Hindawi
Journal of Function Spaces
Volume 2023, Article ID 8001131, 16 pages
https://doi.org/10.1155/2023/8001131

Research Article

Quantitative Weighted Bounds for Littlewood-Paley Functions Generated by Fractional Heat Semigroups Related with Schrödinger Operators

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Received 19 December 2022; Revised 8 March 2023; Accepted 10 March 2023; Published 24 March 2023

Academic Editor: Andrea Scapellato

Let $L=-\Delta+V$ be a Schrödinger operator on $\mathbb{R}^n$, where Δ denotes the Laplace operator $\sum_{i=1}^n \partial^2/\partial x_i^2$ and V is a nonnegative potential belonging to a certain reverse Hölder class $RH_q(\mathbb{R}^n)$ with $q>n/2$. In this paper, by the regularity estimate of the fractional heat kernel related with L, we establish the quantitative weighted boundedness of Littlewood-Paley functions generated by fractional heat semigroups related with the Schrödinger operators.

1. Introduction

Let $L=-\Delta+V$ be a Schrödinger operator, where Δ denotes the Laplace operator $\sum_{i=1}^n \partial^2/\partial x_i^2$ and the potential $V\geq 0$. The Schrödinger operators originate from the famous Schrödinger equations which are used to describe the microparticle state in quantum mechanics. Due to the profound background in mathematic physics, the theory of Schrödinger operators has attracted the attention of a lot of mathematicians (see Simon [1] for a summary on the development of Schrödinger operators). Since 1990s, by the aid of the technology of PDEs and the functional analysis, there has been a considerable development on the theory of harmonic analysis associated with Schrödinger operators. In 1995, under the assumption that the potential $V\in RH_q, q>n/2$, Shen [2] gave the estimates of the fundamental solution for L. Here, a locally L^q-integrable function V is said to belong to $V\in RH_q$ on $\mathbb{R}^n$, $1<q<\infty$, if there exists $C>0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V^q(y)dy\right)^{1/q}\leq \frac{C}{|B|}\int_B V(y)dy \tag{1}$$

holds. The analogue for magnetic Schrödinger operators and the generalized Schrödinger operators are established by Shen in [3, 4], respectively. In 1999, via the semigroup maximal function, Dziubański and Zienkiewicz [5] introduced the Hardy space $H^1_L(\mathbb{R}^n)$ related with L, where $V\in RH_q$, $q>n/2$. By the aid of the local Hardy spaces, the atomic characterization and the Riesz transform characterization of $H^1_L(\mathbb{R}^n)$ were obtained in [5] (see also [6] for the theory of $H^p_L(\mathbb{R}^n)$). As the dual space of $H^1_L(\mathbb{R}^n)$, the bounded mean oscillation space related with L denoted by $BMO_L(\mathbb{R}^n)$ was introduced by Dziubański et al. in [7]. For further progress on this topic, we refer the reader to Bongioanni et al. [8, 9], Duong et al. [10], Duong et al. [11], Guo et al. [12], Li [13], Lin and Liu [14], Ma et al. [15], Tang [16], Yang et al. [17] and the references therein.

In recent years, the quantitative weighted bounds for operators in the Schrödinger settings have been investigated by many researchers. Li et al. [18] established the quantitative weighted boundedness of maximal functions, maximal heat semigroups, and fractional integral operators related to L. In 2020, Zhang and Yang [19] showed that the quantitative weighted boundedness for Littlewood-Paley functions in the Schrödinger setting. Bui et al. [20] investigated the quantitative boundedness for square functions with new class of weights. In [21], Bui et al. studied the quantitative estimates for some singular integrals associated with critical functions. Wen and Wu [22] obtained the quantitative weighted strong and weak-type estimates for variation operators related to heat semigroups associated with L.

Inspired by the results in [18, 19, 22], in this paper, we investigate the quantitative weighted boundedness of Littlewood-Paley functions generated by fractional heat semigroups associated with L. Let $K_t^L(\cdot,\cdot)$ denote the integral kernel of the heat semigroup $\{e^{-tL}\}_{t\geq 0}$. For $\alpha>0$, the subordinative formula (cf. [23]) indicates that there exists a continuous function $\eta_t^\alpha(\cdot)$ on $(0,\infty)$ such that the fractional heat kernel $K_{\alpha,t}^L(\cdot,\cdot)$ related with L can be represented as

$$K_{\alpha,t}^L(x,y)=\int_0^\infty \eta_t^\alpha(s)K_s^L(x,y)ds. \tag{2}$$

Let $\{e^{-tL^\alpha}\}_{t>0}$, $\alpha\in(0,1)$, be the fractional heat semigroup associated with L, i.e.

$$e^{-tL^\alpha}f(x)\coloneqq\int_{\mathbb{R}^n}K_{\alpha,t}^L(x,y)f(y)dy. \tag{3}$$

For the special case $V=0$, $K_{\alpha,t}^L(\cdot,\cdot)$ is exactly the fractional heat kernel related with the Laplace operator. Denote by $\hat{f}$ the Fourier transform of f. The fractional heat semigroup is defined via the Fourier multiplier:

$$\widehat{e^{-t(-\Delta)^\alpha}f}(\xi)=e^{-t|\xi|^{2\alpha}}\hat{f}(\xi), \tag{4}$$

with the integral kernel:

$$K_{\alpha,t}(x)\coloneqq(2\pi)^{-n/2}\int_{\mathbb{R}^n}e^{ix\cdot y-t|y|^{2\alpha}}dy. \tag{5}$$

In the literature, the fractional heat semigroups related with second-order differential operators have been widely used in the study of partial differential equations, harmonic analysis, potential theory, and modern probability theory. For example, the semigroup $\{e^{-t(-\Delta)^\alpha}\}_{t>0}$ is usually applied to construct the linear part of solutions to fluid equations in the mathematic physics, e.g., the generalized Navier-Stokes equation, the quasigeostrophic equation, and the generalized MHD equations. In the field of probability theory, the researchers use $\{e^{-t(-\Delta)^\alpha}\}_{t>0}$ to describe some kind of Markov processes with jumps.

Let $f\in L^2(\mathbb{R}^n)$, $m\in\mathbb{Z}_+$, $\alpha\in(0,1)$, and $x\in\mathbb{R}^n$. Via the higher-order derivatives of $K_{\alpha,t}^L(\cdot,\cdot)$ in the time variable t, the Littlewood-Paley type functions associated with $\{e^{-tL^\alpha}\}_{t>0}$ are defined, respectively, as

$$g_{L^\alpha,m}(f)(x)\coloneqq\left(\int_0^\infty\left|t^m\partial_t^m e^{-tL^\alpha}(f)(x)\right|^2\frac{dt}{t}\right)^{1/2}, \tag{6}$$

$$S_{L^\alpha,m,\beta}(f)(x)\coloneqq\left(\int_0^\infty\int_{|x-y|<\beta t}\left|t^{2\alpha m}\partial_s^m e^{-sL^\alpha}(f)(y)|_{s=t^{2\alpha}}\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2},\quad \beta\in(0,\infty), \tag{7}$$

$$g_{L^\alpha,m,\lambda}^*(f)(x)\coloneqq\left(\int_0^\infty\int_{\mathbb{R}^n}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|t^{2\alpha m}\partial_s^m e^{-sL^\alpha}(f)(y)|_{s=t^{2\alpha}}\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2},\quad \lambda\in(0,\infty). \tag{8}$$

Specially, if $\beta=1$ in (7), we write $S_{L^\alpha,m,\beta}$ as $S_{L^\alpha,m}$. If $V=0$ and $L=-\Delta$ in (6) and (7), then $g_{(-\Delta)^\alpha,m}$ and $S_{(-\Delta)^\alpha,m,\beta}$ related to $-\Delta$ are the classical Littlewood-Paley type functions.

Let $\nabla_x\coloneqq(\partial/\partial_{x_1},\partial/\partial_{x_2},\cdots,\partial/\partial_{x_n})$. Similarly, we can introduce the Littlewood-Paley type functions associated with the spatial gradient of $K_{\alpha,t}^L(\cdot,\cdot)$ as follows:

$$\tilde{g}_{L^\alpha}(f)(x)\coloneqq\left(\int_0^\infty\left|t^{1/(2\alpha)}\nabla_x e^{-tL^\alpha}(f)(x)\right|^2\frac{dt}{t}\right)^{1/2}, \tag{9}$$

$$\tilde{S}_{L^\alpha,\beta}(f)(x)\coloneqq\left(\int_0^\infty\int_{|x-y|<\beta t}\left|t\nabla_x e^{-t^{2\alpha}L^\alpha}(f)(y)\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2},\beta\in(0,\infty), \tag{10}$$

$$\tilde{g}_{L^\alpha,\lambda}^*(f)(x)\coloneqq\left(\int_0^\infty\int_{\mathbb{R}^n}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|t\nabla_x e^{-t^{2\alpha}L^\alpha}(f)(y)\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2},\lambda\in(0,\infty). \tag{11}$$

If $\beta=1$ in (10), we write $\tilde{S}_{L^\alpha,\beta}$ as $\tilde{S}_{L^\alpha}$. If $V=0$ and $L=-\Delta$ in (9) and (10), then $\tilde{g}_{(-\Delta)^\alpha}$ and $\tilde{S}_{(-\Delta)^\alpha,\beta}$ related to $-\Delta$ are the classical Littlewood-Paley-type functions.

We point out that the quantitative weighted boundedness of Littlewood-Paley functions generated by $\{e^{-tL^\alpha}\}_{t>0}$ is not a simple analogue of [19] which deals with the case $\alpha=1$. For $\alpha=1$, it is well known that the heat kernel $K_t(\cdot)$ of the Laplace operator has a good decay properties. Precisely, $K_t(x)=c_n t^{-n/2}e^{-|x|^2/t}$, which indicates that it can be dominated as follows.

$$K_t(x)\leq\frac{C}{t^{n/2}}\frac{1}{\left(1+|x|^2/t\right)^{(n+\epsilon)}}\quad\forall N>0. \tag{12}$$

The arbitrariness of N ensures that the square functions generated with $\{e^{-t(-\Delta)}\}$ can be dominated by the intrinsic Lusin area function: for $\epsilon\in(1,\infty)$ (see [19], Lemma 3.1).

$$g_{-\Delta}(f)(x)\lesssim\tilde{g}_{(1,\epsilon)}(f)(x)\lesssim\tilde{G}_{1,\epsilon}(f)(x)\lesssim G_1(f)(x). \tag{13}$$

For the case $\alpha\in(0,1)$, Miao et al. [24] obtained the following regularity estimate for $K_{\alpha,t}(\cdot)$: $|\nabla_x K_1(x)|\leq C(1+|x|)^{-(n+1)}$ (see ([24], Remark 2.1). By this estimate, it can be only obtained that $g_{(-\Delta)^\alpha}(f)(x)\lesssim\tilde{g}_{(1,1)}(f)(x)$, which indicates that (13) does not hold.

In Lemmas 11 and 12, adopting the subordinate formula (2), we obtain the following revised decay estimate for $K_{\alpha,t}(\cdot)$: there exists a constant C such that

$$\left|t^m \partial_t^m K_{\alpha,t}(x-y)\right| + \left|t^{1/(2\alpha)} \nabla_x K_{\alpha,t}(x-y)\right| \le \frac{Ct}{\left(t^{1/(2\alpha)} + |x-y|\right)^{n+2\alpha}}. \tag{14}$$

The estimates (14) enable us to establish the quantitative weighted bounds for $g_{L^\alpha,m}$, $S_{L^\alpha,m}$, and $g^*_{L^\alpha,m,\lambda}$ with $\alpha \in (1/2, 1)$ (see Theorem 25). By the aid of the quantitative version of the extrapolation theorem in [19], we prove the quantitative weighted boundedness of function $g^*_{L^\alpha,m,\lambda}$ under the assumption that $\lambda \in (2(1+\theta/n),\infty)$, see Theorem 3.12.

The structure of the article is as follows. In Section 2, we give some symbol notations and estimates of kernels which will be used in the sequel. In Section 3, we investigate quantitative weighted bounds for Littlewood-Paley-type functions.

Some notations are as follows: In this article, $C_c^\infty(\mathbb{R}^n)$ denotes the set of all infinitely differential functions with compact supports. $B(x, r)$ is a ball centered at x and with radius r. Given $B = B(x, r)$ and $\lambda > 0$, we will write $\lambda B \coloneqq B(x, \lambda r)$. Unless otherwise indicated, we will denote by C a positive constant, which is different from range to range and depends on the main parameters.

For any constant $c > 1$, denote c' as the conjugate of c, i.e., $1/c' + 1/c = 1$. By $A \lesssim B$, we mean that there is a $C > 0$ such that $A \le CB$. $A \sim B$ means that there exist $C_1, C_2 > 0$ that satisfy $C_1 \le A/B \le C_2$. We write $\mathbb{Z}_+ \coloneqq \mathbb{N} \cup \{0\}$, where $\mathbb{N} \coloneqq \{1, 2, \cdots\}$. For any set E on $\mathbb{R}^n$, we denote $E^c \coloneqq \mathbb{R}^n \setminus E$ and χ_E its characteristic function.

2. Preliminaries

2.1. Some Notations. In this article, a weight denotes a nonnegative locally integrable function. Given a Lebesgue measurable set E and a weight ω, the symbol $|E|$ denotes the Lebesgue measure of E and

$$\omega(E) \coloneqq \int_E \omega(x)dx. \tag{15}$$

For $0 < p < \infty$, the weighted Lebesgue space $L^p(\mathbb{R}^n, \omega)$ is defined as the set of all measurable functions satisfying

$$\|f\|_{L^p(\omega)} \coloneqq \left(\int_{\mathbb{R}^n} |f(y)|^p \omega(y)dy\right)^{1/p} < \infty. \tag{16}$$

In [25], to estimate the fundamental solution of Schrödinger operators, Shen first introduced the following auxiliary function $\rho(\cdot)$.

Definition 1. Assume that $V(\cdot) \in RH_q$ for some $q > n/2$. The function $\rho(\cdot)$ is defined by

$$\rho(x) \coloneqq \sup_{r>0} \left\{ r : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \le 1 \right\}, x \in \mathbb{R}^n. \tag{17}$$

Obviously, $0 < \rho(x) < \infty$ if $V \ne 0$.

The following properties of $\rho(\cdot)$ were obtained by Shen [2].

Lemma 2 (see [2], Lemma 1.4). *Assume that $V \in RH_q$ for some $q > n/2$. There exist constants $N_0 > 0$, $C_1 > 1$, and $C > 0$ such that*

$$\frac{1}{C_1}(1 + |x-y|/\rho(x))^{-N_0} \le \frac{\rho(y)}{\rho(x)} \le C_1(1 + |x-y|/\rho(x))^{N_0/(N_0+1)}. \tag{18}$$

In particular, $\rho(x) \sim \rho(y)$ if $|x-y| \le C\rho(x)$.

Lemma 3 (see [2], page 196). *Assume that $V \in RH_q$, $1 < q < \infty$. For any $r > 0$ and $x \in \mathbb{R}^n$, there exists a constant C_0 such that*

$$\int_{B(x,2r)} V(y)dy \le C_0 \int_{B(x,r)} V(y)dy. \tag{19}$$

In [8], Bongioanni et al. introduced a new class $A_p^{\rho,\theta}(\mathbb{R}^n)$ of weights related with ρ.

Definition 4. Let $p \in (1,\infty)$, $\theta \in [0,\infty)$, and $n \ge 3$. $A_p^{\rho,\theta}(\mathbb{R}^n)$ is defined as the set of all weights $\omega(\cdot)$ satisfying any ball $B = B(x, r)$:

$$[\omega]_{A_p^{\rho,\theta}(\mathbb{R}^n)} \coloneqq \sup_{B\subset\mathbb{R}^n} \left(\frac{1}{\Psi_\theta(B)|B|}\int_B \omega(y)dy\right)\left(\frac{1}{\Psi_\theta(B)|B|}\int_B \omega^{-1/(p-1)}(y)dy\right)^{p/p'} < \infty, \tag{20}$$

where $\Psi_\theta(B) \coloneqq (1 + r/\rho(x))^\theta$.

