Hindawi
Journal of Function Spaces
Volume 2023, Article ID 6692867, 9 pages
https://doi.org/10.1155/2023/6692867

Research Article

The Existence and Multiplicity of Solutions for $p(x)$-Laplacian-Like Neumann Problems

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Received 2 February 2023; Revised 19 June 2023; Accepted 17 August 2023; Published 16 September 2023

Academic Editor: John R. Akeroyd

In the present paper, in view of the variational approach, we discuss the Neumann problems with $p(x)$-Laplacian-like operator and nonstandard growth condition, originated from a capillary phenomena. By using the least action principle and fountain theorem, we prove the existence and multiplicity of solutions to the class of Neumann problems under suitable assumptions.

1. Introduction and Main Result

In this paper, we study existence and multiplicity of solutions for the $p(x)$-Kirchhoff-type equation involving nonstandard growth condition and arising from a capillary phenomena of the following type:

$$\begin{cases} -a(\psi(u))\ \mathrm{div}\left(|\nabla u|^{p(x)-2}\nabla u + \dfrac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\right) = f(x,u), & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $a(t) \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\psi(u) = \int_\Omega (1/p(x))(|\nabla u|^{p(x)} + \sqrt{1+|\nabla u|^{2p(x)}})dx$, and $f(x,u) \in C(\Omega \times \mathbb{R}, \mathbb{R})$; ν is the outward unit normal on $\partial\Omega$. $p \in C_+(\bar{\Omega}) = \{h | h \in C(\bar{\Omega}), h(x) > 1, \forall x \in \bar{\Omega}\}$ with

$$1 < p^- \coloneqq \min_{x\in\bar{\Omega}} p(x) \le p(x) \le p^+ \coloneqq \max_{x\in\bar{\Omega}} p(x) < N. \tag{2}$$

It is worth mentioning that $a(t)$ in problem (1) contains the Kirchhoff-type functions such as $a + bt$, $a > 0$, and $b > 0$. The Kirchhoff-type equation has a strong physical background. We refer the interested readers to [1–3] and the references therein.

It is well-known that $-\Delta_{p(x)} u = -\mathrm{div}\,(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian operator. The study of differential equations and variational problems with nonstandard $p(x)$-growth conditions and (p, q)-Laplacian operator (see [4, 5]) has been a very interesting topic in recent years. They allow the modelling of various phenomena that arise in the study of elastic mechanics and image restoration, and we can refer to [6–8]. Since the left-hand side of problem (1) contains an integral over Ω, it is no longer a pointwise identity. Therefore, it is often called a nonlocal problem. Problem (1) has a rich mathematical and physical background base. For example, when $p(x) \equiv 2$, $a(t) \equiv 1$, problem (1) degenerates into a generalized capillarity equation describing "capillary phenomena."

The problem involving $p(x)$-Laplacian-like operator was firstly studied by Rodrigues based on the variational method (see [9]). He studied the following equation:

$$\begin{cases} -\operatorname{div}\left(\left(1+\dfrac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)-2}\nabla u\right)=\lambda f(x,u), & x\in\Omega,\\ u=0, & x\in\partial\Omega, \end{cases} \tag{3}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $p(x) \in C(\bar{\Omega})$, and $p(x) > 2, \lambda > 0$. The author proved existence and multiplicity of solutions by using mountain pass theorem and fountain theorem. Problem (3) can be used to describe capillarity. Capillarity can be briefly explained by considering the effects of two opposing forces: adhesion, i.e., the attractive (or repulsive) force between the molecules of the liquid and those of the container; and cohesion, i.e., the attractive force between the molecules of the liquid. The study of capillary phenomena has gained some attention recently. This increasing interest is motivated not only by fascination in naturally occurring phenomena such as motion of drops, bubbles, and waves but also its importance in applied fields ranging from industrial and biomedical and pharmaceutical to microfluidic systems. In reference [9–14], the existence and multiplicity of solutions for the Dirichlet boundary value problems with $p(x)$-Laplacian-like operator are also studied.

In recent years, the Neumann problems have been extensively studied, and we can refer to [15–21]. The authors in [20, 21] studied the following p-Kirchhoff-type Neumann problems:

$$\begin{cases} -\left(M\left(\displaystyle\int_\Omega |\nabla u|^p dx\right)\right)^{p-1}\Delta_p u = f(x,u), & x\in\Omega,\\ \dfrac{\partial u}{\partial v}=0, & x\in\partial\Omega, \end{cases} \tag{4}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$ and v is the outward unit normal on $\partial\Omega$. By using the saddle point theorem and abstract linking argument due to Brezis and Nirenberg [22], they showed that problem (4) has at least two nontrivial solutions. In [16], Chung have extended problem (4) to $p(x)$-Kirchhoff-type Neumann problems. Under appropriate assumptions on f, the author proved the existence and multiplicity of solutions by using abstract linking argument. From a mathematical point of view, the extension from the p-Laplace operator $\Delta_p u$ to the $p(x)$-Laplace operator $\Delta_{p(x)} u$ is interesting and not trivial, since the $p(x)$-Laplace operator has a more complicated structure than the p-Laplace operator, for example, they are nonhomogeneous.

Furthermore, Jiang et al. [17] have studied the existence and multiplicity of solutions for problem (4) of the Kirchhoff function $M(t) \equiv 1$ under the Landesman-Lazer type condition by using saddle point theorem and abstract linking argument. Also for further studies on this Landesman-Lazer type condition, we have provided reference for readers in [23–26].

