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Research Article

Sixth-Kind Chebyshev and Bernoulli Polynomial Numerical Methods for Solving Nonlinear Mixed Partial Integrodifferential Equations with Continuous Kernels

Abeer M. Al-Bugami,[1] **Mohamed A. Abdou**,[2] **and Amr M. S. Mahdy**[1,3]

[1]*Department of Mathematics & Statistics, College of Science, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia*
[2]*Department of Mathematics, Faculty of Education, Alexandria University, Alexandria 21526, Egypt*
[3]*Department of Mathematics, Faculty of Science, Zagazig University, P.O. Box 44519, Zagazig, Egypt*

Correspondence should be addressed to Amr M. S. Mahdy; amattaya@tu.edu.sa

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In the present paper, a new efficient technique is described for solving nonlinear mixed partial integrodifferential equations with continuous kernels. Using the separation of variables, the nonlinear mixed partial integrodifferential equation is converted to a nonlinear Fredholm integral equation. Then, using different numerical methods, the Bernoulli polynomial method and the Chebyshev polynomials of the sixth kind, the nonlinear Fredholm integral equation has been reduced into a system of nonlinear algebraic equations. The Banach fixed-point theory is utilized in order to have a conversation about the nonlinear mixed integral equation's solution, namely, its existence and uniqueness. In addition, we talk about the convergence and stability of the solution. Finally, a comparison between the two different methods and some other famous methods is presented through various examples. All the numerical results are calculated and obtained using the Maple software.

1. Introduction

The mixed integral equations (MIEs) in location and time have captured the interest of numerous authors in recent years. In [1], Tohidi et al. investigated a collocation technique for solving the generalized pantograph problem that depended on the Bernoulli operating matrix. Yousefi utilized the Legendre wavelets to evaluate a computational solution to Abel's integral problem in [2]. Mahdy et al. introduced a computational technique for solving MIEs with singular kernels in [3]. Alhazmi and Abdou in [4] used the Toeplitz matrix approach for resolving nonlinear fractional mixed integrodifferential equations. Jan in [5] used the Toeplitz matrix approach for resolving MIE. In [6], Basseem and Alalyani introduced the Toeplitz and Nystrom methods for solving quadratic nonlinear mixed integral equations (NMIEs). In [7], Tahmasbi talked about a new way to use numbers to solve second-order linear Volterra integral equations. Maleknejad and Rahimi discuss a change to block pulse functions to demonstrate how they may be utilized to determine numerical Volterra integral equations of the first kind [8].

Many researchers dealt with the Chebychev polynomials of the first, second, third, and fourth kinds, while the sixth type was not dealt with by many; for example, in [9], Atta et al. presented the solution to the hyperbolic telegraph problem by using shifted CP6K. In [10], Sadri and Aminikhah used CP6K for solving fractional partial differential equations (FPDEs). The solution to fractional integrodifferential equations is obtained by Yaghoubi et al. in their paper [11], which introduces shifted CP6K.

The Bernoulli collocation approach was utilized by Doha et al. in [12] for the purpose of solving hyperbolic telegraph equations. In their study [13], Dascioglu and Sezer looked at the use of BPM in the resolution of high-order modified pantograph models. In [14], Mirzaee utilized BPM for the purpose of solving integral algebraic problems. In [15], Wang et al. provide a description of the Bernoulli polynomial approximation approach for resolving the Volterra integral equations multidimensional with variable-order weakly singular kernels. In [16], Bazm provides a description of the Bernoulli polynomials (BPs) as an approach for finding the numerical solution to certain types of linear and nonlinear integral equations. A description of the identification of the BPs using the Raabe functional equation may be found by Farhi in [17]. Some identities for BPs that involve the Chebyshev polynomials are described by Kim et al. in [18]. In [19], Maleknejad et al. give a numerical solution to the Volterra kind integral problem of the first type with a wavelet base. In [20], Abdou provided the FIE together with its potential kernel and its structural resolvent. For mixed $(1+1)$ dimensional integral equations with substantially symmetric singular kernels, Alhazmi et al. described the computing techniques [21]. Mahdy et al. described the computational method in [22] for resolving mixed Volterra-Fredholm integral equations in three dimensions. Mahdy and Mohamed presented about utilizing the Lucas polynomials to solve the Cauchy integral problems [23]. Pachpatte provided a description of the multidimensional integral equations and inequalities in [24]. Appell et al. discussed integrodifferential equations and partial integral operators in [25]. The Volterra-Fredholm integral equation in two variables is discussed by Pachpatte in [26]. Guenther and Lee discussed integral equations and mathematical physics partial differential equations in [27]. To calculate the nonlinear Volterra-Fredholm integral equations numerically, Brunner applied the combination methods as described in [28]. Al-Bugami talked about how to use numerical methods to solve a two-dimensional mixed integral problem involving cracks on the surface of finite multiple layers of compounds [29]. The mathematical physics special function formulae and theorems are introduced by Magnus et al. in [30]. Costabile et al. developed a novel method for dealing with the Bernoulli polynomials in [31]. For the numerical solution of the mixed Volterra-Fredholm integral equations, He et al. have enhanced block-pulse functions in [32]. The nonlinear one-dimensional Burgers' equation has a spectral solution, which Abd-Elhameed reported in [33] as a novel formula for the derivatives of sixth-kind Chebyshev polynomials. In [34], Lu discussed a few Bernoulli polynomial characteristics and their generalization. See the books Wazwaz in [35] and Rahman in [36] for numerous additional integral equations and their applications using various techniques.

The following is how the rest of this paper is structured. The NMIE and specific cases are presented in Section 2. The existence of the nonlinear MIE and its unique and only solution are topics discussed in Section 3. The convergence and stability of the solution are the topics that will be covered in Section 4. Section 5 separates the variables used in the transformation of the nonlinear MIE into the NFIE. The technique of error has been studied in Section 6. BPM applied for solving the NFIE in Section 7. CP6K applied for solving the NFIE in Section 8. In Section 9, we give some numerical examples for your consideration. In Section 10, we provide the conclusion.

2. The Nonlinear MIE and Special Cases

Take a look at the nonlinear partial integrodifferential equation that is presented below

$$\frac{\partial}{\partial t}\left[\frac{\eta(x,t)-f(x,t)}{\delta(G(t),\eta(x,t))}\right]=F(t)\int_0^1 k(x,y)\gamma(\eta(y,t))dy. \tag{1}$$

Under the condition

$$\eta(x,0)=u(x), \tag{2}$$

where $k(x,y)$, $f(x,t)$, $G(t)$, and $F(t)$ are known functions and $\eta(x,t)$ is the unknown function.

Equation (1) is very important in many applications of nanofluid mechanics, genetic engineering, engineering, biology, applied science, and quantum engineering. This equation has been treated using integral methods, and we discovered that it is easier than treating it with differential methods.

Integrating (1) w.r.t t and using (2), we get

$$\eta(x,t)=f(x,t)+\delta(G(t),\eta(x,t))\left[\frac{u(x)-f(x,0)}{\delta(G(0),\eta(x,0))}\right] + \delta(G(t),\eta(x,t))\int_0^t\int_0^1 F(\tau)k(x,y)\gamma(\eta(y,\tau))dyd\tau. \tag{3}$$

The preceding equation may also be rewritten in the form that is shown below

$$\eta(x,t)=f(x,t)+D(x)\delta(G(t),\eta(x,t)) + \delta(G(t),\eta(x,t))\int_0^t\int_0^1 F(\tau)k(x,y)\gamma(\eta(y,\tau))dyd\tau, \tag{4}$$

at $D(x)=[u(x)-f(x,0)/\delta(G(0),\eta(x,0))]$.

Formula (4) represents a nonlinear MIE in time and position.

The importance of (4) comes from special cases that can be derived from it Special cases:

(i) If $u(x)=f(x,0)$, the nonlinear quadratic integral equation that we have is as follows

$$\eta(x,t)=f(x,t)+\delta(G(t),\eta(x,t))\int_0^t\int_0^1 F(\tau)k(x,y)\gamma(\eta(y,\tau))dyd\tau. \tag{5}$$

For a constant time, Equation (5) yields

$$\Psi(x)=g(x)+\lambda(\Psi(x))\int_0^1 k(x,y)\gamma(\Psi(y))dy. \tag{6}$$

Abdel-Aty et al. [37] discussed numerically the solution of Equation (6) using the Chebyshev polynomial of the first kind. In addition, Abdel-Aty and Abdou [38] discussed the phase lag solution of (6) using the homotopy perturbation method. Awad et al. [39, 40] solved the quadratic integral equation of (6) when the nonlinear term outside the integration takes square power in [39] and cubic power in [40].