Definition 5. The classical Hardy-Littlewood maximal operator M is defined by setting, for any $f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$:

$$Mf(x) \coloneqq \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|}\int_B |f(y)|dy, \tag{21}$$

where the supremum is taken over all balls B of $\mathbb{R}^n$ containing x.

Definition 6. Let $\theta \in [0,\infty)$. The Hardy-Littlewood-type maximal operator M_θ related to L is defined by setting, for any $f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$:

$$M_\theta f(x) \coloneqq \sup_{B\ni x} \frac{1}{\Psi_\theta(B)|B|}\int_B |f(y)|dy, \tag{22}$$

where the supremum is taken over all balls B of $\mathbb{R}^n$ containing x.

In [18], Li et al. obtained the quantitative estimate for the maximal operator M_θ.

Lemma 7 (see [18], Theorem 1.3). *Let $n \geq 3$, $\theta \in [0,\infty)$, and $p \in (1,\infty)$. Then there exists a positive constant C such that for any $\omega \in A_p^{\rho,\eta}(\mathbb{R}^n)$ with $\eta := \theta/p'$ and $f \in L^p(\omega)$:*

$$\|M_\theta(f)\|_{L^p(\omega)} \leq C[\omega]_{A_p^{\rho,\eta}(\mathbb{R}^n)}^{1/(p-1)} \|f\|_{L^p(\omega)}. \quad (23)$$

Specially, if $\theta = 0$, the following result can be deduced from Lemma 7.

Corollary 8 (see [19]). *Let $n \geq 3$ and $p \in (1,\infty)$. There exists a positive constant C such that for any $\omega \in A_p(\mathbb{R}^n)$ and $f \in L^p(\omega)$*

$$\|M(f)\|_{L^p(\omega)} \leq C[\omega]_{A_p(\mathbb{R}^n)}^{1/(p-1)} \|f\|_{L^p(\omega)}. \quad (24)$$

Zhang and Yang [19] established the quantitative version of the extrapolation theorem for $A_p^{\rho,\theta}(\mathbb{R}^n)$ weights.

Lemma 9 (see [19], Lemma 2.6). *Let $n \geq 3$, $p_0 \in (1,\infty)$, $\gamma \in [0,\infty)$ and T be an operator defined on $C_c^\infty(\mathbb{R}^n)$. Suppose that there exist positive constants C and σ such that for any $\omega \in A_{p_0}^{\rho,\gamma}(\mathbb{R}^n)$ and $f \in L^{p_0}(\omega)$*

$$\|T(f)\|_{L^{p_0}(\omega)} \leq C[\omega]_{A_{p_0}^{\rho,\gamma}(\mathbb{R}^n)}^{\sigma} \|f\|_{L^{p_0}(\omega)}. \quad (25)$$

Then for any $p \in (1,\infty)$, there exists a positive constant C such that for any $\omega \in A_p^{\rho,\theta}(\mathbb{R}^n)$

$$\theta = \begin{cases} \gamma/p, p > p_0; \\ \gamma/p', p < p_0; \end{cases} \quad (26)$$

and $f \in L^p(\omega)$

$$\|T(f)\|_{L^p(\omega)} \leq C[\omega]_{A_p^{\rho,\theta}(\mathbb{R}^n)}^{\sigma \max\{1,(p_0-1)/(p-1)\}} \|f\|_{L^p(\omega)}. \quad (27)$$

By the aid of Lemma 2.9, in [19], Zhang and Yang obtained the following conclusion.

Corollary 10 (see [19], Remark 2.7). *Let $\mathcal{F}$ be a given family of pairs (f, g) of nonnegative measurable functions on $\mathbb{R}^n$. Suppose that there exist positive constants C and σ such that for some fixed $p_0 \in [1,\infty)$ and for any $\omega \in A_{p_0}^{\rho,\gamma}(\mathbb{R}^n)$ with $\gamma \in [0,\infty)$*

$$\int_{\mathbb{R}^n} f(x)^{p_0}\omega(x)dx \leq C[\omega]_{A_{p_0}^{\rho,\gamma}(\mathbb{R}^n)}^{p_0\sigma} \int_{\mathbb{R}^n} g(x)^{p_0}\omega(x)dx \quad \forall (f,g) \in \mathcal{F}. \quad (28)$$

Then there exists a positive constant C such that for any $p \in (1,\infty)$ and $\omega \in A_p^{\rho,\theta}(\mathbb{R}^n)$ with θ satisfying (26)

$$\int_{\mathbb{R}^n} f(x)^p\omega(x)dx \leq C[\omega]_{A_p^{\rho,\theta}(\mathbb{R}^n)}^{p\sigma \max\{1,(p_0-1)/(p-1)\}} \int_{\mathbb{R}^n} g(x)^p\omega(x)dx \quad \forall (f,g) \in \mathcal{F}. \quad (29)$$

2.2. Estimations of Fractional Heat Kernels. In this subsection, we list some estimations of kernels used in the proof. Denote by $K_{\alpha,t}^L(\cdot,\cdot)$ the integral kernel of fractional heat semigroups $\{e^{-tL^\alpha}\}_{t>0}$ with $\alpha \in (0,1)$. For the special case $V = 0$, $L = -\Delta$, we denote by $K_{\alpha,t}(\cdot)$ the integral kernel of fractional heat semigroups $\{e^{-t(-\Delta)^\alpha}\}_{t>0}$ with $\alpha \in (0,1)$. In ([24], Lemma 2.1), Miao et al. proved that the kernel $K_{\alpha,t}(\cdot)$ satisfies the following pointwise estimate.

$$K_{\alpha,t}(x) \lesssim \frac{t}{\left(t^{1/(2\alpha)} + |x|\right)^{n+2\alpha}} \quad \forall (x,t) \in \mathbb{R}_+^{n+1}. \quad (30)$$

We denote by $K_t^L(\cdot,\cdot)$ the integral kernel of heat semigroup $\{e^{-tL}\}_{t>0}$. Specially, when $V = 0$ and $L = -\Delta$, we write $K_t(\cdot)$ as the integral kernel of heat semigroup $\{e^{-t(-\Delta)}\}_{t>0}$. The Feynman-Kac formula implies that

$$0 \leq K_t^L(x,y) \leq K_t(x-y) := \frac{1}{(4\pi t)^{n/2}} e^{-|x-y|^2/(4t)}. \quad (31)$$

Now we give the regularity estimate of the fractional heat kernel $K_{\alpha,t}(\cdot)$.

Lemma 11. *Let $\alpha \in (0,1)$, $m \in \mathbb{Z}_+$, and $V \in RH_q$ for some $q > n/2$. For all $x, y \in \mathbb{R}^n$, there exists a positive constant C such that for $t > 0$*

$$\left|t^m \partial_t^m K_{\alpha,t}(x-y)\right| \leq \frac{Ct}{\left(t^{1/(2\alpha)} + |x-y|\right)^{n+2\alpha}}. \quad (32)$$

Proof. Since $\{e^{-t(-\Delta)^\alpha}\}_{t>0}$, $\alpha \in (0,1)$, is a analytic semigroup, then the estimate for $|t^m \partial_t^m K_{\alpha,t}(\cdot)|$ can be deduced by that of $K_{\alpha,t}(\cdot)$ and the Cauchy integral formula. We omit the details. □

Lemma 12. *Let $\alpha \in (0,1)$ and $V \in RH_q$ for some $q > n$. For all $x, y \in \mathbb{R}^n$, there exists a constant $C > 0$ such that for all x, $y \in \mathbb{R}^n$ and $t > 0$*

$$\left|t^{1/(2\alpha)} \nabla_x K_{\alpha,t}(x-y)\right| \leq \frac{Ct}{\left(t^{1/(2\alpha)} + |x-y|\right)^{n+2\alpha}}. \quad (33)$$

Proof. The subordinate formula gives

$$\nabla_x K_{\alpha,t}(x) = \int_0^\infty \eta_t^\alpha(s) \nabla_x K_s(x) ds, \quad (34)$$

where $\eta_t^\alpha(\cdot)$ satisfies (cf. [23])

$$\begin{cases} \eta_t^\alpha(s) = \dfrac{1}{t^{1/\alpha}}\eta_1^\alpha\left(s/t^{1/\alpha}\right); \\ \eta_t^\alpha(s) \lesssim \dfrac{t}{s^{1+\alpha}} \quad \forall s, t > 0; \\ \displaystyle\int_0^\infty s^{-\gamma}\eta_1^\alpha(s)ds < \infty, \gamma > 0; \\ \eta_t^\alpha(s) \simeq \dfrac{t}{s^{1+\alpha}}, \quad \forall s \geq t^{1/\alpha} > 0. \end{cases} \tag{35}$$

A direct computation gives

$$\left|\nabla_x K_t(x)\right| \lesssim \frac{1}{t^{(n+1)/2}} e^{-c|x|^2/t}. \tag{36}$$

On the one hand, by (35) and the change of variables: $s = t^{1/\alpha}u$ and $r^2 = |x|^2/(t^{1/\alpha}u)$, we obtain

$$\begin{aligned} \left|\nabla_x K_{\alpha,t}(x)\right| &\leq \int_0^\infty \frac{t}{s^{1+\alpha}}\frac{1}{s^{(n+1)/2}} e^{-c|x|^2/s} ds \\ &= \int_0^\infty \frac{t}{\left(t^{1/\alpha}u\right)^{1+\alpha+n+1/2}} e^{-c|x|^2/\left(t^{1/\alpha}u\right)} t^{1/\alpha} du \\ &\leq \int_0^\infty \frac{1}{t^{n+1/2\alpha}}\frac{1}{u^{1+\alpha+n+1/2}} e^{-c|x|^2/t^{1/\alpha}u} du \\ &\lesssim \int_0^\infty \frac{1}{t^{n+1/2\alpha}}\left(\frac{t^{1/\alpha}r^2}{|x|^2}\right)^{1+\alpha+n+1/2} e^{-cr^2}\frac{|x|^2}{t^{1/\alpha}r^3} dr \\ &\lesssim \frac{t}{|x|^{2\alpha+n+1}}\int_0^\infty \left(\frac{t^{1/\alpha}r^2}{|x|^2}\right)^{1+\alpha+n+1/2} e^{-cr^2}\frac{|x|^2}{t^{1/\alpha}r^3} dr \\ &\lesssim \frac{t}{|x|^{2\alpha+n+1}}\left\{\int_0^1 r^{2\alpha+n+1}dr + \int_1^\infty r^{2\alpha+n+1}e^{-cr^2}dr\right\} \\ &\lesssim \frac{t}{|x|^{2\alpha+n+1}}. \end{aligned} \tag{37}$$

On the other hand, we can get

$$\begin{aligned} \left|\nabla_x K_{\alpha,t}(x)\right| &\leq \int_0^\infty \frac{1}{t^{1/\alpha}}\eta_1^\alpha\left(\frac{s}{t^{1/\alpha}}\right)\frac{1}{s^{(n+1)/2}} ds \\ &\leq \int_0^\infty \eta_1^\alpha(u)\frac{1}{\left(ut^{1/\alpha}\right)^{(n+1)/2}} du \lesssim \frac{1}{t^{(n+1)/(2\alpha)}}. \end{aligned} \tag{38}$$

Thus, it is easy to see that

$$\left|\nabla_x K_{\alpha,t}(x)\right| \lesssim \min\left\{\frac{t}{|x|^{2\alpha+n+1}}, \frac{1}{t^{(n+1)/(2\alpha)}}\right\}. \tag{39}$$

Case 1: $0 \leq t^{1/(2\alpha)} \leq |x|$. We have $t/|x|^{2\alpha+n+1} \leq 1/t^{(n+1)/(2\alpha)}$. Moreover, it leads to

$$\left|\nabla_x K_{\alpha,t}(x)\right| \lesssim \frac{t}{|x|^{2\alpha+n+1}} \lesssim \frac{t}{|x|^{2\alpha+n}}\frac{1}{t^{1/(2\alpha)}} \lesssim \frac{t}{\left(t^{1/(2\alpha)} + |x|\right)^{2\alpha+n}}\frac{1}{t^{1/(2\alpha)}}. \tag{40}$$

Case 2: $t^{1/(2\alpha)} > |x|$. It holds

$$\left|\nabla_x K_{\alpha,t}(x)\right| \lesssim \frac{1}{t^{(n+1)/(2\alpha)}} = \frac{t}{\left(t^{1/(2\alpha)}\right)^{n+2\alpha}}\frac{1}{t^{1/(2\alpha)}}. \tag{41}$$

□

Lemma 13 (see [26], Proposition 3.3). *Let $\alpha \in (0,1)$, $m \in \mathbb{Z}_+$, and $V \in RH_q$ for some $q > n/2$. For any $N > 0$ and all $x, y \in \mathbb{R}^n$, there exists a positive constant C_N such that for $t > 0$*

$$\left|t^m \partial_t^m K_{\alpha,t}^L(x,y)\right| \leq \frac{C_N t^m}{\left(t^{1/(2\alpha)} + |x-y|\right)^{n+2\alpha m}}\left(1 + \frac{t^{1/(2\alpha)}}{\rho(x)} + \frac{t^{1/(2\alpha)}}{\rho(y)}\right)^{-N}. \tag{42}$$

Lemma 14 (see [26], Proposition 3.6). *Let $\alpha \in (0,1)$, $V \in RH_q$ for some $q > n$ and $t > 0$. For any $N > 0$ and all $x, y \in \mathbb{R}^n$, there exists a constant $C_N > 0$ such that*

$$\left|t^{1/(2\alpha)}\nabla_x K_{\alpha,t}^L(x,y)\right| \leq \frac{C_N t}{\left(t^{1/(2\alpha)} + |x-y|\right)^{n+2\alpha}}\left(1 + \frac{t^{1/(2\alpha)}}{\rho(x)} + \frac{t^{1/(2\alpha)}}{\rho(y)}\right)^{-N}. \tag{43}$$

Lemma 15 (see [27], Proposition 7). *Let $\alpha \in (0,1)$, $m \in \mathbb{Z}_+$, and $t > 0$. For all $x, y \in \mathbb{R}^n$, there exist constants $C > 0$ and $\delta = 2 - n/q$ with $q > n/2$ such that*

$$\begin{aligned} &\left|t^m \partial_t^m K_{\alpha,t}^L(x,y) - t^m \partial_t^m K_{\alpha,t}(x-y)\right| \\ &\leq \begin{cases} C\left(\dfrac{|x-y|}{\rho(x)}\right)^\delta \dfrac{t}{\left(t^{1/(2\alpha)} + |x-y|\right)^{n+2\alpha}}, & t^{1/(2\alpha)} \leq |x-y|; \\ C\left(\dfrac{t^{1/(2\alpha)}}{\rho(x)}\right)^\delta \dfrac{t}{\left(t^{1/(2\alpha)} + |x-y|\right)^{n+2\alpha}}, & t^{1/(2\alpha)} \geq |x-y|. \end{cases} \end{aligned} \tag{44}$$

Lemma 16 (see [27], Proposition 6). *Let $\alpha \in (0,1)$ and $t > 0$. For all $x, y \in \mathbb{R}^n$, there exist constants $C > 0$ and $\delta = 2 - n/q$ with $q > n$ such that*

$$\begin{aligned} &\left|t^{1/(2\alpha)}\nabla_x K_{\alpha,t}^L(x,y) - t^{1/(2\alpha)}\nabla_x K_{\alpha,t}(x-y)\right| \\ &\leq \begin{cases} C\left(\dfrac{|x-y|}{\rho(x)}\right)^\delta \dfrac{t}{\left(t^{1/(2\alpha)} + |x-y|\right)^{n+2\alpha}}, & t^{1/(2\alpha)} \leq |x-y|; \\ C\left(\dfrac{t^{1/(2\alpha)}}{\rho(x)}\right)^\delta \dfrac{t}{\left(t^{1/(2\alpha)} + |x-y|\right)^{n+2\alpha}}, & t^{1/(2\alpha)} \geq |x-y|. \end{cases} \end{aligned} \tag{45}$$

3. Quantitative Boundedness

3.1. Localized Weights and Operators. Bongioanni et al. [8] introduced the following ρ-localized weights.

Definition 17. Let $n \geq 3$ and $p \in (1,\infty)$. The local weight class $A_p^{\rho,loc}(\mathbb{R}^n)$ is defined as the set of all nonnegative locally integrable functions $\omega(\cdot)$ satisfying any ball $B = B(x,r)$

$$[\omega]_{A_p^{\rho,loc}(\mathbb{R}^n)} \coloneqq \sup_{B\in\mathscr{B}_\rho} \left(\frac{1}{|B|}\int_B \omega(y)dy\right)\left(\frac{1}{|B|}\int_B \omega^{-1/(p-1)}(y)dy\right)^{p-1} < \infty, \tag{46}$$

where $\mathscr{B}_\rho \coloneqq \{B(x,r) : x \in \mathbb{R}^n, r \leq \rho(x)\}$.

