Hindawi
Journal of Function Spaces
Volume 2023, Article ID 6833404, 16 pages
https://doi.org/10.1155/2023/6833404

Research Article

Newfangled Linearization Formula of Certain Nonsymmetric Jacobi Polynomials: Numerical Treatment of Nonlinear Fisher's Equation

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Received 18 January 2022; Revised 9 April 2022; Accepted 13 April 2022; Published 21 August 2023

Academic Editor: Richard I. Avery

This article is devoted to deriving a new linearization formula of a class for Jacobi polynomials that generalizes the third-kind Chebyshev polynomials class. In fact, this new linearization formula generalizes some existing ones in the literature. The derivation of this formula is based on employing a new moment formula of this class of polynomials and after that using suitable symbolic computation to reduce the resulting linearization coefficients into simplified forms that do not contain any hypergeometric functions or sums. The new formula is employed along with some other formulas and with the utilization of the spectral tau method to obtain numerical solutions to the nonlinear Fisher equation. The presented method is used to convert the equation governed by its underlying conditions into a nonlinear system of equations. The solution of the resulting system can be obtained through any suitable standard numerical scheme. To demonstrate the efficiency and usefulness of the proposed algorithm, some examples are shown, including comparisons with some existing techniques in the literature.

1. Introduction

The study and the utilization of special functions in general and of orthogonal polynomials, in particular, is a very old and important branch of mathematics. Orthogonal polynomials are fruitfully used for obtaining numerical solutions to all types of differential equations. Among the important polynomials are the classical Jacobi polynomials. These polynomials have important parts in different disciplines. Some applications of Jacobi polynomials in some areas of science and engineering such as integral equations and food engineering can be found in [1–4]. In fact, the class of Jacobi polynomials involves six well-known polynomials. They are the Legendre, Gegenbauer, and the four kinds of Chebyshev polynomials. The existence of four different kinds of Chebyshev polynomials leads to a wide range of outcomes in a variety of fields, such as approximation, interpolation, series expansions, and quadrature and integral equations (see, for example, [5, 6]). These kinds of Chebyshev polynomials have been thoroughly investigated theoretically and numerically. For example, Oloniiju et al. in [7] used the first-kind Chebyshev polynomials to find a pseudospectral solution to a certain multidimensional fractional problem. The authors in [8] established new expressions of the high-order derivatives of Chebyshev polynomials of the third and fourth kinds. In addition, they utilized these formulas to treat numerically specific types of differential equations.

Deriving formulas that are concerned with different special functions and orthogonal polynomials is of interest. In fact, there are many formulas that serve in the numerical treatment of different types of differential equations. Obtaining expressions for the high-order derivatives of orthogonal polynomials in terms of their original ones is very useful in treating numerically the differential equations of different types if the spectral methods are applied. For example, the authors in [9] established new expressions for the high-order derivatives of the fifth-kind Chebyshev polynomials in terms of their original polynomials. These expressions include terminating hypergeometric functions of the type ${}_4F_3(1)$. Moreover, these formulas are employed to treat numerically the convection-diffusion equation. Also, among the important formulas of orthogonal polynomials are the linearization formulas of these polynomials. Because of their importance, linearization problems have attracted the attention of many authors. For example, some articles were devoted to solving the linearization problems of Jacobi polynomials and their special classes. There are different approaches in order to developing these linearization formulas. One can be referred for example to Rahman [10], Chaggara and Koepf [11]. As an application to the linearization formulas, recently, Abd-Elhameed in [12] developed linearization formulas for specific classes of Jacobi polynomials. Furthermore, a certain linearization formula along with the tau spectral method was employed to handle a type of nonlinear Riccati differential equation.

Spectral methods are a class of important methods that treat numerically different types of differential equations. The numerical solution is expressed as a suitable combination of specific polynomials, which is the basic assumption underpinning the implementation of spectral methods. Because of its importance in the area of numerical solutions of differential and integral equations, many types of spectral methods have received a lot of attention. For further information on the numerous applications of spectral approaches in various areas, see [13–15]. Spectral approaches include the Galerkin, tau, and collocation methods. The Galerkin method can be fruitfully utilized for treating several forms of differential equations (see, for instance, [16–20]). Unlike the Galerkin technique, the tau method is more flexible in its application because it does not require selecting basis functions that meet the underlying initial/boundary conditions (see, for example, [21–24]). The collocation method is the most popular method. It may be used to solve any differential equation. The authors in [25–27] used the collocation method to obtain numerical solutions to many types of differential equations.

Fisher's equation arises in various applications like tissue engineering, chemical reactions, and neurophysiology (see [28, 29]). This equation has been treated by both analytic and numerical techniques. For example, Wazwaz and Gorguis in [30] studied analytically the nonlinear Fisher equation by using the Adomian decomposition method. Chandraker et al. in [31] developed implicit numerical techniques for treating Fisher's equation. From a numerical point of view, Haar wavelet's method is applied for solving Fisher's equation in [32]. For some other articles that deal with Fisher's equation and its generalizations and modifications, one can consult [33–36].

Recently, Abd-Elhameed and Alkenedri in [37] investigated two generalized classes of the third- and fourth-kind Chebyshev polynomials. They developed new high-order derivative expressions of these polynomials. In addition and based on these formulas, they obtained spectral solutions to the high-even-order linear and nonlinear boundary value problems. In this article, we are interested in developing some theoretical results concerned with certain generalized third-kind Chebyshev polynomials and after that employing such polynomials to treat numerically the nonlinear Fisher equation.

The five main goals of the current paper can be listed as follows:

(i) Derivation of a new moment formula of the generalized third-kind Jacobi polynomials

(ii) Establishing a new linearization formula of the generalized third-kind Jacobi polynomials

(iii) Deducing some existing moment and linearization formulas in the literature as special cases of our new moment and linearization formulas

(iv) Employing the derived linearization formula in conjunction with the high-order derivative expression of the generalized polynomials to numerically solve the nonlinear Fisher equation using the spectral tau approach

(v) Testing the efficiency and applicability of our proposed algorithm by presenting some examples accompanied by comparisons with some other methods in the literature

The rest of the paper is as follows. In Section 2, some interesting properties concerned with the classical Jacobi polynomials and their shifted ones are presented. Section 3 is devoted to developing new moment formula of the generalized third-kind Chebyshev polynomials. Some specific moment formulas are also deduced by reducing the corresponding moment coefficients via the utilization of Zeilberger's algorithm. Section 4 establishes the main formula in this paper in which we give with proof a new linearization formula of the generalized third-kind Chebyshev polynomials. Section 5 concentrates on proposing a numerical algorithm for solving spectrally the nonlinear Fisher equation. In Section 6, some numerical examples accompanied by comparisons with some other techniques in the literature are displayed. We end the paper with some concluding remarks in Section 7.

2. Some Interesting Properties and Formulas of Jacobi Polynomials

The standard Jacobi polynomial $P_j^{(\lambda,\mu)}(x)$ of degree j can be defined in hypergeometric form as (see, for example, Andrews et al. [38])

$$P_j^{(\lambda,\mu)}(x) = \frac{(\lambda+1)_j}{j!}\,{}_2F_1\left(\left.\begin{matrix}-j, j+\lambda+\mu+1\\ \lambda+1\end{matrix}\right|\frac{1-x}{2}\right). \quad (1)$$

The Jacobi polynomials can be normalized (see [12]); that is, we can define $R_j^{(\lambda,\mu)}(x)$ such that

$$R_j^{(\lambda,\mu)}(x) = 1, \quad \forall j \geq 0, \tag{2}$$

and therefore,

$$R_j^{(\lambda,\mu)}(x) = \frac{P_j^{(\lambda,\mu)}(x)}{P_j^{(\lambda,\mu)}(1)} = \frac{j!}{(\lambda+1)_j} P_j^{(\lambda,\mu)}(x) = {}_2F_1\left(\begin{matrix} -j, j+\lambda+\mu+1 \\ \lambda+1 \end{matrix} \middle| \frac{1-x}{2}\right). \tag{3}$$

The orthogonality property of $R_j^{(\lambda,\mu)}(x)$ on $[-1,1]$ is

$$\int_{-1}^{1} (1-x)^{\lambda}(1+x)^{\mu} R_i^{(\lambda,\mu)}(x) R_j^{(\lambda,\mu)}(x)\, dx = \begin{cases} 0, & j \neq i, \\ h_i^{(\lambda,\mu)}, & j = i, \end{cases} \tag{4}$$

where

$$h_i^{(\lambda,\mu)} = \frac{2^{\lambda+\mu+1}\, i!\Gamma(i+\mu+1)\,[\Gamma(\lambda+1)]^2}{(2i+\lambda+\mu+1)\,\Gamma(i+\lambda+\mu+1)\,\Gamma(i+\lambda+1)}. \tag{5}$$

One comments here that the following six special classes of polynomials can be extracted from $R_j^{(\lambda,\mu)}(x)$ with suitable choices of λ and μ. We have

$$\begin{aligned} C_i^{(\alpha)}(x) &= R_i^{(\alpha-(1/2),\alpha-(1/2))}(x),\ T_i(x) = R_i^{(-(1/2),-(1/2))}(x), \\ U_i(x) &= (i+1)\, R_i^{(1/2,1/2)}(x),\ V_i(x) = R_i^{(-(1/2),1/2)}(x), \\ W_i(x) &= (2i+1)\, R_i^{(1/2,-(1/2))}(x),\ P_i(x) = R_i^{(0,0)}(x), \end{aligned} \tag{6}$$

where $C_i^{(\alpha)}(x)$, $T_i(x)$, $U_i(x)$, $V_i(x)$, $W_i(x)$, and $P_i(x)$ are the ultraspherical, first-, second-, third-, and fourth kinds of Chebyshev polynomials and Legendre polynomials, respectively.