(ii) If $f(x,t)=0, \eta(x,0)=u(x)$, we have

$$\eta(x,t)=D_1(x)\delta(G(t),\eta(x,t)) + \delta(G(t),\eta(x,t))\int_0^t\int_0^1 F(\tau)k(x,y)\gamma(\eta(y,\tau))dyd\tau, \tag{7}$$

where $D_1(x)=[u(x)/\delta(G(0),\eta(x,0))]$.

While when $f(x,t)=0,$' there is a

$$\eta(x,t)=\delta(G(t),\eta(x,t))\int_0^t\int_0^1 F(\tau)k(x,y)\gamma(\eta(y,\tau))dyd\tau. \tag{8}$$

The above formula is called a nonlinear mixed homogeneous quadratic integral equation.

3. Existence and Unique Solution of Equation (4)

In this part, with the help of the Banach fixed point notation, we demonstrate that there is a unique solution to the nonlinear MIE (4). For this aim, we state

Theorem 1 (Banach's Fixed-Point Theorem (Contraction Theorem)). *Let $X=(X,d)$ be a nonempty, complete metric space and $T: X \longrightarrow X$ be a contraction mapping on X. Consequently, T has only one fixed point.*

Definition 2. The norm in $L_2[0,1]\times C[0,T]$ we define the norm of the function $H(x,t)$ in the space $L_2[0,1]\times C[0,T]$ as $f(x,t)\in L_2[0,1]\times C[0,T]$ and its norm is $\|f(x,t)\| = \max_{0\le t\le T}\int_0^t\{\int_0^1 f^2(x,\tau)dx\}^{1/2}d\tau \le E$.

To determine the existence and uniqueness solution of Equation (4), in view of the Banach fixed-point theory, in the design of an integral operator, we write (4) as follows:

$$\eta(x,t)=\bar{W}\eta(x,t)=f(x,t)+D(x)\delta(G(t),\eta(x,t)) + \delta(G(t),\eta(x,t))W\eta(x,t), \tag{9}$$

$$W\eta(x,t)=\int_0^t\int_0^1 F(\tau)k(x,y)\gamma(\eta(y,\tau))dyd\tau. \tag{10}$$

Then, assume the following conditions:

(i) $\forall C\ge \max\{C_1,C_2,C_3\}, |k(x,y)|\le C_1, |F(t)|\le C_2, |G(t)|\le C_3; |D(x)|\le D$

(ii) $f(x,t)\in L_2[0,1]\times C[0,1]$ and its norm is $\|f(x,t)\| = \max_{0\le t\le T}\int_0^t\{\int_0^1 f^2(x,\tau)dx\}^{1/2}d\tau\le E$

(iii) $\forall A\ge\max\{A_1,A_2\}$, the decreasing function $\eta(x,t)$ satisfies

(iii-a) $|\gamma(\eta(x,t))|\le A_1|\eta(x,t)|$

(iii-b) $|\gamma\eta_1(x,y)-\gamma\eta_2(x,y)|\le A_2|\eta_1(x,t)-\eta_2(x,t)|$

(iv) $\forall B\ge\max\{B_1,B_2\}$, the decreasing function $\eta(x,t)$ satisfies

(iv-a) $|\delta(G(t),\eta(x,t))|\le B_1|\eta(x,t)|$

(iv-b) $|\delta(G(t),\eta_1(x,t))-\delta(G(t),\eta_2(x,t))|\le B_2|\eta_1(x,t) - \eta_1(x,t)|$

Theorem 3. *Under the conditions (i)–(iv), the nonlinear mixed integral problem has a single unique solution*

$$T<\frac{1-BD}{ABC^2}. \tag{11}$$

In order to demonstrate that the theorem is true, we must first demonstrate Lemma 4.

Lemma 4. *The integral operator (9) is bounded.*

Proof. Since

$$\bar{W}\eta(x,t)=f(x,t)+D(x)\delta(G(t),\eta(x,t))+\delta(G(t),\eta(x,t))W\eta(x,t). \tag{12}$$

Then

$$\|\bar{W}\eta(x,t)\|\le\|f(x,t)\|+\|D(x)\delta(G(t),\eta(x,t))\| + \|\delta(G(t),\eta(x,t))W\eta(x,t)\|. \tag{13}$$

Using conditions (i), (ii), (iii-a), and (iv-a), we follow

$$\|\bar{W}\eta(x,t)\|\le E+DB\|\eta(x,t)\|+B\|\eta(x,t)\|\|W\eta(x,t)\|, \quad \|W\eta(x,t)\|\le\max C^2AT\|\eta(x,t)\|. \tag{14}$$

Hence, we have

$$\|\bar{W}\eta(x,t)\|\le E+DB\|\eta(x,t))\|+ABC^2T\|\eta(x,t)\|^2. \tag{15}$$

Since, $\eta(x,t)$ is a decreasing function, consequently, $\|\eta(x,t))\|^2\le\|\eta(x,t)\|$

$$\|\bar{W}\eta(x,t)\|\le E+\left(DB+\mathrm{ABC}^2T\right)\|\eta(x,t))\|. \tag{16}$$

The last inequality (16) demonstrates that the operator $\bar{W}$ maps the ball S_r into itself, where

$$r=\frac{E}{\left(1-\left(BD+ABC^2T\right)\right)}. \tag{17}$$

Since, $r > 0$ and $E > 0$, therefore, we must have

$$T < \frac{1 - BD}{(ABC^2)}. \tag{18}$$

□

Lemma 5. *The integral operator (9) is continuous.*

Proof. To prove the continuous of the integral operator, we assume

$$\begin{aligned}\bar{W}(\eta_1(x,t) - \eta_2(x,t)) = D(x)\{\delta(G(t), \eta_1(x,t)) \\ - \delta(G(t), \eta_2(x,t))\} + (\delta(G(t), \eta_1(x,t)) \\ \cdot W\eta_1(x,t) - \delta(G(t), \eta_2(x,t))W\eta_2(x,t)).\end{aligned} \tag{19}$$

Adapting the above equation to take the form

$$\begin{aligned}\left\|\bar{W}(\eta_1(x,t) - \eta_2(x,t))\right\| \le \|D(x)[\delta(G(t), \eta_1(x,t)) \\ - \delta(G(t), \eta_2(x,t))\| \\ + \|\delta(G(t), \eta_1(x,t))\|\|W[\eta_1(x,t) \\ - \eta_2(x,t)\| + \|\delta(G(t), \eta_1(x,t)) \\ - \delta(G(t), \eta_2(x,t))\|\|W(\eta_2(x,t))\|.\end{aligned} \tag{20}$$

Using the norm properties and with the aid of conditions (i)–(iv), and given this, it is safe to suppose that $\eta_1(x,t)$ and $\eta_2(x,t)$ for $t \in [0, T]$ and $T < 1$ are two monotonic decreasing functions, thus we are able to

$$\begin{aligned}\left\|\bar{W}(\eta_1(x,t) - \eta_2(x,t))\right\| \le BD\|\eta_1(x,t) - \eta_2(x,t)\| \\ + BAC^2T\|\eta_1(x,t) \\ - \eta_2(x,t)\|\|\eta_1(x,t) - \eta_2(x,t)\|.\end{aligned} \tag{21}$$

Therefore, the inequality stated above leads to

$$\left\|\bar{W}(\eta_1(x,t) - \eta_2(x,t))\right\| \le \left[BD + BAC^2T\right]\|\eta_1(x,t) - \eta_2(x,t)\|. \tag{22}$$

which leads to the continuity condition. And then under the condition

$$BD + BAC^2T < 1. \tag{23}$$

A contraction operator is denoted by the value (9) in the integral operator. □

4. Solution Convergence and Stability

Assume that the family of solutions takes the form $\eta(x,t) = \{\eta_0(x,t), \eta_1(x,t), \cdots, \eta_{n-1}(x,t), \eta_n(x,t), \cdots\} = \{\eta_i(x,t)\}_{i=1}^{\infty}$, $\eta_0(x,t) = f(x,t) + D(x)\delta(G(t), \eta_0(x,t))$, and $\|\eta_0(x,t)\| \le \|f(x,t)\| + |D(x)|\|G(t), \eta_0(x,t)\| \le E + AD\|\eta_0(x,t)\| \le E/1 - AD$, $(AD < 1)$.