Lemma 18 (see [5], Lemma 3). *There exists a sequence $\{x_j\}_{j\in\mathbb{N}}$ of points in $\mathbb{R}^n$ such that the family $\{B_j \coloneqq B(x_j,\rho(x_j))\}_{j\in\mathbb{N}}$ of balls satisfies that*

$$\bigcup_{j\in\mathbb{N}} B_j = \mathbb{R}^n. \tag{47}$$

For any $\tau \in [1,\infty)$, there exist positive constants C and N such that for any $x \in \mathbb{R}^n$, $\sum_{j\in\mathbb{N}} \chi_{\tau B_j}(x) \leq C\tau^N$.

We list some propositions about ρ-localized weights.

Lemma 19 (see [8], Proposition 3, Corollary 1 and Lemma 1). *Let $n \geq 3$ and $1 \leq p < \infty$.*

For any $\tau > 1$, $A_p^{\tau\rho,loc}(\mathbb{R}^n) = A_p^{\rho,loc}(\mathbb{R}^n)$ with $[\omega]_{A_p^{\rho,loc}} \sim [\omega]_{A_p^{\tau\rho,loc}}$.

Let $\theta \in [0,\infty)$ and $\omega \in A_p^{\rho,\theta}(\mathbb{R}^n)$. Then it is easy to see that $A_p^{\rho,\theta}(\mathbb{R}^n) \subset A_p^{\rho,loc}(\mathbb{R}^n)$, and there exists a constant $C > 0$ such that for any $\omega \in A_p^{\rho,\theta}(\mathbb{R}^n)$, $[\omega]_{A_p^{\rho,loc}}(\mathbb{R}^n) \leq C[\omega]_{A_p^{\rho,\theta}}(\mathbb{R}^n)$.

Let B_0 be a ball in $\mathbb{R}^n$. Assume that $\omega_0 \in A_p(B_0)$. Then ω_0 has an extension $\omega \in A_p(\mathbb{R}^n)$ on $\mathbb{R}^n$ such that for any $x \in B_0$, $\omega_0(x) = \omega(x)$ and $[\omega_0]_{A_p(B_0)} \sim [\omega]_{A_p(\mathbb{R}^n)}$, where the implicit positive equivalence constants are independent of ω_0 and p.

The radial maximal function and the nontangential maximal function related to $(-\Delta)^\alpha$ are defined, respectively, by setting, for any $f \in C_c^\infty(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$

$$\begin{aligned} \mathscr{M}_{(-\Delta)^\alpha}(f)(x) &\coloneqq \sup_{t\in(0,\infty)} \left|e^{-t(-\Delta)^\alpha}(f)(x)\right|, \\ M^*_{(-\Delta)^\alpha,\beta}(f)(x) &\coloneqq \sup_{(y,t)\in\Gamma_\beta(x)} \left|e^{-t^{2\alpha}(-\Delta)^\alpha}(f)(y)\right|, \end{aligned} \tag{48}$$

where $\alpha \in (0,1)$, $\beta \in (0,\infty)$, and $\Gamma_\beta(x) \coloneqq \{(y,t) \in \mathbb{R}_+^{n+1} : |x-y| < \beta t\}$ denotes the cone of aperture β with vertex x.

For any $x \in \mathbb{R}^n$, let $B_x = B(x,\rho(x))$. For any $\beta \in (0,\infty)$, $m \in \mathbb{Z}_+$, and $\alpha \in (0,1)$, we define the following ρ-localized $\mathscr{M}_{(-\Delta)^\alpha}$, $M^*_{(-\Delta)^\alpha,\beta}$, $g_{(-\Delta)^\alpha,m}$, $S_{(-\Delta)^\alpha,m,\beta}$, $\tilde{g}_{(-\Delta)^\alpha}$, and $\tilde{S}_{(-\Delta)^\alpha,\beta}$, respectively, by setting, for any $f \in L^1_{loc}(\mathbb{R}^n)$

$$\begin{cases} \mathscr{M}^{loc}_{(-\Delta)^\alpha} f(x) \coloneqq \mathscr{M}_{(-\Delta)^\alpha}\left(f\chi_{B_x}\right)(x); \\ M^{*,loc}_{(-\Delta)^\alpha,\beta} f(x) \coloneqq M^*_{(-\Delta)^\alpha,\beta}\left(f\chi_{2B_x}\right)(x); \\ g^{loc}_{(-\Delta)^\alpha,m} f(x) \coloneqq g_{(-\Delta)^\alpha,m}\left(f\chi_{B_x}\right)(x); \\ S^{loc}_{(-\Delta)^\alpha,m,\beta} f(x) \coloneqq S_{(-\Delta)^\alpha,m,\beta}\left(f\chi_{2B_x}\right)(x); \\ \tilde{g}^{loc}_{(-\Delta)^\alpha} f(x) \coloneqq \tilde{g}_{(-\Delta)^\alpha}\left(f\chi_{B_x}\right)(x); \\ \tilde{S}^{loc}_{(-\Delta)^\alpha,\beta} f(x) \coloneqq \tilde{S}_{(-\Delta)^\alpha,\beta}\left(f\chi_{2B_x}\right)(x). \end{cases} \tag{49}$$

Proposition 20. *Let $n \geq 3$, $\beta \in [1,\infty)$, $\alpha \in (0,1)$, and $p \in (1,\infty)$. There exists $C > 0$ such that for any $\omega \in A_p^{\rho,loc}(\mathbb{R}^n)$ and $f \in L^p(\omega)$*

$$\left\|\mathscr{M}^{loc}_{(-\Delta)^\alpha}(f)\right\|_{L^p(\omega)} \leq C[\omega]^{1/(p-1)}_{A_p^{\rho,loc}(\mathbb{R}^n)} \|f\|_{L^p(\omega)}, \tag{50}$$

$$\left\|M^{*,loc}_{(-\Delta)^\alpha,\beta}(f)\right\|_{L^p(\omega)} \leq C\beta^n[\omega]^{1/(p-1)}_{A_p^{\rho,loc}(\mathbb{R}^n)} \|f\|_{L^p(\omega)}. \tag{51}$$

Proof. We first prove (51). Let $\{B_j \coloneqq B(x_j,\rho(x_j))\}_{j\in\mathbb{N}}$ be the family of balls given in Lemma 18. Let $\tau = 1 + C_1 2^{1+N_0/(N_0+1)}$ with positive constants C_1 and N_0 same as in Lemma 2. For any $j \in \mathbb{N}$, let $\widetilde{B_j} \coloneqq \tau B_j = B(x_j,\tau\rho(x_j))$ and $\omega \in A_p^{\rho,loc}(\mathbb{R}^n)$. Via Lemma 2, for any $x \in B_j$, we have

$$2\rho(x) + \rho\left(x_j\right) \leq 2C_1 2^{N_0/(N_0+1)}\rho\left(x_j\right) + \rho\left(x_j\right) = \tau\rho\left(x_j\right), \tag{52}$$

that is, $2B_x = B(x,2\rho(x)) \subset \tilde{B}_j$ (cf. ([19], Lemma 3.6). For any $j \in \mathbb{N}$, $\omega|_{\tilde{B}_j}$ has an extension on $\mathbb{R}^n$ denoted by ω_j such that $\omega_j \in A_p(\mathbb{R}^n)$, for any $x \in \tilde{B}_j$, $\omega_j(x) = \omega(x)$ and

$$\left[\omega_j\right]_{A_p(\mathbb{R}^n)} \sim \left[\omega|_{\tilde{B}_j}\right]_{A_p\left(\tilde{B}_j\right)} \lesssim [\omega]_{A_p^{\rho,loc}(\mathbb{R}^n)}, \tag{53}$$

where the implicit positive constants are independent of *j*. We have

$$\begin{aligned} M^{*,loc}_{(-\Delta)^\alpha,\beta}(f)(x) &= M^*_{(-\Delta)^\alpha,\beta}\left(f\chi_{2B_x}\right)(x)\chi_{\bigcup_{j\in\mathbb{N}} B_j}(x) \\ &\leq \sum_{j=1}^\infty M^*_{(-\Delta)^\alpha,\beta}\left(f\chi_{\tilde{B}_j}\right)(x)\chi_{B_j}(x). \end{aligned} \tag{54}$$

It follows from (30) that

$$\begin{aligned} M^*_{(-\Delta)^\alpha,\beta}(f)(x) &= \sup_{(y,t)\in\Gamma_\beta(x)} \left|e^{-t^{2\alpha}(-\Delta)^\alpha}(f)(y)\right| \\ &\lesssim \sup_{(y,t)\in\Gamma_\beta(x)} \left|\int_{\mathbb{R}^n} \frac{t^{2\alpha}}{(t+|y-z|)^{n+2\alpha}} f(z)dz\right| \\ &\lesssim \sup_{(y,t)\in\Gamma_\beta(x)} \left\{\frac{t^{2\alpha}}{t^{n+2\alpha}}\int_{B(y,\beta t)} f(z)dz + \sum_{k=0}^\infty \frac{t^{2\alpha}}{\left(2^k\beta t\right)^{n+2\alpha}}\int_{|y-z|\sim 2^k\beta t} f(z)dz\right\} \\ &\lesssim \beta^n M(f)(x) + \sum_{k=0}^\infty \frac{1}{\left(2^k\beta\right)^{2\alpha}} M(f)(x) \lesssim \beta^n M(f)(x). \end{aligned} \tag{55}$$

For any $\omega \in A_p^{\rho,loc}(\mathbb{R}^n)$ and $f \in L^p(\omega)$, combining with Corollary 8, Lemma 18, and (53), we obtain

$$\begin{aligned}\left\|M^{*,loc}_{(-\Delta)^\alpha,\beta}(f)\right\|^p_{L^p(\omega)} &\le \int_{\mathbb{R}^n}\sum_{j\in\mathbb{N}}\chi_{B_j}(x)\left|M^*_{(-\Delta)^\alpha,\beta}\left(f\chi_{\tilde{B}_j}\right)(x)\right|^p\omega(x)dx\\ &\lesssim \sum_{j\in\mathbb{N}}\int_{B_j}\left|\beta^n M\left(f\chi_{\tilde{B}_j}\right)(x)\right|^p\omega(x)dx\\ &\lesssim \beta^{np}\sum_{j\in\mathbb{N}}\int_{\mathbb{R}^n}\left|M\left(f\chi_{\tilde{B}_j}\right)(x)\right|^p\omega_j(x)dx\\ &\lesssim \beta^{np}\sum_{j\in\mathbb{N}}\left[\omega_j\right]^{p/(p-1)}_{A_p(\mathbb{R}^n)}\int_{\tilde{B}_j}|f(x)|^p\omega_j(x)dx\\ &\lesssim \beta^{np}[\omega]^{p/(p-1)}_{A_p^{\rho,loc}(\mathbb{R}^n)}\int_{\mathbb{R}^n}|f(x)|^p\omega(x)dx.\end{aligned} \tag{56}$$

Next, we consider (50). Note that

$$\begin{aligned}\mathscr{M}_{(-\Delta)^\alpha}(f)(x) &= \sup_{t\in(0,\infty)}\left|e^{-t(-\Delta)^\alpha}(f)(x)\right|\\ &\lesssim \sup_{t\in(0,\infty)}\left|\int_{\mathbb{R}^n}\frac{t}{\left(t^{1/(2\alpha)}+|x-y|\right)^{n+2\alpha}}f(y)dy\right|\\ &\lesssim \sup_{t\in(0,\infty)}\left\{\frac{t}{t^{(n/2\alpha)+1}}\int_{B\left(x,t^{1/(2\alpha)}\right)}f(y)dy\right.\\ &\quad\left.+\sum_{k=0}^{\infty}\frac{t}{\left(2^k t^{1/(2\alpha)}\right)^{n+2\alpha}}\int_{|x-y|\sim 2^k t^{1/(2\alpha)}}f(y)dy\right\}\\ &\lesssim M(f)(x)+\sum_{k=0}^{\infty}\frac{1}{\left(2^k\right)^{2\alpha}}M(f)(x)\lesssim M(f)(x).\end{aligned} \tag{57}$$

The rest of the proof of (50) is similar to that of (51), so we omit the details.

Wilson [28] introduced the following intrinsic Littlewood-Paley functions. For any $r \in (0,1]$, let $\mathscr{C}_r(\mathbb{R}^n)$ be the set of all function ϕ defined on $\mathbb{R}^n$, such that

$$\begin{cases}\operatorname{supp}\ \phi \subset \{x\in\mathbb{R}^n : |x|\le 1\};\\ |\phi(x_1)-\phi(x_2)|\le |x_1-x_2|^r \quad \forall x_1,x_2\in\mathbb{R}^n;\\ \int_{\mathbb{R}^n}\phi(x)dx = 0.\end{cases} \tag{58}$$

For any $r \in (0,1]$, $\beta \in (0,\infty)$, and $f \in L^1_{loc}(\mathbb{R}^n)$, define

$$\begin{aligned} g_r(f)(x) &:= \left(\int_0^\infty (A_r(f)(x,t))^2\frac{dt}{t}\right)^{1/2},\\ G_{r,\beta}(f)(x) &:= \left(\int_0^\infty\int_{|x-y|<\beta t}(A_r(f)(y,t))^2\frac{dydt}{t^{n+1}}\right)^{1/2},\end{aligned} \tag{59}$$

where

$$A_r(f)(y,t) := \sup_{\phi\in\mathscr{C}_r(\mathbb{R}^n)}|\phi_t \times f(y)|, \tag{60}$$

with $\phi_t(x) = t^{-n}\phi(x/t)$. In particular, if $\beta = 1$, $G_{r,\beta} = G_r$.

For any $r \in (0,1]$, $\beta \in (0,\infty)$, and $\alpha \in (0,1)$, let $\mathscr{C}_{(r,2\alpha)}(\mathbb{R}^n)$ be the set of all functions ϕ defined on $\mathbb{R}^n$ such that for any $x, \tilde{x} \in \mathbb{R}^n$

$$\begin{cases}|\phi(x)|\le (1+|x|)^{-n-2\alpha};\\ |\phi(x)-\phi(\tilde{x})|\le |x-\tilde{x}|^r\left\{(1+|x|)^{-n-2\alpha}+(1+|\tilde{x}|)^{-n-2\alpha}\right\};\\ \int_{\mathbb{R}^n}\phi(x)dx = 0.\end{cases} \tag{61}$$

For any $r \in (0,1]$, $\beta \in (0,\infty)$, $\alpha \in (0,1)$, and $f \in C_c^\infty(\mathbb{R}^n)$, define

$$\begin{aligned}\tilde{g}_{(r,2\alpha)}(f)(x) &:= \left(\int_0^\infty\left(\tilde{A}_{(r,2\alpha)}(f)(x,t)\right)^2\frac{dt}{t}\right)^{1/2},\\ \tilde{G}_{(r,2\alpha),\beta}(f)(x) &:= \left(\int_0^\infty\int_{|x-y|<\beta t}\left(\tilde{A}_{(r,2\alpha)}(f)(y,t)\right)^2\frac{dydt}{t^{n+1}}\right)^{1/2},\end{aligned} \tag{62}$$

where

$$\tilde{A}_{(r,2\alpha)}(f)(y,t) := \sup_{\phi\in\mathscr{C}_{(r,2\alpha)}(\mathbb{R}^n)}|\phi_t\times f(y)|, \tag{63}$$

with $\phi_t(x) = t^{-n}\phi(x/t)$. In particular, if $\beta = 1$, $\tilde{G}_{(r,2\alpha),\beta} = \tilde{G}_{(r,2\alpha)}$.