Also, the following relation is noted:

$$R_i^{(\lambda,\mu)}(-x) = \frac{(-1)^i\,\Gamma(\lambda+1)\,\Gamma(i+\mu+1)}{\Gamma(\mu+1)\,\Gamma(i+\lambda+1)} R_i^{(\mu,\lambda)}(x). \tag{7}$$

Now, consider the special class of Jacobi polynomials $R_i^{(\alpha,\alpha+1)}(x)$. It is clear that this class reduces to the class of third-kind Chebyshev polynomials for the case corresponding to $\alpha = -(1/2)$.

Now, define the shifted Jacobi polynomials class on $[0,1]$ as

$$J_i^{(\alpha)}(z) = R_i^{(\alpha,\alpha+1)}(2z-1). \tag{8}$$

The orthogonality relation of $J_i^{(\alpha)}(z)$ on $[0,1]$ is given by

$$\int_0^1 w_1(z) J_i^{(\alpha)}(z)\, J_j^{(\alpha)}(z) dz = h_i \delta_{i,j}, \tag{9}$$

where $w_1(z) = (1-z)^{\alpha} z^{\alpha+1}$, δ_{ij} is the well-known Kronecker delta function, and h_i is given by

$$h_i = \frac{i!\Gamma(\alpha+1)]^2}{2\,\Gamma(i+2\alpha+2)}. \tag{10}$$

The classical Jacobi polynomials and their special ones are investigated in a variety of books (see, for example, Andrews et al. [38] and Mason and Handscomb [6]).

The following three lemmas are of fundamental importance to derive our proposed results in the upcoming sections.

Lemma 1 (see [37]). *Let j be a nonnegative integer. The polynomials $R_j^{(\alpha,\alpha+1)}(x)$ have the following power form representation:*

$$R_j^{(\alpha,\alpha+1)}(x) = \sum_{r=0}^{\lfloor j/2 \rfloor} A_{r,j} x^{j-2r} + \sum_{r=0}^{\lfloor (j-1)/2 \rfloor} B_{r,j} x^{j-2r-1}, \tag{11}$$

where

$$\begin{aligned} A_{r,j} &= \frac{(-1)^r\, 2^{1+j-2r+2\alpha} j!\Gamma(1+\alpha)\,\Gamma((3/2)+j-r+\alpha)}{\sqrt{\pi}\, r!(j-2r)!\Gamma(2+j+2\alpha)}, \\ B_{r,j} &= \frac{(-1)^{1+r}\, j!2^{j-2r+2\alpha}\,\Gamma(1+\alpha)\,\Gamma((1/2)+j-r+\alpha)}{\sqrt{\pi}\, r!(j-2r-1)!\Gamma(2+j+2\alpha)}, \end{aligned} \tag{12}$$

where $\lfloor z \rfloor$ denotes the well-known floor function.

Lemma 2 (see [37]). *For every nonnegative integer j, the following inversion formula for the polynomials $R_j^{(\alpha,\alpha+1)}(x)$ holds:*

$$x^j = \sum_{i=0}^{\lfloor j/2 \rfloor} Q_{i,j} R_{j-2i}^{(\alpha,\alpha+1)}(x) + \sum_{i=0}^{\lfloor (j-1)/2 \rfloor} \bar{Q}_{i,j} R_{j-2i-1}^{(\alpha,\alpha+1)}(x), \tag{13}$$

where

$$\begin{aligned} Q_{i,j} &= \frac{2^{-1-j-2\alpha}\sqrt{\pi}\, j!\Gamma(2-2i+j+2\alpha)}{i!(j-2i)!\Gamma(1+\alpha)\,\Gamma((3/2)-i+j+\alpha)}, \\ \bar{Q}_{i,j} &= \frac{2^{-1-j-2\alpha}\sqrt{\pi}\, j!\Gamma(1-2i+j+2\alpha)}{i!(j-2i-1)!\Gamma(1+\alpha)\,\Gamma((3/2)-i+j+\alpha)}. \end{aligned} \tag{14}$$

Lemma 3 (see [37]). *The qth derivative of the shifted Jacobi polynomial $J_i^{(\alpha)}(z)$ is linked by their original ones by the relation*

$$D^q J_i^{(\alpha)}(z) = \sum_{j=0}^{i-q} d_{j,i,q} J_j^{(\alpha)}(z), \tag{15}$$

where the coefficients $d_{j,i,q}$ are given by

$$d_{j,i,q}=\frac{2^{2q}\,i!\,\Gamma(j+2\alpha+2)}{j!(q-1)!\,\Gamma(i+2\alpha+2)}\begin{cases}\dfrac{((i-j+q-2)/2)!\,\Gamma((i+j+q+2\alpha+3)/2)}{((i-j-q)/2)!\,\Gamma((i+j-q+2\alpha+3)/2)}, & (i+j+q)\text{ even},\\[2mm] \dfrac{((i-j+q-1)/2)!\,\Gamma((i+j+q+2\alpha+2)/2)}{((i-j-q-1)/2)!\,\Gamma((i+j-q+2\alpha+4)/2)}, & (i+j+q)\text{ odd}.\end{cases} \quad (16)$$

3. New Moment Formula of the Jacobi Polynomials $R_i^{(\alpha,\alpha+1)}(x)$

This section is devoted to the implementation of a new moment formula of the Jacobi polynomials $R_j^{(\alpha,\alpha+1)}(x)$. The derivation of this formula is based on the power-form representation of these polynomials along with their inversion formula.

Theorem 4. *Let m an n be positive integers. One has*

$$x^m R_j^{(\alpha,\alpha+1)}(x)=\sum_{p=0}^{\lfloor (j+m)/2\rfloor} U_{p,j,m}R_{j+m-2p}^{(\alpha,\alpha+1)}(x)+\sum_{p=0}^{\lfloor (1/2)(j+m-1)\rfloor}\bar{U}_{p,j,m}R_{j+m-2p-1}^{(\alpha,\alpha+1)}(x), \quad (17)$$

where

$$U_{p,j,m}=\frac{j!\,\Gamma(2+j+m-2p+2\alpha)}{2^m\,(j+m-2p)!\,\Gamma(2+j+2\alpha)}\times\sum_{\ell=0}^{p}\frac{(-1)^\ell (j-2\ell+m-1)!\,\Gamma((1/2)+j-\ell+\alpha)}{\ell!(j-2\ell)!(p-\ell)!\,\Gamma((3/2)+j-\ell+m-p+\alpha)}\times\left((j-2\ell)(\ell-p)+\frac{1}{2}(j-2\ell+m)(1+2j-2\ell+2\alpha)\right), \quad (18)$$

$$\bar{U}_{p,j,m}=\frac{j!\,\Gamma(1+j+m-2p+2\alpha)}{2^{m+1}\,(j+m-2p-1)!\,\Gamma(2+j+2\alpha)}\times\sum_{\ell=0}^{p}\frac{(-1)^\ell\,(m+2\ell m+2jp-4\ell p+2m\alpha)\,(j-2\ell+m-1)!\,\Gamma((1/2)+j-\ell+\alpha)}{\ell!(j-2\ell)!(p-\ell)!\,\Gamma((3/2)+j-\ell+m-p+\alpha)}. \quad (19)$$

Proof. The analytic formula of $R_j^{(\alpha,\alpha+1)}(x)$ enables one to write

$$x^m R_j^{(\alpha,\alpha+1)}(x)=\frac{j!\,\Gamma(1+\alpha)}{\sqrt{\pi}\,\Gamma(2+j+2\alpha)}\cdot\left(\sum_{r=0}^{\lfloor (j/2)\rfloor}\frac{(-1)^r\,2^{1+j-2r+2\alpha}\,\Gamma((3/2)+j-r+\alpha)}{r!(j-2r)!}x^{j+m-2r}+\sum_{r=0}^{\lfloor (j-1)/2\rfloor}\frac{(-1)^{1+r}\,2^{j-2r+2\alpha}\,\Gamma((1/2)+j-r+\alpha)}{r!(j-2r-1)!}x^{j+m-2r-1}\right). \quad (20)$$

In virtue of (13) and after doing some lengthy manipulations, we can write

$$x^m R_j^{(\alpha)}(x)=\sum_{p=0}^{\lfloor (j+m)/2\rfloor} U_{p,j,m}R_{j+m-2p}^{(\alpha,\alpha+1)}(x)+\sum_{p=0}^{\lfloor (1/2)(j+m-1)\rfloor}\bar{U}_{p,j,m}R_{j+m-2p-1}^{(\alpha,\alpha+1)}(x), \quad (21)$$

where $U_{p,j,m}$ and $\bar{U}_{p,j,m}$ are as given in (18) and (19). This ends the proof. □