Pick up two functions $\{\eta_n(x,t), \eta_{n-1}(x,t)\} \in \{\eta_i(x,t)\}_{i=0}^{\infty}$

$$\begin{aligned}\eta_n(x,t) &= f(x,t) + D(x)\delta(G(t), \eta_n(x,t)) \\ &\quad + \delta(G(t), \eta_n(x,t))\int_0^t\int_0^1 F(\tau)k(x,y)\gamma(\eta_{n-1}(y,\tau))dyd\tau, \\ \eta_{n-1}(x,t) &= f(x,t) + D(x)\delta(G(t), \eta_{n-1}(x,t)) \\ &\quad + \delta(G(t), \eta_n(x,t))\int_0^t\int_0^1 F(\tau)k(x,y)\gamma(\eta_{n-2}(y,\tau))dyd\tau.\end{aligned} \tag{24}$$

Thus, we have

$$\begin{aligned}\eta_n(x,t) - \eta_{n-1}(x,t) = D(x)[\delta(G(t), \eta_n(x,t)) - \delta(G(t), \eta_{n-1}(x,t))] \\ + \delta(G(t), \eta_n(x,t))\int_0^t\int_0^1 F(\tau)k(x,y)\gamma(\eta_{n-1}(y,\tau)) \\ \cdot dyd\tau - \delta(G(t), \eta_{n-1}(x,t))\int_0^t\int_0^1 F(\tau)k(x,y) \\ \cdot \gamma(\eta_{n-1}(y,\tau))dyd\tau + \delta(G(t), \eta_{n-1}(x,t)) \\ \cdot \int_0^t\int_0^1 F(\tau)k(x,y)\gamma(\eta_{n-1}(y,\tau)) \\ \cdot dyd\tau - \delta(G(t), \eta_{n-1}(x,t)) \\ \cdot \int_0^t\int_0^1 F(\tau)k(x,y)\gamma(\eta_{n-2}(y,\tau))dyd\tau.\end{aligned} \tag{25}$$

Adapting the above equation

$$\begin{aligned}\|\eta_n(x,t) - \eta_{n-1}(x,t)\| \le BD\|\eta_n(x,t) - \eta_{n-1}(x,t)\| \\ + ABC^2T\|\eta_n(x,t)\|\|\eta_n(x,t) \\ - \eta_{n-1}(x,t)\|ABC^2T\|\eta_n(x,t)\|\|\eta_{n-1}(x,t) \\ - \eta_{n-2}(x,t)\|.\end{aligned} \tag{26}$$

Let

$$\Psi_n = \eta_n(x,t) - \eta_{n-1}(x,t). \tag{27}$$

Thus, we have

$$\eta_n(x,t) = \sum_{j=0}^{n}\Psi_j(x,t), \|\Psi_0(x,t)\| = \|\eta_0(x,t)\| \le \frac{E}{1 - AD}. \tag{28}$$

The formula presented above can be adapted to the following

$$\begin{aligned}\left\|\Psi_j(x,t)\right\| &\le \rho\left\|\Psi_{j-1}(x,t)\right\|, \\ \rho &= \frac{ABC^2T}{1 - (BD + ABC^2T)}, \\ \|\Psi_0(x,t)\| &\le \frac{E}{1 - AD}.\end{aligned} \tag{29}$$

By induction, we have

$$\left\|\Psi_j(x,t)\right\| \le [\rho]^j \|\Psi_0(x,t)\|. \tag{30}$$

Finally, we have

$$\left\|\Psi_j(x,t)\right\| \le \left[\frac{E}{1-BD}\right][\rho]^j. \tag{31}$$

This limit ensures that the sequence $\{\Psi_j(x;t)\}$ uniformly converges consistently. Furthermore, the response to inequality (31) directs us to conclude which the sequence $\{\eta_j(x;t)\}$ is a solution that converges on itself. Therefore, given that

$$\eta(x;t) = \lim_{n\longrightarrow\infty} \eta_n(x;t) = \lim_{n\longrightarrow\infty} \Sigma_{j=0}^{n} \Psi_j(x;t). \tag{32}$$

Because $\Psi_j(x;t)$ are all continuous, $\eta(x;t)$ has to be continuous; thus, the infinite series represented by Equation (32) is uniformly continuous.

5. Separation of Variables

In this part of the article, the strategy of separation of variables is used to convert the nonlinear MIE (4) into the FIE in position. In Equation (4), let us suppose the unknown function and the known function each have one of the next types, respectively:

$$\eta(x,t) = \eta(x)\zeta(t), f(x,t) = g(x)\zeta(t), \zeta(0) \ne 0, \tag{33}$$

during which $\zeta(t)$ and $g(x)$ have known functions and $\eta(x)$ is an unknown function.

Substituting from (33) into (4) we obtain

$$\begin{aligned}\eta(x)\zeta(t) &= g(x)\zeta(t) + D(x)\delta_1(G(t),\zeta(t))\delta_2(\eta(x)) \\ &\quad + \delta_1(G(t),\zeta(t))\delta_2(\eta(x))\int_0^t F(\tau)\gamma_1(\zeta(\tau))d\tau \\ &\quad \times \int_0^1 k(x,y)\gamma_2(\eta(y))dy.\end{aligned} \tag{34}$$

The solution of Equation (34) can be found by dividing both sides by $\zeta(t)$ we get

$$\begin{aligned}\eta(x) &= g(x) + D(x)\delta_2(\eta(x))\frac{\delta_1(G(t),\zeta(t))}{\zeta(t)} \\ &\quad + \left[\frac{\delta_1(G(t),\zeta(t))}{\zeta(t)}\int_0^t F(\tau)\gamma_1(\zeta(\tau))d\tau\right. \\ &\quad \times \delta_2(\eta(x))\int_0^1 k(x,y)\gamma_2(\eta(y))dy.\end{aligned} \tag{35}$$

It is also possible to express the value (35) using a format

$$\begin{aligned}\eta(x) &= g(x) + D(x)\delta_2(\eta(x))I(t) \\ &\quad + \delta_2(\eta(x))\Omega(t)\int_0^1 k(x,y)\gamma_2(\eta(y))dy,\end{aligned} \tag{36}$$

where

$$\begin{aligned}I(t) &= \frac{\delta_1(G(t),\zeta(t))}{\zeta(t)}, \\ \Omega(t) &= I(t)\int_0^t F(\tau)\gamma_1(\zeta(\tau))d\tau.\end{aligned} \tag{37}$$

Equation (36) represents the NFIE of the second kind.

6. The Technique of Error

Assume the numerical solution called $\eta_m(x,t)$ for $f_m(x,t)$, hence, the solution can be approximatively expressed as follows:

$$\begin{aligned}\eta_m(x,t) &= f_m(x,t) + D(x)\delta_m(G(t),\eta(x,t)) \\ &\quad + \delta_m(G(t),\eta(x,t))\int_0^t\int_0^1 F(\tau)k(x,y) \\ &\quad \cdot \gamma_m(\eta(y,\tau))dyd\tau.\end{aligned} \tag{38}$$

The corresponding error becomes

$$\begin{aligned}[\eta(x,t) - \eta_m(x,t)] &= [f(x,t) - f_m(x,t)] \\ &\quad + D(x)[\delta(G(t),\eta(x,t)) - \delta_m(G(t),\eta(x,t))] \\ &\quad + \left\{\delta(G(t),\eta(x,t))\int_0^t\int_0^1 F(\tau)k(x,y)\gamma(\eta(y,\tau))dyd\tau\right. \\ &\quad \left. - \delta_m(G(t),\eta(x,t))\int_0^t\int_0^1 F(\tau)k(x,y)\gamma_m(\eta(y,\tau))dyd\tau\right\}.\end{aligned} \tag{39}$$

Let the estimated error be

$$R_m(x,t) = \eta(x,t) - \eta_m(x,t). \tag{40}$$

Hence, we have

$$\begin{aligned}R_m(x,t) &= g(x,t) + D(x)\delta_{\text{error}}(G(t),\eta(x,t)) \\ &\quad + \delta_{\text{error}}(G(t),\eta(x,t))\int_0^t\int_0^1 F(\tau)k(x,y)\gamma_m(\eta(y,\tau))dyd\tau \\ &\quad + \delta_m(G(t),\eta(x,t))\int_0^t\int_0^1 F(\tau)k(x,y)[\gamma(\eta(y,\tau)) \\ &\quad - \gamma_m(\eta(y,\tau))]dyd\tau.\end{aligned} \tag{41}$$

where

$$\delta_{\text{error}}(G(t),\eta(x,t)) = \delta(G(t),\eta(x,t)) - \delta_m(G(t),\eta(x,t)). \tag{42}$$

Table 1: Comparison among absolute errors of Example 1, $N = 5$ by BPM and CP6K.