However, we find that the $p(x)$-Laplacian-like Neumann problems with Landesman-Lazer type condition is lagged behind. We point out that the main difficulty arises from the fact that the first eigenvalue of the $p(x)$-Laplacian is not isolated. In this case, we can use the technique of decomposing the space $W^{1,p(x)}(\Omega)$ in this paper.

Motivated by the above papers and the results, the main purpose of this paper is to study the existence and multiplicity of solutions for problem (1). Assuming that the following conditions are met.

(a_1) There is a constant $a_0 > 0$, such that $a(t) \geq a_0$, for all $t \geq 0$.

(a_2) There is a constant $\theta \geq 1$, such that

$$\hat{a}(t) \geq \frac{1}{\theta} a(t)t, \text{for all } t \geq 0, \tag{5}$$

where $\hat{a}(t) = \int_0^t a(s)ds$.

(F_1) $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$, and there exists a constant $C_0 > 0$ such that

$$|f(x,t)| \leq C_0\left(1+|t|^{\alpha(x)-1}\right), \forall (x,t) \in \Omega\times\mathbb{R}, \tag{6}$$

where $\alpha(x) \in C_+(\bar{\Omega})$, $1 < \alpha(x) < p^*(x) = Np(x)/N - p(x)$.

(F_2) $\lim_{|t|\longrightarrow\infty} F(x,t) = -\infty$ uniformly for a.e. $x \in \Omega$, where $F(x,t) = \int_0^t f(x,s)ds$.

(F_3) $\liminf_{|t|\longrightarrow\infty} (F(x,t)/|t|^{p^-}) \leq 0$ uniformly for a.e. $x \in \Omega$.

(F_4) (Landesman-Lazer type condition) Whenever $\{u_n\} \subset W^{1,p(x)}(\Omega)$ is such that $\|u_n\| \longrightarrow \infty$ and $|\bar{u}_n||\Omega|^{1/p^-}/\|u_n\| \longrightarrow 1$ as $n \longrightarrow \infty$, then

$$\limsup_{n\longrightarrow\infty} \int_\Omega f(x,u_n)\frac{\bar{u}_n}{|\bar{u}_n|}dx > 0. \tag{7}$$

(F_5) $\limsup_{|t|\longrightarrow\infty} F(x,t)/|t|^{\theta p^+} = +\infty$ uniformly for a.e. $x \in \Omega$.

(F_6) $f(x,-t) = -f(x,t)$ for all $(x,t) \in \Omega \times \mathbb{R}$.

The main results of this paper are as follows.

Theorem 1. *Suppose that $1 < p^- \leq p^+ < \alpha^- \leq \alpha^+ < \min\{p_-^*, N\}$. If (a_1), (a_2), (F_1), and (F_2) hold, the problem (1) has at least one solution.*

Theorem 2. *Suppose that $1 < p^- \leq p^+ < \alpha^- \leq \alpha^+ < \min\{p_-^*, N\}$. If (a_1), (a_2), (F_1), (F_3), (F_4), (F_5), and (F_6) hold, the problem (1) has infinitely many nontrivial solutions.*

In the present paper, first of all, we use the least action principle method to study the existence of solution for the problem (1). Furthermore, we give some new solvability results for the problem (1) under the Landesman-Lazer type condition. By imposing additional assumptions on f, we establish the existence of infinitely many solutions by using fountain theorem, due to Bartsch in [27].

2. Preliminaries

We start with some preliminary basic results on the theory of Lebesgue-Sobolev spaces $L^{p(x)}$ and $W^{1,p(x)}$ with variable exponent; for more details, we refer the reader to the book by Musielak [28] and Fan and Zhao [29].

Let $\zeta(\Omega)$ denoted the set of all measurable real functions defined on Ω. For any $p \in C_+(\bar{\Omega})$, the variable exponent Lebesgue space $L^{p(x)}$ is defined as

$$L^{p(x)}(\Omega) = \left\{ u | u \in \zeta(\Omega): \int_\Omega |u(x)|^{p(x)} dx < \infty \right\}, \quad (8)$$

with the norm

$$\|u\|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \le 1 \right\}. \quad (9)$$

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined as

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega): |\nabla u| \in L^{p(x)}(\Omega) \right\}, \quad (10)$$

with the norm

$$\|u\| = \|u\|_{W^{1,p(x)}(\Omega)} = |u|_{p(x)} + |\nabla u|_{p(x)}. \quad (11)$$

Proposition 3 (see [29]).

(i) For any $u \in L^{p(x)}(\Omega)$, $v \in L^{q(x)}(\Omega)$ with $(1/p(x)) + (1/q(x)) = 1$, the inequality holds as follows

$$\left| \int_\Omega uv dx \right| \le \left(\frac{1}{p^-} + \frac{1}{q^-} \right) |u|_{p(x)} |v|_{q(x)}. \quad (12)$$

(ii) If $1 \le q(x) \in C_+(\bar{\Omega})$ and $q(x) \le p^*(x)$ (respectively $q(x) < p^*(x)$) for any $x \in \bar{\Omega}$, then $W^{1,p(x)}(\Omega)$ is embedded continuously (respectively, compactly) in $L^{q(x)}(\Omega)$.

