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Research Article

Existence and Nonexistence for Boundary Problem Involving the p-Biharmonic Operator and Singular Nonlinearities

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This article concerns the existence and the nonexistence of solution for the following boundary problem involving the p-biharmonic operator and singular nonlinearities, $\Delta_p^2 u = |u|^{\gamma-1}u + \mu(|u|^{-1-\alpha}/|x|^{\beta})u$ in Ω and $u = \partial u/\partial n = 0$ on $\partial\Omega$, where $4 < 2p < N$, $0 \in \Omega$, $-\infty < \mu < \mu_*, = (N-2p)(1-\alpha)/pN$, $p < \gamma < p^* = pN/N-2p$ is the critical Sobolev exponent, $0 \leq \beta < N(\gamma+\alpha)/(\gamma+1)$, $0 < \alpha < 1$. Under some sufficient conditions on coefficients, we prove the existence of at least one nontrivial solutions in E by using variational methods. By using the Pohozaev identity type, we show the nonexistence of positive solution when $\Omega \subset \mathbb{R}^N$ be a bounded, smoothandstrictlystar-shapeddomain, $\beta = 0$ and $\gamma \geq \gamma_*, = \mathrm{pN}(1-\alpha)/(\mathrm{N}-2\mathrm{p})(1-\alpha) - \mu\mathrm{Np} > \mathrm{p}^* = \mathrm{pN}/\mathrm{N}-2\mathrm{p}$.

1. Introduction and Main Results

The main purpose of this article is to investigate the existence and nonexistence of nontrivial solutions of the following problem:

$$\begin{cases} \Delta_p^2 u = |u|^{\gamma-1}u + \mu \dfrac{|u|^{-1-\alpha}}{|x|^{\beta}} u, & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where n is the exterior normal vector to $\partial\Omega$, Ω is a smooth-bounded domain in $\mathbb{R}^N$,$4 < 2p < N$, $0 \in \Omega$,$-\infty < \mu < \mu_* = (N-2p)(1-\alpha)/Np$, $p < \gamma < p^*$ with $p^* := pN/(N-2p)$ is the critical Sobolev exponent, $0 \leq \beta < N(\gamma+\alpha)/(\gamma+1)$, $0 < \alpha < 1$. $\Delta_p^2 u := \Delta(|\Delta u|^{p-2}\Delta u)$ is the operator of the fourth order, so-called the p-biharmonic (or p-bilaplacian) operator. For $p = 2$, the linear operator $\Delta_2^2 = \Delta^2 = \Delta.\Delta$ is the iterated Laplacian that to a multiplicative positive constant appears often in the equations of Navier-Stokes as being a viscosity coefficient, and its reciprocal operator noted $(\Delta^2)^{-1}$ is the celebrated Green's operator see [1].

The fourth-order differential equations with nonlinearity arise in the study of deflections of elastic beams on nonlinear elastic foundations. It furnishes a model to study travelling waves in suspension bridges. Lazer and McKenna [2] gave a survey of results in this direction. This fourth-order semilinear elliptic problem can be considered as an analogue of a class of second-order problems which have been studied by many authors (see [3] and references therein). Thus, they become very significant in engineering and physics and many authors considered this type of equation in recent years, and we refer to [4–11] and the references therein. For this reason, the existence of solutions of p-biharmonic equations has been studied by several authors; see [12–15]. Li [16] establish the existence of at least two distinct weak solutions for the following singular elliptic problems involving a p-biharmonic operator, subject to Navier boundary conditions in a smooth-bounded domain in $\mathbb{R}^N$.

$$\begin{cases} \Delta_p^2 u + \dfrac{|u|^{q-1}u}{|x|^{2q}} = \lambda f(x,u), & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases} \quad (2)$$

with f being a Carathéodory function. Wang [17] studied the existence and multiplicity to p-biharmonic equation with the Sobolev–Hardy term under the Dirichlet boundary conditions and the Navier boundary conditions, respectively.

$$\begin{cases} \Delta_p^2 u = \lambda \dfrac{|u|^{r-1}u}{|x|^s} + f(x,u), & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial n} = 0 \text{ or}, & \text{on } \partial\Omega. \\ u = \Delta u = 0, & \end{cases} \quad (3)$$

Before giving our main results, we state here some definitions, notations, and known results.