x	$T = 0.0012$		$T = 0.22$		$T = 0.4$		$T = 0.62$		$T = 0.8$	
	BPM	CP6K	BPM	CP6K	BPM	CP6K	BPM	CP6K	BPCM	CP6K
0	1.48×10^{-11}	5.22×10^{-13}	4.35×10^{-11}	3.21×10^{-13}	1.88×10^{-10}	4.46×10^{-12}	5.13×10^{-10}	1.54×10^{-11}	7.37×10^{-10}	1.29×10^{-11}
0.2	1.45×10^{-11}	5.26×10^{-13}	4.28×10^{-11}	3.07×10^{-13}	1.84×10^{-10}	4.61×10^{-12}	4.90×10^{-10}	1.56×10^{-11}	7.25×10^{-10}	1.69×10^{-11}
0.4	1.54×10^{-11}	5.93×10^{-13}	4.57×10^{-11}	3.50×10^{-13}	1.95×10^{-10}	5.23×10^{-12}	5.003×10^{-10}	1.76×10^{-11}	7.78×10^{-10}	3.04×10^{-11}
0.6	1.67×10^{-11}	8.82×10^{-13}	4.85×10^{-11}	5.55×10^{-13}	2.08×10^{-10}	7.62×10^{-12}	5.11×10^{-10}	2.62×10^{-11}	8.44×10^{-10}	5.73×10^{-11}
0.8	1.82×10^{-11}	1.66×10^{-12}	4.98×10^{-11}	1.06×10^{-12}	2.16×10^{-10}	1.39×10^{-11}	5.13×10^{-10}	4.92×10^{-11}	9.05×10^{-10}	1.04×10^{-11}
1	2.04×10^{-11}	1.66×10^{-12}	5.19×10^{-11}	2.04×10^{-12}	2.30×10^{-10}	2.74×10^{-11}	5.27×10^{-10}	9.79×10^{-11}	9.94×10^{-10}	1.79×10^{-11}

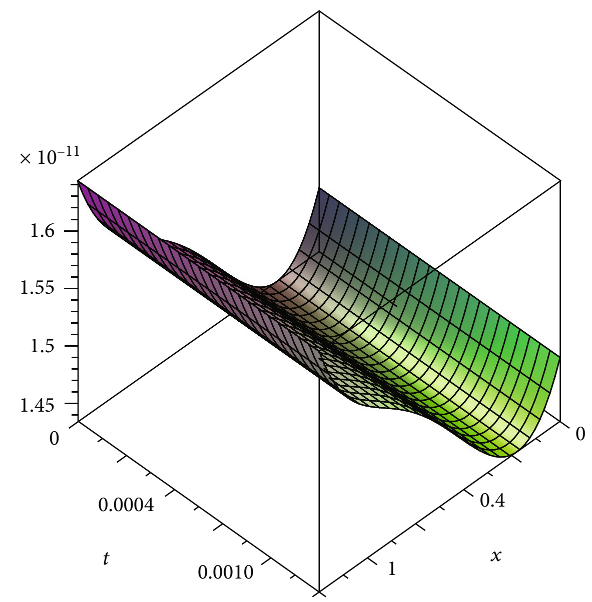

Figure 1: BPM's absolute errors for Example 1 at $T = 0.001$.

Write the above formula in the form

$$R_m(x,t) = g(x,t) + D(x)\delta_{\text{error}}(G(t),\eta(x,t)) + \delta_{\text{error}}(G(t),\eta(x,t))\int_0^t\int_0^1 F(\tau)k(x,y) \cdot \gamma(\eta(y,\tau))dyd\tau + \delta_m(G(t),\eta(x,t)) \cdot \int_0^t\int_0^1 F(\tau)k(x,y)[\gamma(\eta(y,\tau)) - \gamma_m(\eta(y,\tau))]]dyd\tau,$$

$$(g(x,t) = F(x,t) - F_m(x,t)), \quad (43)$$

Hence, the absolute value of the error convergent is given using the next inequality

$$\|R_m(x,t)\| \le \frac{\|F(x,t) - F_m(x,t)\|}{\left(1 - \left(BD + ABC^2T\right)\right)}. \quad (44)$$

The inequality (44) as $m \longrightarrow \infty$, $R \longrightarrow$ zero. From inequality (44), the error $R_m(x,t)$ is convergent under the condition

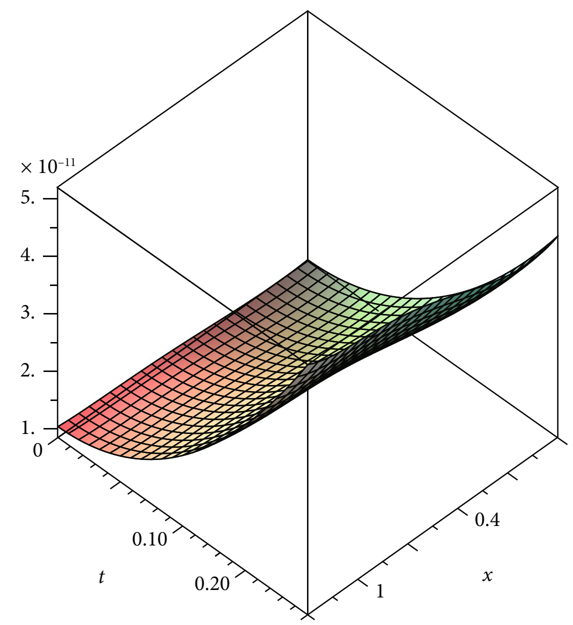

Figure 2: For illustrated Example 1, determined by BPM, the absolute error at $T = 0.2$.

$$T < \frac{1 - BD}{ABC^2}. \quad (45)$$

7. Bernoulli's Polynomial Method

Bernoulli's polynomials play an important part in a vast range of mathematical fields and subfields, including, for example, the mathematics thesis [37] and complex differential equations [38].

Any of the Bernoulli polynomials can be displayed by using the formula [38].

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} B_k, \quad (46)$$

where $\binom{n}{k} = n!/k!(n-k)!$ and $B_n(x)$ has the Bernoulli polynomial of n^{th} degree.

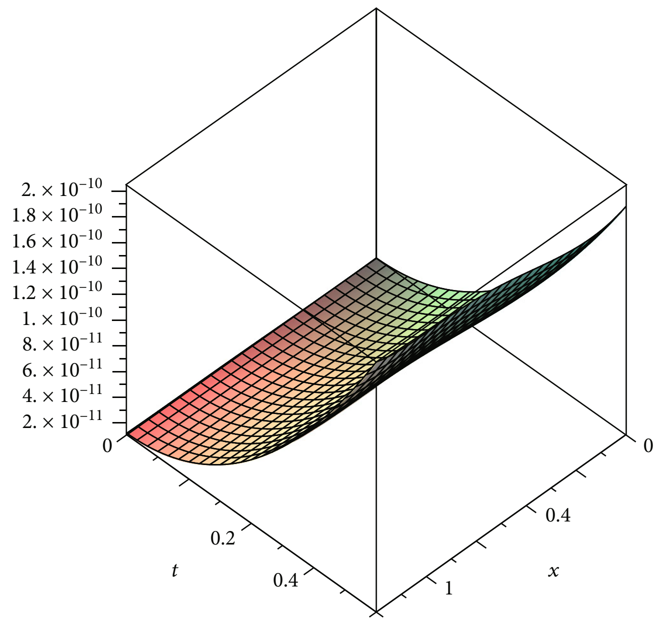

Figure 3: For illustrated Example 1, determined by BPM, the absolute error at $T = 0.4$.

Specifically, if $x = 0$ in (46), after that $B_n(0) = B_n$ which is a description of the Bernoulli numbers and $B_0 = 1$.

The following is the formula that shall be used to calculate the Bernoulli numbers:

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k = (1-n)x^n, n = 0, 1, 2, \cdots,. \quad (47)$$

The initial few Bernoulli polynomials shall be represented as

$$\begin{aligned} &B_0(x) = 1, B_1(x) = x - \frac{1}{2}, B_2(x) = x^2 - x + \frac{1}{6}, B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \\ &B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}, B_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x, \\ &B_6(x) = x^6 - 3x^5 + \frac{5}{2}x^4 + \frac{1}{2}x^2 - \frac{1}{42}. \end{aligned} \quad (48)$$

The following is a truncated version of the estimated solution to Equation (36) that was provided by BP:

$$\eta(x) \simeq \eta_M(x) = \sum_{i=0}^{M} a_i B_i(x), \quad (49)$$

where $a_i, i = 0, 1, \cdots, M$ have the unknown coefficients Bernoulli and $B_i(x)$ have BP described in the definition (46).