Let us now recall the modular function $\rho_{p(x)}(u)$: $L^{p(x)}(\Omega) \longrightarrow \mathbb{R}$ which plays an important role in the variable order Lebesgue spaces and which is defined by

$$\rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)} dx. \quad (13)$$

Proposition 4 (see [30]). *For any $u_n, u \in L^{p(x)}(\Omega)$, then the following properties hold:*

$$|u|_{p(x)} < 1 (= 1; > 1) \Leftrightarrow \rho_{p(x)}(u) < 1 (= 1; > 1),$$

$$|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^-}, |u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^+},$$

$$\lim_{n\longrightarrow\infty} |u_n|_{p(x)} = 0 \Leftrightarrow \lim_{n\longrightarrow\infty} \rho_{p(x)}(u_n) = 0, \lim_{n\longrightarrow\infty} |u_n - u| = 0 \Leftrightarrow \lim_{n\longrightarrow\infty} \rho_{p(x)}(u_n - u) = 0. \quad (14)$$

We can split $W^{1,p(x)}(\Omega)$ in the following way. Let

$$V_2 = \left\{ u \in W^{1,p(x)}(\Omega): \int_\Omega u dx = 0 \right\}. \quad (15)$$

For each $u \in W^{1,p(x)}(\Omega)$, denote $\tilde{u}(x) = u(x) - \bar{u}$, where $\bar{u} = 1/|\Omega| \int_\Omega u(x) dx$, $\bar{u} \in \mathbb{R}$ and $\tilde{u} \in V_2$. Note V_2 is a closed linear subspace of $W^{1,p(x)}(\Omega)$ with codimension 1. Then, $W^{1,p(x)}(\Omega) = V_2 \oplus \mathbb{R}$ (see [31]).

The following proposition plays an important role in our proof.

Proposition 5. *(see [31], Proposition 2.6) there is a positive constant k, such that*

$$|u|_{p(x)} \le k |\nabla u|_{p(x)}, \forall u \in V_2. \quad (16)$$

We define the norm by

$$\|u\|_V = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{\nabla u(x)}{\lambda} \right|^{p(x)} dx + \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \le 1 \right\}. \quad (17)$$

We conclude that $\|u\|$ and $\|u\|_V$ are equivalent norms. Invoking Proposition 5, it is easy to see that $\|u\|_V$ and $|\nabla u|_{p(x)}$ are equivalent norms in V_2.

The modular function $\rho(u)$: $V_2 \longrightarrow \mathbb{R}$ define by

$$\rho(u) = \int_\Omega |\nabla u(x)|^{p(x)} dx + \int_\Omega |u(x)|^{p(x)} dx. \quad (18)$$

A similar derivation of [29] has the following proposition.

Proposition 6. *Set $u \in V_2$. Then, the following properties hold the following:*

$$\|u\| < 1(= 1; > 1) \Leftrightarrow \rho(u) < 1(= 1; > 1),$$
$$\|u\| < 1 \Rightarrow \|u\|^{p^+} \le \rho(u) \le \|u\|^{p^-}, \|u\| > 1 \Rightarrow \|u\|^{p^-} \le \rho(u) \le \|u\|^{p^+},$$
$$\lim_{n\longrightarrow\infty} \|u_n\| = 0 \Leftrightarrow \lim_{n\longrightarrow\infty} \rho(u_n) = 0, \lim_{n\longrightarrow\infty} \|u_n - u\| = 0 \Leftrightarrow \lim_{n\longrightarrow\infty} \rho(u_n - u) = 0. \tag{19}$$

Proposition 7 (see [9, 30]). $\psi(u) \in C^1(W^{1,p(x)}(\Omega), \mathbb{R})$ *and its derivative is given by*

$$\left\langle \psi'(u), v \right\rangle = \int_\Omega \left(|\nabla u|^{p(x)-2} + \frac{|\nabla u|^{2p(x)-2}}{\sqrt{1 + |\nabla u|^{2p(x)}}} \right) \nabla u \nabla v dx, \tag{20}$$

For all $u, v \in W^{1,p(x)}(\Omega)$ and the following properties hold the following:

(i) ψ is convex and sequentially weakly lower semicontinuous

(ii) $\psi' : W^{1,p(x)}(\Omega) \longrightarrow (W^{1,p(x)}(\Omega))^*$ is a mapping of type (S_+), that is, if $u_n \rightharpoonup u$ in $W^{1,p(x)}(\Omega)$ and $\limsup_{n\longrightarrow\infty} \langle \psi'(u_n), u_n - u \rangle \le 0$ imply $u_n \longrightarrow u$ in $W^{1,p(x)}(\Omega)$

(iii) $\psi' : W^{1,p(x)}(\Omega) \longrightarrow (W^{1,p(x)}(\Omega))^*$ is a strictly monotone operator and homeomorphism

Definition 8. We say that $u \in W^{1,p(x)}(\Omega)$ is a weak solution of problem (1), and if for any $v \in W^{1,p(x)}(\Omega)$, it satisfies the following

$$a(\psi(u)) \int_\Omega \left(|\nabla u|^{p(x)-2} + \frac{|\nabla u|^{2p(x)-2}}{\sqrt{1 + |\nabla u|^{2p(x)}}} \right) \nabla u \nabla v dx - \int_\Omega f(x, u) v dx = 0. \tag{21}$$

The energy functional $I : W^{1,p(x)}(\Omega) \longrightarrow \mathbb{R}$ associated to problem (1) is defined as

$$I(u) = \widehat{a}(\psi(u)) - \int_\Omega F(x, u) dx, \tag{22}$$

where $\widehat{a}(t) = \int_0^t a(s) ds$. Obviously, $I \in C^1(W^{1,p(x)}, \mathbb{R})$ functional and

$$\left\langle I'(u), v \right\rangle = a(\psi(u)) \int_\Omega \left(|\nabla u|^{p(x)-2} + \frac{|\nabla u|^{2p(x)-2}}{\sqrt{1 + |\nabla u|^{2p(x)}}} \right) \nabla u \nabla v dx - \int_\Omega f(x, u) v dx, \tag{23}$$

for all $u, v \in W^{1,p(x)}(\Omega)$. It is well-known that the weak solutions of problem (1) correspond to the critical point of the functional I on $W^{1,p(x)}(\Omega)$.