1.1. Notations. Set $E = W_0^{2,p}(\Omega) \cap L^\infty(\Omega)$ a Banach space, with the norm $\|u\| = \|\Delta u\|_p$ for $1 \le p \le \infty$.

We consider the following approximation equation:

$$\begin{cases} \Delta_p^2 u = |u|^{\gamma-1}u + \dfrac{\mu}{|x|^\beta (u+\theta)^\alpha}, & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases} \quad (4)$$

for any $\theta > 0$ such that $u + \theta > 0$. The energy functional of (4) J_θ is defined by

$$J_\theta(u) := \frac{1}{p}\|u\|^p - \frac{1}{\gamma+1}\int_\Omega |u|^{\gamma+1}dx - \frac{\mu}{1-\alpha}\int_\Omega \frac{(u+\theta)^{1-\alpha} - \theta^{1-\alpha}}{|x|^\beta}dx. \quad (5)$$

for all $u \in E$.

We note that J_θ is a C^1 – function on $E = W_0^{2,p}(\Omega) \cap L^\infty(\Omega)$.

A point $u \in E$ is a weak solution of the Equation (1) if it satisfies

$$\left\langle J_\theta'(u), \varphi \right\rangle := \int_\Omega F_p(\Delta u)\Delta\varphi dx - \int_\Omega |u|^{\gamma-1}u\varphi dx - \mu\int_\Omega \frac{\varphi}{(u+\theta)^\alpha |x|^\beta}dx = 0, \text{for all } \varphi \in E, \quad (6)$$

where $F_p(s) = |s|^{p-2}s$ with $p > 2$.

Here, $\langle .,. \rangle$ denotes the product in the duality $E', E(E'$ the dual of $E)$.

In our work, we research the critical points as the minimizers of the energy functional associated to problem (1).

Let

$$S := \inf_{u \in E\setminus\{0\}} \frac{\|u\|^p}{\left(\int_\Omega |u|^{\gamma+1}dx\right)^{p/(\gamma+1)}}. \quad (7)$$

From [18], S is achieved.

Let

$$A = \left[\frac{2\pi^{N/2}(q+\beta)}{N\Gamma(N/2)(q+\beta) - \alpha(q+1)}\right]^{(q+\beta)/(q+1)} R_0^{N/(q+1)(q+\beta)-\alpha} > 0, \quad (8)$$

with

$$0 \le \alpha < \frac{N}{q+1}(q+\beta). \quad (9)$$

Now, we can state our main results.

Theorem 1. *Assume that $4 < 2p < N$, $0 \in \Omega, p < \gamma < p^*$ with $p^* := pN/(N-2p)$ is the critical Sobolev exponent, $0 \le \beta < N(\gamma+\alpha)/(\gamma+1)$, $0 < \alpha < 1$ and $\mu \le 0$; then, problem (1) has at least a nontrivial solution in E.*

Theorem 2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded, smooth, and strictly star-shaped domain, $0 \le \mu < \mu_* = (N-2p)(1-\alpha)/Np$ and $\beta = 0$, with $\gamma \ge \gamma_* = Np(1-\alpha)/((N-2p)(1-\alpha) - \mu Np) > p^* = Np/(N-2p)$. Then, problem (1) has no positive solutions in E.*

This paper is organized as follows. In Section 2, we give some preliminaries. Sections 3 and 4 are devoted to the proofs of Theorems 1 and 2.

2. Some Preliminary Results

Definition 3 [1]. An operator $A: E \longrightarrow E'$ is hemicontinuous if

$$\lambda \mapsto \langle A(u_1 + \lambda u_2), u_3 \rangle, \quad (10)$$

is continuous from $\mathbb{R}$ to $\mathbb{R}$, for all $u_1, u_2, u_3 \in E$.