Lerner [29] investigated the sharp A_p bounds for g_r and G_r. □

Lemma 21 (see [29], Theorem 1.1). *Let $p \in (1,\infty)$ and $r \in (0,1]$. There exists a constant $C > 0$ such that for any $\omega \in A_p(\mathbb{R}^n)$ and $f \in L^p(\omega)$*

$$\|g_r(f)\|_{L^p(\omega)} + \|G_r(f)\|_{L^p(\omega)} \le C[\omega]^{\max\{1/2,1/(p-1)\}}_{A_p(\mathbb{R}^n)}\|f\|_{L^p(\omega)}. \tag{64}$$

Lemma 22 (see [28], Theorem 6.3). *Let $0 < \bar{r} \le r$ and $\bar{r} < \epsilon$. There is a constant $C = C(r,\bar{r},\epsilon,n)$ such that, for all f having $|f|(1+|x|)^{-n-\epsilon} \in L^1(\mathbb{R}^n)$*

$$\tilde{G}_{(r,\epsilon)}(f) \le CG_{\bar{r}}(f), \tag{65}$$

pointwise.

Proposition 23. *Let $n \ge 3$, $\beta \in [1,\infty)$, $\alpha \in (1/2,1)$, $m \in \mathbb{Z}_+$, and $p \in (1,\infty)$. There exists $C > 0$ such that for any $\omega \in$*

$A_p^{\rho,loc}(\mathbb{R}^n)$ *and* $f\in L^p(\omega)$

$$\left\|g^{loc}_{(-\Delta)^\alpha,m}(f)\right\|_{L^p(\omega)}\le C[\omega]^{\max\{1/2,1/(p-1)\}}_{A_p^{\rho,loc}(\mathbb{R}^n)}\|f\|_{L^p(\omega)}, \quad (66)$$

$$\left\|S^{loc}_{(-\Delta)^\alpha,m,\beta}(f)\right\|_{L^p(\omega)}\le C\beta^{3n/2+1}[\omega]^{\max\{1/2,1/(p-1)\}}_{A_p^{\rho,loc}(\mathbb{R}^n)}\|f\|_{L^p(\omega)}. \quad (67)$$

Proof. For $S_{(-\Delta)^\alpha,m,\beta}$, we denote $\psi_t(x):=s^m\partial_s^m K_{\alpha,s}(x)$ with $s=t^{2\alpha}$. Thus

$$S_{(-\Delta)^\alpha,m,\beta}(f)(x)=\left(\int_0^\infty\int_{|x-y|<\beta t}|\psi_t\times f(y)|^2\frac{dydt}{t^{n+1}}\right)^{1/2},$$
$$g_{(-\Delta)^\alpha,m}(f)(x)=\left(\int_0^\infty|\psi_{t^{1/(2\alpha)}}\times f(x)|^2\frac{dt}{t}\right)^{1/2}\simeq\left(\int_0^\infty|\psi_\tau\times f(x)|^2\frac{d\tau}{\tau}\right)^{1/2}, \quad (68)$$

where $\tau=t^{1/(2\alpha)}$. On the one hand

$$\int_{\mathbb{R}^n}\psi_t(x)dx=s^m\partial_s^m\int_{\mathbb{R}^n}K_{\alpha,s}(x)|_{s=t^{2\alpha}}dx=0, \quad (69)$$

on the other hand

$$\int_{\mathbb{R}^n}\psi_t(x)dx=t^{-n}\int_{\mathbb{R}^n}\psi(x/t)dx=\int_{\mathbb{R}^n}\psi(u)du, \quad (70)$$

where $u=x/t$. Furthermore, it follows from Lemma 11 that

$$\psi_t(x)\lesssim\frac{1}{(1+|x|/t)^{n+2\alpha}}\frac{1}{t^n}\quad\text{and}\quad\psi(x)\lesssim\frac{1}{(1+|x|)^{n+2\alpha}}. \quad (71)$$

□

For any $x,\tilde{x}\in\mathbb{R}^n$, we divide into two cases.

Case 1: if $|x-\tilde{x}|>1$, we obtain

$$|\psi(x)-\psi(\tilde{x})|\le|\psi(x)|+|\psi(\tilde{x})|\le|x-\tilde{x}|\left\{(1+|x|)^{-n-2\alpha}+(1+|\tilde{x}|)^{-n-2\alpha}\right\}. \quad (72)$$

Case 2: if $|x-\tilde{x}|\le 1$, we consider $|x|\le 2$. Then $|\tilde{x}|<3$, $(1+|x|)^{-n-2\alpha}\sim 1$, and $(1+|\tilde{x}|)^{-n-2\alpha}\}\sim 1$. Hence

$$|\psi(x)-\psi(\tilde{x})|\lesssim|x-\tilde{x}|\lesssim|x-\tilde{x}|\left\{(1+|x|)^{-n-2\alpha}+(1+|\tilde{x}|)^{-n-2\alpha}\right\}. \quad (73)$$

When $|x|>2$, for any $\varepsilon\in[0,1]$, we have $|x+\varepsilon(x-\tilde{x})|\sim|x|$. Moreover

$$\begin{aligned}|\psi(x)-\psi(\tilde{x})|&\le|x-\tilde{x}|\cdot|\nabla\psi(x+\varepsilon(x-\tilde{x}))|\\&\lesssim|x-\tilde{x}|\left\{(1+|x|)^{-n-2\alpha}+(1+|\tilde{x}|)^{-n-2\alpha}\right\}.\end{aligned} \quad (74)$$

Therefore, we obtain $\psi(x)\in\mathscr{C}_{(1,2\alpha)}$ for any $\alpha\in(0,1)$, $g_{(-\Delta)^\alpha,m}(f)(x)\lesssim\tilde{g}_{(1,2\alpha)}(f)(x)$, and $S_{(-\Delta)^\alpha,m,\beta}(f)(x)\tilde{G}_{(1,2\alpha),\beta}(f)(x)$. Combining with Lemma 22 and the fact (from [30], Exercise 6.4) that $\tilde{g}_{(1,2\alpha)}(f)\sim\tilde{G}_{(1,2\alpha)}(f)$, we have $\alpha\in(1/2,1)$, $g_{(-\Delta)^\alpha,m}(f)(x)\lesssim G_1(f)(x)$. Via Lemma 21, similar to the proof of Proposition 20, we can get (66).

For any $f\in C_c^\infty(\mathbb{R}^n)$ and $x\in\mathbb{R}^n$

$$\tilde{G}_{(1,2\alpha),\beta}(f)(x)=\left(\int_0^\infty\int_{|x-y|<u}\left(\tilde{A}_{(1,2\alpha)}(f)(y,u/\beta)\right)^2\beta^n\frac{dydu}{u^{n+1}}\right)^{1/2}, \quad (75)$$

where $u=\beta t$. Note that for any $u\in(0,\infty)$ and $y\in B(x,u)$

$$\begin{aligned}\tilde{A}_{(1,2\alpha)}(f)(y,u/\beta)&=\sup_{\phi\in\mathscr{C}_{(1,2\alpha)}(\mathbb{R}^n)}\left|\phi_{u/\beta}\times f(y)\right|\\&=\sup_{\phi\in\mathscr{C}_{(1,2\alpha)}(\mathbb{R}^n)}\int_{\mathbb{R}^n}\frac{\beta^n}{u^n}\phi\left(\frac{y-z}{u/\beta}\right)f(z)dz\\&=\sup_{\phi\in\mathscr{C}_{(1,2\alpha)}(\mathbb{R}^n)}\beta^{n+1}\int_{\mathbb{R}^n}\phi_u(y-z)f(z)dz\\&=\beta^{n+1}\tilde{A}_{(1,2\alpha)}(f)(y,u).\end{aligned} \quad (76)$$

It follows from Lemma 22 that for any $\alpha\in(1/2,1)$

$$\begin{aligned}\tilde{G}_{(1,2\alpha),\beta}(f)(x)&=\left(\int_0^\infty\int_{|x-y|<u}\left(\beta^{n+1}\tilde{A}_{(1,2\alpha)}(f)(y,u)\right)^2\beta^n\frac{dydu}{u^{n+1}}\right)^{1/2}\\&=\beta^{3n/2+1}\tilde{G}_{(1,2\alpha)}(f)(x)\lesssim\beta^{3n/2+1}G_1(f)(x).\end{aligned} \quad (77)$$

Using Lemma 21, we obtain for any $\omega\in A_p(\mathbb{R}^n)$ and $f\in L^p(\omega)$

$$\begin{aligned}\left\|S_{(-\Delta)^\alpha,m,\beta}(f)\right\|_{L^p(\omega)}&\lesssim\beta^{3n/2+1}\|G_1(f)\|_{L^p(\omega)}\\&\lesssim\beta^{3n/2+1}[\omega]^{\max\{1/2,1/(p-1)\}}_{A_p(\mathbb{R}^n)}\|f\|_{L^p(\omega)}.\end{aligned} \quad (78)$$

Similar to the proof of Proposition 20, we can get (67).

Proposition 24. *Let* $n\ge 3$, $\beta\in[1,\infty)$, $\alpha\in(1/2,1)$, *and* $p\in(1,\infty)$. *There exists* $C>0$ *such that for any* $\omega\in A_p^{\rho,loc}(\mathbb{R}^n)$ *and* $f\in L^p(\omega)$

$$\left\|\tilde{g}^{loc}_{(-\Delta)^\alpha}(f)\right\|_{L^p(\omega)}\le C[\omega]^{\max\{1/2,1/(p-1)\}}_{A_p^{\rho,loc}(\mathbb{R}^n)}\|f\|_{L^p(\omega)}, \quad (79)$$

$$\left\|\tilde{S}^{loc}_{(-\Delta)^\alpha,\beta}(f)\right\|_{L^p(\omega)}\le C\beta^{3n/2+1}[\omega]^{\max\{1/2,1/(p-1)\}}_{A_p^{\rho,loc}(\mathbb{R}^n)}\|f\|_{L^p(\omega)}. \quad (80)$$

Proof. For $\tilde{S}_{(-\Delta)^\alpha,\beta}$, we denote $\psi_t(x):=s^{1/(2\alpha)}\nabla_x K_{\alpha,s}(x)$ with $s=t^{2\alpha}$. Thus

$$\tilde{S}_{(-\Delta)^\alpha,\beta}(f)(x) = \left(\int_0^\infty \int_{|x-y|<\beta t} |\psi_t \times f(y)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2},$$
$$\tilde{g}_{(-\Delta)^\alpha}(f)(x) = \left(\int_0^\infty |\psi_{t^{1/(2\alpha)}} \times f(x)|^2 \frac{dt}{t}\right)^{1/2} \simeq \left(\int_0^\infty |\psi_\tau \times f(x)|^2 \frac{d\tau}{\tau}\right)^{1/2}, \tag{81}$$

where $\tau = t^{1/(2\alpha)}$. On the one hand, for $s = t^{2\alpha}$

$$\begin{aligned}\int_{\mathbb{R}^n} \psi_t(x)dx &= \int_{\mathbb{R}^n} s^{1/(2\alpha)} \nabla_x K_{\alpha,s}(x) e^{-i2\pi x\cdot 0} dx \\ &= s^{1/(2\alpha)} \widehat{\nabla_x K_{\alpha,s}}(0) \\ &= s^{1/(2\alpha)} i2\pi \cdot 0 \cdot \int_{\mathbb{R}^n} \nabla_x K_{\alpha,s}(x)dx = 0.\end{aligned} \tag{82}$$

On the other hand

$$\int_{\mathbb{R}^n} \psi_t(x)dx = t^{-n} \int_{\mathbb{R}^n} \psi(x/t)dx = \int_{\mathbb{R}^n} \psi(u)du, \tag{83}$$

where $u = x/t$. Furthermore, it follows from Lemma 12 that

$$\psi_t(x) \lesssim \frac{1}{(1+|x|/t)^{n+2\alpha}} \frac{1}{t^n} \quad \text{and} \quad \psi(x) \lesssim \frac{1}{(1+|x|)^{n+2\alpha}}. \tag{84}$$

Similar to the proof of Proposition 24, we have $\psi(x) \in \mathscr{C}_{(1,2\alpha)}$ for any $\alpha \in (0,1)$, $\tilde{g}_{(-\Delta)^\alpha}(f)(x) \lesssim \tilde{g}_{(1,2\alpha)}(f)(x)$, and $\tilde{S}_{(-\Delta)^\alpha,\beta}(f)(x) \lesssim \tilde{G}_{(1,2\alpha),\beta}(f)(x)$. Following the process of the proof of Proposition 23, for $\alpha \in (1/2,1)$, we can get (79) and (80). □

3.2. Main Results

Theorem 25. *Let $n \geq 3$, $p \in (1,\infty)$, and $m \in \mathbb{Z}_+$. Assume that $\theta \in [0, 4/9(2\alpha-\delta))$ with $\alpha \in (1/2,1)$ and $\delta = 2 - n/q$, where $q \in (n/2, n/(2(1-\alpha)))$, and $\lambda \in (3+(2/n)\max\{N,1\},\infty)$, where $N > 2\alpha m(N_0+1)$ and N_0 satisfies (18). There exists a constant $C>0$ such that for any $\omega \in A_p^{\rho,\theta}(\mathbb{R}^n)$ and $f \in L^p(\omega)$*

$$\begin{aligned}&\|g_{L^\alpha,m}(f)\|_{L^p(\omega)} + \|S_{L^\alpha,m}(f)\|_{L^p(\omega)} + \|g^*_{L^\alpha,m,\lambda}(f)\|_{L^p(\omega)} \\ &\quad \leq C[\omega]_{A_p^{\rho,\theta}(\mathbb{R}^n)}^{\max\{1/2,1/(p-1)\}} \|f\|_{L^p(\omega)}.\end{aligned} \tag{85}$$

Proof. We split the proof into three steps.

Step 1: for any $\omega \in A_p^{\rho,\theta}(\mathbb{R}^n)$, $\theta \in [0, 4/9(2\alpha-\delta))$ and $f \in L^p(\omega)$

$$\|g_{L^\alpha,m}(f)\|_{L^p(\omega)} \leq C[\omega]_{A_p^{\rho,\theta}(\mathbb{R}^n)}^{\max\{1/2,1/(p-1)\}} \|f\|_{L^p(\omega)}. \tag{86}$$

Firstly, for $m \in \mathbb{Z}_+$, $\alpha \in (0,1)$, $B_x = B(x,\rho(x))$, and $f \in C_c^\infty(\mathbb{R}^n)$, we divide

$$\begin{aligned}g_{L^\alpha,m}(f)(x) &\leq g_{L^\alpha,m}\left(f\chi_{B_x}\right)(x) + g_{L^\alpha,m}\left(f\chi_{B_x^c}\right)(x) \\ &=: g^{loc}_{L^\alpha,m}(f)(x) + g^{glob}_{L^\alpha,m}(f)(x).\end{aligned} \tag{87}$$

To deal with $g^{glob}_{L^\alpha,m}$, via Lemma 13, we obtain

$$\begin{aligned}g^{glob}_{L^\alpha,m}(f)(x) &\lesssim \left(\int_0^\infty \left|\int_{|x-y|>\rho(x)} \frac{t^m}{\left(t^{1/(2\alpha)}+|x-y|\right)^{n+2\alpha m}} \left(1+\frac{t^{1/(2\alpha)}}{\rho(x)}\right)^{-N} f(y)dy\right|^2 \frac{dt}{t}\right)^{1/2} \\ &\lesssim \left(\int_0^\infty \frac{t^{2m}}{\left(1+\left(t^{1/(2\alpha)}/\rho(x)\right)\right)^{2N}} \left(\sum_{k=0}^\infty \int_{|x-y|\sim 2^k\rho(x)} \frac{1}{\left(t^{1/(2\alpha)}+2^k\rho(x)\right)^{n+2\alpha m}} |f(y)|dy\right)^2 \frac{dt}{t}\right)^{1/2} \\ &\lesssim \left(\int_0^\infty \frac{t^{2m}}{\left(1+\left(t^{1/(2\alpha)}/\rho(x)\right)\right)^{2N}} \left(\sum_{k=0}^\infty \frac{1}{\left(2^k\rho(x)\right)^{n+2\alpha m}} \frac{2^{k\overline{\theta}}}{\left(1+2^k\right)^{\overline{\theta}}} \int_{|x-y|\sim 2^k\rho(x)} |f(y)|dy\right)^2 \frac{dt}{t}\right)^{1/2} \\ &\lesssim M_{\overline{\theta}}f(x) \left(\int_0^\infty \frac{t^{2m}}{(\rho(x))^{4\alpha m}\left(1+\left(t^{1/(2\alpha)}/\rho(x)\right)\right)^{2N}} \left(\sum_{k=0}^\infty \frac{1}{\left(2^k\right)^{2\alpha m-\overline{\theta}}}\right)^2 \frac{dt}{t}\right)^{1/2} \\ &\lesssim M_{\overline{\theta}}f(x) \left(\int_0^\infty \frac{u^{4\alpha m-1}}{(1+u)^{2N}} du\right)^{1/2} \lesssim M_{\overline{\theta}}f(x) \left(\int_0^1 u^{4\alpha m-1} du + \int_1^\infty \frac{1}{(1+u)^{2(N-2\alpha m)}} \frac{du}{u}\right)^{1/2} \\ &\lesssim M_{\overline{\theta}}f(x),\end{aligned} \tag{88}$$

where $u = t^{1/(2\alpha)}/\rho(x)$, $\overline{\theta} \in [0, 2\alpha m)$ and $N > 2\alpha m$.