Definition 9. Let X be a Banach space and $\phi \in C^1(X, \mathbb{R})$. We say that ϕ satisfies the (PS) condition if any sequence $\{u_n\} \subset X$ such that $\phi(u_n) \longrightarrow c$ and $\phi'(u_n) \longrightarrow 0$ in X as $n \longrightarrow \infty$ has a convergent subsequence.

Let $X = V_2$ is a reflexive and separable Banach space, and then, there exists $\{e_j\}_{j=1}^\infty \subset X$ and $\{e_j^*\}_{j=1}^\infty \subset X^*$ such that

$$X = \overline{\operatorname{span}\{e_j | j = 1, 2, \cdots\}} \quad X^* = \overline{\operatorname{span}\left\{e_j^* \middle| j = 1, 2, \cdots\right\}},$$
$$\left\langle e_i^*, e_j \right\rangle = \begin{cases} 1 & (\text{if } i = j), \\ 0 & (\text{if } i \ne j). \end{cases} \tag{24}$$

For convenience, we write

$$X_j = \operatorname{span}\{e_j\}, Y_k = \bigoplus_{j=1}^k X_j, Z_k = \bigoplus_{j=k}^\infty X_j. \tag{25}$$

Lemma 10 (see [32] fountain theorem). *Let $X = V_2$ be a Banach space, $\phi \in C^1(X, \mathbb{R})$ is an even functional and satisfies the (PS) condition. If for every $k \in N$, there exist $\rho_k > r_k > 0$ such that*

(A_1): $a_k = \inf_{u \in Z_k, \|u\| = r_k} \phi(u) \longrightarrow +\infty as\, k \longrightarrow +\infty$;

(A_2): $b_k = \max_{u \in Y_k, \|u\| = \rho_k} \phi(u) \le 0$.

Then, ϕ has a sequence of critical values tending to $+\infty$.

3. The Proof of Theorem 1.1

In this section, the existence of at least one solution for problem (1) is obtained in $W^{1,p(x)}(\Omega)$. We start with one auxiliary result.

Lemma 11 (see [29]). *By Sobolev's inequality, there exists a positive constant C such that*

$$\int_\Omega |u| dx \le C\|u\|, \int_\Omega |u|^{\alpha(x)} dx \le C\left(\|u\|^{\alpha^-} + \|u\|^{\alpha^+}\right). \tag{26}$$

for all $u \in W^{1,p(x)}(\Omega)$.

Proof of Theorem 1.1. The proof is divided into two steps as follows.

Firstly, we show that I is coercive. By (F_1) and (F_2), there exists a constant ω and $G \in C(\mathbb{R}, \mathbb{R})$, which is subadditive, that is

$$G(s + t) \le G(s) + G(t), \tag{27}$$

for all $s, t \in \mathbb{R}$, and coercive, so

$$G(t) \longrightarrow +\infty, \tag{28}$$

as $|t| \longrightarrow \infty$, and satisfies that

$$G(t) \le |t| + 4, \tag{29}$$

for all $t \in \mathbb{R}$, such as

$$F(x,t) \le -G(t) + \omega, \tag{30}$$

for all $t \in \mathbb{R}$ and $x \in \Omega$.

In fact, since $F(x,t) \longrightarrow -\infty$ as $|t| \longrightarrow \infty$ uniformly for all $x \in \Omega$, there exists a sequence of positive integers (n_k) with $n_{k+1} > 2n_k$ for all positive integer k such that

$$F(x,t) \le -k, \tag{31}$$

for all $|t| \ge n_k$ and all $x \in \Omega$. Let $n_0 = 0$ and define

$$G(t) = k + 2 + \frac{|t| - n_{k-1}}{n_k - n_{k-1}}, \tag{32}$$

for $n_{k-1} \le |t| < n_k$, where $k \in N$.

By the definition of G, we have

$$k + 2 \le G(t) \le k + 3, \tag{33}$$

for $n_{k-1} \le |t| < n_k$. By (F_1), one has

$$|F(x,t)| \le C_0 |t| + \frac{C_0}{\alpha(x)} |t|^{\alpha(x)}, \tag{34}$$

for all $t \in \mathbb{R}$ and $x \in \Omega$. It follows that

$$|F(x,t)| \le C_0 n_1 + \frac{C_0}{\alpha(x)} n_1^{\alpha(x)} \le -G(t) + \omega, \tag{35}$$

for all $|t| \le n_1$ and $x \in \Omega$ by (33), where

$$\omega = C_0 n_1 + \frac{C_0}{\alpha^-} n_1^{p_-^*} + 4, \tag{36}$$

with $1 < \alpha(x) < p_-^*$ and $n_1 > 1$. In fact, when $k \ge 2$, we have

$$F(x,t) \le -k \le -(k-1) = -(k+3) + 4 \le -G(t) + \omega, \tag{37}$$

for all $n_{k-1} \le |t| \le n_k$ and $x \in \Omega$.

