems with Riemann-Stieltjes Integrals

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In this paper, the existence result of at least two positive solutions is obtained for a nonlinear Riemann-Liouville fractional differential equation subject to nonlocal boundary conditions, where fractional derivatives and Riemann-Stieltjes integrals are involved. The nonlinearity possesses singularities on both its time and space variables. The discussion is based on the fixed point index theory on cones.

1. Introduction

We consider the existence of at least two positive solutions for the following nonlinear fractional differential equation with integral boundary value conditions

$$\begin{cases} D_{0+}^{\alpha}u(t) + f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, & D_{0+}^{\beta_0}u(1) = \sum_{i=1}^{m}\int_0^1 D_{0+}^{\beta_i}u(t)\mathrm{d}H_i(t), \end{cases} \tag{1}$$

where $\alpha \in \mathbb{R}, \alpha \in (n-1, n], n, m \in \mathbb{N}, n \geq 3, \beta_i \in \mathbb{R}$ for all $i = 0, 1, \cdots, m, 0 \leq \beta_1 < \beta_2 < \cdots < \beta_m \leq \beta_0 < \alpha - 1, \beta_0 \geq 1, f \in C((0,1) \times (0,+\infty), [0,+\infty))$. D_{0+}^{μ} represents the Riemann-Liouville derivative of order μ (for $\mu = \alpha, \beta_0, \beta_1, \cdots, \beta_m$). The integrals involved in the boundary conditions are Riemann-Stieltjes integrals; here, $H_i (i = 1, 2, \cdots, m)$ are functions of bounded variation. For more general integral conditions, see [1] and the references therein. The nonlinearity f permits singularities at both $t = 0, 1$ and $u = 0$.

In recent years, there has been a gradual increase in the investigation of fractional differential equations and systems of fractional differential equations with nonlocal boundary value conditions owing to their better descriptions in very important phenomena in science and technology than in integers. For fractional calculus and its applications in nonlocal problems, see monographs [2–6] and papers [7–25] and the references therein. Very recently, by means of the fixed point theory, principal characteristic value, and fixed point theorems together with height functions, Tudorache et al. [7, 9] investigated the existence of one, two, or three positive solutions for BVP (1). Existence results can be found in [15–19] for the system of fractional differential equations with boundary conditions related to BVP (1).

As is well known, Riemann-Stieltjes integral boundary conditions are more general, and they include many special cases such as two-point, three-point, and other classical integral conditions or a combination of them. As a consequence, boundary conditions in BVP (1) are generalizations of those adopted in the literature [10–14], which can be listed below:

$$\begin{aligned} &u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad D_{0+}^{p}u(1) = \sum_{i=1}^{m} a_i D_{0+}^{q}u(1), \\ &u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad D_{0+}^{\beta}u(1) = \lambda\int_0^{\eta} D_{0+}^{\beta}h(t)u(t)\mathrm{d}t, \\ &u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad D_{0+}^{p}u(1) = \int_0^1 D_{0+}^{q}u(t)\mathrm{d}H(t), \\ &u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad D_{0+}^{\beta}u(1) = \lambda\int_0^{\eta} D_{0+}^{\gamma}u(t)\mathrm{d}H(t). \end{aligned} \tag{2}$$

We make an effort in this paper to investigate the existence of multiple positive solutions for BVP (1). By a positive solution of BVP (1), we mean a function $u \in C[0,1]$ satisfying BVP (1) with $u(t) > 0$ for all $t \in (0,1]$. This paper admits the following features. Firstly, compared with [16–18], the nonlinearity f in this paper possesses singularities not only on the time but also on the space variables. Secondly, compared with [7–9], super linear conditions on the nonlinearity at 0 and ∞ are imposed to obtain the existence of at least two positive solutions. Thirdly, conditions given in this paper are shown to be easy to verify by an example. The main tools employed in this paper are the cone theory and fixed point index theorems on cones.