Corollary 5. *The moment formula of Chebyshev polynomials of the third kind is given explicitly as*

$$x^m V_j(x)=\frac{1}{2^m}\sum_{p=0}^{m}\binom{m}{p}V_{j+m-2p}(x). \quad (22)$$

Proof. Setting $\alpha=-(1/2)$ in (17) gives the following formula:

$$x^m V_j(x)=\frac{1}{2^m}\left(\sum_{p=0}^{\lfloor (j+m)/2\rfloor} H_{p,j,m}R_{j+m-2p}^{(\alpha,\alpha+1)}(x)+\sum_{p=0}^{\lfloor (1/2)(j+m-1)\rfloor}\bar{H}_{p,j,m}R_{j+m-2p-1}^{(\alpha,\alpha+1)}(x)\right), \quad (23)$$

where

$$H_{p,j,m} = \sum_{\ell=0}^{p} \frac{(-1)^{1+\ell}\left(-j^2 + \ell(m-2p) + j(2\ell - m + p)\right)(j-\ell-1)!(j-2\ell+m-1)!}{\ell!(j-2\ell)!(p-\ell)!(j-\ell+m-p)!}, \quad (24)$$

$$\bar{H}_{p,j,m} = \sum_{\ell=0}^{p} \frac{(-1)^{\ell}(\ell(m-2p)+jp)(j-\ell-1)!(j-2\ell+m-1)!}{(j-2\ell)!\ell!(p-\ell)!(j-\ell+m-p)!}. \quad (25)$$

Regarding the sum in (24), the utilization of Zeilberger's algorithm (see [39]) enables one to obtain the following recurrence relation for $H_{p,j,m}$:

$$(p+1)H_{p+1,j,m} - (m-p)H_{p,j,m}, H_{0,j,m} = 1, \quad (26)$$

which can be handled quickly to provide

$$H_{p,j,m} = \binom{m}{p}. \quad (27)$$

In addition, it is easy to demonstrate the following formula:

$$\bar{H}_{p,j,m} = 0. \quad (28)$$

Now, the two sums in (27) and (28) along with formula (23) lead to the following simplified moment formula:

$$x^m V_j(x) = \frac{1}{2^m}\sum_{p=0}^{m}\binom{m}{p} V_{j+m-2p}(x). \quad (29)$$

□

Remark 6. It is worth mentioning here that the moment formula (22) is similar to that obtained in Ref. [40].

Corollary 7. *For the case corresponds to α = 1/2, the following moment formula holds for all nonnegative integers m and j:*

$$x^m R_j^{(1/2,3/2)}(x) = \frac{1}{2^m(j+1)(j+2)} \cdot \left(\sum_{p=0}^{\lfloor (j+m)/2 \rfloor}\binom{m}{p}(2+m+j(3+j+m-2p)-3p) \cdot R_{j+m-2p}^{(1/2,3/2)}(x) + \sum_{p=0}^{\lfloor (1/2)(j+m-1) \rfloor}\binom{m}{p}(m-p)R_{j+m-2p+1}^{(1/2,3/2)}(x)\right). \quad (30)$$

Proof. Setting $\alpha = 1/2$ in the moment formula (17) gives the following formula:

$$x^m R_j^{(1/2,3/2)}(x) = \left(\sum_{p=0}^{\lfloor (j+m)/2 \rfloor} M_{p,j,m} R_{j+m-2p}^{(1/2,3/2)}(x) + \sum_{p=0}^{\lfloor 1/2(j+m-1) \rfloor} \bar{M}_{p,j,m} R_{j+m-2p-1}^{(1/2,3/2)}(x)\right), \quad (31)$$

where

$$M_{p,j,m} = \frac{(1+j+m-2p)(2+j+m-2p)}{2^m(j+1)(j+2)} \times \sum_{\ell=0}^{p} \frac{(-1)^{\ell}\left(j^2+m-\ell(2+m-2p)+j(1-2\ell+m-p)\right)(j-\ell)!(j-2\ell+m-1)!}{\ell!(p-\ell)!(j-2\ell)!(j-\ell+m-p+1)!}, \quad (32)$$

$$\bar{M}_{p,j,m} = \frac{(j+m-2p)(1+j+m-2p)}{2^m(j+1)(j+2)} \times \sum_{\ell=0}^{p} \frac{(-1)^{\ell}(m+\ell m+jp-2\ell p)(j-\ell)!(j-2\ell+m-1)!}{\ell!(j-2\ell)!(p-\ell)!(j-\ell+m-p+1)!}. \quad (33)$$

Regarding the two summations that appear in equations (32) and (33), set

$$S_{p,j,m} = \sum_{\ell=0}^{p} \frac{(-1)^{\ell}\left(j^2+m-\ell(2+m-2p)+j(1-2\ell+m-p)\right)(j-\ell)!(j-2\ell+m-1)!}{(j-2\ell)!\ell!(p-\ell)!(j-\ell+m-p+1)!},$$
$$\bar{S}_{p,j,m} = \sum_{\ell=0}^{p} \frac{(-1)^{\ell}(m+\ell m+jp-2\ell p)(j-\ell)!(j-2\ell+m-1)!}{(j-2\ell)!\ell!(p-\ell)!(j-\ell+m-p+1)!} \quad (34)$$

and utilize Zeilberger's algorithm ([39]) to show that the following two recurrence relations are, respectively, satisfied by $S_{p,j,m}$ and $\bar{S}_{p,j,m}$:

$$\begin{aligned}(p+1)(-2p+j+m-1)(-2p+j+m)\\ \cdot(-3p+m+j(-2p+j+m+3)+2)\\ \cdot S_{p+1,j,m}+(p-m)(-2p+j+m+1)\\ \cdot(-2p+j+m+2)(-3p+m+j(-2p+j+m+1)-1)\\ \cdot S_{p,j,m}=0,\end{aligned} \tag{35}$$

with the following initial condition:

$$\begin{aligned}S_{0,j,m}=\frac{j+1}{m+j+1},\\ (p+1)(-2p+j+m-2)(-2p+j+m-1)\\ \cdot\bar{S}_{p+1,j,m}-(-p+m-1)(-2p+j+m)\\ \times(-2p+j+m+1)\bar{S}_{p,j,m}=0,\end{aligned} \tag{36}$$

with the following initial condition:

$$\bar{S}_{0,j,m}=\frac{m}{(m+j)(m+j+1)}. \tag{37}$$

The above two recurrence relations can be directly solved to give

$$\begin{aligned}S_{p,j,m}=\frac{(m-p+1)_p\,(-3p+m+j(-2p+j+m+3)+2)}{p!(-2p+j+m+1)\,(-2p+j+m+2)},\\ \bar{S}_{p,j,m}=\frac{(m-p)_{p+1}}{p!(-2p+j+m+1)\,(-2p+j+m)},\end{aligned} \tag{38}$$

and consequently, the linearization coefficients $M_{p,j,m}$ and $\bar{M}_{p,j,m}$ reduce to the following expressions:

$$\begin{aligned}M_{p,j,m}=\frac{(2+m+j(3+j+m-2p)-3p)(1+m-p)_p}{2^m\,(j+1)\,(j+2)\,p!},\\ \bar{M}_{p,j,m}=\frac{(m-p)_{p+1}}{2^m\,(j+1)\,(j+2)\,p!},\end{aligned} \tag{39}$$

and therefore, the linearization formula (30) can be obtained. □

4. Linearization Formula of $R_i^{(\alpha,\alpha+1)}(x)$

In this section and based on the moment formula that was derived in the previous section, we present and prove a new linearization formula of the generalized third-kind Chebyshev polynomials $R_i^{(\alpha,\alpha+1)}(x)$.

Theorem 8. *For all nonnegative integers i and j, the following linearization formula is valid:*

$$R_i^{(\alpha,\alpha+1)}R_j^{(\alpha,\alpha+1)}=\sum_{p=0}^{2\,\min(i,j)}H_{p,i,j}R_{i+j-p}^{(\alpha,\alpha+1)}(x), \tag{40}$$

where

$$H_{p,i,j}=\frac{2^{2\alpha+1}\,i!j!\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(2+i+2\alpha)\,\Gamma(2+j+2\alpha)\,\Gamma((3/2)+\alpha)}\times\begin{cases}\dfrac{\Gamma((3/2)+i-(p/2)+\alpha)\Gamma((3/2)+j-(p/2)+\alpha)\Gamma((3+p)/2+\alpha)\Gamma(2+i+j-(p/2)+2\alpha)}{(i-(p/2))!(j-(p/2))!(p/2)!\Gamma((3/2)+i+j-(p/2)+\alpha)}, & p\ \text{even},\\ -\dfrac{\Gamma(1+i-(p/2)+\alpha)\Gamma(1+j-(p/2)+\alpha)\Gamma(1+(p/2)+\alpha)\Gamma((5/2)+i+j-(p/2)+2\alpha)}{(i-(p+1/2))!(j-(p+1/2))!((p-1)/2)!\,\Gamma(2+i+j-(p/2)+\alpha)}, & p\ \text{odd}.\end{cases} \tag{41}$$

Proof. Starting with the power form representation of $R_i^{(\alpha,\alpha+1)}(x)$ yields

$$R_i^{(\alpha,\alpha+1)}(x)\,R_j^{(\alpha,\alpha+1)}(x)=\sum_{r=0}^{\lfloor j/2\rfloor}A_{r,j}\,x^{j-2r}\,R_i^{(\alpha,\alpha+1)}(x)+\sum_{r=0}^{\lfloor (j-1)/2\rfloor}B_{r,j}\,x^{j-2r-1}R_i^{(\alpha,\alpha+1)}(x). \tag{42}$$