It is obvious that G is continuous and coercive. Moreover, one has

$$G(t) \le |t| + 4, \tag{38}$$

for all $t \in \mathbb{R}$. In fact, for every $t \in \mathbb{R}$, there exists $k \in N$ such that

$$n_{k-1} \le |t| < n_k, \tag{39}$$

which implies that

$$G(t) \le k + 3 = (k-1) + 4 \le n_{k-1} + 4 \le |t| + 4, \tag{40}$$

for all $t \in \mathbb{R}$ by (33) and the fact that $n_k \ge k$ for all integers $k \ge 0$.

Now, we only need to prove the subadditivity of G. Let

$$n_{k-1} \le |s| < n_k, n_{j-1} \le |t| < n_j, \tag{41}$$

and $m = \max\{k, j\}$. Then, we have

$$|s + t| \le |s| + |t| < n_k + n_j \le 2n_m < n_{m+1}. \tag{42}$$

Hence, by (33), we obtain

$$G(s+t) \le m + 4 \le k + 2 + j + 2 \le G(s) + G(t), \tag{43}$$

which shows that G is subadditive.

Let $\tilde{u}(x) = u(x) - \bar{u}$, $\bar{u} = 1/|\Omega| \int_\Omega u(x)dx$. So $\nabla u = \nabla(\tilde{u} + \bar{u}) = \nabla \tilde{u}$. By conditions (a_1), (a_2), (16), (26)–(30), and Proposition 6, we have

$$\begin{aligned}
I(u) &= \widehat{a}(\psi(u)) - \int_\Omega F(x,u)dx \ge \frac{1}{\theta} a(\psi(u))\psi(u) \\
&\quad - \int_\Omega F(x,u)dx \ge \frac{a_0}{\theta}\psi(u) + \int_\Omega G(u)dx \\
&\quad - \int_\Omega \omega dx \ge \frac{2a_0}{\theta p^+}\int_\Omega |\nabla u|^{p(x)}dx \\
&\quad + \left(\int_\Omega G(\bar{u})dx - \int_\Omega G(-\tilde{u})dx\right) \\
&\quad - \int_\Omega \omega dx \ge \frac{2a_0}{\theta p^+}\left(\frac{1}{2}\int_\Omega |\nabla u|^{p(x)}dx + \frac{1}{2}\int_\Omega |\nabla u|^{p(x)}dx\right) \\
&\quad + \int_\Omega G(\bar{u})dx - \int_\Omega |\tilde{u}|dx - 4|\Omega| \\
&\quad - \int_\Omega \omega dx \ge \frac{2a_0}{\theta p^+}\left(\frac{1}{2}\int_\Omega |\nabla \tilde{u}|^{p(x)}dx + \frac{1}{2k}\int_\Omega |\tilde{u}|^{p(x)}dx\right) \\
&\quad + (G(\bar{u}) - 4 - \omega)|\Omega| - \int_\Omega |\tilde{u}|dx \ge \frac{a_0}{\theta p^+}\min\left\{1, \frac{1}{k}\right\}\|\tilde{u}\|^{p^-} \\
&\quad - C\|\tilde{u}\| + (G(\bar{u}) - 4 - \omega)|\Omega|,
\end{aligned} \tag{44}$$

Note that $\|u\| \longrightarrow \infty$, we have $|\bar{u}| + \|\tilde{u}\| \longrightarrow \infty$. Since G is coercive and $p^- > 1$, which implies that I is coercive.

Next, we show that the I is weakly lower semicontinuous. Let

$$\begin{cases} u_n \rightharpoonup u, & \text{in } W^{1,p(x)}(\Omega), \\ u_n \longrightarrow u, & \text{in } L^{p(x)}(\Omega), \quad 1 \le p(x) < p^*(x), \\ u_n(x) \longrightarrow u(x), & \text{a.e.in } \Omega. \end{cases} \tag{45}$$

Since $F(x, u_n(x)) \longrightarrow F(x, u(x))$ as $n \longrightarrow \infty$, for a.e. $x \in \Omega$. By Fatou Lemma, we have

$$\limsup_{n\longrightarrow\infty} \int_\Omega F(x, u_n(x))dx \le \int_\Omega F(x, u(x))dx. \tag{46}$$

Then

$$\liminf_{n\longrightarrow\infty} I(u_n) \geq \liminf_{n\longrightarrow\infty} \widehat{a}(\psi(u_n)) - \limsup_{n\longrightarrow\infty}\int_\Omega F(x,u_n)dx \geq \widehat{a}(\psi(u)) - \int_\Omega F(x,u)dx = I(u). \tag{47}$$

So, the I is weakly lower semicontinuous.

Hence, by the least action principle (see [17]), I has a minimum. So problem (1) has at least one solution in $W^{1,p(x)}$, which completes our proof. □

4. The Proof of Theorem 1.2

In this section, we will show that problem (1) has infinitely many nontrivial solutions by using Lemma 10. Firstly, we need to prove lemma as follows.