2. Preliminaries and Several Lemmas

First, we introduce some useful lemmas from [7, 9] which will be used in the latter. For notational convenience, denote

$$\Delta = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta_0)} - \sum_{i=1}^{m} \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta_i)} \int_0^1 s^{\alpha-\beta_i-1} \mathrm{d}H_i(s). \quad (3)$$

Consider the fractional differential equation

$$\begin{cases} D_{0+}^{\alpha} u(t) + x(t) = 0, & 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, & D_{0+}^{\beta_0} u(1) = \sum_{i=1}^{m} \int_0^1 D_{0+}^{\beta_i} u(t) \mathrm{d}H_i(t), \end{cases} \quad (4)$$

where $x \in C(0,1) \cap L^1(0,1)$.

Lemma 1 (see [7, 9]). *If $\Delta \neq 0$, then the unique solution $u \in C[0,1]$ of problems (4) is given by*

$$u(t) = \int_0^1 \mathcal{G}(t,s) x(s) ds, \quad t \in [0,1], \quad (5)$$

where

$$\mathcal{G}(t,s) = g_1(t,s) + \frac{t^{\alpha-1}}{\Delta} \sum_{i=1}^{m} \int_0^1 g_{2i}(\tau,s) dH_i(\tau), \quad (6)$$

$$g_1(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-\beta_0-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-1}(1-s)^{\alpha-\beta_0-1}, & 0 \le t \le s \le 1, \end{cases} \quad (7)$$

$$g_{2i}(t,s) = \frac{1}{\Gamma(\alpha-\beta_i)} \begin{cases} t^{\alpha-\beta_i-1}(1-s)^{\alpha-\beta_0-1} - (t-s)^{\alpha-\beta_i-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-\beta_i-1}(1-s)^{\alpha-\beta_0-1}, & 0 \le t \le s \le 1, \end{cases} \quad (8)$$

for all $(t,s) \in [0,1] \times [0,1]$, $i = 1, 2, \cdots, m$.

Lemma 2 (see [7, 9]). *We suppose that $\Delta > 0$. Then, the Green function $\mathcal{G}$ given by (6) is a continuous function on $[0,1] \times [0,1]$ and satisfies the following inequalities:*

(i) $\mathcal{G}(t,s) \le \mathcal{J}(s)$ for all $t, s \in [0,1]$, where

$$\mathcal{J}(s) = h_1(s) + \frac{1}{\Delta} \sum_{i=1}^{m} \int_0^1 g_{2i}(\tau,s) dH_i(\tau), \quad s \in [0,1],$$
$$h_1(s) = \frac{1}{\Gamma(\alpha)} (1-s)^{\alpha-\beta_0-1} \left(1 - (1-s)^{\beta_0}\right), \quad s \in [0,1] \quad (9)$$

(ii) $\mathcal{G}(t,s) \ge t^{\alpha-1} \mathcal{J}(s)$ for all $t, s \in [0,1]$

(iii) $\mathcal{G}(t,s) \le \sigma t^{\alpha-1}$ for all $t, s \in [0,1]$, where

$$\sigma = \frac{1}{\Gamma(\alpha)} + \frac{1}{\Delta} \sum_{i=1}^{m} \frac{1}{\Gamma(\alpha-\beta_i)} \int_0^1 \tau^{\alpha-\beta_i-1} dH_i(\tau). \quad (10)$$

We make the following assumptions:

(H_1) $\alpha \in \mathbb{R}$, $\alpha \in (n-1, n]$, $n, m \in \mathbb{N}$, $n \ge 3$, $\beta_i \in \mathbb{R}$ for all $i = 0, 1, \cdots, m$, $0 \le \beta_1 < \beta_2 < \cdots < \beta_m \le \beta_0 < \alpha - 1$, $\beta_0 \ge 1$, $H_i : [0,1] \longrightarrow \mathbb{R} (i = 1, 2, \cdots, m)$ are nondecreasing functions and $\Delta > 0$.

(H_2) $f \in C((0,1) \times (0,+\infty), [0,+\infty))$.

