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Research Article

New Fixed Point Theorems for $\theta - \omega -$ Contraction on (λ, μ)-Generalized Metric Spaces

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In this paper, we consider a new extension of the Banach contraction principle, which is called the $\theta - \omega -$ contraction inspired by the concept of $\theta -$ contraction in (λ, μ)-generalized metric spaces and to study the existence and uniqueness of fixed point for the mappings in metric space. Moreover, we discuss some illustrative examples to highlight the improvements that were made, and we also give an iterated application of linear integral equations.

1. Introduction

Fixed point theory is an important and fascinating subject, and it provides essential tools for solving problems arising in various branches of mathematical analysis, see [1–5]. Fixed point theory guarantees the uniqueness and existence of the solution of integral and differential equations.

In 1922, a Polish mathematician Banach introduced a contraction principle [6], which was one of the most applicable results in mathematics. In recent times, many generalizations and improvement of the Banach contraction principle have appeared in the literature (see [7–13]).

The metric function has been generalized many times by modifying the associated axioms. Specifically, Bakhtin [14] and Czerwik [15] presented b-metric spaces in a way that the triangle inequality was replaced by the b-triangle inequality: $\rho(\nu, \mu) \leq s(\rho(\nu, \varsigma) + \rho(\varsigma, \mu))$ for all pairwise distinct points ν, μ, ρ and $s \geq 1$. One of these generalizations was given by Branciari [16]. Any metric space is a generalized metric space, but in general, generalized metric space might not be a metric space for details [17, 18]. Various fixed point results have been established on such spaces; for example, Nazam et al. [19–21] have proved some fixed point and common fixed point results in partial metric and S-metric spaces. For more details, see [12, 22, 23]).

Huang and Zhang [24] redefined the concept of K-metric spaces and convergence in an ordered Banach space E with a normal solid cone. Shatanawi et al. in [23] defined E-metric spaces and characterized the cone metric spaces in a more general way by defining ordered normed spaces. Also, Mehmood et al. in [25] deduced more results about E-metric spaces.

Jleli et al. [11, 22] introduced the notion of θ-contraction and proved a fixed point theorem which generalizes the Banach contraction principle in a different way than in the known results from the literature. Later, Kari et al. [26] proved new type of fixed-point theorems in a rectangular metric space and generalized asymmetric metric space by using a modified generalized θ-contraction maps.

In 2014, Hussain and Salimi [27] introduced the notion of an $\alpha -$ GF-contraction and stated fixed point theorems for $\alpha -$ GF-contractions. On the other hand, Hussain et al. [28] establish some new fixed-point theorems for generalized $(\lambda - \mu) - G - F$-contractions mappings in complete b-metric spaces.

In this paper, we introduce the notion of a generalized $(\lambda-\mu)-\theta-\omega$-contraction to generalize an θ-contraction in generalized metric space. Also, examples are given to illustrate the obtained results we derive some useful corollaries of these results.

2. Preliminaries

Definition 1 (see [16]). Let C be a nonempty set and $\rho : C \times C \longrightarrow \mathbb{R}^+$ be a mapping such that for all $z, t \in C$ and for all distinct points $m, n \in C$, each of them different from z and t

(i) $\rho(z,t)=0 \Leftrightarrow z=t$

(ii) $\rho(z,t)=\rho(t,z) \forall z, t \in C$

(iii) $\rho(z,t) \leq \rho(z,m)+\rho(m,n)+\rho(n,t)$. Then, (C,ρ) is called a generalized metric space

Definition 2 (see [16]). Let (C,ρ) be a generalized metric space and $\{\alpha_m\}_{m\in\mathbb{N}}$ be a sequence in C, and $\alpha \in C$. Then,

(i) We say that the sequence $\{\alpha_m\}_{m\in\mathbb{N}}$ converges to α if and only if

$$\lim_{m\longrightarrow+\infty} \rho(\alpha,\alpha_m)=0 \tag{1}$$

(ii) We say that $\{\alpha_m\}_{m\in\mathbb{N}}$ is the Cauchy if

$$\lim_{m,p\longrightarrow+\infty} \rho(x_m,x_p)=0 \tag{2}$$

Lemma 3 (see [16]). *Let (C,ρ) be a generalized metric space and $\{\alpha_m\}_m$ be a Cauchy sequence with pairwise disjoint elements in C. If $\{\alpha_m\}_m$ converges to both $\alpha \in C$ and $\beta \in C$, then $\alpha=\beta$.*

Definition 4 (see [16]). The generalized metric space is said to be complete if every Cauchy sequence $\{\alpha_m\}_m$ *in* C converges to an $\alpha \in C$.