Based on the moment formula in (17), the last relation turns into

$$R_i^{(\alpha,\alpha+1)}(x)R_j^{(\alpha,\alpha+1)}(x)=\sum_{r=0}^{\lfloor j/2\rfloor}A_{r,j}\left(\sum_{p=0}^{\lfloor (1/2)(i+j)\rfloor-r}U_{p,i,j-2r}R_{i+j-2r-2p}^{(\alpha,\alpha+1)}(x)+\sum_{p=0}^{\lfloor (1/2)(i+j-1)\rfloor-r}\bar{U}_{p,i,j-2r}\,R_{i+j-2r-2p-1}^{(\alpha,\alpha+1)}(x)\right)+\sum_{r=0}^{\lfloor (j-1)/2\rfloor}B_{r,j}\left(\sum_{p=0}^{\lfloor (1/2)(i+j-1)\rfloor-r}U_{p,i,j-2r-1}R_{i+j-2r-2p-1}^{(\alpha,\alpha+1)}(x)+\sum_{p=0}^{\lfloor (1/2)(i+j)\rfloor-r-1}\bar{U}_{p,i,j-2r-1}R_{i+j-2r-2p-2}^{(\alpha,\alpha+1)}(x)\right). \tag{43}$$

After some algebraic computations, equation (43) can be rewritten in the following form:

$$R_i^{(\alpha,\alpha+1)}(x)R_j^{(\alpha,\alpha+1)}(x)=\sum_1+\sum_2, \tag{44}$$

where

$$\sum_1=\sum_{r=0}^{\lfloor j/2\rfloor}A_{r,j}\sum_{p=0}^{\lfloor (1/2)(i+j)\rfloor-r}U_{p,i,j-2r}R_{i+j-2r-2p}^{(\alpha,\alpha+1)}(x)+\sum_{r=0}^{\lfloor (j-1)/2\rfloor}B_{r,j}\sum_{p=0}^{\lfloor (1/2)(i+j)\rfloor-r-1}\bar{U}_{p,i,j-2r-1}R_{i+j-2r-2p-2}^{(\alpha,\alpha+1)}(x), \tag{45}$$

and

$$\sum_2=\sum_{r=0}^{\lfloor j/2\rfloor}A_{r,j}\sum_{p=0}^{\lfloor 1/2(i+j-1)\rfloor-r}U_{p,i,j-2r}R_{i+j-2r-2p-1}^{(\alpha,\alpha+1)}(x)+\sum_{r=0}^{\lfloor (j-1)/2\rfloor}B_{r,j}\sum_{p=0}^{\lfloor (1/2)(i+j-1)\rfloor-r}U_{p,i,j-2r-1}R_{i+j-2r-2p-1}^{(\alpha,\alpha+1)}(x). \tag{46}$$

After expanding and rearranging the terms in (45) and (46), they can be written in the following expressions:

$$\sum_1=\sum_{p=0}^{\lfloor (i+j)/2\rfloor}\left\{\sum_{\ell=0}^{p}\left(A_{\ell,j}U_{p-\ell,i,j-2\ell}+B_{\ell,j}\bar{U}_{p-\ell-1,i,j-2\ell-1}\right)\right\}R_{i+j-2p}^{(\alpha,\alpha+1)}(x),$$
$$\sum_2=\sum_{p=0}^{\lfloor (1/2)(i+j-1)\rfloor}\left\{\sum_{\ell=0}^{p}\left(A_{\ell,j}\bar{U}_{p-\ell,i,j-2\ell}+B_{\ell,j}U_{p-\ell,i,j-2\ell-1}\right)\right\}R_{i+j-2p-1}^{(\alpha,\alpha+1)}(x). \tag{47}$$

Now and in order to obtain the linearization coefficients of the linearization formula of $R_i^{(\alpha,\alpha+1)}(x)$ in a reduced formula that is free of any sums, we employ symbolic computation. For such purpose, set

$$M_{p,i,j}=\sum_{\ell=0}^{p}\left(A_{\ell,j}U_{p-\ell,i,j-2\ell}+B_{\ell,j}\bar{U}_{p-\ell-1,i,j-2\ell-1}\right),$$
$$\bar{M}_{p,i,j}=\sum_{\ell=0}^{p}\left(A_{\ell,j}\bar{U}_{p-\ell,i,j-2\ell}+B_{\ell,j}U_{p-\ell,i,j-2\ell-1}\right). \tag{48}$$

It can be shown by symbolic computation and, in particular, Zeilberger's algorithm that the following two recurrence relations, each of order one, are satisfied, respectively, by $M_{p,i,j}$ and $\bar{M}_{p,i,j}$:

$$(p+1)(-2p+2i+2\alpha+1)(-p+i+j+2\alpha+1)\cdot(-2p+2j+2\alpha+1)M_{p+1,i,j}-(i-p)(j-p)\cdot(2p+2\alpha+3)(-2p+2i+2j+2\alpha+1)M_{p,i,j}=0, \tag{49}$$

with the following initial value:

$$M_{0,i,j}=\frac{2^{1+2\alpha}\,\Gamma(1+\alpha)\,\Gamma((3/2)+i+\alpha)\,\Gamma((3/2)+j+\alpha)\Gamma(2+i+j+2\alpha)}{\sqrt{\pi}\,\Gamma((3/2)+i+j+\alpha)\,\Gamma(2+i+2\alpha)\,\Gamma(2+j+2\alpha)},$$
$$(p+1)(-2p+2i+2\alpha-1)(-p+i+j+2\alpha+1)\cdot(-2p+2j+2\alpha-1)\bar{M}_{p+1,i,j}-(i-p-1)(j-p-1)\cdot(2p+2\alpha+3)(-2p+2i+2j+2\alpha+1)\bar{M}_{p,i,j}=0, \tag{50}$$

with the following initial value:

$$\bar{M}_{0,i,j}=\frac{-2^{1+2\alpha}\,i\,j\,\Gamma(1+\alpha)\,\Gamma((1/2)+i+\alpha)\Gamma((1/2)+j+\alpha)\,\Gamma(2+i+j+2\alpha)}{\sqrt{\pi}\,\Gamma((3/2)+i+j+\alpha)\,\Gamma(2+i+2\alpha)\,\Gamma(2+j+2\alpha)}. \tag{51}$$

The above two recurrence relations can be directly solved to give

$$M_{p,i,j}=\frac{-2^{1+2\alpha}\,i!j!\Gamma(1+\alpha)\,\Gamma((3/2)+i-p+\alpha)\,\Gamma((3/2)+j-p+\alpha)\,\Gamma((3/2)+p+\alpha)\,\Gamma(2+i+j-p+2\alpha)}{\sqrt{\pi}\,p!(i-p)!(j-p)!\Gamma((3/2)+\alpha)\,\Gamma((3/2)+i+j-p+\alpha)\,\Gamma(2+i+2\alpha)\,\Gamma(2+j+2\alpha)}, \tag{52}$$

$$\bar{M}_{p,i,j}=\frac{-2^{1+2\alpha}i!j!\Gamma(\alpha+1)\Gamma((1/2)+i-p+\alpha)\Gamma((1/2)+j-p+\alpha)\Gamma((3/2)+p+\alpha)\Gamma(2+i+j-p+2\alpha)}{\sqrt{\pi}p!(i-p-1)!(j-p-1)!\Gamma((3/2)+\alpha)\Gamma((3/2)+i+j-p+\alpha)\Gamma(2+i+2\alpha)\Gamma(2+j+2\alpha)}. \tag{53}$$

Therefore, the following linearization formula is obtained:

$$R_i^{(\alpha,\alpha+1)} R_j^{(\alpha,\alpha+1)} = \sum_{p=0}^{\lfloor (i+j)/2 \rfloor} M_{p,i,j} R_{i+j-2p}^{(\alpha,\alpha+1)} + \sum_{p=0}^{\lfloor (1/2)(i+j-1) \rfloor} \bar{M}_{p,i,j} R_{i+j-2p-1}^{(\alpha,\alpha+1)}(x), \tag{54}$$

here the linearization coefficients $M_{p,i,j}$ and $\bar{M}_{p,i,j}$ are, respectively, given by (52) and (53). Formula (54) can be written alternatively as

$$R_i^{(\alpha,\alpha+1)} R_j^{(\alpha,\alpha+1)} = \sum_{p=0}^{2\ \min(i,j)} H_{p,i,j} R_{i+j-p}^{(\alpha,\alpha+1)}(x), \tag{55}$$

with the following linearization coefficients $H_{p,i,j}$:

$$H_{p,i,j} = \frac{2^{1+2\alpha}\, i!j!\Gamma(1+\alpha)}{\sqrt{\pi}\,\Gamma(2+i+2\alpha)\,\Gamma(2+j+2\alpha)\,\Gamma((3/2)+\alpha)} \times \begin{cases} \dfrac{\Gamma((3/2)+i-(p/2)+\alpha)\Gamma((3/2)+j-(p/2)+\alpha)\Gamma((3+p)/2+\alpha)\Gamma(2+i+j-(p/2)+2\alpha)}{(i-(p/2))!(j-(p/2))!((p/2))!\Gamma((3/2)+i+j-(p/2)+\alpha)}, & p \text{ even}, \\ -\dfrac{\Gamma(1+i-(p/2)+\alpha)\Gamma(1+j-(p/2)+\alpha)\Gamma(1+(p/2)+\alpha)\Gamma((5/2)+i+j-(p/2)+2\alpha)}{(i-(p+1/2))!(j-(p+1/2))!((p-1)/2)!\,\Gamma(2+i+j-(p/2)+\alpha)}, & p \text{ odd}. \end{cases} \tag{56}$$

This completes the proof of Theorem 8. □

In the following two corollaries, we give two specific linearization formulas of the linearization formula (40).