(H_3) There exist $a, b \in C((0,1), [0,+\infty)) \cap L^1[0,1]$, $g \in C((0,+\infty), [0,+\infty))$ such that

$$f(t,u) \le a(t) g(u) + b(t), \quad \forall t \in (0,1), u \in (0,+\infty),$$
$$\widetilde{a_{pq}}^{*} = \int_0^1 a(t) g_{pq}(t) dt < +\infty, \quad \widetilde{b}^{*} = \int_0^1 b(t) dt < +\infty, \quad (11)$$

for any $q \ge p > 0$, where

$$g_{pq}(t) = \max \left\{ g(u) : t^{\alpha-1} p \le u \le q \right\}. \quad (12)$$

(H_4) There exists $c \in C((0,1), [0,+\infty))$ such that

$$\frac{f(t,u)}{c(t) u} \longrightarrow +\infty as\, u \longrightarrow +\infty, \quad (13)$$

uniformly for $t \in (0,1)$, and

$$c^{*} = \int_0^1 c(t) dt < +\infty. \quad (14)$$

(H_5) There exists $d \in C((0,1), [0,+\infty))$ such that

$$\frac{f(t,u)}{d(t)} \longrightarrow +\infty as\, u \longrightarrow 0^{+}, \quad (15)$$

uniformly for $t \in (0,1)$, and

$$d^{*} = \int_0^1 d(t) dt < +\infty. \quad (16)$$

In addition, considering the boundedness of $\mathcal{J}(t), t \in [0,1]$*, it is easy to know that*

$$a_{pq}^* = \int_0^1 \mathcal{J}(t)a(t)g_{pq}(t)dt < +\infty, \quad b^* = \int_0^1 \mathcal{J}(t)b(t)dt < +\infty,$$
$$a_p^* = \int_0^1 \mathcal{J}(t)a(t)g_{pp}(t)dt < +\infty, \tag{17}$$

where

$$g_{pp}(t) = \max\left\{g(u): t^{\alpha-1}p \le u \le p\right\}, \quad \text{for all } p > 0. \tag{18}$$

Let $E = C[0,1]$ *be the traditional Banach space of all continuous functions defined on* $[0,1]$ *with the maximum norm* $\|u\| = \max_{0\le t\le 1}|u(t)|$ *and* P *the cone*

$$P = \left\{u \in E : u(t) \ge t^{\alpha-1}\|u\|, t \in [0,1]\right\}. \tag{19}$$

Denote $P_{pq} = \{u \in P : p \le \|u\| \le q\}, P_r = \{u \in P : \|u\| \le r\}, \partial P_r = \{u \in P : \|u\| = r\}$ *for* $q > p > 0, r > 0$.

Define the operator T *as follows:*

$$(Tu)(t) = \int_0^1 \mathcal{G}(t,s)f(s,u(s))ds, \quad 0 \le t \le 1. \tag{20}$$

Clearly, $T : P\setminus\{0\} \longrightarrow C[0,1]$.

Lemma 3. *Suppose that* (H_1)*–*(H_3) *hold; then, for any* $q > p > 0$, $T : P_{pq} \longrightarrow P$ *is completely continuous.*

Proof. For any $u \in P_{pq}$, we have $p \le \|u\| \le q$. It follows from the definition of cone P that

$$t^{\alpha-1}p \le u(t) \le q, \quad \forall t \in [0,1]. \tag{21}$$

By (H_2), (H_3), and Lemma 2, we get that

$$f(t,u(t)) \le a(t)g_{pq}(t) + b(t), \quad \forall t \in (0,1), u \in P_{pq}, \tag{22}$$

$$(Tu)(t) = \int_0^1 \mathcal{G}(t,s)f(s,u(s))\mathrm{d}s \le \int_0^1 \mathcal{J}(s)f(s,u(s)) \cdot \mathrm{d}s \le \int_0^1 \mathcal{J}(s)\left[a(s)g_{pq}(s) + b(s)\right] \cdot \mathrm{d}s = a_{pq}^* + b^*, \quad \forall t \in [0,1], \tag{23}$$

which means that T is well defined. For any $t \in [0,1]$, we have from (23) that

$$(Tu)(t) = \int_0^1 \mathcal{G}(t,s)f(s,u(s))\mathrm{d}s \le \int_0^1 \mathcal{J}(s)f(s,u(s))\mathrm{d}s. \tag{24}$$

Therefore,

$$\|Tu\| \le \int_0^1 \mathcal{J}(s)f(s,u(s))\mathrm{d}s. \tag{25}$$

On the other hand, it follows from Lemma 2 and (25) that

$$(Tu)(t) = \int_0^1 \mathcal{G}(t,s)f(s,u(s))\mathrm{d}s \ge t^{\alpha-1}\int_0^1 \mathcal{J}(s)f(s,u(s))\mathrm{d}s \ge t^{\alpha-1}\|Tu\|, \quad \forall t \in [0,1]. \tag{26}$$

Thus, we have proven that T maps P_{pq} into P.