In [11, 22], authors defined the following collections functions. Let Θ_c be the family of all functions θ: $(0,+\infty) \longrightarrow (1,+\infty)$ such that

$(\theta_1)\theta$ is increasing

(θ_2) For each sequence $(\alpha_m) \subset (0,+\infty)$

$$\lim_{n\longrightarrow 0} \alpha_m = 0 \Leftrightarrow \lim_{n\longrightarrow\infty} \theta(\alpha_m)=1 \tag{3}$$

(θ_{3C}) is continuous.

Let Θ_G be the family of all functions $\theta : (0,+\infty) \longrightarrow (1,+\infty)$ such that

$(\theta_1)\theta$ is increasing

(θ_2) for each sequence $(\alpha_m) \subset (0,+\infty)$

$$\lim_{m\longrightarrow 0} \alpha_m = 0 \Leftrightarrow \lim_{m\longrightarrow\infty} \theta(\alpha_m)=1 \tag{4}$$

(θ_{3G}) there exist $r \in (0,1)$ and $l \in (0,+\infty]$ such that $\lim_{l\longrightarrow 0}\theta(l)-1/l^r = l$.

Definition 5. Let (C,ρ) be a generalized metric space and $T : C \longrightarrow C$ be a mapping. T is said to be a θ-contraction if there exist $\theta \in \Theta_G$ and $s \in (0,1)$ such that for any u, $v \in C$,

$$\rho(Tu,Tv)>0 \Rightarrow \theta[\rho(Tu,Tv)] \leq [\theta(M(u,v))]^s, \tag{5}$$

where

$$M(u,v)=\max\{\rho(u,v),\rho(Tu,v),\rho(u,Tv)\}. \tag{6}$$

Theorem 6 (see [11]). *Let (C,ρ) be a complete generalized metric space and let $T : C \longrightarrow C$ be a $\theta-$contraction. Then, T has a unique fixed point.*

Remark 7. The sets Θ_G and Θ_C are different.

Example 8. Define $\theta : (0,+\infty) \longrightarrow (1,+\infty)$ by

$$\theta(t)=\begin{cases}\sqrt{t}+1 \text{ if } t \in \left(0,\dfrac{1}{2}\right], \\ e^t \text{ if } t \in \left[\dfrac{1}{2},+\infty\right).\end{cases} \tag{7}$$

Then, $\theta \in \Theta_G$, but for any $t>0$,

$$\begin{aligned}&\lim_{t\longrightarrow\frac{1}{2}^-}\theta(t)=\sqrt{\frac{1}{2}}+1.\\&\lim_{t\longrightarrow\frac{1}{2}^+}\theta(t)=e^{1/2}\end{aligned} \tag{8}$$

Since $\sqrt{(1/2)}+1 \neq e^{1/2}$, so, θ does not satisfy the condition (θ_{3C}), then, $\theta \notin \Theta_{C.}$

Example 9 (see [29]). Define $\theta : (0,+\infty) \longrightarrow (1,+\infty)$ by

$$\theta(t)=e^{e^{-1/t^p}}, p>0. \tag{9}$$

Then, $\theta \in \Theta_C$, but for any $r>0$,

$$\begin{aligned}\lim_{t\longrightarrow 0^+}\frac{\theta(t)-1}{t^r}&=\lim_{t\longrightarrow 0^+}\frac{e^{e^{-1/t^p}}-1}{t^r}=\lim_{t\longrightarrow 0^+}\frac{e^{-1/t^p}}{t^r}\\&=\lim_{t\longrightarrow 0^+}\frac{1/t^r}{1/e^{1/t^p}}=0.\end{aligned} \tag{10}$$