Lemma 12. *Suppose that the conditions of Theorem 2 hold. Then, I satisfies the* (PS) *condition.*

Proof. Let $\{u_n\} \subset W^{1,p(x)}(\Omega)$ be a sequence such that $\{I(u_n)\}$ is bounded and $I'(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$. Firstly, we prove that $\{u_n\}$ is bounded in $W^{1,p(x)}(\Omega)$. Arguing by contradiction, we assume that any subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$), we have $\|u_n\| \longrightarrow +\infty$ as $n \longrightarrow +\infty$. Let $v_n = u_n/\|u_n\|$. Note that $\|v_n\| = 1$ and $\{v_n\}$ are bounded in $W^{1,p(x)}(\Omega)$. Now, we can find $v \in W^{1,p(x)}(\Omega)$ and a subsequence of $\{v_n\}$ (still denoted by $\{v_n\}$), with

$$\begin{cases} v_n \rightharpoonup v, & \text{in } W^{1,p(x)}(\Omega), \\ v_n \longrightarrow v, & \text{in } L^{p(x)}(\Omega), \quad 1 \leq p(x) < p^*(x), \\ v_n(x) \longrightarrow v(x), & \text{a.e.in } \Omega. \end{cases} \tag{48}$$

By conditions (a_1) and (a_2), we have

$$\widehat{a}(\psi(u)) = \int_0^{\psi(u)} a(s)ds \geq a_0\psi(u). \tag{49}$$

By (F_3), for any $\varepsilon > 0$, there is a constant $\xi > 0$ such that

$$F(x,t) \leq \frac{\varepsilon}{p^+}|t|^{p^-} \text{ for all } |t| > \xi \text{ and all } x \in \Omega. \tag{50}$$

There exists a positive constant $C_1 > 0$ such that

$$F(x,t) \leq \frac{\varepsilon}{p^+}|t|^{p^-} + C_1 \text{ for all } t \in \mathbb{R} \text{ and all } x \in \Omega. \tag{51}$$

Then, by (51), one deduces that

$$\begin{aligned} \frac{I(u_n)}{\|u_n\|^{p^-}} &= \frac{1}{\|u_n\|^{p^-}}\left(\widehat{a}\left(\int_\Omega \frac{|\nabla u_n|^{p(x)} + \sqrt{1+|\nabla u_n|^{2p(x)}}}{p(x)}dx\right) - \int_\Omega F(x,u_n)dx\right) \\ &\geq \frac{1}{\|u_n\|^{p^-}}\left(\left(a_0\int_\Omega \frac{|\nabla u_n|^{p(x)} + \sqrt{1+|\nabla u_n|^{2p(x)}}}{p(x)}dx\right) - \int_\Omega F(x,u_n)dx\right) \\ &\geq \frac{1}{\|u_n\|^{p^-}}\left(\frac{2a_0}{p^+}\int_\Omega |\nabla u_n|^{p(x)}dx - \frac{\varepsilon}{p^+}\int_\Omega |u_n|^{p^-}dx - \int_\Omega C_1 dx\right) \\ &\geq \frac{2a_0}{p^+} - \left(\frac{2a_0}{p^+} + \frac{\varepsilon}{p^+}\right)\int_\Omega |v_n|^{p^-}dx - \frac{C_1|\Omega|}{\|u_n\|^{p^-}}. \end{aligned} \tag{52}$$

In view of (52) and the fact that $\|u_n\| \longrightarrow \infty$ as $n \longrightarrow \infty$, one has

$$0 \geq \frac{2a_0}{p^+} - \left(\frac{2a_0}{p^+} + \frac{\varepsilon}{p^+}\right)\int_\Omega |v|^{p^-}dx. \tag{53}$$

Let $\varepsilon \longrightarrow 0$, we have

$$\int_\Omega |v|^{p^-}dx \geq 1. \tag{54}$$

On the other hand, by the weakly lower semicontinuous of the norm, one has

$$\|v\| \leq \liminf_{n\longrightarrow+\infty}\|v_n\| = 1, \tag{55}$$

then

$$\int_\Omega |\nabla v|^{p^-}dx + \int_\Omega |v|^{p^-}dx = \|v\|^{p^-} \leq 1. \tag{56}$$

From (54) and (56), we derive that

$$\int_\Omega |\nabla v|^{p^-}dx = 0. \tag{57}$$

Therefore, $|\nabla v(x)| = 0$ for all $x \in \Omega$ which yields $v \in \mathbb{R}$. It follows that $|v|^{p^-} = 1/|\Omega|$ and hence that

$$\begin{aligned} \lim_{n\longrightarrow+\infty}\frac{|\bar{u}_n|^{p^-}}{\|u_n\|^{p^-}} &= \lim_{n\longrightarrow+\infty}\left|\frac{1}{|\Omega|}\int_\Omega \frac{u_n}{\|u_n\|}dx\right|^{p^-} \\ &= \lim_{n\longrightarrow+\infty}\left|\frac{1}{|\Omega|}\int_\Omega |v_n|dx\right|^{p^-} = \left|\frac{1}{|\Omega|}\int_\Omega |v|dx\right|^{p^-} \\ &= \left(\frac{1}{|\Omega|}\int_\Omega |v|dx\right)^{p^-} = \frac{1}{|\Omega|}. \end{aligned} \tag{58}$$

which means that $|\bar{u}_n||\Omega|^{1/p^-}/\|u_n\| \longrightarrow 1$ as $n \longrightarrow \infty$. By condition (F_4) and $\|u_n\| \longrightarrow \infty$ as $n \longrightarrow \infty$, one deduces

$$\limsup_{n\longrightarrow\infty}\int_\Omega f(x,u_n)\frac{\bar{u}_n}{|\bar{u}_n|}dx > 0. \tag{59}$$