Definition 4 [1]. An operator $A: E \longrightarrow E'$ is " calculus of variations," if it is bounded and if it can be represented by

$d_1) A(u) = \tilde{A}(u,u)$ where

$$\tilde{A} : E \times E \longrightarrow E', \quad (u,\hat{u}) \mapsto \tilde{A}(u,\hat{u}), \quad (11)$$

is an operator satisfying the following properties.

$d_2)$

$$\begin{cases} \forall u \in E; \hat{u} \mapsto \tilde{A}(u,\hat{u}) \text{ hemicontinuous, bounded of } E \longrightarrow E' \\ \left\langle \tilde{A}(u,u) - \tilde{A}(u,\hat{u}), (u-\hat{u}) \right\rangle \ge 0. \end{cases} \quad (12)$$

(d_3) For all: $\hat{u} \in E$; $u \mapsto \tilde{A}(u,\hat{u})$ is bounded, hemicontinuous of $E \longrightarrow E'$.

(d_4) If $(u_n)_{n\in\mathbb{N}}$ converges weakly to u in E and if

$$\left\langle \tilde{A}(u_n,u_n)-\tilde{A}(u_n,u),(u_n-u)\right\rangle \longrightarrow 0 \text{ as } n\longrightarrow\infty, \quad (13)$$

then, for all $\hat{u}\in E$, the sequence definite by $\tilde{A}(u_n,\hat{u})$ converges to $\tilde{A}(u,\hat{u})$ in E'.

(d_5) If $(u_n)_{n\in N}$ converges weakly to u in E, and if $\tilde{A}(u_n,\hat{u})$ converges weakly to φ in E', then

$$\left\langle \tilde{A}(u_n,\hat{u}),u_n\right\rangle_{E',E} \longrightarrow \langle\varphi,u\rangle_{E',E} \text{ as } n\longrightarrow\infty. \quad (14)$$

3. Existence Results

The energy functional J_θ associated to Equation (4) is defined on E by

$$J_\theta(u) := \frac{1}{p}\|u\|^p - \frac{1}{\gamma+1}\int_\Omega |u|^{\gamma+1}dx - \frac{\mu}{1-\alpha}\int_\Omega \frac{(u+\theta)^{1-\alpha}-\theta^{1-\alpha}}{|x|^\beta}dx. \quad (15)$$

To prove Theorem 1, we use the three following lemmas.

Set $J_\theta(u):=A(u)-B(u)$ with $A(u)=1/p\int_\Omega|\Delta u|^p dx = (1/p)\|u\|^p$ and $B(u)=1/\gamma+1\int_\Omega|u|^{\gamma+1}dx+(\mu/1-\alpha)\int_\Omega((u+\theta)^{1-\alpha}-\theta^{1-\alpha}/|x|^\beta)dx$.

Firstly, we need the following Lemmas.

Lemma 5. *J_θ is $C^1(E,\mathbb{R})$.*

Proof. Let $R_0>0$ such that $\Omega\subset B(0,R_0)=\{x\in\mathbb{R}^N : |x|<R_0\}$. If $u\in E$ and by the Hölder inequality, we obtain

$$\int_\Omega \frac{|u|^{1-\alpha}}{|x|^\beta}dx \le C_1\|u\|^{1-\alpha}, \quad (16)$$

with $C_1 = AS_1^{(-(1-\alpha))/p}$,

where

$$S_1 := \inf_{u\in E\setminus\{0\}} \frac{\|u\|^p}{\left(\int_\Omega\left(|u|^{(1-\alpha)}/|x|^\beta\right)dx\right)^{p/(1-\alpha)}}. \quad (17)$$

It is enough to show that A and B are C^1 on E.

According to an algebraic relation of Li and Tang [3] and the Hölder inequality, one has

$$\begin{aligned}\|A'(u_1)-A'(u_2)\|_{E'} &\le \|F_p(\Delta u_1)-F_p(\Delta u_2)\|_{p'} \\ &\le C\|\Delta u_1-\Delta u_2\|_p\left(\|\Delta u_1\|_p+\|\Delta u_2\|_p\right)^{p-2},\end{aligned} \quad (18)$$

where $F_p(t)=|t|^{p-2}t$.