Next, we estimate $g^{loc}_{L^\alpha,m}$. Write

$$\begin{aligned}g^{loc}_{L^\alpha,m}(f)(x) &\lesssim \left(\int_0^{(\rho(x))^{2\alpha}} \left|t^m \partial_t^m e^{-tL^\alpha}\left(f\chi_{B_x}\right)(x)\right|^2 \frac{dt}{t}\right)^{1/2} \\ &\quad + \left(\int_{(\rho(x))^{2\alpha}}^\infty \left|t^m \partial_t^m e^{-tL^\alpha}\left(f\chi_{B_x}\right)(x)\right|^2 \frac{dt}{t}\right)^{1/2} \\ &\lesssim I_1(x) + I_2(x) + I_3(x),\end{aligned} \tag{89}$$

where

$$\begin{cases} I_1(x) := \left(\int_0^{(\rho(x))^{2\alpha}} \left|t^m \partial_t^m \left(e^{-tL^\alpha} - e^{-t(-\Delta)^\alpha}\right)\left(f\chi_{B_x}\right)(x)\right|^2 \frac{dt}{t}\right)^{1/2}; \\ I_2(x) := \left(\int_0^{(\rho(x))^{2\alpha}} \left|t^m \partial_t^m e^{-t(-\Delta)^\alpha}\left(f\chi_{B_x}\right)(x)\right|^2 \frac{dt}{t}\right)^{1/2}; \\ I_3(x) := \left(\int_{(\rho(x))^{2\alpha}}^\infty \left|t^m \partial_t^m e^{-tL^\alpha}\left(f\chi_{B_x}\right)(x)\right|^2 \frac{dt}{t}\right)^{1/2}. \end{cases} \tag{90}$$

For $I_2(x)$, we have $I_2(x) \leq g^{loc}_{(-\Delta)^\alpha,m}(f)(x)$. To deal with $I_1(x)$, it follows from Lemma 15 that for $\delta = 2 - n/q$ with $q > n/2$

$$\begin{aligned}&\left|t^m \partial_t^m \left(e^{-tL^\alpha} - e^{-t(-\Delta)^\alpha}\right)\left(f\chi_{B_x}\right)(x)\right| \\ &\quad \lesssim \int_{B_x} \max\left\{\left(\frac{|x-y|}{\rho(x)}\right)^\delta, \left(\frac{t^{1/(2\alpha)}}{\rho(x)}\right)^\delta\right\} \frac{t}{\left(|x-y|+t^{1/(2\alpha)}\right)^{n+2\alpha}} |f(y)|dy,\end{aligned} \tag{91}$$

which indicates

$$
\begin{aligned}
I_1(x) \lesssim & \left(\int_0^{(\rho(x))^{2\alpha}} \left(\int_{|x-y|<t^{1/(2\alpha)}} \left(\frac{t^{1/(2\alpha)}}{\rho(x)}\right)^{\delta} \frac{t}{\left(|x-y|+t^{1/(2\alpha)}\right)^{n+2\alpha}} |f(y)| dy\right)^2 \frac{dt}{t}\right)^{1/2} \\
& + \left(\int_0^{(\rho(x))^{2\alpha}} \left(\int_{t^{1/(2\alpha)} \le |x-y|<\rho(x)} \left(\frac{|x-y|}{\rho(x)}\right)^{\delta} \frac{t}{\left(|x-y|+t^{1/(2\alpha)}\right)^{n+2\alpha}} |f(y)| dy\right)^2 \frac{dt}{t}\right)^{1/2}.
\end{aligned} \tag{92}
$$

For $t^{1/(2\alpha)} < \rho(x)$

$$
\int_{|x-y|<t^{1/(2\alpha)}} \left(\frac{t^{1/(2\alpha)}}{\rho(x)}\right)^{\delta} \frac{t}{\left(|x-y|+t^{1/(2\alpha)}\right)^{n+2\alpha}} |f(y)| dy \lesssim \left(\frac{t^{1/(2\alpha)}}{\rho(x)}\right)^{\delta} R^{loc}_{(-\Delta)^{\alpha}}(|f|)(x),
$$

$$
\begin{aligned}
& \int_{t^{1/(2\alpha)} \le |x-y|<\rho(x)} \left(\frac{|x-y|}{\rho(x)}\right)^{\delta} \frac{t}{\left(|x-y|+t^{1/(2\alpha)}\right)^{n+2\alpha}} |f(y)| dy \\
& \lesssim t \left(\frac{t^{1/(2\alpha)}}{\rho(x)}\right)^{\delta} \sum_{k=0}^{\infty} \frac{2^{k\delta}}{\left(2^k t^{1/(2\alpha)} + t^{1/(2\alpha)}\right)^{n+2\alpha}} \int_{|x-y|\sim 2^k t^{1/2a}} |f(y)| dy \\
& \lesssim t \left(\frac{t^{1/(2\alpha)}}{\rho(x)}\right)^{\delta} \sum_{k=0}^{\infty} \frac{2^{k\delta}}{\left(2^k t^{1/(2\alpha)}\right)^{n+2\alpha}} \frac{\left(1+2^k\right)^{\overline{\theta}}}{\left(1+2^k t^{1/(2\alpha)}/\rho(x)\right)^{\overline{\theta}}} \int_{|x-y|\sim 2^k t^{1/2a}} |f(y)| dy \\
& \lesssim \left(\frac{t^{1/(2\alpha)}}{\rho(x)}\right)^{\delta} M_{\overline{\theta}} f(x) \sum_{k=0}^{\infty} \frac{1}{2^{k(2\alpha-\delta-\overline{\theta})}} \lesssim \left(\frac{t^{1/(2\alpha)}}{\rho(x)}\right)^{\delta} M_{\overline{\theta}} f(x),
\end{aligned} \tag{93}
$$

where $\overline{\theta} \in [0, 2\alpha - \delta)$. It holds

$$
\begin{aligned}
I_1(x) & \lesssim \left(\int_0^{(\rho(x))^{2\alpha}} \left(\frac{t^{1/(2\alpha)}}{\rho(x)}\right)^{2\delta} \frac{dt}{t}\right)^{1/2} \left\{\mathscr{M}^{loc}_{(-\Delta)^{\alpha}}(|f|)(x) + M_{\overline{\theta}} f(x)\right\} \\
& \lesssim \left(\int_0^1 u^{2\delta-1} du\right)^{1/2} \left\{\mathscr{M}^{loc}_{(-\Delta)^{\alpha}}(|f|)(x) + M_{\overline{\theta}} f(x)\right\} \\
& \lesssim \mathscr{M}^{loc}_{(-\Delta)^{\alpha}}(|f|)(x) + M_{\overline{\theta}} f(x),
\end{aligned} \tag{94}
$$

where $u = t^{1/(2\alpha)}/\rho(x)$. For the term $I_3(x)$, we can get

$$
\begin{aligned}
I_3(x) & \lesssim \left(\int_{(\rho(x))^{2\alpha}}^{\infty} \left(\int_{B_x} \frac{t^m}{\left(t^{1/(2\alpha)}+|x-y|\right)^{2\alpha m+n}} \left(1+\frac{t^{1/(2\alpha)}}{\rho(x)}\right)^{-N} |f(y)| dy\right)^2 \frac{dt}{t}\right)^{1/2} \\
& \lesssim \left(\int_{(\rho(x))^{2\alpha}}^{\infty} \left(1+\frac{t^{1/(2\alpha)}}{\rho(x)}\right)^{-2N} \left(\frac{t^m}{\left(t^{1/2a}\right)^{2\alpha m+n}} \frac{2^{\overline{\theta}}(\rho(x))^n}{\left(1+\rho(x)/\rho(x)\right)^{\overline{\theta}}(\rho(x))^n} \int_{B_x} |f(y)| dy\right)^2 \frac{dt}{t}\right)^{1/2} \\
& \lesssim M_{\overline{\theta}} f(x) \left(\int_{(\rho(x))^{2\alpha}}^{\infty} \left(1+\frac{t^{1/(2\alpha)}}{\rho(x)}\right)^{-2N} t^{-n/a} (\rho(x))^{2n} \frac{dt}{t}\right)^{1/2} \\
& \lesssim M_{\overline{\theta}} f(x) \left(\int_1^{\infty} (1+u)^{-2N} u^{-2n} \frac{du}{u}\right)^{1/2} \lesssim M_{\overline{\theta}} f(x),
\end{aligned} \tag{95}
$$

where $u = t^{1/(2\alpha)}/\rho(x)$. Thus, we obtain

$$
g^{loc}_{L^{\alpha},m}(f)(x) \lesssim g^{loc}_{(-\Delta)^{\alpha},m}(f)(x) + \mathscr{M}^{loc}_{(-\Delta)^{\alpha}}(|f|)(x) + M_{\overline{\theta}} f(x). \tag{96}
$$

Furthermore, it holds

$$
g_{L^{\alpha},m}(f)(x) \lesssim g^{loc}_{(-\Delta)^{\alpha},m}(f)(x) + \mathscr{M}^{loc}_{(-\Delta)^{\alpha}}(|f|)(x) + M_{\overline{\theta}} f(x). \tag{97}
$$

Let $\gamma = (2/3)\overline{\theta} \in [0, (2/3)(2\alpha - \delta))$. By Lemmas 7 and 19 and Propositions 20 and 23, we can get for any $\omega \in A_3^{\rho,\gamma}$, $\alpha \in (1/2, 1)$, and $f \in L^p(\omega)$

$$
\begin{aligned}
\left\| g_{L^{\alpha},m}(f) \right\|_{L^3(\omega)} & \lesssim \left\| g^{loc}_{(-\Delta)^{\alpha},m}(f) \right\|_{L^3(\omega)} + \left\| \mathscr{M}^{loc}_{(-\Delta)^{\alpha}}(|f|) \right\|_{L^3(\omega)} \\
& + \left\| M_{\overline{\theta}} f \right\|_{L^3(\omega)} \lesssim \left\{ [\omega]^{1/2}_{A_3^{\rho,loc}} + [\omega]^{1/2}_{A_3^{\rho,loc}} + [\omega]^{1/2}_{A_3^{\rho,\gamma}} \right\} \\
& \cdot \|f\|_{L^3(\omega)} \lesssim [\omega]^{1/2}_{A_3^{\rho,\gamma}} \|f\|_{L^3(\omega)}.
\end{aligned} \tag{98}
$$

The (86) can be deduced from Lemma 9 and the density of $C_c^{\infty}(\mathbb{R}^n)$ in $L^p(\omega)$.

Step 2: for any $\omega \in A_p^{\rho,\theta}(\mathbb{R}^n)$, $\theta \in [0, 4/9(2\alpha - \delta))$ and $f \in L^p(\omega)$

$$
\left\| g^{*}_{L^{\alpha},m,\lambda}(f) \right\|_{L^p(\omega)} \le C[\omega]^{\max\{1/2, 1/(p-1)\}}_{A_p^{\rho,\theta}(\mathbb{R}^n)} \|f\|_{L^p(\omega)}. \tag{99}
$$

For $m \in \mathbb{Z}_+$, $\alpha \in (0,1)$, and $f \in C_c^{\infty}(\mathbb{R}^n)$, we have

$$
\begin{aligned}
\left(g^{*}_{L^{\alpha},m,\lambda}(f)(x)\right)^2 & \lesssim \int_0^{\infty} \int_{B(x,t)} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left| t^{2\alpha m} \partial_s^m e^{-sL^{\alpha}}(f)(y)|_{s=t^{2\alpha}} \right|^2 \frac{dydt}{t^{n+1}} \\
& + \sum_{j=1}^{\infty} \int_0^{\infty} \int_{|x-y|\sim 2^j t} \left(\frac{t}{t+2^j t}\right)^{\lambda n} \left| t^{2\alpha m} \partial_s^m e^{-sL^{\alpha}}(f)(y)|_{s=t^{2\alpha}} \right|^2 \frac{dydt}{t^{n+1}} \\
& \lesssim \left(S_{L^{\alpha},m}(f)(x)\right)^2 + \sum_{j=1}^{\infty} 2^{-j\lambda n} \left(S_{L^{\alpha},m,2^j}(f)(x)\right)^2 \\
& \lesssim \sum_{j=0}^{\infty} 2^{-j\lambda n} \left(S_{L^{\alpha},m,2^j}(f)(x)\right)^2.
\end{aligned} \tag{100}
$$

To deal with $S_{L^{\alpha},m,2^j}$, let $\beta \in [1,\infty)$ and $B_x = B(x, \rho(x))$. Split

$$
\begin{aligned}
S_{L^{\alpha},m,\beta}(f)(x) & \le S_{L^{\alpha},m,\beta}\left(f\chi_{2B_x}\right)(x) + S_{L^{\alpha},m,\beta}\left(f\chi_{(2B_x)^c}\right)(x) \\
& =: S^{loc}_{L^{\alpha},m,\beta}(f)(x) + S^{glob}_{L^{\alpha},m,\beta}(f)(x).
\end{aligned} \tag{101}
$$

Firstly, we further split $S^{glob}_{L^{\alpha},m,\beta}(f)(x)$ as $S^{glob}_{L^{\alpha},m,\beta}(f)(x) \le J_1(x) + J_2(x)$, where

$$
\begin{cases}
J_1(x) := \left(\displaystyle\int_0^{\rho(x)/\beta} \int_{|x-y|<\beta t} \left| t^{2\alpha m} \partial_s^m e^{-sL^{\alpha}}\left(f\chi_{(2B_x)^c}\right)(y)|_{s=t^{2\alpha}} \right|^2 \frac{dydt}{t^{n+1}}\right)^{1/2}; \\
J_2(x) := \left(\displaystyle\int_{\rho(x)/\beta}^{\infty} \int_{|x-y|<\beta t} \left| t^{2\alpha m} \partial_s^m e^{-sL^{\alpha}}\left(f\chi_{(2B_x)^c}\right)(y)|_{s=t^{2\alpha}} \right|^2 \frac{dydt}{t^{n+1}}\right)^{1/2}.
\end{cases} \tag{102}
$$

Via Lemma 13, we obtain

$$\begin{aligned} J_1(x) &\lesssim \left(\int_0^{\rho(x)/\beta}\int_{|x-y|<\beta t}\left(\int_{|x-y|>\rho(x)}\frac{t^{2\alpha m}}{(t+|x-y|)^{n+2\alpha m}}\left(1+\frac{t}{\rho(x)}\right)^{-N}|f(y)|dy\right)^2\frac{dydt}{t^{n+1}}\right)^{1/2} \\ &\lesssim \left(\int_0^{\rho(x)/\beta}\int_{|x-y|<\beta t}\left(\sum_{k=2}^{\infty}\int_{2^{k-1}\rho(x)<|x-z|\le 2^k\rho(x)}\frac{t^{2\alpha m}}{\left(t+2^k\rho(x)\right)^{n+2\alpha m}}\frac{1}{(1+t/\rho(x))^N}|f(y)|dy\right)^2\frac{dydt}{t^{n+1}}\right)^{1/2}. \end{aligned} \tag{103}$$