Corollary 9. *Setting* $\alpha = -(1/2)$ *in (40) leads to the following linearization formula:*

$$V_i(x)\, V_j(x) = \sum_{p=0}^{2\ \min(i,j)} (-1)^p\, V_{i+j-p}(x). \tag{57}$$

Remark 10. The linearization formula (57) was previously obtained in [41], but here, it is derived in an alternative approach.

Corollary 11. *Setting* $\alpha = 1/2$ *in (40) leads to the following linearization formula:*

$$R_i^{(1/2,3/2)} R_j^{(1/2,3/2)} = \sum_{p=0}^{2\ \min(i,j)} H_{p,i,j}\, R_{i+j-p}^{(1/2,3/2)}(x), \tag{58}$$

with

$$H_{p,i,j} = (1/8(i+1)(i+2)(j+1)(j+2)) \times \begin{cases} (2+2i-p)(2+2j-p)(4+2i+2j-p)(2+p), & p \text{ even}, \\ -(1+2i-p)(1+2j-p)(5+2i+2j-p)(p+1), & p \text{ odd}. \end{cases} \tag{59}$$

Remark 12. The linearization formula (57) *was previously obtained in* [42].

Remark 13. The basic linearization formula (40) *in Theorem8holds for the shifted Jacobi polynomials* $J_i^{(\alpha)}(x)$, *only if* x *is replaced by* $(2x-1)$ *. The following theorem exhibits this formula in an appropriate form.*

Theorem 14. *For all nonnegative integers i and j, the following linearization formula is valid:*

$$J_i^{(\alpha)}(x)\, J_j^{(\alpha)}(x) = \sum_{p=|i-j|}^{i+j} G_{p,i,j}\, J_p^{(\alpha)}(x), \tag{60}$$

with the following linearization coefficients $G_{p,i,j}$:

$$G_{p,i,j} = \frac{2^{2\alpha+1}\, i!j!\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma((3/2)+\alpha)\,\Gamma(i+2\alpha+2)\,\Gamma(j+2\alpha+2)} \times \begin{cases} \dfrac{\Gamma((1/2)(i+j-p+3)+\alpha)\,\Gamma((1/2)(i-j+p+3)+\alpha)}{((1/2)(i+j-p))!\,((1/2)(i-j+p))!\,((1/2)(-i+j+p))!} \times \dfrac{\Gamma((1/2)(-i+j+p+3)+\alpha)\,\Gamma((1/2)(i+j+p+4\alpha+4))}{\Gamma((1/2)(i+j+p+3)+\alpha)}, & (i+j-p) \text{ even}, \\ \dfrac{-\Gamma(1+(1/2)(i+j-p)+\alpha)\,\Gamma((1/2)(i-j+p+2)+\alpha)}{\Gamma((1/2)(i+j-p+1))\Gamma((1/2)(i-j+p+1))} \times \dfrac{\Gamma((1/2)(2-i+j+p)+\alpha)\Gamma((1/2)(i+j+p+4\alpha+5))}{\Gamma((1/2)(-i+j+p+1))\Gamma((1/2)(i+j+p+4)+\alpha)}, & (i+j-p) \text{ odd}. \end{cases} \tag{61}$$

Input Given κ, M, η, ξ_0 and ξ_1.
Step 1. Evaluate the derivatives coefficients $d_{j,r,1}, d_{j,s,2}$ from relation (16), and the linearization. coefficients $G_{\bar{v},\bar{s},s}, G_{v,\bar{r},r}$ from relation (61).
Step 2. Assume an approximate solution in the form: $\mathscr{V}_M = \sum_{r=0}^{M}\sum_{s=0}^{M} v_{rs} J_s^{(\alpha)}(x) J_r^{(\alpha)}(t)$.
Step 3. Compute the residuals: $RE(x,t), RI(x), RB_0(t), RB_1(t)$ via Eqs. (69-72z).
Step 4. Apply the tau method to obtain the system in ((74)-(77)).
Step 5. Employ Newtons' iterative method *-FindRoot-* to obtain the coefficients v_{rs}.
Step 6. Find the double expansion: $\sum_{r=0}^{M}\sum_{s=0}^{M} v_{rs} J_s^{(\alpha)}(x) J_r^{(\alpha)}(t)$.
Output the approximate solution: $\mathscr{V}_M$.

ALGORITHM 1: Coding algorithm for the proposed scheme.

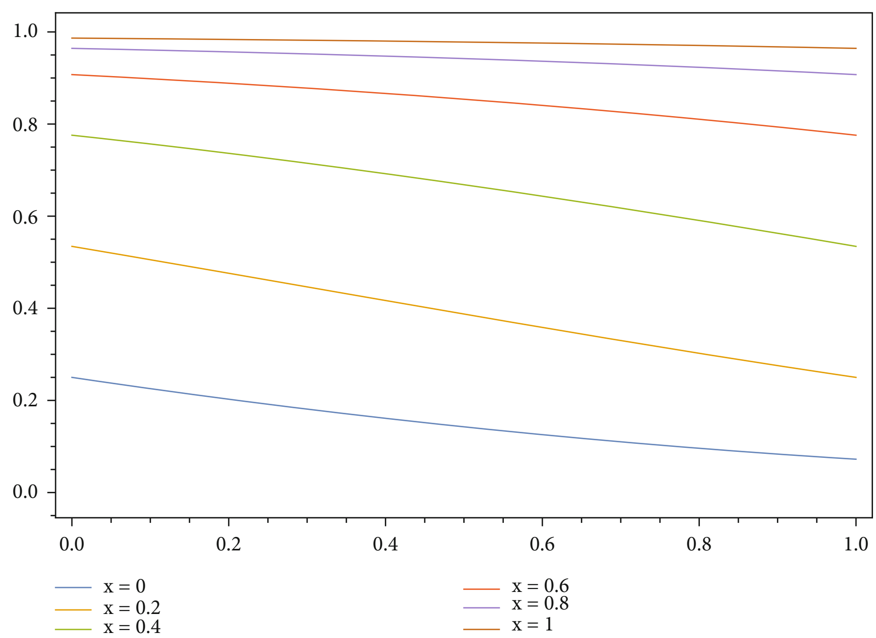

FIGURE 1: Solution of Example 1 at different values of x.

5. The Proposed Algorithm for the Numerical Treatment of the Nonlinear Fisher Equation

In this section, we are interested in obtaining a numerical algorithm for solving the nonlinear Fisher equation. More precisely, we will employ the established linearization formula along with the derivatives formula of the shifted polynomials $J_i^{(\alpha)}(x)$ to obtain a numerical solution based on applying the tau method. We denote our algorithm by the nonsymmetric Jacobi Tau Method (NJTM).

Now, consider the Fisher differential equation [32]:

$$\frac{\partial \mathscr{V}}{\partial t} = \frac{\partial^2 \mathscr{V}}{\partial x^2} + \kappa \mathscr{V}(1-\mathscr{V}), \quad (x,t) \in \Omega = (0,1) \times (0,1), \tag{62}$$

governed by the following initial and boundary conditions:

$$\mathscr{V}(x,0) = \eta(x), \quad x \in (0,1), \tag{63}$$

$$\mathscr{V}(0,t) = \xi_0(t), \quad \mathscr{V}(1,t) = \xi_1(t), \quad t \in (0,1), \tag{64}$$

where κ is the positive coefficient of kinematic viscosity and η, ξ_0 and ξ_1 are prescribed known continuous functions. Now, for solving (62) governed by the conditions (63) and (64), we utilize the spectral tau method.