So, θ does not satisfy the condition (θ_{3G}), then, $\theta \notin \Theta_G$

Definition 10 (see [28]). Let $T : C \longrightarrow C$ and $\lambda,\mu : C \times C \longrightarrow [0,+\infty)$. We say that T is a triangular (λ,μ)-admissible mapping if

$(T_1)\lambda(k,l) \geq 1 \Rightarrow \lambda(Tk,Tl) \geq 1$, for all $k, l \in C$

$(T_2)\mu(k,l) \leq 1 \Rightarrow \mu(Tk,Tl) \leq 1$, for all $k, l \in C$

$$(\mathrm{T}_3)\begin{cases}\lambda(k,l)\geq 1\\ \lambda(l,m)\geq 1\end{cases}\Rightarrow \lambda(k,m)\geq 1 \text{ for all } k,l,m\in C$$

$$(\mathrm{T}_4)\begin{cases}\mu(k,l)\leq 1\\ \lambda(l,m)\leq 1\end{cases}\Rightarrow \mu(k,m)\leq 1 \text{ for all } k,l,m\in C.$$

Definition 11 (see [28]). Let (C,ρ) be a generalized metric space and let $\lambda,\mu{:}C\times C\longrightarrow[0,+\infty)$ be two mappings. Then,

$(k)T$ is a λ-continuous mapping on (C,ρ), if for a given point $\alpha\in C$ and sequence (α_m) in $C,\alpha_m\longrightarrow\alpha$ and $\lambda(\alpha_m,\alpha_{m+1})\geq 1$ for all $m\in\mathbb{N}$, then $T\alpha_m\longrightarrow T\alpha$

$(l)T$ is a μ subcontinuous mapping on (C,ρ), if for given point $\alpha\in C$ and sequence (α_m) in $C,\alpha_m\longrightarrow\alpha$ and $\mu(\alpha_m,\alpha_{m+1})\leq 1$ for all $m\in\mathbb{N}$, then $T\alpha_m\longrightarrow T\alpha$

$(m)T$ is a (λ,μ)-continuous mapping on (C,ρ), if for given point $\alpha\in C$ and sequence (α_m) in C such that $\alpha_m\longrightarrow\alpha$ with $\lambda(\alpha_m,\alpha_{m+1})\geq 1$ and $\mu(\alpha_m,\alpha_{m+1})\leq 1$ for all $m\in\mathbb{N}$, we have $T\alpha_m\longrightarrow T\alpha$

Definition 12 (see [30]). Let (C,ρ) be a rectangular b-metric space and let $\lambda,\mu{:}C\times C\longrightarrow[0,+\infty)$ be two mappings. Then, the space C is said to be

$(k)\lambda$-complete, if every Cauchy sequence (α_m) in C with $\lambda(\alpha_m,\alpha_{m+1})\geq 1$ for all $m\in\mathbb{N}$, converges in C

$(l)\mu$ subcomplete, if every Cauchy sequence (α_m) in C with $\mu(\alpha_m,\alpha_{m+1})\leq 1$ for all $m\in\mathbb{N}$, converges in C

$(m)(\lambda,\mu)$-complete, if every Cauchy sequence (α_m) in C with $\lambda(\alpha_m,\alpha_{m+1})\geq 1$ and $\mu(\alpha_m,\alpha_{m+1})\leq 1$ for all $m\in\mathbb{N}$, converges in C

Definition 13 (see [30]). Let (C,ρ) be a generalized metric space and let $\lambda,\mu:C\times C\longrightarrow[0,+\infty)$ be two mappings. Then, the space C is said to be

$(k)\lambda$-regular, if $\alpha_m\longrightarrow\alpha$, $\lambda(\alpha_m,\alpha_{m+1})\geq 1$ for all $m\in\mathbb{N}$, implies $\lambda(\alpha_m,\alpha)\geq 1$ for all $m\in\mathbb{N}$