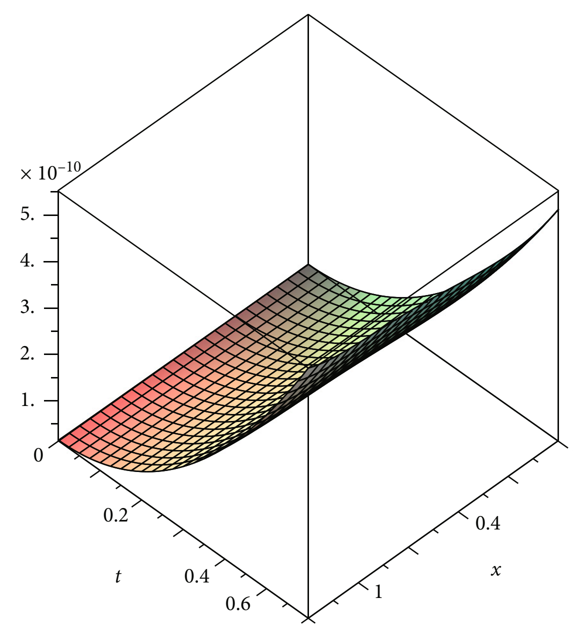

Figure 4: For illustrated Example 1, determined by BPM, the absolute error at $T = 0.6$.

Substituting from (49) into (36), we obtain

$$\begin{aligned} \sum_{i=0}^{M} a_i B_i(x) = g(x) + D(x)\delta_2 \left(\sum_{i=0}^{M} a_i B_i(x) \right) I(t) \\ + \delta_2 \left(\sum_{i=0}^{M} a_i B_i(x) \right) \Omega(t) \int_0^1 k(x,y)\gamma_2 \\ \cdot \left(\sum_{i=0}^{M} a_i B_i(y) \right) dy. \end{aligned} \quad (50)$$

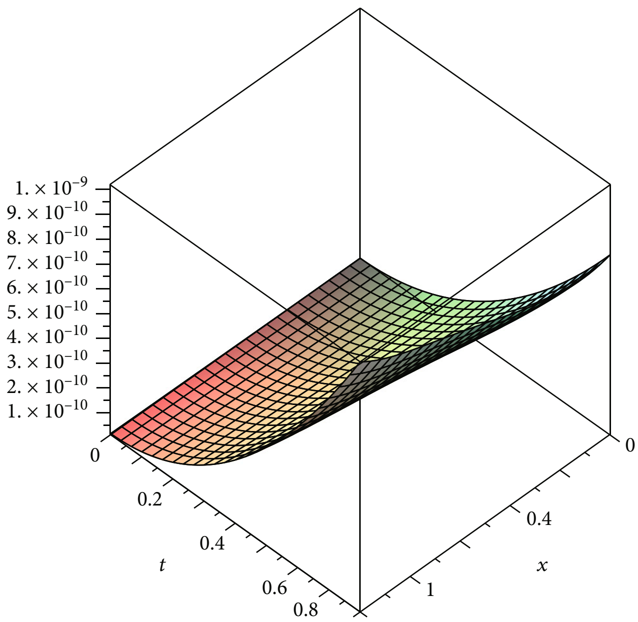

FIGURE 5: For illustrated Example 1, determined by BPM, the absolute error at $T = 0.8$.

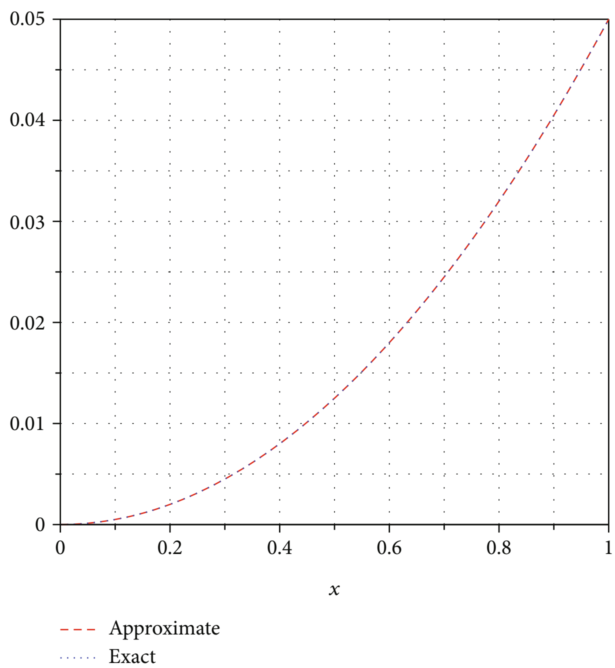

FIGURE 6: Accurate and approximate resolution of Example 1 by BPM, $T = 0.2$.

Using the collocation points, we can continue

$$x_i = a + \frac{(b-a)i}{M}, i = 0, 1, 2, \cdots, M. \tag{51}$$

The results that we get are as follows: $(M+1)$ nonlinear algebraic equations (SNAE) with $(M+1)$ unknowns

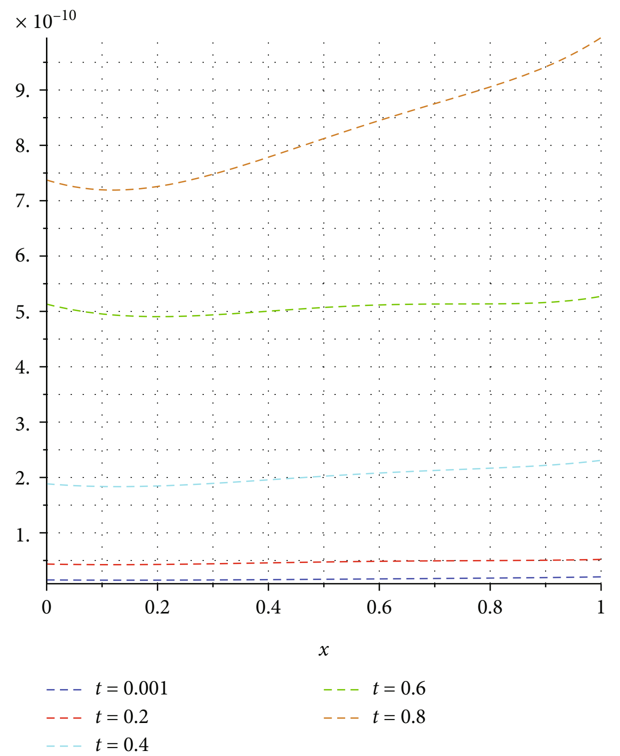

FIGURE 7: Absolute errors were made in Example 1 by BPM.

$$\begin{aligned}&\sum_{i=0}^{M} a_i B_i(x_i) - D(x_i)\delta_2\left(\sum_{i=0}^{M} a_i B_i(x_i)\right) I(t)\\&\quad + \delta_2\left(\sum_{i=0}^{M} a_i B_i(x_i)\right)\Omega(t)\int_0^1 k(x_i, y)\gamma_2\\&\quad\cdot\left(\sum_{i=0}^{M} a_i B_i(y)\right)dy = g(x_i).\end{aligned} \tag{52}$$

After solving the above system, we obtain the approximate solution, and by using (33), we display the numerical solution of (4) which corresponds to the nonlinear partial integrodifferential Equation (1).

8. Sixth Type of Chebyshev Polynomials

In the following part of the essay, CP6K is applied to solve NFIE (36).

Sixth-type Chebyshev polynomials $Y_j(\xi)$ have orthogonal functions on the interval $[-1, 1]$ defined according to the recurrence connection that is as follows:

$$\begin{aligned}Y_j(\xi) &= \xi Y_{j-1} - \frac{j(j+1) + (-1)^j(2j+1) + 1}{4j(j+1)} Y_{j-2}(\xi), Y_0(\xi)\\&= 1, Y_1(\xi) = \xi, j \geq 2.\end{aligned} \tag{53}$$

These polynomials have an orthogonal relationship on the

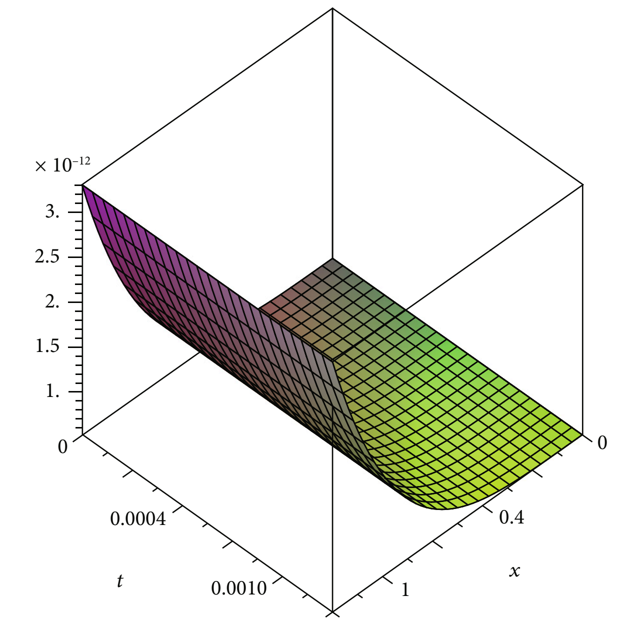

Figure 8: The absolute error, for Example 1 calculated by CP6K, is $T = 0.001$.