Therefore A is C^1 on E and B is C^1, since $g(u)=|u|^{\gamma-1}u$ is subcritical and by (16). Thus, J_θ is C^1 on E. □

Lemma 6. *J_θ satisfies the Palais-Smale condition.*

Proof. Let (u_n) be a Palais-Smale sequence in E (i.e., $J_\theta(u_n)$ is bounded and $J'_\theta(u_n)\longrightarrow 0$). (u_n) is bounded in E. In fact,

$$\int_\Omega\left(|\Delta u_n|^p-|u_n|^{\gamma+1}\right)dx \longrightarrow 0, \text{ as } n\longrightarrow\infty, \quad (19)$$

$$\int_\Omega\left(\frac{1}{p}|\Delta u_n|^p-\frac{\mu}{\gamma+1}|u_n|^{\gamma+1}\right) \text{ is bounded.} \quad (20)$$

Combining (19), (20), and (16), we obtain that (u_n) is bounded when $p<\gamma<p^*, 0\le\beta<N(\gamma+\alpha)/(\gamma+1), 0<\alpha<1$ and $\mu\le 0$.

The embedding $E\hookrightarrow L^\gamma(\Omega)$ is compact for $p<\gamma<p^*$; then, there exists a subsequence, still denoted by (u_n), which converges in $L^\gamma(\Omega)$. Show that (u_n) converges in E.

Set

$$I_{n,m} = \int_\Omega\left[F_p(\Delta u_n)-F_p(\Delta u_m)\right](\Delta u_n-\Delta u_m). \quad (21)$$

Then, we can write $I_{n,m}$ in the form

$$\begin{aligned}I_{n,m} = &\left\langle J'_\theta(u_n)-J'_\theta(u_m),u_n-u_m\right\rangle \\ &+[B(x,u_n)-B(x,u_m)](u_n-u_m)dx.\end{aligned} \quad (22)$$

In addition, F_p verifies an algebraic relation [1], from where

$$|\Delta u_n-\Delta u_m|^p \le C\left\{\left[F_p(\Delta u_n)-F_p(\Delta u_m)\right](\Delta u_n-\Delta u_m)\right\}, \quad (23)$$

thus,

$$\|u_n-u_m\|^p \le CI_{n,m}, \quad (24)$$

and therefore

$$\lim_{n,m\longrightarrow+\infty}\|u_n-u_m\|=0. \quad (25)$$

Then, (u_n) converges strongly in E. □

Lemma 7. *Suppose $4<2p<N$, $0\in\Omega, p<\gamma<p^*$ with $p^*:=pN/(N-2p)$ is the critical Sobolev exponent, $0\le\beta<N(\gamma+\alpha)/(\gamma+1)$, $0<\alpha<1$ and $\mu\le0$; then, there exist ε and η positive constants such that*

(i) We have

$$J_\theta(u)\ge\eta>0 \text{ when } \varepsilon=\|u\| \text{ small.} \quad (26)$$

(ii) *There exists $v \in \mathcal{M}_\rho$ when $\|v\| > \varepsilon$, with $\varepsilon = \|u\|$, such that $J_\theta(v) \le 0$.*

Proof.

(i) We have

$$\int_\Omega |u|^{\gamma+1} dx \le \|u\|^{\gamma+1} S^{-(\gamma+1)/p}, \\ \int_\Omega \frac{|u|^{1-\alpha}}{|x|^\beta} dx \le \|u\|^{1-\alpha} A S_1^{(-(1-\alpha))/p}. \quad (27)$$

Then, we get

$$J_\theta(u) \ge \frac{1}{p}\|u\|^p - \frac{S^{-(\gamma+1)/p}}{\gamma+1}\|u\|^{\gamma+1} - \mu \frac{A S_1^{(-(1-\alpha))/p}}{1-\alpha}\|u\|^{1-\alpha}. \quad (28)$$

$J_\theta(u) \ge \eta > 0$ when $\varepsilon = \|u\|$ small.