In the following, we are in the position to show that T is completely continuous. First, we prove that T is continuous. For $u_n, \bar{u} \in P_{pq}$ with $\|u_n - \bar{u}\| \longrightarrow 0 (n \longrightarrow \infty)$, we have $\lim_{n\longrightarrow\infty} u_n(t) = \bar{u}(t), t \in [0,1]$. By (H_1), we know

$$\lim_{n\longrightarrow\infty} f(t,u_n(t)) = f(t,\bar{u}(t)), \quad 0 < t < 1. \tag{27}$$

Similar to (22), for $u_n, \bar{u} \in P_{pq}$, we have

$$f(t,u_n(t)) \le a(t)g_{pq}(t) + b(t), f(t,\bar{u}(t)) \le a(t)g_{pq}(t) + b(t), \quad \forall t \in (0,1). \tag{28}$$

Thus,

$$|\mathcal{G}(t,s)f(s,u_n(s)) - \mathcal{G}(t,s)f(s,\bar{u}(s))| \le 2\mathcal{J}(s)\left[a(s)g_{pq}(s) + b(s)\right] = \sigma(s) \in L^1[0,1]. \tag{29}$$

It follows from (27), (29), (H_3), and the Lebesgue-dominated convergence theorem that

$$\lim_{n\longrightarrow\infty}\|Tu_n - T\bar{u}\| \le \lim_{n\longrightarrow\infty} \max_{t\in[0,1]}\int_0^1 |\mathcal{G}(t,s)f(s,u_n(s)) - \mathcal{G}(t,s)f(s,\bar{u}(s))|\mathrm{d}s = 0, \tag{30}$$

which means that T is continuous.

Next, we will show that T is a compact operator. Let V be a bounded set in P_{pq}. For any $u \in V$, we have $p \le \|u\| \le q$. Similar to (23), we know

$$(Tu)(t) \le a_{pq}^* + b^*, \quad \forall t \in [0,1], u \in P_{pq}, \tag{31}$$

which means that $T(V)$ is bounded uniformly. In the following, we shall prove that $T(V)$ is equicontinuous. To this end, we estimate $(Tu)'$ for $u \in V$.

$$
\begin{aligned}
\|(Tu)'(t)\| &= \int_0^1 \frac{\partial \mathscr{G}(t,s)}{\partial t} f(s,u(s))\mathrm{d}s \\
&= \int_0^1 \left(\frac{\partial g_1(t,s)}{\partial t} + \frac{(\alpha-1)t^{\alpha-2}}{\Delta} \sum_{i=1}^m \int_0^1 g_{2i}(\tau,s)\mathrm{d}H_i(\tau) \right) \\
&\cdot f(s,u(s))\mathrm{d}s \le (\alpha-1)\int_0^1 \left[\frac{1}{\Gamma(\alpha)} \left(t^{\alpha-2}(1-s)^{\alpha-\beta_0-1} + (t-s)^{\alpha-2} \right) \right. \\
&\left. + \frac{t^{\alpha-2}}{\Delta} \sum_{i=1}^m \frac{1}{\Gamma(\alpha-\beta_i)} \tau^{\alpha-\beta_i-1}(1-s)^{\alpha-\beta_0-1}\mathrm{d}H_i(\tau) \right] f(s,u(s)) \\
&\cdot \mathrm{d}s \le (\alpha-1)\int_0^1 \left(\frac{2}{\Gamma(\alpha)} + \frac{1}{\Delta} \sum_{i=1}^m \frac{1}{\Gamma(\alpha-\beta_i)} (H_i(1)-H_i(0)) \right) \\
&\cdot \left(a(s)g_{pq}(s) + b(s) \right)\mathrm{d}s \le (\alpha-1) \\
&\cdot \left(\frac{2}{\Gamma(\alpha)} + \frac{1}{\Delta} \sum_{i=1}^m \frac{1}{\Gamma(\alpha-\beta_i)} (H_i(1)-H_i(0)) \right) \\
&\cdot \left(\widetilde{a_{pq}}^* + \widetilde{b^*} \right) \triangleq \mathfrak{F}, \quad \forall t \in [0,1].
\end{aligned} \tag{32}
$$