Since $y\in B(x,\beta t)$ and $2^{k-1}\rho(x)<|x-z|$, it holds

$$|y-z|\ge|x-z|-|y-x|\ge 2^{k-1}\rho(x)-\beta t\ge 2^{k-2}\rho(x). \tag{104}$$

By the fact that $\beta\ge 1$ and Lemma 2, we can get for any $y\in B(x,\beta t)$

$$\begin{aligned} (1+t/\rho(y))^{-1} &\le (1/\beta+t/\rho(y))^{-1} \\ &\lesssim \frac{\beta}{1+(t\beta/\rho(x))(1/C)(1+|x-y|/\rho(x))^{-N_0/(N_0+1)}} \\ &\lesssim \frac{\beta(1+|x-y|/\rho(x))^{N_0/(N_0+1)}}{1+t\beta/\rho(x)} \\ &\lesssim \frac{\beta}{(1+t\beta/\rho(x))^{1/(N_0+1)}}, \end{aligned} \tag{105}$$

where N_0 is the same constant in Lemma 2. Hence, it leads to

$$\begin{aligned} J_1(x) &\lesssim \beta^N\left(\int_0^{\rho(x)/\beta}\int_{|x-y|<\beta t}\frac{t^{4\alpha m}}{(1+t\beta/\rho(x))^{2N/(N_0+1)}}\right. \\ &\quad\left.\cdot\left(\sum_{k=2}^{\infty}\frac{1}{\left(2^{k-2}\rho(x)\right)^{n+2\alpha m}}\frac{2^{k\overline{\theta}}}{\left(1+2^k\right)^{\overline{\theta}}}\int_{|x-y|\sim 2^k\rho(x)}|f(y)|dy\right)^2\frac{dydt}{t^{n+1}}\right)^{1/2} \\ &\lesssim \beta^{N-2\alpha m}M_{\overline{\theta}}f(x)\left(\int_0^{\rho(x)/\beta}\int_{|x-y|<\beta t}\frac{(t\beta/\rho(x))^{4\alpha m}}{(1+t\beta/\rho(x))^{2N/(N_0+1)}}\left(\sum_{k=2}^{\infty}\frac{1}{\left(2^k\right)^{2\alpha m-\overline{\theta}}}\right)^2\frac{dydt}{t^{n+1}}\right)^{1/2} \\ &\lesssim \beta^{N-2\alpha m+n/2}M_{\overline{\theta}}f(x)\left(\int_0^1\frac{u^{4\alpha m-1}}{(1+u)^{2N/(N_0+1)}}du\right)^{1/2} \\ &\lesssim \beta^{N-2\alpha m+n/2}M_{\overline{\theta}}f(x)\left(\int_0^1 u^{4\alpha m-1}du\right)^{1/2}\lesssim\beta^{N-2\alpha m+n/2}M_{\overline{\theta}}f(x), \end{aligned} \tag{106}$$

where $u=t\beta/\rho(x)$, $\overline{\theta}\in[0,2\alpha m)$. Next, we estimate $J_2(x)$.

$$J_2(x)\lesssim\left(\int_{\rho(x)/\beta}^{\infty}\int_{|x-y|<\beta t}\left\{\sum_{k=2}^{\infty}\int_{2^{k-1}\rho(x)<|x-z|\le 2^k\rho(x)}\left|\partial_s^m K_{\alpha,s}^L(y,z)|_{s=t^{2\alpha}}\right|\cdot|f(z)|dz\right\}^2\frac{dydt}{t^{n+1}}\right)^{1/2}. \tag{107}$$

We consider two cases.

Case 1: $|y-z|<\beta t$. For this case, $2\beta t>|y-z|+|y-x|\ge|x-z|>2^{k-1}\rho(x)$. It follows from Lemma 13 and (105) that

$$\begin{aligned} &\sum_{k=2}^{\infty}\int_{|x-z|\sim 2^k\rho(x)}\left|\partial_s^m K_{\alpha,s}^L(y,z)|_{s=t^{2\alpha}}\right|\cdot|f(z)|dz \\ &\lesssim\sum_{k=2}^{\infty}\int_{|x-z|\sim 2^k\rho(x)}\frac{t^{2\alpha m}}{(t+|y-z|)^{2\alpha m+n}}\left(1+\frac{t}{\rho(y)}\right)^{-N}|f(z)|dz \\ &\lesssim\sum_{k=2}^{\infty}\beta^N\left(1+\frac{t\beta}{\rho(x)}\right)^{-N/(N_0+1)}\frac{1}{t^n}\int_{|x-z|\sim 2^k\rho(x)}|f(z)|dz \\ &\lesssim\beta^N\sum_{k=2}^{\infty}\frac{1}{(1+(t\beta/\rho(x)))^{N/(N_0+1)}}\left(\frac{\rho(x)}{t}\right)^n \\ &\quad\cdot\frac{2^{kn}\left(1+2^k\right)^{\overline{\theta}}}{\left(2^k\rho(x)\right)^n\left(1+2^k\right)^{\overline{\theta}}}\int_{|x-z|\sim 2^k\rho(x)}|f(z)|dz \\ &\lesssim\beta^{N+n}\left(\frac{\rho(x)}{t\beta}\right)^{\tau}M_{\overline{\theta}}(f)(x)\sum_{k=2}^{\infty}\frac{2^{k(n+\overline{\theta})}}{\left(1+2^{k-2}\right)^{N/(N_0+1)}}\frac{1}{\left(2^{k-2}\right)^{n-\tau}} \\ &\lesssim\beta^{N+n}\left(\frac{\rho(x)}{t\beta}\right)^{\tau}M_{\overline{\theta}}(f)(x)\sum_{k=2}^{\infty}\frac{1}{\left(2^k\right)^{N/(N_0+1)-\overline{\theta}-\tau}} \\ &\lesssim\beta^{N+n}\left(\frac{\rho(x)}{t\beta}\right)^{\tau}M_{\overline{\theta}}(f)(x), \end{aligned} \tag{108}$$

where $N>(N_0+1)(\overline{\theta}+\tau)$ and $\tau:=2\alpha m-\overline{\theta}\in(0,n)$.

Case 2: $|y-z|\ge\beta t$. Note that $2|y-z|\ge|y-z|+\beta t\ge|y-z|+|x-y|\ge|x-z|>2^{k-1}\rho(x)$. By Lemma 13 and (105), it holds

$$\begin{aligned} &\sum_{k=2}^{\infty}\int_{|x-z|\sim 2^k\rho(x)}\left|\partial_s^m K_{\alpha,s}^L(y,z)|_{s=t^{2\alpha}}\right|\cdot|f(z)|dz \\ &\lesssim\beta^N\sum_{k=2}^{\infty}\int_{|x-z|\sim 2^k\rho(x)}\frac{t^{2\alpha m}}{\left(2^{k-2}\rho(x)\right)^{2\alpha m+n}}\left(1+\frac{t\beta}{\rho(x)}\right)^{-(N/(N_0+1))}|f(z)|dz \\ &\lesssim\beta^N\left(1+\frac{t\beta}{\rho(x)}\right)^{-(N/(N_0+1))}\left(\frac{t}{\rho(x)}\right)^{2\alpha m}\sum_{k=2}^{\infty}\frac{1}{\left(2^{k-2}\right)^{2\alpha m}\left(2^{k-2}\rho(x)\right)^n}\int_{|x-z|\sim 2^k\rho(x)}|f(z)|dz \\ &\lesssim\beta^{N-2\alpha m}\left(1+\frac{t\beta}{\rho(x)}\right)^{-(N/(N_0+1))}\left(\frac{t\beta}{\rho(x)}\right)^{2\alpha m} \\ &\quad\cdot\sum_{k=2}^{\infty}\frac{1}{\left(2^k\right)^{2\alpha m-\overline{\theta}}}M_{\overline{\theta}}(f)(x)\lesssim\beta^{N-2\alpha m}\left(1+\frac{t\beta}{\rho(x)}\right)^{-(N/(N_0+1))}\left(\frac{t\beta}{\rho(x)}\right)^{2\alpha m}M_{\overline{\theta}}(f)(x), \end{aligned} \tag{109}$$

where $\overline{\theta}\in(0,2\alpha m)$. Furthermore, we can get

$$\begin{aligned} J_2(x) &\lesssim\beta^{n+N}M_{\overline{\theta}}(f)(x)\left(\int_{\rho(x)/\beta}^{\infty}\int_{|x-y|<\beta t}\left\{\left(\frac{\rho(x)}{t\beta}\right)^{2\tau}\right.\right. \\ &\quad\left.\left.+\frac{1}{(1+t\beta/\rho(x))^{2N/(N_0+1)}}\left(\frac{t\beta}{\rho(x)}\right)^{4\alpha m}\right\}\frac{dydt}{t^{n+1}}\right)^{1/2} \\ &\lesssim\beta^{3n/2+N}M_{\overline{\theta}}(f)(x)\left(\int_{\rho(x)/\beta}^{\infty}\left\{\left(\frac{\rho(x)}{t\beta}\right)^{2\tau}\right.\right. \\ &\quad\left.\left.+\frac{1}{(1+t\beta/\rho(x))^{2N/(N_0+1)}}\left(\frac{t\beta}{\rho(x)}\right)^{4\alpha m}\right\}\frac{dt}{t}\right)^{1/2} \\ &\lesssim\beta^{3n/2+N}M_{\overline{\theta}}(f)(x)\left(\int_1^{\infty}\left\{u^{-2\tau}+\frac{u^{4\alpha m}}{(1+u)^{2N/(N_0+1)}}\right\}\frac{du}{u}\right)^{1/2} \\ &\lesssim\beta^{3n/2+N}M_{\overline{\theta}}(f)(x), \end{aligned} \tag{110}$$

where $u = t\beta/\rho(x)$ and $N > 2\alpha m(N_0+1)$. Hence, it is easy to see that

$$S^{glob}_{L^\alpha,m,\beta}(f)(x) \lesssim \beta^{3n/2+N} M_{\overline{\theta}}(f)(x). \tag{111}$$

Next, we estimate $S^{loc}_{L^\alpha,m,\beta}(f)$. Split

$$\begin{aligned} S^{loc}_{L^\alpha,m,\beta}(f)(x) &\lesssim \left(\int_0^{\rho(x)/\beta}\int_{|x-y|<\beta t}\left|t^{2\alpha m}\partial_s^m e^{-sL^\alpha}\left(f\chi_{2B_x}\right)(y)|_{s=t^{2\alpha}}\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2} \\ &\quad + \left(\int_{\rho(x)/\beta}^{\infty}\int_{|x-y|<\beta t}\left|t^{2\alpha m}\partial_s^m e^{-sL^\alpha}\left(f\chi_{2B_x}\right)(y)|_{s=t^{2\alpha}}\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2} \\ &\lesssim K_1(x)+K_2(x)+K_3(x), \end{aligned} \tag{112}$$

where

$$\begin{cases} K_1(x) := \left(\int_0^{\rho(x)/\beta}\int_{|x-y|<\beta t}\left|t^{2\alpha m}\partial_s^m\left(e^{-sL^\alpha}-e^{-s(-\Delta)^\alpha}\right)\Big|_{s=t^{2\alpha}}\left(f\chi_{2B_x}\right)(y)\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2}; \\ K_2(x) := \left(\int_0^{\rho(x)/\beta}\int_{|x-y|<\beta t}\left|t^{2\alpha m}\partial_s^m e^{-s(-\Delta)^\alpha}\Big|_{s=t^{2\alpha}}\left(f\chi_{2B_x}\right)(y)\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2}; \\ K_3(x) := \left(\int_{\rho(x)/\beta}^{\infty}\int_{|x-y|<\beta t}\left|t^{2\alpha m}\partial_s^m e^{-sL^\alpha}\Big|_{s=t^{2\alpha}}\left(f\chi_{2B_x}\right)(y)\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2}. \end{cases} \tag{113}$$

For $K_2(x)$, it can be seen that $K_2(x) \le S^{loc}_{(-\Delta)^\alpha,m,\beta}(f)(x)$. To deal with $K_1(x)$, using Lemma 15, we obtain

$$\left|t^{2\alpha m}\partial_s^m\left(e^{-sL^\alpha}-e^{-s(-\Delta)^\alpha}\right)\Big|_{s=t^{2\alpha}}\left(f\chi_{2B_x}\right)(y)\right| \lesssim K_{1,1}(x)+K_{1,2}(x), \tag{114}$$

where

$$\begin{cases} K_{1,1}(x) := \int_{|y-z|\ge t}\left(\frac{|y-z|}{\rho(y)}\right)^\delta \frac{t^{2\alpha}}{(|y-z|+t)^{n+2\alpha}}\left|\left(f\chi_{2B_x}\right)(z)\right|dz; \\ K_{1,2}(x) := \int_{0<|y-z|<t}\left(\frac{t}{\rho(y)}\right)^\delta \frac{t^{2\alpha}}{(|y-z|+t)^{n+2\alpha}}\left|\left(f\chi_{2B_x}\right)(z)\right|dz. \end{cases} \tag{115}$$

Since $0 < t\beta < \rho(x)$, $|x-y| < t\beta$, we have $\rho(x) \sim \rho(y)$. For $K_{1,1}(x)$, we can get

$$\begin{aligned} K_{1,1}(x) &\lesssim \sum_{k=0}^{\infty}\int_{|y-z|\sim 2^k t}\left(\frac{2^k t}{\rho(y)}\right)^\delta \frac{t^{2\alpha}}{\left(2^k t+t\right)^{n+2\alpha}}\left|\left(f\chi_{2B_x}\right)(z)\right|dz \\ &\lesssim \left(\frac{t}{\rho(x)}\right)^\delta \sum_{k=0}^{\infty}2^{k\delta}\frac{t^{2\alpha}}{\left(2^k t\right)^{n+2\alpha}}\frac{\left(1+2^k\right)^{\overline{\theta}}}{\left(1+\left(2^k t/\rho(x)\right)\right)^{\overline{\theta}}}\int_{|y-z|\sim 2^k t}|f(z)|dz \\ &\lesssim M_{\overline{\theta}}(f)(x)\left(\frac{t}{\rho(x)}\right)^\delta \sum_{k=0}^{\infty}\frac{1}{\left(2^k\right)^{2\alpha-\delta-\overline{\theta}}} \lesssim M_{\overline{\theta}}(f)(x)\left(\frac{t}{\rho(x)}\right)^\delta, \end{aligned} \tag{116}$$

where $\overline{\theta} \in [0, 2\alpha-\delta)$. For $K_{1,2}(x)$, it holds

$$K_{1,2}(x) \le \left(\frac{t}{\rho(y)}\right)^\delta M^{*,loc}_{(-\Delta)^\alpha,\beta}(f)(x). \tag{117}$$