$(l)\mu-$ subregular, if $\alpha_m\longrightarrow\alpha$, $\mu(\alpha_m,\alpha_{m+1})\leq 1$ for all $m\in\mathbb{N}$, implies $\mu(\alpha_m,\alpha)\leq 1$ for all $m\in\mathbb{N}$

$(m)(\lambda,\mu)$-regular, if $\alpha_m\longrightarrow\alpha$, $\lambda(\alpha_m,\alpha_{m+1})\geq 1$ and $\mu(\alpha_m,\alpha_{m+1})\leq 1$ for all $m\in\mathbb{N}$, imply that $\lambda(\alpha_m,\alpha)\geq 1$ and $\mu(\alpha_m,\alpha)\leq 1$ for all $m\in\mathbb{N}$

3. Main Results

In this section, we introduce a new notion of generalized $\theta-\omega$-contraction in the context of $(\lambda-\mu)$-generalized metric spaces as follows.

Definition 14. Let Ω denote the set of all functions $\omega:\boldsymbol{R}_+^{\mathbf{5}}\longrightarrow\mathbb{R}_+$ satisfying the following: for all $u_1,u_2,u_3,u_4,u_5\in\mathbb{R}_+$ with $u_1u_2u_3u_4u_5=0$, there exists $\delta\in]0,1[$ such that $\omega(u_1,u_2,u_3,u_4,u_5)=\delta$.

Example 15. If $\omega(u_1,u_2,u_3,u_4,u_5)=\min\{u_1,u_2,u_3,u_4,u_5\}+\delta$ where $\delta\in]0,1[$, then, $\omega\in\Omega$.

Example 16. If $\omega(u_1,u_2,u_3,u_4,u_5)=\min\{u_1,u_2,u_3,u_4,u_5\}/\max\{u_1,u_2,u_3,u_4,u_5+1\}+\delta$ where $\delta\in]0,1[$, then, $\omega\in\Omega$.

Definition 17. Let (C,ρ) be a $(\lambda-\mu)$ -complete generalized metric space, and let T be a self-mapping on C, where λ, $\mu:C\times C\longrightarrow[0,+\infty)$ are two functions. We say that T is an $(\lambda,\mu)-\omega-\theta$ -contraction, if for all $x,y\in C$ with $(\lambda(x,y)\geq 1$ and $\mu(x,y)\leq 1)$ and $\rho(Tx,Ty)>0$, we have

$$\theta(\rho(Tx,Ty))\leq[\theta(M(x,y))]^{\omega\left(\rho(x,Tx),\rho(y,Ty),\rho(x,y),\rho(y,Tx),\rho\left(T^2x,y\right)\right)},\tag{11}$$

where $\theta\in\Theta_C$, $\omega\in\Omega$, and

$$M(x,y)=\max\{\rho(x,y),\rho(x,Tx),\rho(y,Ty),\rho(Tx,y),\times\rho\left(T^2x,y\right),\rho\left(T^2x,Ty\right),\rho\left(T^2x,Tx\right)\}.\tag{12}$$

Example 18. Let $C=[1,+\infty)$ and $\theta(t)=e^t$for all $t\in]0,+\infty[$. So, $\theta\in\Theta_C$.

Define $\rho:C\times C\longrightarrow[0,+\infty)$ by

$$\rho(x,y)=|x-y|.\tag{13}$$

Then, (C,ρ) is a complete generalized metric space. Define $T:C\longrightarrow C$ by

$$\begin{aligned}&T(t)=\sqrt{t}\text{ for all } t\in[1,+\infty),\\&\lambda(x,y)=1,\\&\mu(x,y)=1,\\&\omega(t_1,t_2,t_3,t_4,t_5)=\frac{\min\{t_1,t_2,t_3,t_4,t_5\}}{\max\{t_1,t_2,t_3,t_4,t_5\}+1}\text{ for all } t_1,t_2,t_3,t_4,t_5\in\mathbb{R}_+.\end{aligned}\tag{14}$$

Then, T is an (λ,μ)-continuous triangular $(\lambda,\mu)-$ admissible mapping.