Thus, for any $0 \le t_1 \le t_2 \le 1$ and $u \in V$, one has

$$
|Tu(t_1) - Tu(t_2)| = \left| \int_{t_1}^{t_2} (T_n u)'(s)\mathrm{d}s \right| \le \mathfrak{F}|t_1 - t_2|. \tag{33}
$$

Thus, $T(V)$ is equicontinuous. It follows from the Arzelà-Ascoli theorem that $T(V)$ is relatively compact, and then, T is a compact operator. Hence, $T : P_{pq} \longrightarrow P$ is completely continuous. □

Lemma 4 (see [26]). *Let E be a Banach space and $P \subset E$ a cone in E. Assume that $T : P_r \longrightarrow P$ is a compact map such that $Tu \ne u$ for $u \in \partial P_r$,*

(i) If $\|u\| \le \|Tu\|, \forall u \in \partial P_r$, then

$$
i(T, P_r, P) = 0 \tag{34}
$$

(ii) If $\|u\| \ge \|Tu\|, \forall u \in \partial P_r$, then

$$
i(T, P_r, P) = 1. \tag{35}
$$

3. Main Result

Theorem 5. *Assume that (H_1)–(H_5) hold. In addition, there exists $R_0 > 0$ such that*

$$
a_{R_0}^* + b^* < R_0. \tag{36}
$$

Then, the BVP (1) has at least two positive solutions u^* and u^{**} with $0 < \|u^*\| < R_0 < \|u^{**}\|$.

Proof. It follows from Lemma 3 that for any $q > p > 0$, the operator $T : P_{pq} \longrightarrow P$ is completely continuous. In the following, we shall prove that T has two different fixed points u^* and u^{**} in P satisfying $0 < \|u^*\| < R_0 < \|u^{**}\|$.

Choose $\theta \in (0, 1/2)$. We know from (H_4) that there exists $r_1 > 0$ such that

$$
f(t,u) \ge \theta^{1-\alpha} \left(\int_\theta^{1-\theta} \mathscr{G}(1/2, s)c(s)\mathrm{d}s \right)^{-1} c(t)u, \quad \forall t \in (0,1), u \ge r_1. \tag{37}
$$

Let

$$
R_1 > \max \left\{ \theta^{1-\alpha} r_1, R_0 \right\}. \tag{38}
$$

For $u \in \partial P_{R_1}$, we have, by the construction of cone P, that

$$
u(t) \ge t^{\alpha-1}R_1 \ge \theta^{\alpha-1}R_1 > r_1, \quad \forall t \in [\theta, 1-\theta]. \tag{39}
$$

It follows from (37) to (39) that

$$
\begin{aligned}
(Tu)\left(\frac{1}{2}\right) &= \int_0^1 \mathscr{G}\left(\frac{1}{2}, s\right) f(s,u(s))\mathrm{d}s > \theta^{1-\alpha} \left(\int_\theta^{1-\theta} \mathscr{G}\left(\frac{1}{2}, s\right) c(s)\mathrm{d}s \right)^{-1} \\
&\cdot \int_\theta^{1-\theta} \mathscr{G}\left(\frac{1}{2}, s\right) c(s)u(s)\mathrm{d}s \ge \theta^{1-\alpha} \left(\int_\theta^{1-\theta} \mathscr{G}\left(\frac{1}{2}, s\right) c(s)\mathrm{d}s \right)^{-1} \\
&\cdot \int_\theta^{1-\theta} \mathscr{G}\left(\frac{1}{2}, s\right) c(s)\mathrm{d}s \cdot \theta^{\alpha-1}R_1 = R_1.
\end{aligned} \tag{40}
$$

Thus,

$$
\|Tu\| = \max_{t\in[0,1]} \|(Tu)(t)\| \ge \left\| (Tu)\left(\frac{1}{2}\right) \right\| > R_1 = \|u\|, \quad \forall u \in \partial P_{R_1}. \tag{41}
$$