Now, assume that $\mathscr{V}(x,y) = \mathscr{V} \in L^2(\Omega)$ and let it have the following double series expansion:

$$\mathscr{V} = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} v_{rs} J_s^{(\alpha)}(x) J_r^{(\alpha)}(t). \tag{65}$$

Furthermore, assume an approximate solution to (62) in the form

$$\mathscr{V} \simeq \mathscr{V}_M = \sum_{r=0}^{M}\sum_{s=0}^{M} v_{rs} J_s^{(\alpha)}(x) J_r^{(\alpha)}(t). \tag{66}$$

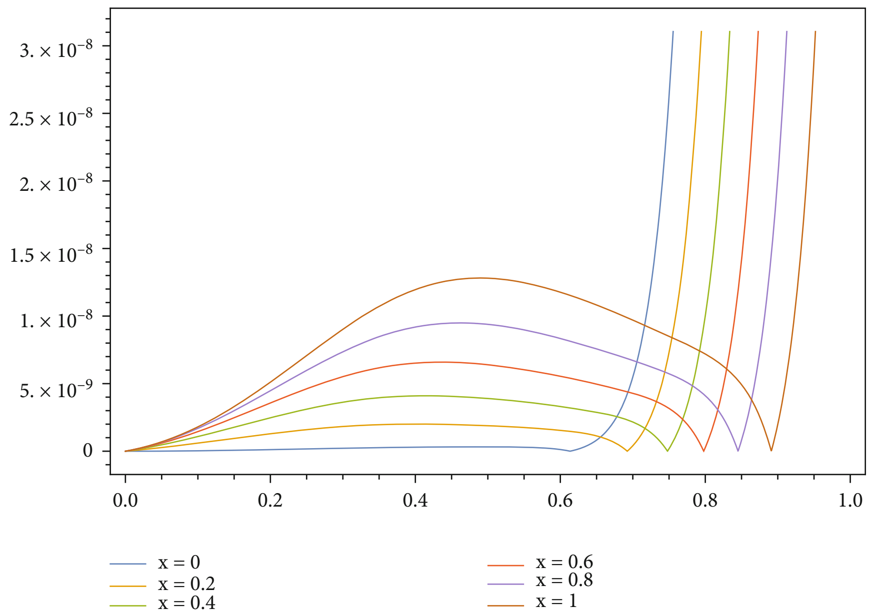

Figure 2: Absolute error of Example 1 at different values of x.

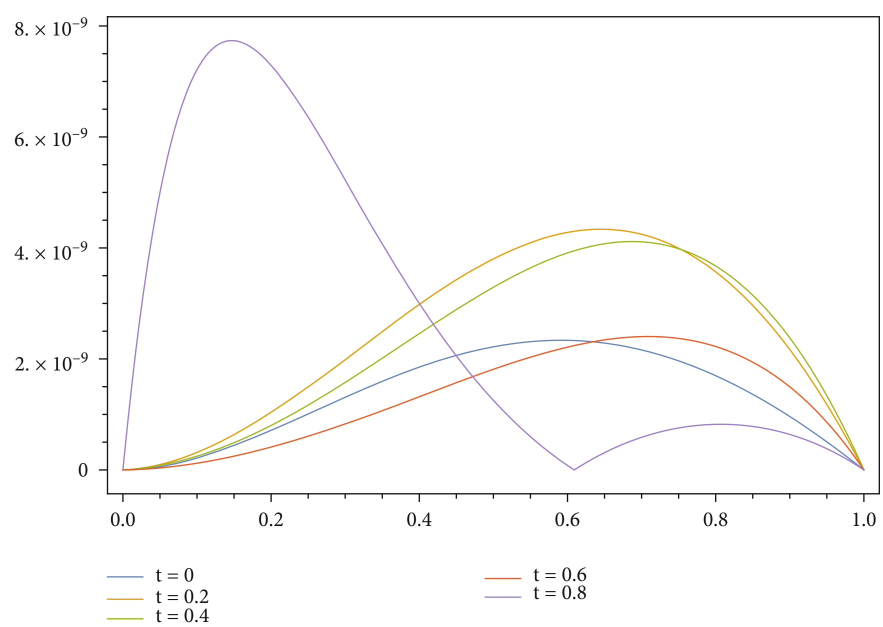

Figure 3: Absolute error of Example 1 at different values of t.

In virtue of formula (15), it can be shown that $\partial \mathscr{V}_M/\partial t$ and $\partial^2 \mathscr{V}_M/\partial x^2$ can be represented as follows:

$$\frac{\partial \mathscr{V}_M}{\partial t} = \sum_{r=1}^{M}\sum_{s=0}^{M}\sum_{j=0}^{r-1} d_{j,r,1}\, v_{rs} J_s^{(\alpha)}(x) J_j^{(\alpha)}(t),$$
$$\frac{\partial^2 \mathscr{V}_M}{\partial x^2} = \sum_{r=0}^{M}\sum_{s=2}^{M}\sum_{j=0}^{s-2} d_{j,s,2}\, v_{rs} J_j^{(\alpha)}(x) J_r^{(\alpha)}(t). \quad (67)$$

Table 1: MAE for Example 1.

M	$\alpha = 0$	$\alpha = 1/2$	$\alpha = 1$
4	2.35×10^{-2}	3.48×10^{-2}	5.61×10^{-2}
6	3.41×10^{-6}	5.47×10^{-6}	8.35×10^{-6}
8	4.73×10^{-8}	5.27×10^{-8}	9.37×10^{-8}

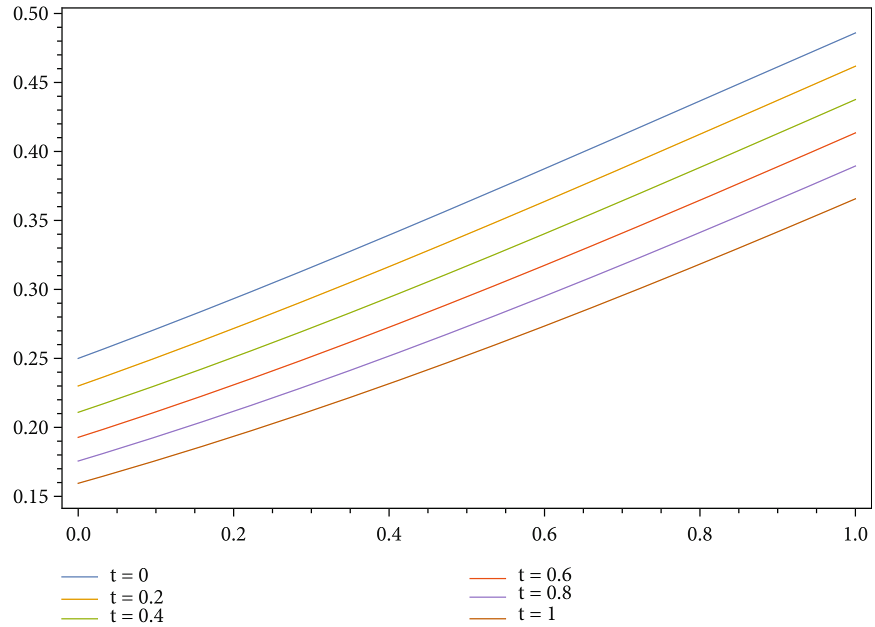

FIGURE 4: Solution of Example 2 at different values of t.

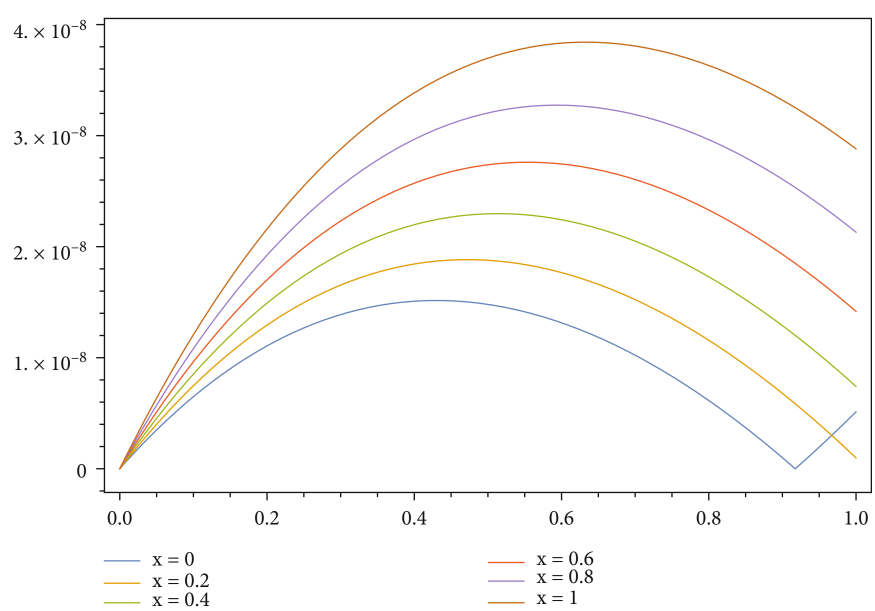

FIGURE 5: Absolute error of Example 2 at different values of x.