Case 1. $0\leq x\leq y$

$$\begin{aligned}&\rho(Tx,Ty)=\left(\sqrt{y}-\sqrt{x}\right),\\&M(\rho(x,y))=\max\left\{\rho(x,y),\rho\left(x,\sqrt{x}\right),\rho\left(y,\sqrt{y}\right),\rho\left(y,\sqrt{x}\right),\times\rho\left(\sqrt{\sqrt{x}},y\right),\rho\left(\sqrt{\sqrt{x}},\sqrt{y}\right),\rho\left(\sqrt{\sqrt{x}},\sqrt{x}\right)\right\}.\end{aligned}\tag{15}$$

Since $x\leq y$, we get

$$M(x,y)=\max\left\{(y-x),\left(x-\sqrt{x}\right),\left(y-\sqrt{y}\right),\left(y-\sqrt{x}\right),\times\left(y-\sqrt{\sqrt{x}}\right),\left(\sqrt{y}-\sqrt{\sqrt{x}},\right)\left(\sqrt{x}-\sqrt{\sqrt{x}}\right)\right\}.\tag{16}$$

Thus,

$$M(x,y)\geq y-x,\tag{17}$$

which implies that

$$\theta(M(x,y)) \geq \theta(\rho(x,y)) = e^{(y-x)}. \tag{18}$$

Thus,

$$\begin{aligned}
&\theta(\rho(x,y))^{\omega\left(\rho(x,Tx),\rho(y,Ty),\rho(x,Ty),\rho(y,Tx),\rho\left(T^2x,y\right)\right)}\\
&= e^{(y-x)\omega(x-\sqrt{x},y-\sqrt{y},|x-\sqrt{y}|,y-\sqrt{x},y-\sqrt{\sqrt{x}})}\\
&= e^{(y-x)(\min\{x-\sqrt{x},y-\sqrt{y},|x-\sqrt{y}|,y-\sqrt{x},y\sqrt{\sqrt{x}}\}/\max\{x-\sqrt{x},y-\sqrt{y},|x-\sqrt{y}|,y-\sqrt{x},y-\sqrt{\sqrt{x}})\}+1)}\\
&\leq e^{(y-x)}.
\end{aligned} \tag{19}$$

Thus,

$$\theta(\rho(Tx,Ty)) \leq \theta(M(\rho(x,y)))^{\omega\left(\rho(x,y),\rho(x,Tx),\rho(y,Ty),\rho(y,Tx),\rho\left(T^2x,y\right)\right)}. \tag{20}$$

Case 2. $x > y > 0$. Similarly, we conclude that

$$\theta(\rho(Tx,Ty)) \leq \theta(M(\rho(x,y)))^{\omega\left(\rho(x,y),\rho(x,Tx),\rho(y,Ty),\rho(y,Tx),\rho\left(T^2x,y\right)\right)}. \tag{21}$$

Hence, T is an $(\lambda, \mu) - \omega - \theta$-contraction.

Theorem 19. *Let (C, ρ) be a (λ, μ) – complete generalized metric space. Let $T : C \longrightarrow C$, satisfying the following conditions:*

(I) T is a triangular (λ, μ) – admissible mapping

(II) T is an $(\lambda, \mu) - \theta - \omega$ – contraction

(III) There exists $x_0 \in C$ such that $\lambda(x_0, Tx_0) \geq 1$ and $\mu(x_0, Tx_0) \leq 1$

(IV) T is an (λ, μ) – continuous

Then, T has a fixed point. Moreover, T has a unique fixed point when $\lambda(k, l) \geq 1$ and $\mu(k, l) \leq 1$ for all fixed points $k, l \in C$.

Proof. Let $x_0 \in C$ such that $\lambda(x_0, Tx_0) \geq 1$ and $\mu(x_0, Tx_0) \leq 1$.