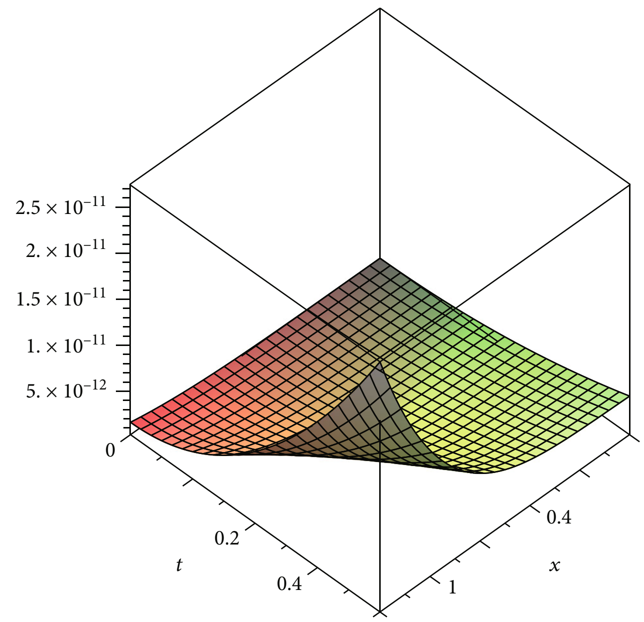

Figure 10: At $T = 0.4$, the errors absolute that was estimated by CP6K for the illustrated Example 1.

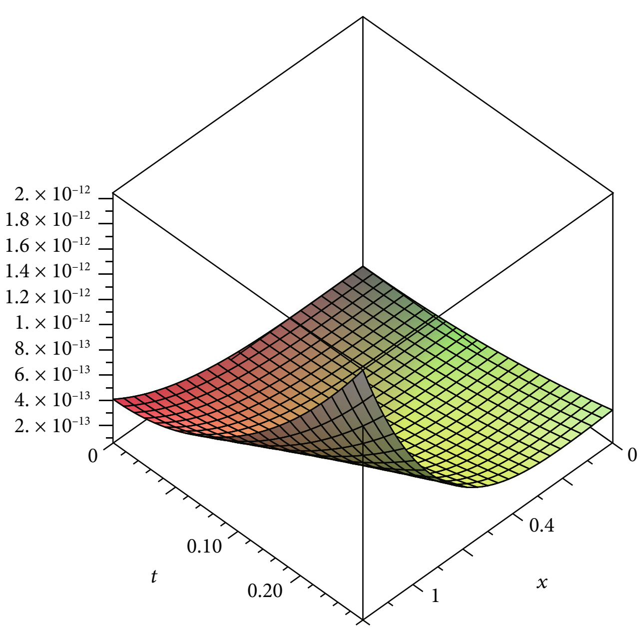

Figure 9: The error, for Example 1 calculated by CP6K, is $T = 0.2$.

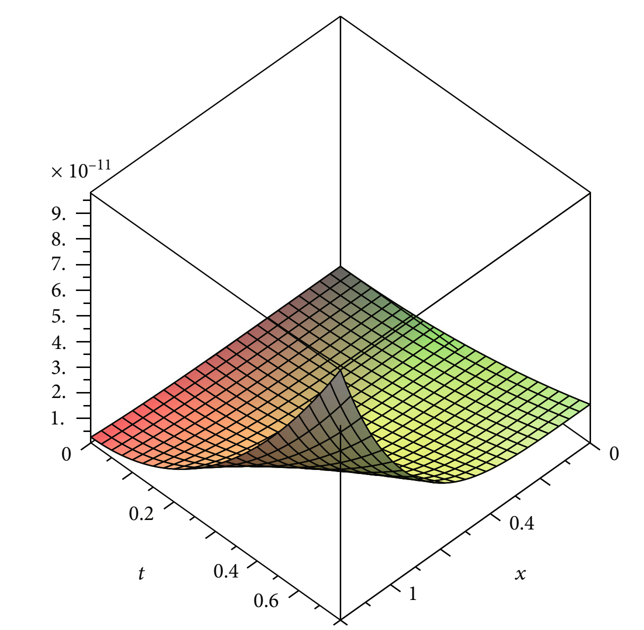

Figure 11: At $T = 0.6$, the errors absolute that was estimated by CP6K for the illustrated Example 1.

interval $[-1, 1]$, regarding the weight function is $\omega_2(\xi) = \xi^2 \sqrt{1-\xi^2}$ and satisfies

$$\int_{-1}^{1} \xi^2 \sqrt{1-\xi^2}\, Y_i(\xi) Y_j(\xi)\, d\xi = \begin{cases} \Theta_i, i = j, \\ 0, i \neq j, \end{cases} \tag{54}$$

where

$$\Theta_i = \frac{\pi}{2^{2i+3}} \begin{cases} 1, i\text{even}, \\ i + 3/i + 1, i\text{odd}. \end{cases} \tag{55}$$

In their paper [41], Abd-Elhameed and Youssri demonstrated that CP6K can be represented trigonometrically as follows:

$$Y_j(\cos\theta) = \begin{cases} \sin((j+2)\theta)/2^j \sin(2\theta), j\text{even}, \\ \sin((j+1)\theta) + (j+1)\cos(\theta)\sin((j+2)\theta)/2^{j+1}(j+1)\cos^2(\theta)\sin(\theta), j\text{odd}. \end{cases} \tag{56}$$

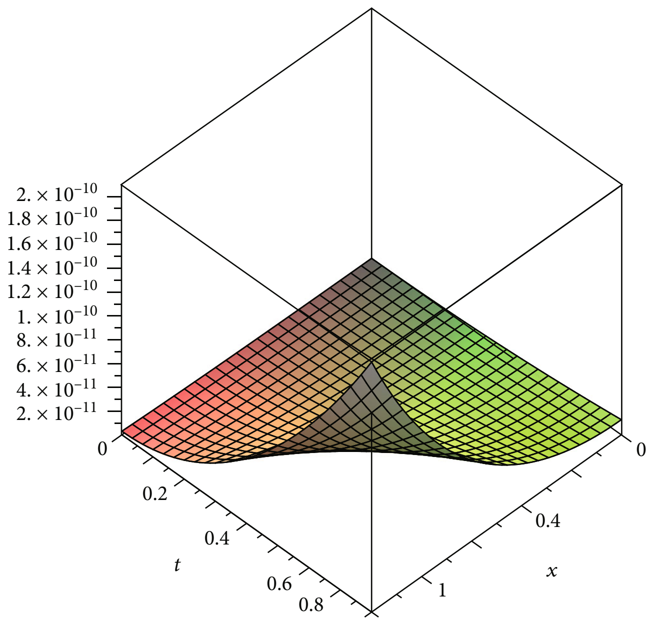

Figure 12: At $T = 0.8$, the errors absolute that was estimated by CP6K for the illustrated Example 1.

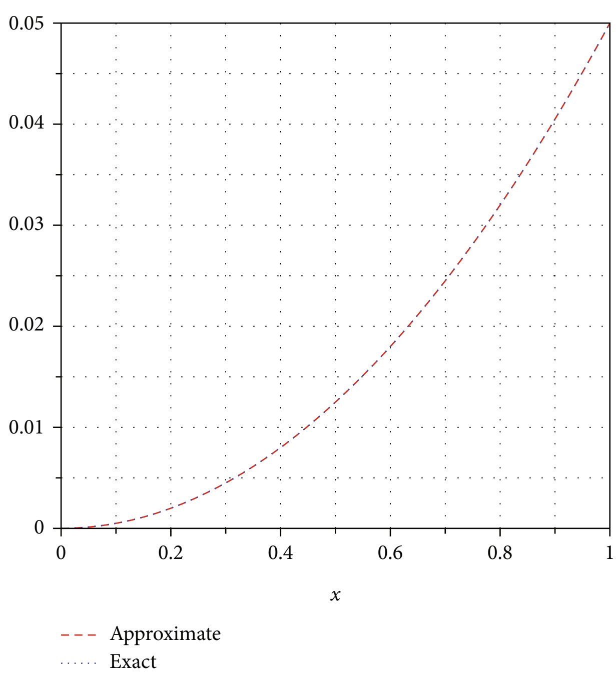

Figure 13: Accurate and approximate resolution of Example 1 by CP6K, $T = 0.2$.