Hence, we obtain

$$\begin{aligned} K_1(x) &\lesssim \left(\int_0^{\rho(x)/\beta}\int_{|x-y|<\beta t}\left(\frac{t}{\rho(x)}\right)^{2\delta}\frac{dydt}{t^{n+1}}\right)^{1/2} \cdot \left\{M_{\overline{\theta}}(f)(x)+M^{*,loc}_{(-\Delta)^\alpha,\beta}(f)(x)\right\} \\ &\lesssim \left(\int_0^{\rho(x)/\beta}\left(\frac{t}{\rho(x)}\right)^{2\delta}(\beta t)^n\frac{dt}{t^{n+1}}\right)^{1/2} \cdot \left\{M_{\overline{\theta}}(f)(x)+M^{*,loc}_{(-\Delta)^\alpha,\beta}(f)(x)\right\} \\ &\lesssim \beta^{n/2}\left(\int_0^{\rho(x)/\beta}\left(\frac{t}{\rho(x)}\right)^{2\delta}\frac{dt}{t}\right)^{1/2} \cdot \left\{M_{\overline{\theta}}(f)(x)+M^{*,loc}_{(-\Delta)^\alpha,\beta}(f)(x)\right\} \\ &\lesssim \beta^{n/2}\left(\int_0^{1/\beta}u^{2\delta-1}du\right)^{1/2}\left\{M_{\overline{\theta}}(f)(x)+M^{*,loc}_{(-\Delta)^\alpha,\beta}(f)(x)\right\} \\ &\lesssim \beta^{n/2}\left\{M_{\overline{\theta}}(f)(x)+M^{*,loc}_{(-\Delta)^\alpha,\beta}(f)(x)\right\}, \end{aligned} \tag{118}$$

where $u = t/\rho(x)$. For $K_3(x)$, note that $|y-z| \le |x-y|+|x-z| \le \beta t + 2\rho(x) < 3\beta t$. Via Lemma 13 and (105), we can get

$$\begin{aligned} K_3(x) &\lesssim \left(\int_{\rho(x)/\beta}^{\infty}\int_{|x-y|<\beta t}\left(\int_{\mathbb{R}^n}\frac{t^{2\alpha m}}{(t+|y-z|)^{n+2\alpha m}} \cdot \left(1+\frac{t}{\rho(y)}\right)^{-N}\left|\left(f\chi_{2B_x}\right)(z)\right|dz\right)^2\frac{dydt}{t^{n+1}}\right)^{1/2} \\ &\lesssim \left(\int_{\rho(x)/\beta}^{\infty}\int_{|x-y|<\beta t}\beta^{2N}\left(1+\frac{t\beta}{\rho(x)}\right)^{-2N/(N_0+1)} \cdot \left(\int_{\mathbb{R}^n}\frac{t^{2\alpha m}}{(t+|y-z|)^{n+2\alpha m}}\left|\left(f\chi_{2B_x}\right)(z)\right|dz\right)^2\frac{dydt}{t^{n+1}}\right)^{1/2} \\ &\lesssim \left(\int_{\rho(x)/\beta}^{\infty}\int_{|x-y|<\beta t}\beta^{2N}\left(1+\frac{t\beta}{\rho(x)}\right)^{-2N/(N_0+1)} \cdot \left(\frac{t^{2\alpha m}}{t^{n+2\alpha m}}\frac{3^{\overline{\theta}}(2\rho(x))^n}{(1+2)^{\overline{\theta}}(2\rho(x))^n}\int_{|x-z|<2\rho(x)}|f(z)|dz\right)^2\frac{dydt}{t^{n+1}}\right)^{1/2} \\ &\lesssim \beta^N M_{\overline{\theta}}(f)(x)\left(\int_{\rho(x)/\beta}^{\infty}\int_{|x-y|<\beta t}\left(1+\frac{t\beta}{\rho(x)}\right)^{-2N/(N_0+1)}\left(\frac{3^{\overline{\theta}}(2\rho(x))^n}{t^n}\right)^2\frac{dydt}{t^{n+1}}\right)^{1/2} \\ &\lesssim \beta^{N+n}M_{\overline{\theta}}(f)(x)\left(\int_{\rho(x)/\beta}^{\infty}\int_{|x-y|<\beta t}\left(1+\frac{t\beta}{\rho(x)}\right)^{-2N/(N_0+1)}\left(\frac{\rho(x)}{t\beta}\right)^{2n}\frac{dydt}{t^{n+1}}\right)^{1/2} \\ &\lesssim \beta^{N+3n/2}M_{\overline{\theta}}(f)(x)\left(\int_{\rho(x)/\beta}^{\infty}\left(1+\frac{t\beta}{\rho(x)}\right)^{-2N/(N_0+1)}\left(\frac{\rho(x)}{t\beta}\right)^{2n}\frac{dt}{t}\right)^{1/2} \\ &\lesssim \beta^{N+3n/2}M_{\overline{\theta}}(f)(x)\left(\int_1^{\infty}(1+u)^{-2N/(N_0+1)}u^{-2n}\frac{du}{u}\right)^{1/2} \lesssim \beta^{N+3n/2}M_{\overline{\theta}}(f)(x), \end{aligned} \tag{119}$$

where $u = t\beta/\rho(x)$. Furthermore, it is easy to see that

$$S^{loc}_{L^\alpha,m,\beta}(f)(x) \lesssim \beta^{n/2}\left\{M_{\overline{\theta}}(f)(x) + M^{*,loc}_{(-\Delta)^\alpha,\beta}(f)(x)\right\} + S^{loc}_{(-\Delta)^\alpha,m,\beta}(f)(x) + \beta^{3n/2+N}M_{\overline{\theta}}(f)(x). \quad (120)$$

Thus, for $N > 2\alpha m(N_0+1) = (\overline{\theta}+\tau)(N_0+1)$ and $\overline{\theta} \in [0, 2\alpha-\delta)$

$$S_{L^\alpha,m,\beta}(f)(x) \lesssim \beta^{n/2}M^{*,loc}_{(-\Delta)^\alpha,\beta}(f)(x) + S^{loc}_{(-\Delta)^\alpha,m,\beta}(f)(x) + \beta^{3n/2+N}M_{\overline{\theta}}(f)(x). \quad (121)$$

Let $\gamma := 2\overline{\theta}/3 \in [0, (2/3)(2\alpha-\delta))$ and $\omega \in A_3^{\rho,\gamma}(\mathbb{R}^n)$. By the fact that $\lambda \in (3+(2/n)\max\{N,1\},\infty)$, where $N > 2\alpha m(N_0+1)$ and N_0 satisfies (18), we can get

$$\begin{aligned}\left\|g^*_{L^\alpha,m,\lambda}(f)\right\|_{L^3(\omega)} &\lesssim \sum_{j=0}^{\infty} 2^{-j\lambda n/2}\left\|S_{L^\alpha,m,2^j}(f)\right\|_{L^3(\omega)} \\ &\lesssim \sum_{j=0}^{\infty} 2^{-j\lambda n/2}\left\{\left(2^j\right)^{n/2}\left\|M^{*,loc}_{(-\Delta)^\alpha,\beta}(f)\right\|_{L^3(\omega)} + \left(2^j\right)^{3n/2+N}\left\|M_{\overline{\theta}}(f)\right\|_{L^3(\omega)} + \left\|S^{loc}_{(-\Delta)^\alpha,m,\beta}(f)(f)\right\|_{L^3(\omega)}\right\} \\ &\lesssim \sum_{j=0}^{\infty} 2^{-j\lambda n/2}\left\{\left(2^j\right)^{3n/2}[\omega]^{1/2}_{A_3^{\rho,loc}} + \left(2^j\right)^{3/n+N}[\omega]^{1/2}_{A_3^{\rho,\gamma}} + \left(2^j\right)^{3n/2+1}[\omega]^{\max\{(1/2,1)/(3-1)\}}_{A_3^{\rho,loc}}\right\}\|f\|_{L^3(\omega)} \\ &\lesssim \sum_{j=0}^{\infty} 2^{-j\lambda n/2}\left(2^j\right)^{3n/2+\max\{N,1\}}[\omega]^{1/2}_{A_3^{\rho,\gamma}}\|f\|_{L^3(\omega)} \\ &\lesssim [\omega]^{1/2}_{A_3^{\rho,\gamma}}\|f\|_{L^3(\omega)}.\end{aligned} \quad (122)$$

Via Lemma 9 and the fact that $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\omega)$, we obtain (99).

Step 3: for any $\omega \in A_p^{\rho,\theta}(\mathbb{R}^n)$, $\theta \in [0, (4/9)(2\alpha-\delta))$ and $f \in L^p(\omega)$

$$\left\|S_{L^\alpha,m}(f)\right\|_{L^p(\omega)} \le C[\omega]^{\max\{1/2,1/(p-1)\}}_{A_p^{\rho,\theta}(\mathbb{R}^n)}\|f\|_{L^p(\omega)}. \quad (123)$$

Since

$$\begin{aligned}g^*_{L^\alpha,m,\lambda}(f)(x) &\ge \left(\int_0^\infty\int_{|x-y|<t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|t^{2\alpha m}\partial_s^m e^{-sL^\alpha}(f)(y)|_{s=t^{2\alpha}}\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2} \\ &\ge \left(\int_0^\infty\int_{|x-y|<t}\frac{1}{2^{\lambda n}}\left|t^{2\alpha m}\partial_s^m e^{-sL^\alpha}(f)(y)|_{s=t^{2\alpha}}\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2} \simeq S_{L^\alpha,m}(f)(x).\end{aligned} \quad (124)$$

Combining with (99), we can get (123). This completes the proof of Theorem 25. □

Theorem 26. *Let $n \ge 3$, $p \in (1,\infty)$, and $m \in \mathbb{Z}_+$. Assume that $\theta \in [0, (4/9)(2\alpha-\delta))$ with $\alpha \in (1/2, 1)$ and $\delta = 2 - n/q$, where $q \in (n, n/(2(1-\alpha)))$, and $\lambda \in (3+(2/n)\max\{N,1\},\infty)$, where $N > 2\alpha m(N_0+1)$ and N_0 satisfies (18). There exists a constant $C > 0$ such that for any $\omega \in A_p^{\rho,\theta}(\mathbb{R}^n)$ and $f \in L^p(\omega)$*

$$\left\|\tilde{g}_{L^\alpha}(f)\right\|_{L^p(\omega)} + \left\|\tilde{S}_{L^\alpha}(f)\right\|_{L^p(\omega)} + \left\|\tilde{g}^*_{L^\alpha,\lambda}(f)\right\|_{L^p(\omega)} \le C[\omega]^{\max\{1/2,1/(p-1)\}}_{A_p^{\rho,\theta}(\mathbb{R}^n)}\|f\|_{L^p(\omega)}. \quad (125)$$

Proof. Similar to the proof of Theorem 25, we can prove this theorem by the aid of Lemmas 14 and 16 and Proposition 24. Thus, we omit the details. □

Lemma 27 (see [19], Lemma 4.1). *Let $p \in (1,\infty)$, $1/p + 1/p' = 1$, $\theta \in [0,\infty)$, $n \ge 3$. Assume that $\omega \in A_p^{\rho,\theta}(\mathbb{R}^n)$. There exists a constant $C > 0$ such that for any $\lambda \in [1,\infty)$ and ball $B(x_B, r_B)$ of $\mathbb{R}^n$*

$$\omega(B(x_B,\lambda r_B)) \le C[\omega]_{A_p^{\rho,\theta}(\mathbb{R}^n)}\lambda^{pn}\left(1+\frac{\lambda r_B}{\rho(x_B)}\right)^{p\theta}\omega(B(x_B,r_B)). \quad (126)$$

Theorem 28. *Let $n \ge 3$, $p \in (1,\infty)$, and $m \in \mathbb{Z}_+$. Assume that $\lambda \in (2(1+\theta/n),\infty)$ and $\theta \in [0, (4/9)(2\alpha-\delta))$, where $\alpha \in (1/2, 1)$, $\delta = 2 - n/q$ with $q \in (n/2, n/2(1-\alpha))$. There exists a constant $C > 0$ such that for any $\omega \in A_p^{\rho,\theta}(\mathbb{R}^n)$ and $f \in L^p(\omega)$*

$$\left\|g^*_{L^\alpha,m,\lambda}(f)\right\|_{L^p(\omega)} \le C[\omega]^{1/2\max\{1,1/(p-1)\}+\max\{1/2,1/(p-1)\}}_{A_p^{\rho,\theta}(\mathbb{R}^n)}\|f\|_{L^p(\omega)}. \quad (127)$$

Proof. Let $\mu \in [0,\infty)$. We first introduce an auxiliary function $\tilde{S}^\mu_{L^\alpha,m}$, which is defined by setting, for any $f \in C_c^\infty(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$

$$\tilde{S}^\mu_{L^\alpha,m}(f)(x) := \left(\int_0^\infty\int_{|x-y|<t}\left(1+\frac{t}{\rho(y)}\right)^\mu\left|t^{2\alpha m}\partial_s^m e^{-sL^\alpha}(f)(y)|_{s=t^{2\alpha}}\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2}. \quad (128)$$

It is easy to see that $\tilde{S}^\mu_{L^\alpha,m}(f)(x) \lesssim I(x) + II(x)$, where

$$\begin{cases} I(x) := \left(\int_0^{\rho(x)}\int_{|x-y|<t}\left(1+\frac{t}{\rho(y)}\right)^\mu\left|t^{2\alpha m}\partial_s^m e^{-sL^\alpha}(f)(y)|_{s=t^{2\alpha}}\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2}; \\ II(x) := \left(\int_{\rho(x)}^\infty\int_{|x-y|<t}\left(1+\frac{t}{\rho(y)}\right)^\mu\left|t^{2\alpha m}\partial_s^m e^{-sL^\alpha}(f)(y)|_{s=t^{2\alpha}}\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2}. \end{cases} \quad (129)$$

For $I(x)$, noting that $|x-y| < t < \rho(x)$, we have $\rho(x) \sim \rho(y)$. Thus

$$I(x) \le \left(\int_0^{\rho(x)} \int_{|x-y|<t} 2^{\mu} \left| t^{2\alpha m} \partial_s^m e^{-sL^\alpha}(f)(y)|_{s=t^{2\alpha}} \right|^2 \frac{dydt}{t^{n+1}} \right)^{1/2} \lesssim S_{L^\alpha,m}(f)(x). \tag{130}$$

For $II(x)$, it follows from Lemma 13 that

$$II(x) \le \left(\int_{\rho(x)}^{\infty} \int_{|x-y|<t} \left(\int_{\mathbb{R}^n} \left(1 + \frac{t}{\rho(y)}\right)^{-N+\mu/2} \frac{t^{2\alpha m}}{(t+|y-z|)^{n+2\alpha m}} |f(z)| dz \right)^2 \frac{dydt}{t^{n+1}} \right)^{1/2} \le II_1(x) + II_2(x), \tag{131}$$

where

$$\begin{cases} II_1(x) := \left(\int_{\rho(x)}^{\infty} \int_{|x-y|<t} \left(\int_{2B_x} \left(1 + \frac{t}{\rho(y)}\right)^{-N+\mu/2} \frac{t^{2\alpha m}}{(t+|y-z|)^{n+2\alpha m}} |f(z)| dz \right)^2 \frac{dydt}{t^{n+1}} \right)^{1/2}; \\ II_2(x) := \left(\int_{\rho(x)}^{\infty} \int_{|x-y|<t} \left(\int_{(2B_x)^c} \left(1 + \frac{t}{\rho(y)}\right)^{-N+\mu/2} \frac{t^{2\alpha m}}{(t+|y-z|)^{n+2\alpha m}} |f(z)| dz \right)^2 \frac{dydt}{t^{n+1}} \right)^{1/2}. \end{cases} \tag{132}$$

For $II_1(x)$, similar to the proof of Theorem 25, we have

$$\begin{aligned} II_1(x) &\lesssim \left(\int_{\rho(x)}^{\infty} \int_{|x-y|<t} \frac{1}{(1+t/\rho(x))^{(2N-\mu)/(N_0+1)}} \left(\frac{\rho(x)}{t}\right)^{2n} \frac{dydt}{t^{n+1}} \right)^{1/2} M_{\bar{\theta}}(f)(x) \\ &\lesssim \left(\int_1^{\infty} \frac{1}{(1+u)^{(2N-\mu)/(N_0+1)}} u^{-2n} \frac{du}{u} \right)^{1/2} M_{\bar{\theta}}(f)(x) \lesssim M_{\bar{\theta}}(f)(x), \end{aligned} \tag{133}$$

where $u = t/\rho(x)$. To deal with $II_2(x)$, similarly, for $\tau = 2\alpha m - \bar{\theta} \in (0, n)$, $N > (N_0+1)(\bar{\theta}+\tau) + \mu/2$ and $\bar{\theta} \in [0, 2\alpha m)$, we obtain

$$\begin{aligned} II_2(x) &\lesssim \left\{ \left(\int_{\rho(x)}^{\infty} \int_{|x-y|<t} \left(\frac{\rho(x)}{t}\right)^{2\tau} \frac{dydt}{t^{n+1}} \right)^{1/2} + \left(\int_{\rho(x)}^{\infty} \int_{|x-y|<t} \left(1 + \frac{t}{\rho(x)}\right)^{-(2N-\mu)/(N_0+1)} \left(\frac{t}{\rho(x)}\right)^{4\alpha m} \frac{dydt}{t^{n+1}} \right)^{1/2} \right\} \\ &\cdot M_{\bar{\theta}}(f)(x) \lesssim \left\{ \left(\int_1^{\infty} u^{-2\tau} \frac{du}{u} \right)^{1/2} + \left(\int_1^{\infty} \frac{u^{4\alpha m}}{(1+u)^{(2N-\mu)/(N_0+1)}} \frac{du}{u} \right)^{1/2} \right\} \\ &\cdot M_{\bar{\theta}}(f)(x) \lesssim M_{\bar{\theta}}(f)(x), \end{aligned} \tag{134}$$

where $u = t/\rho(x)$. Thus, $II(x) \lesssim M_{\bar{\theta}}(f)(x)$ and

$$\tilde{S}^{\mu}_{L^\alpha,m}(f)(x) \lesssim S_{L^\alpha,m}(f)(x) + M_{\bar{\theta}}(f)(x). \tag{135}$$