Hence, by Lemma 4,

$$
i\left(T, P_{R_1}, P\right) = 0. \tag{42}
$$

By condition (H_4), there exists $r_2 > 0$ such that

$$
f(t,u) \ge \left(\int_\theta^{1-\theta} \mathscr{G}\left(\frac{1}{2}, s\right) d(s)\mathrm{d}s \right)^{-1} d(t)R_0, \quad \forall t \in (0,1), 0 < u < r_2. \tag{43}
$$

Choose

$$
0 < R_2 < \min \{r_2, R_0\}. \tag{44}
$$

For $u \in P_{R_2}$, we have

$$
0 < R_2 t^{\alpha-1} \le u(t) \le \|u\| = R_2 < r_2, \quad \forall t \in (0,1). \tag{45}
$$

Consequently, we have from (43) to (45) and (H_5) that

$$(Tu)\left(\frac{1}{2}\right) = \int_0^1 \mathcal{G}\left(\frac{1}{2}, s\right) f(s, u(s))\mathrm{d}s \geq \left(\int_\theta^{1-\theta} \mathcal{G}\left(\frac{1}{2}, s\right) d(s)\mathrm{d}s\right)^{-1} \cdot \int_\theta^{1-\theta} \mathcal{G}\left(\frac{1}{2}, s\right) d(s) R_0 \mathrm{d}s \geq \left(\int_\theta^{1-\theta} \mathcal{G}\left(\frac{1}{2}, s\right) d(s)\mathrm{d}s\right)^{-1} \cdot \int_\theta^{1-\theta} \mathcal{G}\left(\frac{1}{2}, s\right) d(s)\mathrm{d}s \cdot R_0 = R_0 > R_2. \tag{46}$$

Thus,

$$\|Tu\| = \max_{t\in[0,1]} \|(Tu)(t)\| \geq \left\|(Tu)\left(\frac{1}{2}\right)\right\| > R_2 = \|u\|, \quad \forall u \in P_{R_2}. \tag{47}$$

As a consequence, we get

$$i(T, P_{R\_2}, P) = 0. \tag{48}$$

On the other hand, for $u \in \partial P_{R_0}$, by (H_3), Lemma 2, and (36), we get

$$(Tu)(t) = \int_0^1 \mathcal{G}(t,s) f(s, u(s))\mathrm{d}s \leq \int_0^1 \mathcal{J}(s) f(s, u(s))\mathrm{d}s \leq \int_0^1 \mathcal{J}(s) \cdot \left[a(s) g_{R_0 R_0}(s) + b(s)\right]\mathrm{d}s \leq a^*_{R_0} + b^* < R_0, \quad \forall t \in [0,1], \tag{49}$$

i.e.,

$$\|Tu\| \leq \|u\|, \quad u \in \partial P_{R_0}. \tag{50}$$

Then, Lemma 4 guarantees that

$$i\left(T, P_{R_0}, P\right) = 1. \tag{51}$$

It follows from (42), (48), (51) and the additivity of the fixed point index that

$$i\left(T, P_{R_1} \setminus \overset{\circ}{P}_{R_0}, P\right) = -1, \quad i\left(T, P_{R_0} \setminus \overset{\circ}{P}_{R_2}, P\right) = 1. \tag{52}$$

Hence, T has two distinct fixed points u^* and u^{**} belonging to $P_{R_0} \setminus P_{R_2}{}^\circ$ and $P_{R_1} \setminus P_{R_0}{}^\circ$, respectively, with $0 < R_2 < \|u^*\| < R_0 < \|u^{**}\| \leq R_1$. □

4. An Example

Example 1. Consider the following fractional differential equations with nonlocal boundary value problems

$$\begin{cases} D_{0+}^{11/3} u(t) + \dfrac{1}{75\sqrt[5]{t^2(1-t)^4}}\left(u^3 + u^{-1/7}\right) + \dfrac{1}{25\sqrt[4]{t(1-t)^3}} = 0, & 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, & D_{0+}^{13/6} u(1) = \displaystyle\int_0^1 D_{0+}^{2/3} u(t)\mathrm{d}t + D_{0+}^{5/3} u\left(\frac{1}{2}\right). \end{cases} \tag{53}$$

Conclusion: BVP (53) has at least two positive solutions u^* and u^{**} with $0 < \|u^*\| < 5 < \|u^{**}\|$.