Lemma 6 (see [42]). *Let m be any nonnegative integer. Consequently, CP6K can be explained in the power form shown below*

$$Y_{2m}(\xi) = \frac{\Gamma(m+3/2)}{(2m+1)!}\sum_{i=0}^{m}\frac{(-1)^i\binom{m}{i}(2m-i+1)!}{\Gamma(m-i+3/2)}\xi^{2m-2i}, \quad (57)$$

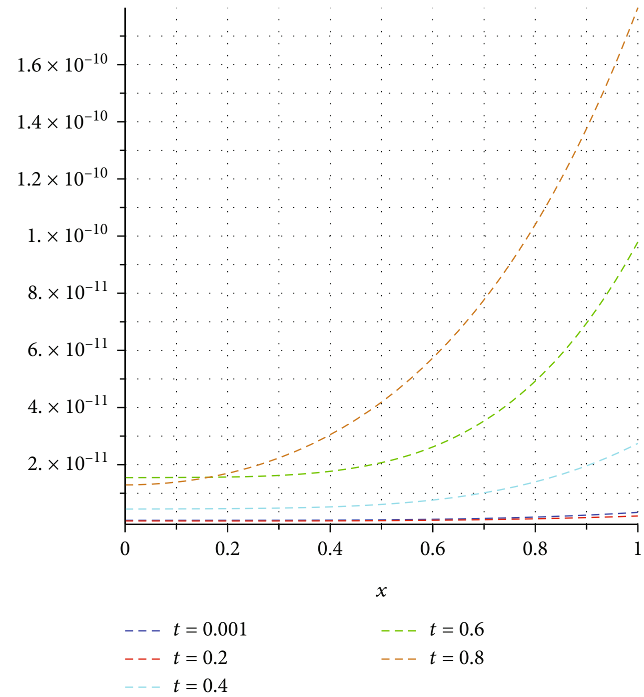

Figure 14: The errors absolute were made of Example 1 by CP6K.

Table 2: CPU time of Example 1.

CPU time	CP6K	BPCM
$T = 0.001$	0.06 s	0.07 s
$T = 0.2$	0.06 s	0.06 s
$T = 0.4$	0.09 s	0.12 s
$T = 0.6$	0.09 s	0.09 s
$T = 0.8$	0.06 s	0.10 s

$$Y_{2m+1}(\xi) = \frac{\Gamma(m+5/2)}{(2m+2)!}\sum_{i=0}^{m}\frac{(-1)^i\binom{m}{i}(2m-i+2)!}{\Gamma(m-i+5/2)}\xi^{2m-2i+1}. \quad (58)$$

Lemma 7 (see [42]). *Let p be any nonnegative integer. Then the inversion formulas of CP6K are given as*

$$\xi^{2p} = (2p+1)!\sum_{i=0}^{p}\frac{2^{1-2i}(p-i+1)!}{i!(2p-i+2)!}Y^{2p-2i}(\xi), \quad (59)$$

$$\xi^{2p+1} = (2p+1)!(2p+3)\sum_{i=0}^{p}\frac{2^{1-2i}(p-i+1)!}{i!(2p-i+3)!}Y^{2p-2i+1}(\xi). \quad (60)$$

Let $\eta(x) \in L_2[-1,1]$, then $\eta(x)$ could be generally described with terms similar $Y_i(x)$ as next

Table 3: Comparison between absolute errors of Example 2, $N = 5$ by BPM and CP6K.

x	$T = 0.001$		$T = 0.2$		$T = 0.4$		$T = 0.6$		$T = 0.8$	
	BPM	CP6K	BPM	CP6K	BPM	CP6K	BPM	CP6K	BPM	CP6K
0	6.99×10^{-11}	9.99×10^{-11}	5.73×10^{-11}	0	2.01×10^{-11}	0	9.32×10^{-11}	5.48×10^{-11}	1.79×10^{-11}	4.49×10^{-11}
0.2	4.70×10^{-11}	9.58×10^{-11}	3.85×10^{-11}	4.59×10^{-13}	3.53×10^{-11}	8.31×10^{-15}	8.06×10^{-11}	5.300×10^{-11}	4.71×10^{-11}	1.25×10^{-11}
0.4	4.83×10^{-11}	7.92×10^{-11}	3.91×10^{-11}	2.81×10^{-12}	3.38×10^{-11}	3.90×10^{-14}	8.01×10^{-11}	5.12×10^{-11}	1.07×10^{-10}	4.27×10^{-11}
0.6	1.55×10^{-10}	3.82×10^{-11}	1.25×10^{-10}	1.27×10^{-11}	4.12×10^{-11}	9.82×10^{-14}	1.34×10^{-10}	6.10×10^{-11}	2.19×10^{-10}	6.24×10^{-12}
0.8	3.75×10^{-10}	4.60×10^{-11}	2.99×10^{-10}	4.04×10^{-11}	1.96×10^{-10}	1.58×10^{-13}	2.41×10^{-10}	1.01×10^{-10}	4.01×10^{-10}	3.71×10^{-10}
1	5.81×10^{-10}	2.006×10^{-10}	4.56×10^{-10}	1.006×10^{-10}	3.51×10^{-10}	1.44×10^{-13}	3.24×10^{-10}	2.00×10^{-10}	6.11×10^{-10}	1.35×10^{-9}

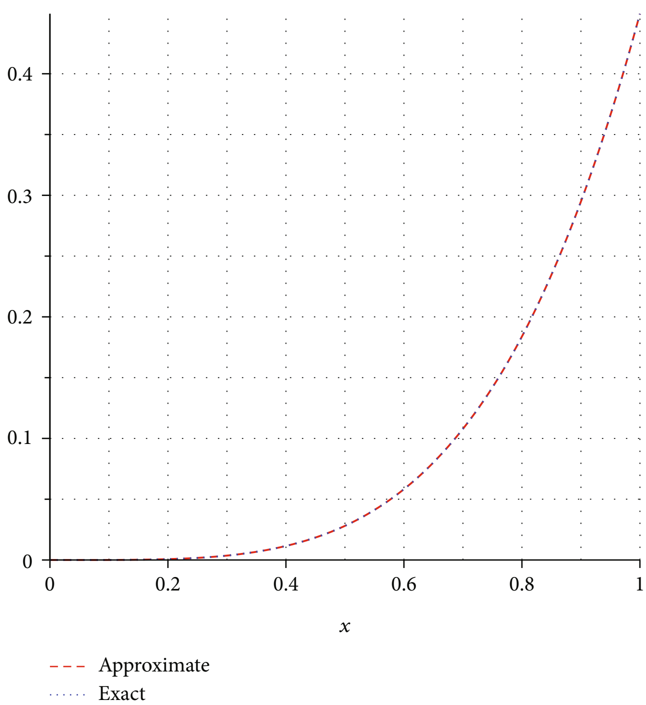

Figure 15: Accurate and approximate resolution of Example 2 by BPM, $T = 0.8$.

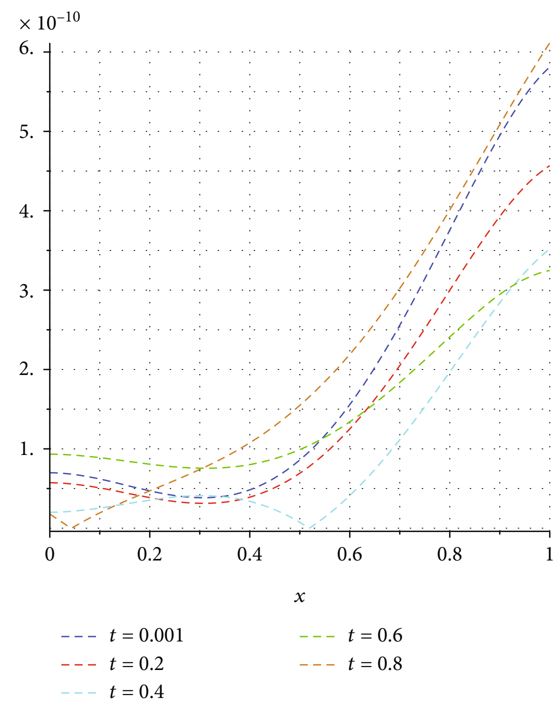

Figure 16: Absolute errors of Example 2 by BPM.

$$\eta(x) \approx \eta_M(x) = \sum_{i=0}^{M} c_i Y_i(x), \quad (61)$$

where c_i are constants $i = 0, 1, \cdots, M$.