Using (67), Lemma 9 and Theorem 25, we can get that for $\omega \in A_p^{\rho,\theta}(\mathbb{R}^n)$, $\theta \in [0, (4/9)(2\alpha - \delta))$, where $\alpha \in (1/2, 1)$ and $\delta = 2 - n/q$ with $q > n/2$, $1 < p < \infty$, and $f \in L^p(\omega)$

$$\left\| \tilde{S}^{\mu}_{L^\alpha,m}(f) \right\|_{L^p(\omega)} \lesssim [\omega]_{A_p^{\rho,\theta}(\mathbb{R}^n)}^{\max\{1/2, 1/(p-1)\}} \|f\|_{L^p(\omega)}. \tag{136}$$

Next, we claim that for any $\beta \in [1,\infty)$, $\theta \in [0,\infty)$, $\omega \in A_p^{\rho,\theta}(\mathbb{R}^n)$, and $f \in L^p(\omega)$

$$\left\| S_{L^\alpha,m,\beta}(f) \right\|_{L^p(\omega)} \lesssim \beta^{n+\theta} [\omega]_{A_p^{\rho,\theta}(\mathbb{R}^n)}^{1/2 \max\{1, 1/(p-1)\}} \left\| \tilde{S}^{2\theta}_{L^\alpha,m}(f) \right\|_{L^p(\omega)}. \tag{137}$$

Indeed, for any $\omega \in A_2^{\rho,\theta}(\mathbb{R}^n)$ and $f \in C_c^{\infty}(\mathbb{R}^n)$, using Lemma 27, we have

$$\begin{aligned} \left\| S_{L^\alpha,m,\beta}(f) \right\|_{L^2(\omega)}^2 &= \int_{\mathbb{R}^n} \int_0^{\infty} \int_{|x-y|<\beta t} \left| t^{2\alpha m} \partial_s^m e^{-sL^\alpha}(f)(y) \Big|_{s=t^{2\alpha}} \right|^2 \frac{dydt}{t^{n+1}} \omega(x) dx \\ &= \int_0^{\infty} \int_{\mathbb{R}^n} \omega(B(y, \beta t)) \left| t^{2\alpha m} \partial_s^m e^{-sL^\alpha}(f)(y) \Big|_{s=t^{2\alpha}} \right|^2 \frac{dydt}{t^{n+1}} \\ &\lesssim \int_0^{\infty} \int_{\mathbb{R}^n} [\omega]_{A_2^{\rho,\theta}(\mathbb{R}^n)} \beta^{2n} \\ &\quad \cdot \left(1 + \frac{\beta t}{\rho(y)}\right)^{2\theta} \omega(B(y,t)) \left| t^{2\alpha m} \partial_s^m e^{-sL^\alpha}(f)(y) \Big|_{s=t^{2\alpha}} \right|^2 \frac{dydt}{t^{n+1}} \\ &\lesssim \beta^{2(n+\theta)} [\omega]_{A_2^{\rho,\theta}(\mathbb{R}^n)} \int_0^{\infty} \int_{\mathbb{R}^n} \left(1 + \frac{t}{\rho(y)}\right)^{2\theta} \\ &\quad \cdot \omega(B(y,t)) | t^{2\alpha m} \partial_s^m e^{-sL^\alpha}(f)(y)|_{s=t^{2\alpha}} |^2 \frac{dydt}{t^{n+1}} \\ &\simeq \beta^{2(n+\theta)} [\omega]_{A_2^{\rho,\theta}(\mathbb{R}^n)} \left\| \tilde{S}^{2\theta}_{L^\alpha,m}(f) \right\|_{L^2(\omega)}^2. \end{aligned} \tag{138}$$

Hence

$$\left\| S_{L^\alpha,m,\beta}(f) \right\|_{L^2(\omega)} \lesssim \beta^{n+\theta} [\omega]_{A_2^{\rho,\theta}(\mathbb{R}^n)}^{1/2} \left\| \tilde{S}^{2\theta}_{L^\alpha,m}(f) \right\|_{L^2(\omega)}. \tag{139}$$

Choose $\mathscr{F}$ to be the family of all pairs $(\tilde{f}, \tilde{g}) = (S_{L^\alpha,m,\beta}(f), \beta^{n+\theta} \tilde{S}^{2\theta}_{L^\alpha,m}(f))$. Since

$$\int_{\mathbb{R}^n} \left(S_{L^\alpha,m,\beta}(f)(x) \right)^2 \omega(x) dx \lesssim [\omega]_{A_2^{\rho,\theta}(\mathbb{R}^n)} \int_{\mathbb{R}^n} \left(\beta^{n+\theta} \tilde{S}^{2\theta}_{L^\alpha,m}(f)(x) \right)^2 \omega(x) dx, \tag{140}$$

it can be deduced from Corollary 10 that for $p \in (1,\infty)$

$$\begin{aligned} \int_{\mathbb{R}^n} \left(S_{L^\alpha,m,\beta}(f)(x) \right)^p \omega(x) dx &\lesssim [\omega]_{A_p^{\rho,\theta}(\mathbb{R}^n)}^{p/2 \max\{1, 1/(p-1)\}} \\ &\cdot \int_{\mathbb{R}^n} \left(\beta^{n+\theta} \tilde{S}^{2\theta}_{L^\alpha,m}(f)(x) \right)^p \omega(x) dx, \end{aligned} \tag{141}$$

which implies (137) holds. Hence

$$g^*_{L^\alpha,m,\lambda}(f)(x) \lesssim \sum_{j=0}^{\infty} 2^{-j\lambda n/2} S_{L^\alpha,m,2^j}(f)(x). \tag{142}$$

Furthermore, for any $\omega \in A_p^{\rho,\theta}(\mathbb{R}^n)$, $f \in L^p(\omega)$, $\lambda \in (2(1 + \theta/n),\infty)$, and $\theta \in [0, (4/9)(2\alpha - \delta))$

$$\begin{aligned}\left\|g^*_{L^\alpha,m,\lambda}(f)\right\|_{L^p(\omega)} &\lesssim \sum_{j=0}^{\infty} 2^{-j\lambda n/2}\left\|S_{L^\alpha,m,2^j}(f)\right\|_{L^p(\omega)}\\ &\lesssim \sum_{j=0}^{\infty} 2^{-j\lambda n/2}\left(2^j\right)^{n+\theta}[\omega]_{A_p^{\rho,\theta}(\mathbb{R}^n)}^{1/(2\max\{(1,1)/(p-1)\})}\left\|\tilde{S}^{2\theta}_{L^\alpha,m}(f)\right\|_{L^p(\omega)}\\ &\lesssim \sum_{j=0}^{\infty} 2^{-j\lambda n/2}\left(2^j\right)^{n+\theta}[\omega]_{A_p^{\rho,\theta}(\mathbb{R}^n)}^{\max\{(1/2,1)/(p-1)\}}[\omega]_{A_p^{\rho,\theta}(\mathbb{R}^n)}^{1/(2\max\{(1/2,1)/(p-1)\})}\|f\|_{L^p(\omega)}\\ &\lesssim [\omega]_{A_p^{\rho,\theta}(\mathbb{R}^n)}^{1/(2\max\{(1/2,1)/(p-1)\})+\max\{(1/2,1)/(p-1)\}}\|f\|_{L^p(\omega)},\end{aligned} \tag{143}$$

which completes the proof of Theorem 28. □

Theorem 29. *Let $n \geq 3$, $p \in (1,\infty)$, and $m \in \mathbb{Z}_+$. Assume that $\theta \in [0, (4/9)(2\alpha - \delta))$ with $\alpha \in (1/2, 1)$ and $\delta = 2 - n/q$, where $q \in (n, n/(2(1-\alpha)))$, and $\lambda \in (2(1+(\theta/n)),\infty)$. There exists a constant $C > 0$ such that for any $\omega \in A_p^{\rho,\theta}(\mathbb{R}^n)$ and $f \in L^p(\omega)$,*

$$\left\|\tilde{g}^*_{L^\alpha,\lambda}(f)\right\|_{L^p(\omega)} \leq C[\omega]_{A_p^{\rho,\theta}(\mathbb{R}^n)}^{1/2\max\{1,1/(p-1)\}+\max\{1/2,1/(p-1)\}}[\omega]_{A_p^{\rho,\theta}(\mathbb{R}^n)}^{\max\{1/2,1/(p-1)\}}\|f\|_{L^p(\omega)}. \tag{144}$$

Proof. Similar to the proof of Theorem 28, we can prove this theorem via Lemma 14 and Theorem 26. Hence, we omit the details. □

Data Availability

The findings in this research do not make use of data.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work was in part supported by the National Science Foundation of China (Nos. 11871293, 12071272) and Natural Science Foundation of Shandong Province (No. ZR2020MA004).

References

[1] B. Simon, "Schrödinger operators in the twentieth century," *Journal of Mathematical Physics*, vol. 41, no. 6, pp. 3523–3555, 2000.

[2] Z. Shen, "L^p estimates for Schrödinger operators with certain potentials," *Annales de l'institut Fourier*, vol. 45, pp. 513–546, 1995.

[3] Z. Shen, "Estimates in L_p for magnetic Schrödinger operators," *Indiana University Mathematics Journal*, vol. 45, pp. 817–841, 1996.

[4] Z. Shen, "On fundamental solutions of generalized Schrodinger operators," *Journal of Functional Analysis*, vol. 167, no. 2, pp. 521–564, 1999.

[5] J. Dziubański and J. Zienkiewicz, "Hardy space H^1 associated to Schrödinger operator with potential satisfying reverse Hölder inequality," *Revista Matemática Iberoamericana*, vol. 15, pp. 279–296, 1999.

[6] J. Dziubański and J. Zienkiewicz, "H^p spaces associated with Schrödinger operators with potentials from reverse Hölder classes," *Colloquium Mathematicum*, vol. 98, pp. 5–38, 2003.

[7] J. Dziubański, G. Garrigós, T. Martínez, L. Torrea, and J. Zienkiewicz, "BMO spaces related to Schrödinger operators with potentials satisfying a reverse Hölder inequality," *Mathematische Zeitschrift*, vol. 249, pp. 329–356, 2005.

[8] B. Bongioanni, E. Harboure, and O. Salinas, "Classes of weights related to Schrodinger operators," *Journal of Mathematical Analysis and Applications*, vol. 373, no. 2, pp. 563–579, 2011.

[9] B. Bongioanni, E. Harboure, and O. Salinas, "Commutators of Riesz transforms related to Schrödinger operators," *Journal of Fourier Analysis and Applications*, vol. 17, no. 1, pp. 115–134, 2011.

[10] X. Duong, E. Ouhabaz, and L. Yan, "Endpoint estimates for Riesz transforms of magnetic Schrödinger operators," *Arkiv för Matematik*, vol. 44, no. 2, pp. 261–275, 2006.

[11] X. Duong, L. Yan, and C. Zhang, "On characterization of Poisson integrals of Schrodinger operators with BMO traces," *Journal of Functional Analysis*, vol. 266, no. 4, pp. 2053–2085, 2014.

[12] Z. Guo, P. Li, and L. Peng, "L_p boundedness of commutators of Riesz transforms associated to Schrödinger operator," *Journal of Mathematical Analysis and Applications*, vol. 341, pp. 421–432, 2008.

[13] H.-Q. Li, "Estimations L_p des opérateurs de Schrödinger sur les groupes nilpotents," *Journal of Functional Analysis*, vol. 161, pp. 152–218, 1999.

[14] C. Lin and H. Liu, "$BMO_L(\mathbb{H}^n)$ spaces and Carleson measures for Schrödinger operators," *Advances in Mathematics*, vol. 228, pp. 1631–1688, 2011.

[15] T. Ma, P. Stinga, J. Torrea, and C. Zhang, "Regularity properties of Schrodinger operators," *Journal of Mathematical Analysis and Applications*, vol. 388, no. 2, pp. 817–837, 2012.

[16] L. Tang, "Weighted norm inequalities for Schrödinger type operators," *Forum Mathematicum*, vol. 27, no. 4, pp. 2491–2532, 2015.

[17] D. Yang, D. Yang, and Y. Zhou, "Localized Morrey-Campanato spaces on metric measure spaces and applications to Schrüdinger operators," *Nagoya Mathematical Journal*, vol. 198, pp. 77–119, 2010.

[18] J. Li, R. Rahm, and B. Wick, "A_p weights and quantitative estimates in the Schrödinger setting," *Mathematische Zeitschrift*, vol. 293, pp. 259–283, 2019.

[19] J. Zhang and D. Yang, "Quantitative boundedness of Littlewood-Paley functions on weighted Lebesgue spaces in the Schrodinger setting," *Journal of Mathematical Analysis and Applications*, vol. 484, no. 2, article 123731, 2020.

[20] T. Bui, T. Bui, and X. Duong, "Quantitative estimates for square functions with new class of weights," *Potential Analysis*, vol. 57, no. 4, pp. 545–569, 2022.

[21] T. Bui, T. Bui, and X. Duong, "Quantitative weighted estimates for some singular integrals related to critical functions," *Journal of Geometric Analysis*, vol. 31, no. 10, pp. 10215–10245, 2021.

[22] Y. Wen and H. Wu, "Quantitative weighted bounds for variation operators associated with heat semigroups in the Schrödinger setting," *Rocky Mountain Journal of Mathematics*, 2022.

[23] A. Grigor'yan, "Heat kernels and function theory on metric measure spaces," *Contemporary Mathematics*, vol. 338, pp. 143–172, 2003.

[24] C. Miao, B. Yuan, and B. Zhang, "Well-posedness of the Cauchy problem for the fractional power dissipative equations," *Nonlinear Analysis*, vol. 68, no. 3, pp. 461–484, 2008.

[25] Z. Shen, "On the Neumann problem for Schrödinger operators in Lipschitz domains," *Indiana University Mathematics Journal*, vol. 43, no. 1, pp. 143–176, 1994.

[26] P. Li, Z. Wang, T. Qian, and C. Zhang, "Regularity of fractional heat semigroup associated with Schrödinger operators," *Fractal and Fractional*, vol. 6, no. 2, p. 112, 2022.

[27] Z. Wang, P. Li, and C. Zhang, "Boundedness of operators generated by fractional semigroups associated with Schrödinger operators on Campanato type spaces via $T1$ theorem," *Banach Journal of Mathematical Analysis*, vol. 15, no. 4, pp. 1–37, 2021.

[28] J. Wilson, "The intrinsic square function," *Revista Matemática Iberoamericana*, vol. 23, pp. 771–791, 2007.

[29] A. Lerner, "Sharp weighted norm inequalities for Littlewood-Paley operators and singular integrals," *Advances in Mathematics*, vol. 226, no. 5, pp. 3912–3926, 2011.

[30] J. Wilson, *Weighted Littlewood-Paley Theory and Exponential-Square Integrability*, vol. 1924 of Lecture Notes in Mathematics, Springer, Berlin, 2008.