Define a sequence $\{x_m\}$ by $x_m = T^m x_0 = Tx_{m-1}$. Since T is a triangular (λ, μ)-admissible mapping, then $\lambda(x_0, x_1) = \lambda(x_0, Tx_0) \geq 1 \Rightarrow \lambda(Tx_0, Tx_1) \geq 1 = \lambda(x_1, x_2)$ and $\mu(x_0, x_1) = \mu(x_0, Tx_0) \leq 1 \Rightarrow \mu(Tx_0, Tx_1) \leq 1 = \mu(x_1, x_2)$.

Continuing this process, we have $\lambda(x_{m-1}, x_m) \geq 1$ and $\mu(x_{m-1}, x_m) \leq 1$, for all $n \in \mathbb{N}$. By (T_3) and (T_4), one has

$$\lambda(x_m, x_n) \geq 1 \text{ and } \mu(x_m, x_n) \leq 1, \text{for } all\ m, n \in \mathbb{N}, m \neq n. \tag{22}$$

Suppose that there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = Tx_{n_0}$. Then, x_{n_0} is a fixed point of T, and we have nothing to prove. Hence, we assume that $x_n \neq Tx_n$, i.e., $\rho(x_{n-1}, x_n) > 0$ for all $n \in \mathbb{N}$. We have

$$x_n \neq x_m, \text{for } all\ m, n \in \mathbb{N}, m \neq n. \tag{23}$$

Indeed, suppose that $x_n = x_m$ for some $m = n + k > n$, so we have

$$x_{n+1} = Tx_n = Tx_m = x_{m+1}. \tag{24}$$

Denote $\rho_m = \rho(x_m, x_{m+1})$. Then, (11) implies that

$$\begin{aligned}
\theta(\rho_n) = \theta(\rho_m) &= \theta(\rho(Tx_{m-1}, Tx_m))\\
&\leq [\theta(M(x_{m-1}, x_m))]^{\omega(\rho(x_{m-1},x_m),\rho(x_m,x_{m+1}),\rho(x_{m-1},x_m),\rho(x_m,x_m),\rho(x_{m+1},x_m))}\\
&= [\theta(M(x_{m-1}, x_m))]^{\omega(\rho_{m-1},\rho_{m-1},\rho_m,0,\rho_{m+1})},
\end{aligned} \tag{25}$$

where

$$\begin{aligned}
M(x_{m-1}, x_m) = \{&\rho(x_{m-1}, x_m), \rho(x_{m-1}, x_m), \rho(x_m, x_{m+1}),\\
&\times \rho(x_m, x_m), \rho(x_{m+1}, x_m), \rho(x_{m+1}, x_{m+1}),\\
&\times \rho(x_{m+1}, x_m)\}.
\end{aligned} \tag{26}$$

Then,

$$M(x_{m-1}, x_m) = \max\{\rho(x_{m-1}, x_m), \rho(x_m, x_{m+1})\}, \tag{27}$$

and there exists $\delta \in]0, 1[$ such that

$$\omega(\rho_{m-1}, \rho_{m-1}, \rho_m, 0, \rho_{m+1}) = \delta. \tag{28}$$

Thus,

$$\theta(\rho_n) \leq [\theta(M(x_{m-1}, x_m))]^{\delta}. \tag{29}$$

Let

$$M(x_{m-1}, x_m) = \rho(x_m, x_{m+1}). \tag{30}$$

Then, we have

$$\theta(\rho_m) \leq [\theta(\rho_m)]^{\delta} < \theta(\rho_m), \tag{31}$$

which is a contradiction, so

$$\begin{aligned}
M(x_{m-1}, x_m) &= \rho(x_{m-1}, x_m).\\
\rho_n = \rho_m &< \rho_{m-1}
\end{aligned} \tag{32}$$

Continuing this process, we get $\rho_n = \rho_m < \rho_{m-1} < \rho_{m-2} < .. < \rho_n$, which is a contradiction. Thus, as follows, we can assume that (22) and (23) hold.