(ii) $J_\theta(0) = 0$ and $J_\theta(w) > 0$ for an element $w \ge 0\,;\, w \not\equiv 0$; then,

$$\lim_{t \longrightarrow +\infty} J(tw) = -\infty, \quad (29)$$

Indeed
Let t be such that $t > 0$, (t sufficiently large)

$$J_\theta(tw) = \frac{t^p}{p}\int_\Omega |\Delta w|^p dx - \frac{t^{\gamma+1}}{\gamma+1}\int_\Omega |w|^{\gamma+1} dx \\ + - \mu \frac{t^{1-\alpha}}{1-\alpha}\int_\Omega \frac{(w+\theta)^{1-\alpha} - \theta^{1-\alpha}}{|x|^\beta} dx. \quad (30)$$

However $\gamma > p$, we obtain

$$\lim_{t \longrightarrow +\infty} J_\theta(tw) = -\infty. \quad (31)$$

Owing to the fact that J_θ is continuous, then there exists $v_0 \ne 0$ such that $J_\theta(v_0) < 0$. □

Proof of Theorem 8. From Lemmas 5–7, we deduce that there exists $u \in E - \{0\}$ such that

$$\int_\Omega F_p(\Delta u).\Delta v - \int_\Omega |u|^{\gamma-1} u v dx \\ - \mu \int_\Omega \frac{v}{(u+\theta)^\alpha |x|^\beta} dx = 0, \forall v \in E. \quad (32)$$

□

Finally, for every $\theta \in (0,1)$, problem (4) has solution $u_\theta \in \mathcal{H}$ such that $J_\theta(u_\theta) = 0$. Thus, there exist $\{\theta_n\} \subset (0,1)$ with $\theta_n \longrightarrow 0$ as $n \longrightarrow \infty$. Then, we get $u = \lim_{n\longrightarrow\infty} u_{\theta_n}$.

4. Nonexistence Result

By a Pohozaev-type identity, one shows the nonexistence of positive solutions of (1) when $0 \le \mu < \mu_* = (N-2p)(1-\alpha)/Np$ and $\beta = 0$, with $\gamma \ge \gamma_* = Np(1-\alpha)/((N-2p)(1-\alpha) - \mu Np) > p^* = Np/(N-2p)$.

The Proof Theorem 10 uses an identity of the Pohozaev type which we state in the following Lemma.

Lemma 9. *Let $u \in E$ be a positive solution of (1).*
Then, the following identity is checked

$$\left(\frac{N-2p}{p}\right)\int_\Omega |\Delta u|^p dx - \frac{N}{\gamma}\int_\Omega |u|^\gamma dx - \frac{N\mu}{1-\alpha}\int_\Omega |u|^{1-\alpha} dx \\ = -\left(\frac{p-1}{p}\right)\int_{\partial\Omega} |\Delta u|^p \left(x \cdot \vec{n}\right) d\sigma, \quad (33)$$

where $\vec{n}$ is the normal exterior at $\partial\Omega$.

Proof. Multiplying Equation (1) by $(x \cdot \nabla u)$ and integrating on Ω, we obtain

$$\int_\Omega g(u)(x \cdot \nabla u) dx = \int_\Omega (\Delta w)(x \cdot \nabla u) dx \text{ where } w \\ = |\Delta u|^{p-2}\Delta u \text{ and } g(u) = |u|^\gamma + |u|^{1-\alpha}. \quad (34)$$

Set

$$A_1(u) = \int_\Omega g(u)(x \cdot \nabla u) dx \text{ and } A_2(u) = \int_\Omega (\Delta w)(x \cdot \nabla u) dx. \quad (35)$$

Calculation of $A_1(u)$

$$\int_\Omega g(u)(x.\nabla u) dx = \sum_{j=1}^N \int_\Omega g(u)\left(x_j \frac{\partial u}{\partial x_j}\right) d\sigma, \\ = \sum_{j=1}^N \int_\Omega \frac{\partial}{\partial x_i}(G(u)) x_j d\sigma. \quad (36)$$