Proof. In this problem, $\alpha = 11/3, n = 4, m = 2, \beta_0 = 13/6, \beta_1 = 2/3, \beta_2 = 5/3, H_1(t) = t$ for all $t \in [0,1]$, $H_2(t) = \{0$ for $t \in [0, 1/2)$; 1 for $t \in [1/2, 1]\}$. By simple computation, we have $\Delta = 1.852483495372207 > 0$. It is clear that ($H_1$) and ($H_2$) are satisfied. Furthermore,

$$h_1(s) = \frac{1}{\Gamma(11/3)}(1-s)^{1/2}\left(1 - (1-s)^{13/6}\right), \quad s \in [0,1], \tag{54}$$

$$g_{21}(t,s) = \frac{1}{\Gamma(3)} \begin{cases} t^2(1-s)^{1/2} - (t-s)^2, & 0 \leq s \leq t \leq 1, \\ t^2(1-s)^{1/2}, & 0 \leq t \leq s \leq 1, \end{cases} \tag{55}$$

$$g_{22}(t,s) = \frac{1}{\Gamma(2)} \begin{cases} t(1-s)^{1/2} - (t-s), & 0 \leq s \leq t \leq 1, \\ t(1-s)^{1/2}, & 0 \leq t \leq s \leq 1, \end{cases} \tag{56}$$

$$\mathcal{J}(s) = \begin{cases} h_1(s) + \dfrac{1}{\Delta}\left\{\dfrac{1}{\Gamma(4)}(1-s)^{1/2} - \dfrac{1}{\Gamma(4)}(1-s)^3 + \dfrac{1}{\Gamma(2)}\left[\dfrac{1}{2}\cdot(1-s)^{1/2} - \left(\dfrac{1}{2} - s\right)\right]\right\}, & 0 \leq s \leq \dfrac{1}{2}, \\ h_1(s) + \dfrac{1}{\Delta}\left\{\dfrac{1}{\Gamma(4)}(1-s)^{1/2} - \dfrac{1}{\Gamma(4)}(1-s)^3 + \dfrac{1}{\Gamma(2)}\cdot\dfrac{1}{2}\cdot(1-s)^{1/2}, \right. & \dfrac{1}{2} < s \leq 1. \end{cases} \tag{57}$$

For any $r > 0$, (H_3) holds for $a(t) = 1/75\sqrt[5]{t^2(1-t)^4}$, $g(u) = u^3 + u^{-1/7}$, $b(t) = 1/25\sqrt[4]{t(1-t)^3}$, and

$$\widetilde{a_{pq}}^* = \int_0^1 a(t) g_{pq}(t)\mathrm{d}t = \int_0^1 \frac{1}{75\sqrt[5]{t^2(1-t)^4}}\left(q^3 + \frac{1}{\sqrt[7]{t^{8/3}p}}\right) \cdot \mathrm{d}t = \frac{1}{75}\left(\int_0^1 t^{-2/5}(1-t)^{-4/5}\mathrm{d}t \cdot q^3 + \int_0^1 t^{-82/105}(1-t)^{-4/5}\mathrm{d}t \cdot p^{-1/7}\right) = \frac{1}{75}\left[\mathrm{B}\left(\frac{3}{5}, \frac{1}{5}\right)q^3 + \mathrm{B}\left(\frac{23}{105}, \frac{1}{5}\right)p^{-1/7}\right] < +\infty. \tag{58}$$

$\tilde{b}^* = \int_0^1 (1/25\sqrt[4]{t(1-t)^3})\mathrm{d}t = (1/25)\mathrm{B}(3/4, 1/4) \approx 0.177715317526335$. Thus, ($H_3$) is verified. Obviously, (H_4) and (H_5) are valid for $c(t) = d(t) = 1/75\sqrt[5]{t^2(1-t)^4}$, $c^* = d^* \approx 0.078296677374705$.