Substituting $x = x_{n-1}$ and $y = x_n$ in (11), for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
\theta(\rho(x_n, x_{n+1})) &\leq [\theta(M(x_{n-1}, x_n))]^{\omega(\rho(x_{n-1},x_n),\rho(x_n,x_{n+1}),\rho(x_{n-1},x_{n+1}),\rho(x_n,x_n),d(x_{n+1},x_n))}\\
&= [\theta(M(x_{n-1}, x_n))]^{\omega(\rho(x_{n-1},x_n),\rho(x_n,x_{n+1}),\rho(x_{n-1},x_{n+1}),0,\rho(x_{n+1},x_n))},
\end{aligned} \tag{33}$$

where

$$M(x_{n-1}, x_n) = \max \{\rho(x_{n-1}, x_n), \rho(x_{n-1}, x_n), \\ \times \rho(x_n, x_{n+1}), \rho(x_{n+1}, x_n), \rho(x_{n+1}, x_n), \\ \times \rho(x_{n+1}, x_{n+1}), \rho(x_{n+1}, x_n)\}. \tag{34}$$

Then,

$$M(x_{n-1}, x_{n+1}) = \max \{\rho(x_{n-1}, x_n), \rho(x_n, x_{n+1})\}, \tag{35}$$

and there exists $\delta \in (0, 1)$ such that

$$\omega[\rho(x_{n-1},x_n),\rho(x_n,x_{n+1}),\rho(x_{n-1},x_{n+1}),0,\rho(x_{n+1},x_n)] = \delta. \tag{36}$$

Let

$$M(x_{n-1}, x_{n+1}) = \rho(x_n, x_{n+1}). \tag{37}$$

Then,

$$\theta[\rho(x_n, x_{n+1})] \le \left[\theta(\rho(x_n, x_{n+1})\right]^{\delta} < \theta(\rho(x_n, x_{n+1})). \tag{38}$$

It is a contradiction. Therefore,

$$M(x_{n-1}, x_{n+1}) = \rho(x_{n-1}, x_n). \tag{39}$$

Using (θ_1), we get

$$\rho(x_n, x_{n+1}) < \rho(x_{n-1}, x_n). \tag{40}$$

Therefore, $\rho(x_{n,}x_{n+1})_{n\in\mathbb{N}}$ is a nonnegative strictly decreasing sequence of real numbers. Consequently, there exists $\alpha \ge 0$ such that

$$\lim_{n\longrightarrow\infty} \rho(x_{n+1,}x_n) = \alpha. \tag{41}$$

Now, we claim that $\alpha = 0$. Arguing by contradiction, we assume that $\alpha > 0$. Since $\rho(x_{n,}x_{n+1})_{n\in\mathbb{N}}$ is a nonnegative strictly decreasing sequence of real numbers, then we have

$$\rho(x_{n,}x_{n+1}) \ge \alpha \text{ for } all\ n \in \mathbb{N}. \tag{42}$$

By property of θ, we get

$$1 < \theta(\alpha) \le \theta(\rho(x_0, x_1))^{\delta^n}. \tag{43}$$

By letting $n \longrightarrow \infty$ in inequality (43), we obtain

$$1 < \theta(\alpha) \le 1. \tag{44}$$

It is a contradiction. Therefore,

$$\lim_{n\longrightarrow\infty} \rho(x_{n,}x_{n+1}) = 0. \tag{45}$$

Substituting $x = x_{n-1}$ and $y = x_{n+1}$ in (11), for all $n \in \mathbb{N}$, we have

$$\theta(\rho(x_n, x_{n+2})) \le [\theta(M(x_{n-1}, x_{n+1}))]^{\omega(\rho(x_{n-1},x_{n+1}),\rho(x_{n-1},x_n),\rho(x_{n+1},x_{n+2}),\rho(x_n,x_{n+1}),d(x_{n+1},x_{n+1}))} \\ = [\theta(M(x_{n-1}, x_{n+1}))]^{\omega(\rho(x_{n-1},x_{n+1}),\rho(x_{n-1},x_n),\rho(x_{n+1},x_{n+2}),\rho(x_n,x_{n+1}),0)}, \tag{46}$$

where

$$M(x_{n-1}, x_{n+1}) = \max \{\rho(x_{n-1}, x_{n+1}), \rho(x_{n-1}, x_n), \\ \times \rho(x_{n+1}, x_{n+2}), \rho(x_{n+1}, x_n), \rho(x_{n+1}, x_{n+1}), \\ \times \rho(x_{n+1}, x_n), d(x_{n+1}, x_n)\}.. \tag{47}$$