According to the divergence theorem,

$$\sum_{j=1}^N \int_\Omega \frac{\partial}{\partial x_i}(G(u)) x_j d\sigma = -\sum_{j=1}^N \int_\Omega G(u) dx + \sum_{j=1}^N \int_\Omega G(u) x_j \vec{n}_j d\sigma, \\ = -N\int_\Omega G(u) dx + \sum_{j=1}^N \int_{\partial\Omega} G(u)\left(x.\vec{n}\right) d\sigma. \quad (37)$$

However,

$$u = 0 \text{ on } \partial\Omega \Rightarrow G(u) = 0 \text{ on } \partial\Omega, \quad (38)$$

from where one deduces from (36) and (37) that

$$A_1(u) = -N\int_\Omega G(u)dx = \frac{-N}{\gamma}\int_\Omega |u|^\gamma dx - \frac{N\mu}{1-\alpha}\int_\Omega |u|^{1-\alpha}dx. \tag{39}$$

Calculation of $A_2(u)$

In [19], Mitidieri established the following relation:

$$\begin{aligned}\int_\Omega \{\Delta v(x\cdot\nabla u) + \Delta u(x\cdot\nabla v)\}dx &= \int_{\partial\Omega}\left\{\frac{\partial v}{\partial \vec{n}}(x\cdot\nabla u) + \frac{\partial u}{\partial \vec{n}}(x\cdot\nabla v) - (\nabla u\nabla v)\left(x\cdot\vec{n}\right)\right\}d\sigma \\ &\quad + (N-2)\int_\Omega(\nabla u\nabla v)dx.\end{aligned} \tag{40}$$

Let us apply to our case with $v = w = |\Delta u|^{p-2}\Delta u$

$$\begin{aligned}A_2(u) &= \int_\Omega\left\{\frac{\partial w}{\partial \vec{n}}(x\cdot\nabla u) + \frac{\partial u}{\partial \vec{n}}(x\cdot\nabla w)\right\}d\sigma \\ &\quad + -\int_{\partial\Omega}(\nabla u\nabla w)\left(x\cdot\vec{n}\right)d\sigma - \int_\Omega \Delta u(x\cdot\nabla w) \\ &\quad + (N-2)\int_\Omega(\nabla u\nabla w)dx.\end{aligned} \tag{41}$$

Set

$$\begin{aligned}H_1 &= \int_{\partial\Omega}\left\{\frac{\partial w}{\partial \vec{n}}(x\cdot\nabla u) + \frac{\partial u}{\partial \vec{n}}(x\cdot\nabla w)\right\}d\sigma, \\ H_2 &= \int_{\partial\Omega}(\nabla u\nabla w)\left(x\cdot\vec{n}\right)d\sigma, \\ H_3 &= \int_\Omega \Delta u(x\cdot\nabla w), \text{ and } H_4 = \int_\Omega(\nabla u\nabla w)dx.\end{aligned} \tag{42}$$

Owing to the fact that $u = \nabla u = 0$ on $\partial\Omega$,
we have

$$H_1 = \int_{\partial\Omega}\left\{\frac{\partial w}{\partial \vec{n}}(x\cdot\nabla u) + \frac{\partial u}{\partial \vec{n}}(x\cdot\nabla w)\right\}d\sigma = 0, \tag{43}$$

$$H_2 = \int_{\partial\Omega}(\nabla u\nabla w)\left(x\cdot\vec{n}\right)d\sigma = 0. \tag{44}$$

Calculations of H_3

We have

$$\begin{aligned}H_3 &= \int_\Omega \Delta u(x\cdot\nabla w)dx, \\ &= \int_\Omega x.\left[\nabla\left(\frac{p-1}{p}\right)|\Delta u|^p\right]dx,\end{aligned} \tag{45}$$

by applying the divergence theorem, we obtain

$$H_3 = \left(\frac{p-1}{p}\right)\int_{\partial\Omega}|\Delta u|^p\left(x\cdot\vec{n}\right)d\sigma - \frac{N(p-1)}{p}\int_\Omega|\Delta u|^p dx. \tag{46}$$