Since $I'(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$, we have

$$\begin{aligned}\int_\Omega f(x,u_n)\frac{\bar{u}_n}{|\bar{u}_n|}dx &= -\left\langle I'(u_n), \frac{\bar{u}_n}{|\bar{u}_n|}\right\rangle + a(\psi(u))\int_\Omega \\ &\quad \cdot \left(|\nabla u|^{p(x)-2} + \frac{|\nabla u|^{2p(x)-2}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)\nabla u \nabla \frac{\bar{u}_n}{|\bar{u}_n|}dx \\ &= -\left\langle I'(u_n), \frac{\bar{u}_n}{|\bar{u}_n|}\right\rangle \longrightarrow 0 \, as \, n \longrightarrow +\infty,\end{aligned} \tag{60}$$

which contradicts to (59). Therefore, we conclude that the sequence $\{u_n\}$ is bounded in $W^{1,p(x)}(\Omega)$.

Secondly, we will prove that $\{u_n\}$ has a convergent subsequence in $W^{1,p(x)}(\Omega)$. Since $\{u_n\}$ is bounded, there exist a subsequence, still denoted by $\{u_n\}$ and $u \in W^{1,p(x)}(\Omega)$ such that

$$\begin{cases} u_n \rightharpoonup u, & \text{in } W^{1,p(x)}(\Omega), \\ u_n \longrightarrow u, & \text{in } L^{\alpha(x)}(\Omega), \quad 1 \le \alpha(x) < p^*(x), \\ u_n(x) \longrightarrow u(x), & \text{a.e.in } \Omega. \end{cases} \tag{61}$$

By (12), (61), and (F_1), we have

$$\begin{aligned}\left|\int_\Omega f(x,u_n)(u_n-u)dx\right| &\le \int_\Omega |f(x,u_n)||u_n-u|dx \\ &\le C_0\int_\Omega \left|1+|u_n|^{\alpha(x)-1}\right||u_n-u|dx \\ &\le 2C_0\left|1+|u_n|^{\alpha(x)-1}\right|_{(\alpha'(x))}|u_n-u|_{\alpha(x)} \longrightarrow 0,\end{aligned} \tag{62}$$

as $n \longrightarrow \infty$. One has

$$\lim_{n\longrightarrow+\infty}\int_\Omega f(x,u_n)(u_n-u)dx = 0. \tag{63}$$

According to $I'(u_n) \longrightarrow 0$, we have $\langle I'(u_n),(u_n-u)\rangle \longrightarrow 0$. Therefore

$$\begin{aligned}\left\langle I'(u_n), u_n-u\right\rangle &= a(\psi(u_n))\int_\Omega\left(|\nabla u_n|^{p(x)-2} + \frac{|\nabla u_n|^{2p(x)-2}}{\sqrt{1+|\nabla u_n|^{2p(x)}}}\right) \\ &\quad \cdot \nabla u_n(\nabla u_n - \nabla u)dx - \int_\Omega f(x,u_n)(u_n-u)dx \longrightarrow 0,\end{aligned} \tag{64}$$

as $n \longrightarrow \infty$. By (63) and condition (a_1), we can deduce from that

$$\int_\Omega\left(|\nabla u_n|^{p(x)-2} + \frac{|\nabla u_n|^{2p(x)-2}}{\sqrt{1+|\nabla u_n|^{2p(x)}}}\right)\nabla u_n(\nabla u_n - \nabla u)dx \longrightarrow 0, \tag{65}$$

as $n \longrightarrow \infty$. Invoking the S_+ condition (see Proposition 7), we can deduce that $u_n \longrightarrow u$ strongly in $W^{1,p(x)}(\Omega)$ as $n \longrightarrow \infty$. So, I satisfies the (PS) condition. The proof is complete. □

The following lemma plays an important role in our proof.

Lemma 13 (see [30]). *If $\alpha \in C_+(\bar{\Omega})$, $\alpha(x) < p^*(x)$ for any $x \in \bar{\Omega}$, denote*

$$\eta_k = \sup\left\{|u|_{\alpha(x)} \,\middle|\, \|u\| = 1, u \in Z_k\right\}, \tag{66}$$

then $\lim_{k\longrightarrow\infty}\eta_k = 0$.

Lemma 14. *Suppose that the conditions of Theorem 2 hold. Then, there exist $\rho_k > r_k > 0$ such that*

(A_1): $a_k = \inf_{u\in Z_k, \|u\|=r_k} I(u) \longrightarrow +\infty \, as \, k \longrightarrow +\infty$;

(A_2): $b_k = \max_{u\in Y_k, \|u\|=\rho_k} I(u) \le 0$.