Substituting from (61) to (36), we are able to get

$$\sum_{i=0}^{M} a_i Y_i(x) = g(x) + D(x)\delta_2 \left(\sum_{i=0}^{M} a_i Y_i(x) \right) I(t) + \delta_2 \left(\sum_{i=0}^{M} a_i Y_i(x) \right) \Omega(t) \int_0^1 k(x, y)\gamma_2 \left(\sum_{i=0}^{M} a_i Y_i(y) \right) dy. \quad (62)$$

Using the collocation points, we can continue

$$x_r = a + \frac{(b-a)r}{M}, r = 0, 1, 2, \cdots, M. \quad (63)$$

The results that we get are as follows: $(M+1)$ SNAE with $(M+1)$ unknowns

$$\sum_{i=0}^{M} c_i Y_i(x_r) - D(x_r)\delta_2 \left(\sum_{i=0}^{M} c_i Y_i(x_r) \right) I(t) + \delta_2 \left(\sum_{i=0}^{M} c_i Y_i(x_r) \right) \Omega(t) \int_0^1 k(x_r, y)\gamma_2 \cdot \left(\sum_{i=0}^{M} a_i Y_i(y) \right) dy = g(x_r). \quad (64)$$

After solving the above system, we obtain the approximate solution, and by using (33), we get the numerical solution of (4) which corresponds to the nonlinear partial integrodifferential Equation (1).

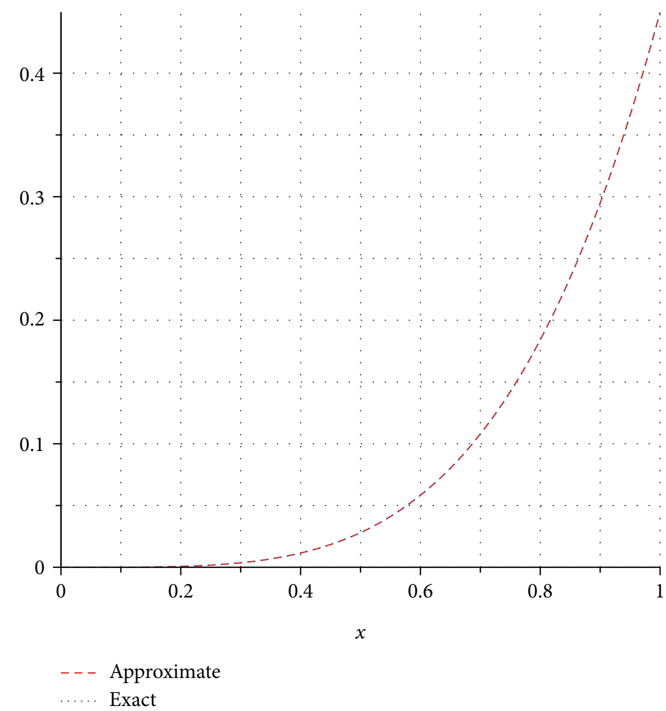

FIGURE 17: Accurate and approximate resolution of Example 2 by CP6K, $T = 0.8$.

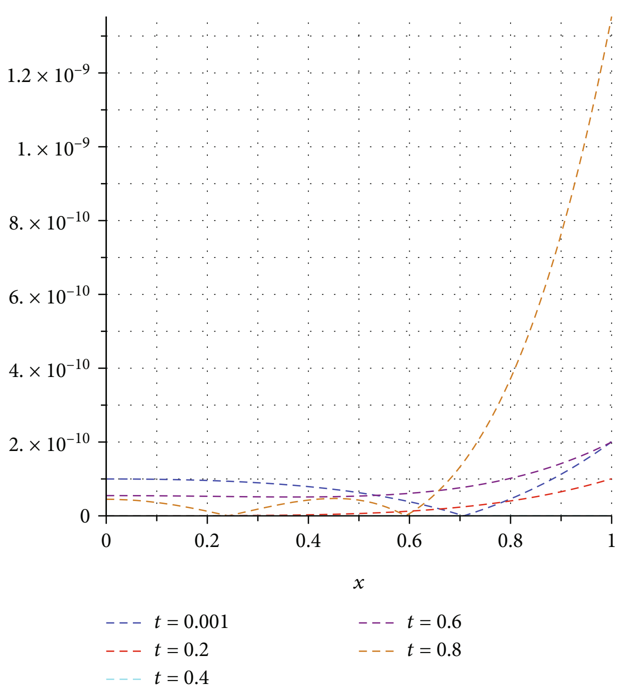

FIGURE 18: Absolute errors of Example 2 by CP6K.

TABLE 4: CPU time of Example 2.

CPU time	CP6K	BPCM
$T = 0.001$	0.04 s	0.06 s
$T = 0.2$	0.04 s	0.07 s
$T = 0.4$	0.07 s	0.09 s
$T = 0.6$	0.04 s	0.09 s
$T = 0.8$	0.04 s	0.09 s

9. Illustrative Examples

In this section, we will illustrate the above results via applying the examples below for various time values.

Example 1. Suppose the nonlinear partial integrodifferential equation:

$$\eta(x,t) = f(x,t) + \left(e^{-t}\eta(x,t)\right)\left[\frac{(u(x) - f(x,0))}{\delta(G(0), u(x))}\right] + \left(e^{-t}\eta(x,t)\right)\int_0^t\int_0^1 \left(1+\tau^2\right)x^2y^2\eta^2(y,\tau)dyd\tau, \tag{65}$$

utilizing an analytical resolution $\eta(x,t) = x^2(0.01 + t^2)$, $G(t) = e^{-t}$, $\delta(G(t), \eta(x,t)) = e^{-t}\eta(x,t)$.

Applying the separation of variables BPM and CP6K for Equation (65) when $N = 5$ for different values of T, the numerical outcomes have been demonstrated in Table 1, Figures 1–5 presented absolute error by using BPM, and Figure 6 presented discussion among the two accurate and approximate solutions at $T = 0.2$ by using BPM. Figure 7 presents a discussion of the absolute errors for various values of the time by using BPM. While Figures 8–12 present absolute error by using CP6K, Figure 13 presents a discussion of the accurate response and an approximate response at $T = 0.2$ through the use of CP6K, and Figure 14 presents a discussion among the absolute errors for various values of the time through the use of CP6K. In Table 2, we have stated that the CPU time is different between the two methods used. Also, the advantages between used methods.

Example 2. Suppose the nonlinear partial integrodifferential equation

$$\eta(x,t) = f(x,t) + ((0.01+t)\eta(x,t))\left[\frac{(u(x) - f(x,0))}{\delta(G(0),u(x))}\right] + ((0.01+t)\eta(x,t))\int_0^t\int_0^1 \left(0.25+\tau^2\right)x^2y^2\eta^2(y,\tau)dyd\tau, \quad (66)$$

with accurate resolution $\eta(x,t) = x^4e^{-t}$, $G(t) = (0.01+t)$, $\delta(G(t),\eta(x,t)) = (0.01+t)\eta(x,t)$.

Applying the separation of variables, BPM and CP6K for Equation (66) at $N = 5$ for various values of T, the results of the numerical analysis can be found in Table 3, as well as Figures 15–18. In Table 4, we have stated the CPU time to be different between the two methods used. Also, the advantages between used methods.

10. Conclusion

In the following paper, we computed the NMPIoDE's numerical solution. Using the separation of variables, the NMPIoDE is converted to NFIE. The NFIE was further condensed into a SNAE utilizing two distinct numerical techniques, BPM and CP6K. The Banach fixed-point theory is applied to studying the nonlinear MIE solution's existence and uniqueness. We additionally discuss the solution's convergence and stability. Finally, using numerous examples, a comparison between the two various approaches and several other well-known approaches is provided. The Maple software was used to obtain all of the numerical results. Comparing the results, it turns out that CP6K is better than BPM, as the error in CP6K is less than the error using BPM, and also because CP6K is orthogonal but BPM is not orthogonal.

Data Availability

To support this study, no data were used.

Additional Points

Future Work. Other methods, such as the Adomian decomposition, the Sumudu transform approach, the homotopy perturbation approach, and the variational iteration approach, will be used to solve the nonlinear NMPIoDE.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed to writing the revision draft, reviewing, and editing.

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