Based on the linearization formula (60), the nonlinear term $(\mathcal{V}_M)^2$ can be written in the following form:

$$(\mathcal{V}_M)^2=\sum_{\bar{r}=0}^{M}\sum_{\bar{s}=0}^{M}\sum_{r=0}^{M}\sum_{s=0}^{M}\sum_{\bar{v}=|\bar{s}-s|}^{\bar{s}+s}\sum_{v=|\bar{r}-r|}^{\bar{r}+r} G_{\bar{v},\bar{s},s}\,G_{v,\bar{r},r}\,v_{\bar{s}\bar{r}}\,v_{rs}J_{\bar{v}}^{(\alpha)}(x)\,J_{v}^{(\alpha)}(t). \tag{68}$$

Now, to apply the tau method to (62) governed by the conditions (63) and (64), we first compute the residual $RE(x,t)$ of the differential equation (62). It is given explicitly as

$$\begin{aligned}RE(x,t)=\frac{\partial\mathcal{V}_M}{\partial t}-\frac{\partial^2\mathcal{V}_M}{\partial x^2}-\kappa\mathcal{V}_M+\kappa(\mathcal{V}_M)^2&=\sum_{r=1}^{M}\sum_{s=0}^{M}\sum_{j=0}^{r-1}d_{j,r,1}\,v_{rs}J_s^{(\alpha)}(x)J_j^{(\alpha)}(t)\\&-\sum_{r=0}^{M}\sum_{s=2}^{M}\sum_{j=0}^{s-2}d_{j,s,2}\,v_{rs}J_j^{(\alpha)}(x)J_r^{(\alpha)}(t)-\kappa\sum_{r=0}^{M}\sum_{s=0}^{M}v_{rs}J_s^{(\alpha)}(x)J_r^{(\alpha)}(t)\\&+\kappa\sum_{\bar{r}=0}^{M}\sum_{\bar{s}=0}^{M}\sum_{r=0}^{M}\sum_{s=0}^{M}\sum_{\bar{v}=|\bar{s}-s|}^{\bar{s}+s}\sum_{v=|\bar{r}-r|}^{\bar{r}+r}G_{\bar{v},\bar{s},s}\,G_{v,\bar{r},r}\,v_{\bar{s}\bar{r}}\,v_{rs}J_{\bar{v}}^{(\alpha)}(x)\,J_{v}^{(\alpha)}(t).\end{aligned} \tag{69}$$

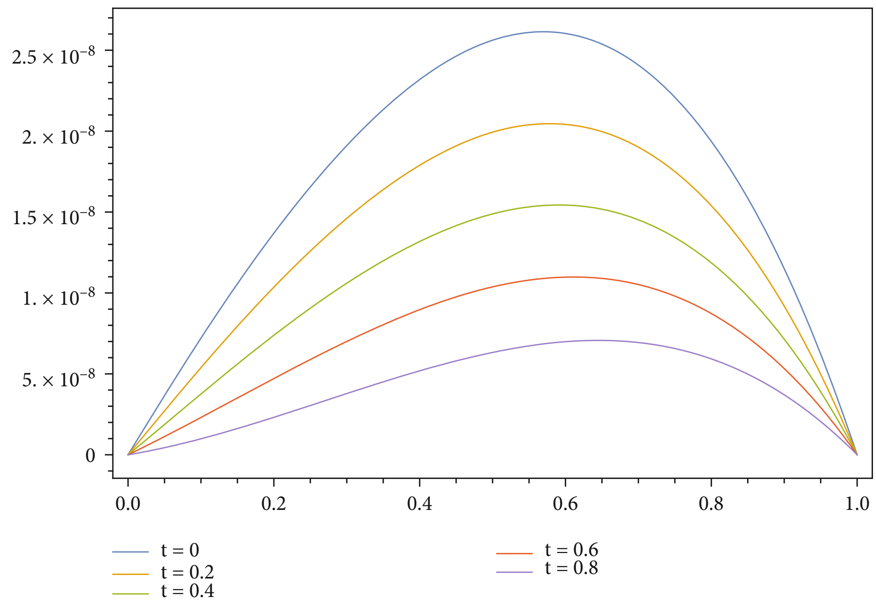

FIGURE 6: Absolute error of Example 2 at different values of t.

On the other hand, the residual of (63) ($RI(x)$) can be written:

$$RI(x) = \mathscr{V}_M(x,0) - \eta(x) = \sum_{r=0}^{M}\sum_{s=0}^{M} v_{rs} J_s^{(\alpha)}(x) J_r^{(\alpha)}(0) - \eta(x). \tag{70}$$

Moreover, the two residuals of the boundary conditions (64) ($RB_0(t)$ and $RB_1(t)$) are given by

$$RB_0(t) = \mathscr{V}_M(0,t) - \xi_0(t) = \sum_{r=0}^{M}\sum_{s=0}^{M} v_{rs} J_s^{(\alpha)}(0) J_r^{(\alpha)}(t) - \xi_0(t), \tag{71}$$

$$RB_1(t) = \mathscr{V}_M(1,t) - \xi_1(t) = \sum_{r=0}^{M}\sum_{s=0}^{M} v_{rs} J_s^{(\alpha)}(1) J_r^{(\alpha)}(t) - \xi_1(t). \tag{72}$$

Now, the application of the tau method leads to the following equations:

$$\begin{aligned}
&\int_0^1\int_0^1 RE(x,t)\, J_r^{(\alpha)}(x) J_s^{(\alpha)}(t) w_1(x) w_1(t)\, dx dt = 0, \quad 0 \le r \le M-1, 0 \le s \le M-1,\\
&\int_0^1 RI(x)\, J_0^{(\alpha)}(x) w_1(x)\, dx = 0,\\
&\int_0^1 RB_0(t)\, J_r^{(\alpha)}(t) w_1(t)\, dt = 0, \quad 0 \le r \le M-1,\\
&\int_0^1 RB_1(t)\, J_r^{(\alpha)}(t) w_1(t)\, dt = 0, \quad 0 \le r \le M-1.
\end{aligned} \tag{73}$$

The substitution by the four residuals in equations (69), (70), (71), and (72) yields, respectively, the following equations:

TABLE 2: MAE for Example 2.

M	$\alpha = 0$	$\alpha = 1/2$	$\alpha = 1$
3	4.24×10^{-3}	8.27×10^{-3}	2.16×10^{-2}
5	5.37×10^{-6}	5.69×10^{-6}	4.88×10^{-5}
7	2.66×10^{-8}	7.28×10^{-8}	8.61×10^{-8}

$$\begin{aligned}
&\sum_{r=1}^{M} d_{s,r,1}\, nu_{rn}\, h_r\, h_{s,1} - \sum_{s=2}^{M} d_{r,s,2}\, v_{ms}\, h_r\, h_s - \kappa\, nu_{rs}\, h_r\, h_s\\
&\quad + \kappa \sum_{\bar{n}=0}^{M}\sum_{\bar{s}=0}^{M}\sum_{n=0}^{M}\sum_{s=0}^{M} G_{r,\bar{s},s}\, G_{r,\bar{n},n}\, v_{\bar{s}\bar{n}}\, v_{rs} h_r\, h_s = 0, \quad 0 \le r \le M-1, 0 \le s \le M-1,
\end{aligned} \tag{74}$$

$$\sum_{n=0}^{M} v_{0n}\, h_0\, J_n^{(\alpha)}(0) = \int_0^1 \eta(x) J_0^{(\alpha)}(x) w_1(x) dx, \tag{75}$$

$$\sum_{s=0}^{M} v_{mr}\, h_r J_s^{(\alpha)}(0) = \int_0^1 \xi_0(t) J_r^{(\alpha)}(t) w_1(t)\, dt, \quad 0 \le r \le M-1, \tag{76}$$

$$\sum_{s=0}^{M} v_{mr} h_r J_s^{(\alpha)}(1) = \int_0^1 \xi_1(t) J_r^{(\alpha)}(t) w_1(t)\, dt, \quad 0 \le r \le M-1, \tag{77}$$

where $J_i^{(\alpha)}(0) = ((-1)^i\,(i+\alpha+1))/(\alpha+1)$ and $J_i^{(\alpha)}(1) = 1$.

The proposed tau approach produces the nonlinear system of equations (74)-(77) with the unknowns $\{v_{rs}\}$ of

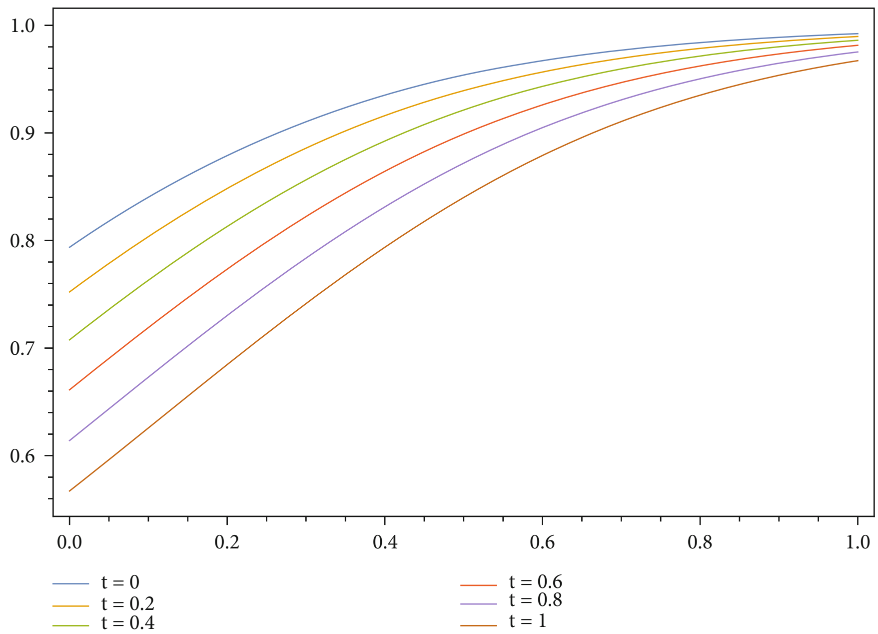

FIGURE 7: Solution of Example 3 at different values of t.