Proof. To prove (A_1). In view of (F_1), we obtain

$$F(x,t) = \int_0^1 f(x,st)tds \le C_0|t| + \frac{C_0}{\alpha(x)}|t|^{\alpha(x)} \le C_0|t| + C|t|^{\alpha(x)}, \tag{67}$$

for all $x \in \Omega$ and $t \in \mathbb{R}$. Furthermore, for any $u \in Z_k \subset V_2$ with $\|u\| > 1$, it follows from conditions (a_1), (a_2), (26), and Proposition 4, and we have

$$\begin{aligned} I(u) &= \hat{a}(\psi(u)) - \int_\Omega F(x,u)dx \\ &\ge \frac{1}{\theta}a(\psi(u))\psi(u) - \int_\Omega F(x,u)dx \\ &\ge \frac{a_0}{\theta}\psi(u) - C_0\int_\Omega |u|dx - C\int_\Omega |u|^{\alpha(x)}dx \\ &\ge \frac{2a_0}{\theta p^+}\int_\Omega |\nabla u|^{p(x)}dx - C\|u\| - C\int_\Omega |u|^{\alpha(x)}dx \\ &\ge \frac{2a_0}{\theta p^+}\|u\|^{p^-} - C\|u\| - C\int_\Omega |u|^{\alpha(x)}dx \\ &\ge \begin{cases} \dfrac{2a_0}{\theta p^+}\|u\|^{p^-} - C\|u\| - C & \left(\text{if } |u|_{\alpha(x)} \le 1\right) \\ \dfrac{2a_0}{\theta p^+}\|u\|^{p^-} - C\|u\| - C\eta_k^{\alpha^+}\|u\|^{\alpha^+} & \left(\text{if } |u|_{\alpha(x)} > 1\right) \end{cases} \\ &\ge \frac{2a_0}{\theta p^+}\|u\|^{p^-} - C\|u\| - C\eta_k^{\alpha^+}\|u\|^{\alpha^+} - C \\ &= \|u\|^{p^-}\left(\frac{2a_0}{\theta p^+} - C\eta_k^{\alpha^+}\|u\|^{\alpha^+-p^-}\right) - C\|u\| - C. \end{aligned} \tag{68}$$

Choose $r_k = (C\eta_k^{\alpha^+}(2\theta p^+/a_0))^{1/p^- - \alpha^+}$. For any $u \in Z_k$ with $\|u\| = r_k$, we infer that

$$I(u) \geq \frac{3a_0}{2\theta p^+} r_k^{p^-} - Cr_k - C. \tag{69}$$

Note that $p^- < \alpha^+$ and $\eta_k \longrightarrow 0$ as $k \longrightarrow +\infty$, we assert $r_k \longrightarrow +\infty$ as $k \longrightarrow +\infty$. We have

$$a_k = \inf_{u \in Z_k, \|u\| = r_k} I(u) \longrightarrow +\infty, \tag{70}$$

as $k \longrightarrow +\infty$. Which implies (A_1).

Next, we will prove (A_2). Let

$$M(t) = \frac{\widehat{a}(t)}{t^\theta} = \frac{\int_0^t a(s)ds}{t^\theta}, \tag{71}$$

by condition (a_2), we have

$$M'(t) = \frac{a(t)t^\theta - \theta t^{\theta-1}\int_0^t a(s)ds}{t^{2\theta}} = \frac{a(t)t^\theta - \theta t^{\theta-1}\widehat{a}(t)}{t^{2\theta}} \leq 0. \tag{72}$$

$M(t)$ is monotonically decreasing with respect to the variable t as $t \geq t_0 > 0$. So $M(t) \leq M(t_0)$, we have $\widehat{a}(t) \leq (\widehat{a}(t_0)/t_0^\theta)t^\theta$ for any $t \in [t_0, +\infty)$, and we have

$$\widehat{a}(t) \leq C_2 t^\theta + C_3, \tag{73}$$

where $C_2 = \widehat{a}(t_0)/t_0^\theta$,

$$C_3 = \max_{t \in [0, t_0]} \widehat{a}(t). \tag{74}$$

By conditions (F_1) and (F_5), for any $H > 0$, there exists $C_4 > 0$, and we have

$$F(x,t) \geq H|t|^{\theta p^+} - C_4, \forall (x,t) \in \Omega \times \mathbb{R}. \tag{75}$$

By (73), (75), and Proposition 4, for any $u \in Y_k$ with $\|u\| = \rho_k > r_k$ large enough, we have

$$\begin{aligned} I(u) &= \widehat{a}(\psi(u)) - \int_\Omega F(x,u)dx \leq C_2(\psi(u))^\theta - \int_\Omega F(x,u)dx + C_3 \\ &\leq C_2\left(\frac{1}{p^-}\int_\Omega \left(2|\nabla u|^{p(x)} + 1\right)dx\right)^\theta - H\int_\Omega |u|^{\theta p^+}dx + C_5 \\ &\leq C_2\left(\frac{2}{p^-}\|u\|^{p^+} + \frac{1}{p^-}|\Omega|\right)^\theta - H\int_\Omega |u|^{\theta p^+}dx + C_5. \end{aligned} \tag{76}$$

Since $\dim Y_k < +\infty$ and equivalence of norm in finite dimensional space, it is clear that $I(u) \longrightarrow -\infty$ when $H > 0$ large enough. Therefore, we conclude

$$b_k = \max_{u \in Y_k, \|u\| = \rho_k} I(u) \leq 0. \tag{77}$$

The proof is complete. □

Proof of Theorem 1.2. Since V_2 is also a reflexive and separable Banach space, we can give the decomposition to V_2 as (25). According to (F_6) and Lemma 12, we deduce that I is an even functional and satisfies (PS) condition in V_2. By Lemma 14, it was proven that if k is large enough, there exist $\rho_k > r_k > 0$ such that (A_1) and (A_2) hold in V_2. Thus, we can deduce that I satisfies all conditions of Lemma 10 in V_2. Therefore, I has an unbounded sequence of critical values, which implies that problem (1) has infinitely many nontrivial critical points in V_2. This completes the Proof of Theorem 1.2. □

Data Availability

No underlying data was collected or produced in this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (no. 11861021).

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