Calculations of H_4

Applying Green's formula generalized, we obtain

$$\begin{aligned}H_4 &= \int_\Omega(\nabla u\nabla w)dx, \\ &= -\int_\Omega w\Delta u dx,\end{aligned} \tag{47}$$

therefore,

$$H_4 = -\int_\Omega|\Delta u|^p dx. \tag{48}$$

From (43), (44), (46), and (48), we obtain

$$\begin{aligned}A_2(u) &= -\frac{(p-1)}{p}\int_{\partial\Omega}|\Delta u|^p\left(x\cdot\vec{n}\right)d\sigma + \frac{N(p-1)}{p}\int_\Omega|\Delta u|^p dx \\ &\quad + -(N-2)\int_\Omega|\Delta u|^p dx.\end{aligned} \tag{49}$$

Thus,

$$A_2(u) = -\left(\frac{N-2p}{p}\right)\int_\Omega|\Delta u|^p dx - \frac{(p-1)}{p}\int_{\partial\Omega}|\Delta u|^p\left(x\cdot\vec{n}\right)d\sigma. \tag{50}$$

From (39) and (50), we deduces that

$$\begin{aligned}&\left(\frac{N-2p}{p}\right)\int_\Omega|\Delta u|^p dx - \frac{N}{\gamma}\int_\Omega|u|^\gamma dx - \frac{N\mu}{1-\alpha}\int_\Omega|u|^{1-\alpha}dx \\ &\quad = -\left(\frac{p-1}{p}\right)\int_{\partial\Omega}|\Delta u|^p\left(x\cdot\vec{n}\right)d\sigma,\end{aligned} \tag{51}$$

□

Proof of Theorem 10 (By the absurdity).

$$\begin{aligned}&\left(\frac{N-2p}{p}\right)\int_\Omega|\Delta u|^p dx - \frac{N}{\gamma}\int_\Omega|u|^\gamma dx - \frac{N\mu}{1-\alpha}\int_\Omega|u|^{1-\alpha}dx \\ &\quad = -\left(\frac{p-1}{p}\right)\int_{\partial\Omega}|\Delta u|^p\left(x\cdot\vec{n}\right)d\sigma,\end{aligned} \tag{52}$$

thus, we obtain

$$\left(\frac{N-2p}{p}-\frac{N}{\gamma}-\frac{N\mu}{1-\alpha}\right)\int_{\Omega}|\Delta u|^{p}dx = -\frac{(p-1)}{p}\int_{\partial\Omega}|\Delta u|^{p}\left(x\cdot\vec{n}\right)d\sigma. \tag{53}$$

Knowing that Ω is strictly star-shaped then,

$$\frac{N-2p}{p}-\frac{N}{\gamma}-\frac{N\mu}{1-\alpha}<0, \tag{54}$$

what contradicts the fact that $\gamma \geq \gamma_{*} = Np(1-\alpha)/((N-2p)(1-\alpha)-\mu Np) > p^{*} = Np/(N-2p)$ when $0 \leq \mu < \mu_{*} = (N-2p)(1-\alpha)/Np$. □

5. Conclusion

In our work, we have searched the critical points as the minimizers of the energy functional associated to the problem. Under some sufficient conditions on coefficients of equation of (1) such that $4 < 2p < N$, $0 \in \Omega, p < \gamma < p^{*}$ with $p^{*} := pN/(N-2p)$ is the critical Sobolev exponent, $0 \leq \beta < N(\gamma+\alpha)/(\gamma+1)$, $0 < \alpha < 1$ and $\mu \leq 0$. In Section 3, we have shown the existence of at least one nontrivial solution in E. In the Section 4, the nonexistence of positive solution at (1) has proved when $\Omega \subset \mathbb{R}^{N}$ can be a bounded, smooth, and stricly star-shaped domain, $0 \leq \mu < \mu_{*} = (N-2p)(1-\alpha)/Np$ and $\beta = 0$, with $\gamma \geq \gamma_{*} = Np(1-\alpha)/((N-2p)(1-\alpha) - \mu Np) > p^{*} = Np/(N-2p)$.

Data Availability

The author declares that the data supporting the findings of this study are available within the article.

Conflicts of Interest

The author declares that they have no conflicts of interest.

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