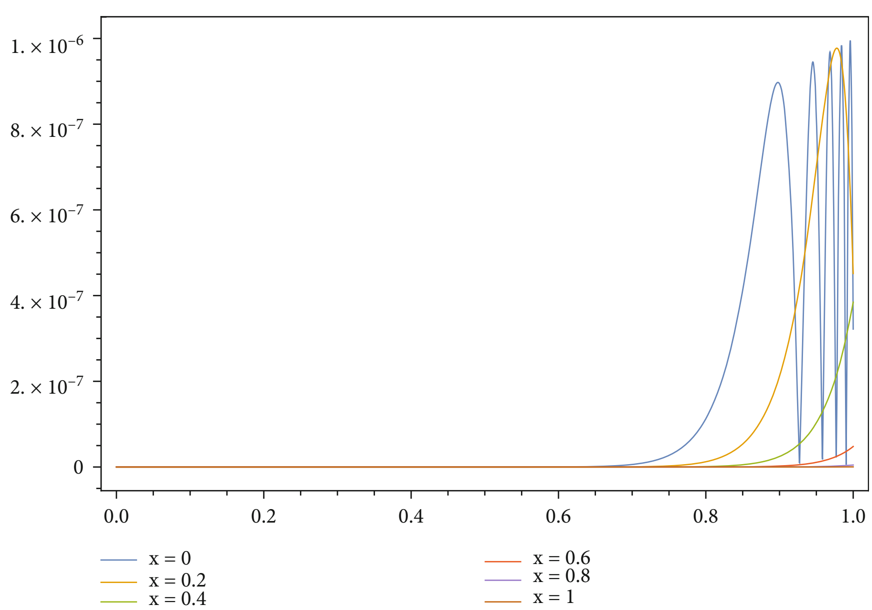

FIGURE 8: Absolute error of Example 3 at different values of x.

dimension M^2. It is necessary to solve this nonlinear system to obtain the desired approximate solution. This can be accomplished by using a suitable numerical method, such as Newton's iterative technique.

Remark 15. It is very imperative to communique here that if the term $\kappa\mathscr{V}(1-\mathscr{V})$ is replaced by $\kappa\mathscr{V}(1-\mathscr{V}^m)$, where m is any positive integer, then the repeated use of linearization formula (59) will generate a system that is similar to ((73))-((76)). This means our algorithm can be extended to solve this generalized Fisher problem. The details are omitted.

Remark 16. To summarize our proposed numerical algorithm, in Algorithm 1, we list in order the steps required to obtain the desired numerical solution.

6. Numerical Experiments and Comparisons

Example 1. Consider the following nonlinear Fisher equation [32]:

$$\frac{\partial\mathscr{V}}{\partial t}=\frac{\partial^2\mathscr{V}}{\partial x^2}+6\mathscr{V}(1-\mathscr{V}),\quad (x,t)\in\Omega=(0,1)\times(0,1), \tag{78}$$

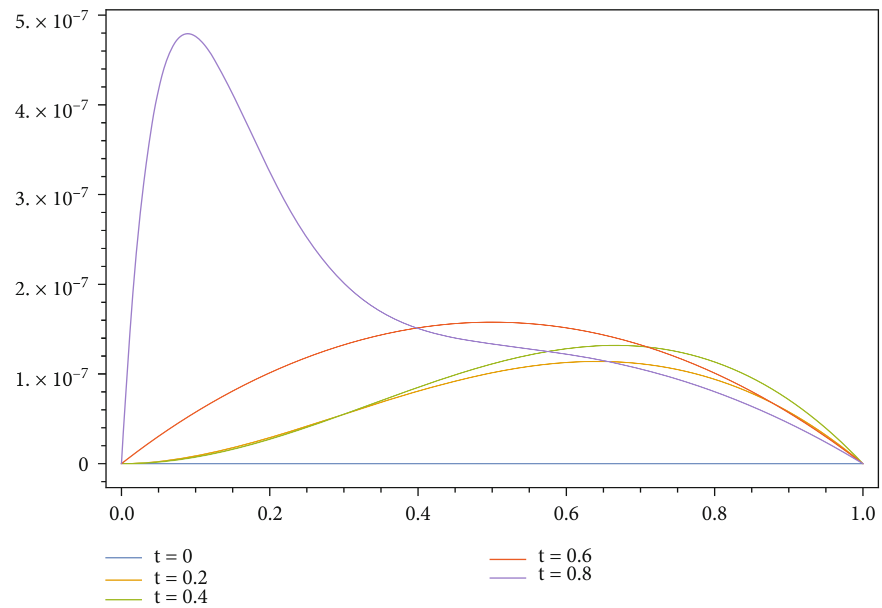

FIGURE 9: Absolute error of Example 3 at different values of t.

TABLE 3: MAE for Example 3.

M	$\alpha = 0$	$\alpha = 1/2$	$\alpha = 1$
4	5.27×10^{-3}	2.48×10^{-3}	6.87×10^{-3}
6	3.41×10^{-5}	4.29×10^{-5}	8.63×10^{-5}
8	4.73×10^{-8}	5.27×10^{-7}	9.37×10^{-7}

along with

$$\mathscr{V}(x,0) = (1+e^{x})^{-2}, \quad x \in (0,1), \tag{79}$$

and the boundary conditions:

$$\begin{aligned}\mathscr{V}(0,t) &= \left(1+e^{-5t}\right)^{-2},\\ \mathscr{V}(1,t) &= \left(1+e^{1-5t}\right)^{-2}, \quad t \in (0,1),\end{aligned} \tag{80}$$

with the exact solution

$$\mathscr{V}(x,t) = \left(1+e^{x-5t}\right)^{-2}. \tag{81}$$

We apply the NJTM for the case corresponding to $\alpha = 0$, $M = 8$. In Figure 1, we depict the approximate solution of Example 1 at different values of x. In Figure 2, we depict the absolute error of Example 1 at different values of x. In Figure 3, we depict the absolute error of Example 1 at different values of t. In Table 1, we report the maximum absolute error (MAE) for different values of M and α.

Example 2. Consider the following nonlinear Fisher equation [32]:

$$\frac{\partial \mathscr{V}}{\partial t} = \frac{\partial^2 \mathscr{V}}{\partial x^2} + \mathscr{V}(1-\mathscr{V}), (x,t) \in \Omega = (0,1) \times (0,1), \tag{82}$$

along with

$$\mathscr{V}(x,0) = \frac{1}{\left(e^{x/\sqrt{6}}+1\right)^2}, \tag{83}$$

and the boundary conditions:

$$\mathscr{V}(0,t) = \frac{1}{\left(e^{-5t/6}+1\right)^2}, \mathscr{V}(1,t) = \frac{1}{\left(e^{(1/\sqrt{6})-(5t/6)}+1\right)^2}, \quad t \in (0,1), \tag{84}$$

with the exact solution

$$\mathscr{V}(x,t) = \frac{1}{\left(e^{(x/\sqrt{6})-(5t/6)}+1\right)^2}. \tag{85}$$

We apply NJTM for the case corresponding to $\alpha = 1/2$, $M = 7$. Figure 4 displays the approximate solutions of Example 2 at different values of t. Additionally, Figure 5 displays the absolute error of Example 2 at different values of x. Figure 6 displays the absolute error of Example 2 at different values of t. Finally, Table 2 reports the MAE for different values of M and α.

Example 3. Consider the following nonlinear Fisher equation ([32, 43]):

$$\frac{\partial \mathcal{V}}{\partial t} = \frac{\partial^2 \mathcal{V}}{\partial x^2} + \mathcal{V}\left(1 - \mathcal{V}^6\right), (x, t) \in \Omega = (0, 1) \times (0, 1), \quad (86)$$

along with

$$\mathcal{V}(x, 0) = \sqrt[3]{\frac{1}{2} - \frac{1}{2} \tanh\left(\frac{3x}{4}\right)}, \quad x \in (0, 1), \quad (87)$$

and the boundary conditions:

$$\begin{aligned} \mathcal{V}(0, t) &= \sqrt[3]{\frac{1}{2} \tanh\left(\frac{15t}{8}\right) + \frac{1}{2}}, \mathcal{V}(1, t) \\ &= \sqrt[3]{\frac{1}{2} - \frac{1}{2} \tanh\left(\frac{3}{4}\left(1 - \frac{5t}{2}\right)\right)}, \quad t \in (0, 1), \end{aligned} \quad (88)$$

with the exact solution

$$\mathcal{V}(x, t) = \sqrt[3]{\frac{1}{2} - \frac{1}{2} \tanh\left(\frac{3}{4}\left(x - \frac{5t}{2}\right)\right)}. \quad (89)$$

We apply NJTM for the case corresponding to $\alpha = 1$, $M = 8$. In Figure 7, we depict the solution of Example 3 at different values of t. Figure 8 shows the absolute error of Example 3 at different values of x, while Figure 9 shows the absolute error of Example 3 at different values of t. The MAE for different values of M and α is reported in Table 3.

7. Concluding Remarks

In this article, a novel linearization formula of a class of Jacobi polynomials that generalizes the third-kind Chebyshev polynomials was established. The establishment of this linearization formula depends on using a new moment formula of these polynomials together with the employment of suitable symbolic computation. The linearization formula and the high-order derivative formula of a certain class of Jacobi polynomials are utilized along with the spectral tau method to develop a new numerical algorithm for treating the nonlinear Fisher equation. We do believe that our theoretical results and the proposed numerical results are new. Furthermore, other types of nonlinear differential equations may be treated using similar techniques. Some illustrative examples were presented accompanied by some comparisons to validate the accuracy and efficiency and the proposed tau algorithm.

Data Availability

No data is associated with this research.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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