Next, we focus on checking (36). Take $R_0 = 5$. By (57), we know that

$$b^* = \int_0^1 \mathcal{J}(s)b(s)ds \le \int_0^1 \left\{ h_1(s) + \frac{1}{\Delta}\left[\frac{1}{\Gamma(4)}(1-s)^{1/2} + \frac{1}{\Gamma(3)}(1-s)^{1/2}\right]\right\} \cdot \frac{1}{25\sqrt[4]{s(1-s)^3}}ds \le \int_0^1 \left\{ \frac{1}{\Gamma(\alpha)} + \frac{1}{\Delta}\left[\frac{1}{\Gamma(4)}(1-s)^{1/2} + \frac{1}{\Gamma(3)}(1-s)^{1/2}\right]\right\} \cdot \frac{1}{25\sqrt[4]{s(1-s)^3}}ds \le \frac{1}{25}\cdot\left[\frac{1}{\Gamma(11/3)}\cdot\int_0^1 s^{-1/4}(1-s)^{-3/4}ds + \frac{1}{\Delta}\left(\frac{1}{\Gamma(4)} + \frac{1}{\Gamma(3)}\right) \cdot \int_0^1 s^{-1/4}(1-s)^{-1/4}ds\right] = \frac{1}{25}\cdot\left[\frac{1}{\Gamma(11/3)}\mathrm{B}\left(\frac{3}{4},\frac{1}{4}\right) + \frac{1}{1.852483495372207} \cdot \left(\frac{1}{\Gamma(4)} + \frac{1}{\Gamma(3)}\right)\cdot \mathrm{B}\left(\frac{3}{4},\frac{3}{4}\right)\right] \approx \frac{1}{25}\left[0.249239737672306\cdot 4.442882938158366 + 0.539815875552012\cdot\left(\frac{1}{6}+\frac{1}{2}\right)\cdot 1.694426169587958\right] \approx 0.068685136355131,$$

$$a^*_{R_0} = \int_0^1 \mathcal{J}(s)a(s)g_{R_0}ds \le \frac{1}{75}\int_0^1 \left[\frac{1}{\Gamma(\alpha)} + \frac{1}{\Delta}\left(\frac{1}{\Gamma(4)} + \frac{1}{\Gamma(3)}\right)(1-s)^{1/2}\right] \cdot \frac{1}{\sqrt[5]{s^2(1-s)^4}}\left(R_0^3 + \frac{1}{\sqrt[7]{s^{8/3}R_0}}\right)ds \le \frac{1}{75} \cdot\left\{\frac{1}{\Gamma(11/3)}\cdot\left[\mathrm{B}\left(\frac{3}{5},\frac{1}{5}\right)\cdot R_0^3 + \mathrm{B}\left(\frac{23}{105},\frac{1}{5}\right)\cdot R_0^{-1/7}\right] + \frac{1}{1.852483495372207}\cdot\left(\frac{1}{\Gamma(4)} + \frac{1}{\Gamma(3)}\right) \cdot\left[\mathrm{B}\left(\frac{3}{5},\frac{7}{10}\right)\cdot R_0^3 + \mathrm{B}\left(\frac{23}{105},\frac{7}{10}\right)\cdot R_0^{-1/7}\right]\right\} \approx \frac{1}{75} \cdot\left[0.249239737672306\cdot\left(5.872250803102905\cdot 5^3 + 9.049460301355431\cdot 5^{-1/7}\right) + 0.539815875552012\cdot\left(\frac{1}{6}+\frac{1}{2}\right)\cdot\left(2.153890871161322\cdot 5^3 + 5.136337054837259\cdot 5^{-1/7}\right)\right] \approx 3.774703981617563. \quad (59)$$

Hence,

$$a^*_{R_0} + b^* = 3.774703981617563 + 0.068685136355131 = 3.843389117972694 < 5 = R_0, \quad (60)$$

which implies that (36) holds. Consequently, our conclusion follows from Theorem 5. □

5. Conclusions

In this paper, we focus on the existence and multiplicity of positive solutions for a class of a higher-order Riemann-Liouville fractional differential equation with Riemann-Stieltjes integrals. The nonlinearity possesses singularities on both its time and space variables. By means of the fixed point index theory on cones, the existence result of at least two positive solutions is obtained. Conditions imposed on the nonlinearity are shown to be easy to verify by an example.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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