Since

$$\rho(x_{n+1}, x_{n+2}) \le \rho(x_n, x_{n+1}) \le \rho(x_{n-1}, x_n), \tag{48}$$

we have

$$M(x_{n-1}, x_{n+1}) = \max \{\rho(x_{n-1}, x_{n+1}), \rho(x_{n-1}, x_n)\}, \tag{49}$$

and there exists $\delta \in (0, 1)$ such that

$$\omega(\rho(x_{n-1}, x_{n+1}), \rho(x_{n-1}, x_n), \rho(x_{n+1}, x_{n+2}), \rho(x_n, x_{n+1}), 0) = \delta. \tag{50}$$

Then,

$$\theta(\rho(x_n, x_{n+2})) \le [\theta(\max \{\rho(x_{n-1}, x_{n+1}), \rho(x_{n-1}, x_n)\})]^{\delta}. \tag{51}$$

Take $a_n = \rho(x_n, x_{n+2})$ and $b_n = \rho(x_n, x_{n+1})$. Thus, one can write

$$\theta(a_n) \le [\theta(\max \{a_{n-1}, b_{n-1}\})]^{\delta}. \tag{52}$$

By (θ_1), we get

$$a_n < \max \{a_{n-1}, b_{n-1}\}. \tag{53}$$

By (40), we have

$$b_n \le b_{n-1} \le \max \{a_{n-1}, b_{n-1}\}, \tag{54}$$

which implies that

$$\max \{a_n, b_n\} \le \max \{a_{n-1}, b_{n-1}\}, \text{ for } all\ n \in \mathbb{N}. \tag{55}$$

Therefore, the sequence $\max \{a_{n-1}, b_{n-1}\}_{n\in\mathbb{N}}$ is a nonnegative strictly decreasing sequence of real numbers. Thus, there exists $\beta \ge 0$ such that

$$\lim_{n\longrightarrow\infty} \max \{a_n, b_n\} = \beta. \tag{56}$$

We assume that $\beta > 0$. By (45) and

$$\limsup_{n\longrightarrow\infty} b_n = \limsup_{n\longrightarrow\infty} \rho(x_{n-1}, x_n) = 0, \tag{57}$$

then

$$\limsup_{n\longrightarrow\infty} a_n = \limsup_{n\longrightarrow\infty} \max\{a_n, b_n\} = \lim_{n\longrightarrow\infty} \max\{a_n, b_n\}. \tag{58}$$

Taking the $\limsup_{n\longrightarrow\infty}$ in (51), and using the properties of θ, we obtain

$$\theta\left(\limsup_{n\longrightarrow\infty} a_n\right) < \theta\left(\lim_{n\longrightarrow\infty} \max\{a_{n-1}, b_{n-1}\}\right). \tag{59}$$

Therefore,

$$\theta(\beta) < \theta(\beta), \tag{60}$$

which is a contradiction. Therefore,

$$\lim_{n\longrightarrow\infty} \rho(x_n, x_{n+2}) = 0. \tag{61}$$

Next, we shall prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, i.e, $\lim_{n\longrightarrow\infty}\rho(x_n, x_m) = 0$, for all $n, m \in \mathbb{N}$. Suppose to the contrary, we assume that there exist $\varepsilon > 0$ and a sequence $\{n_{(k)}\}$ and $\{m_{(k)}\}$ of natural numbers such that $m_{(k)} > n_{(k)} > k$,
and

$$\rho\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \geq \varepsilon \text{ and } \rho\left(x_{m_{(k)-1}}, x_{n_{(k)}}\right) < \varepsilon. \tag{62}$$

Now, using (40), (51), (61), and the quadrilateral inequality, we find

$$\begin{aligned} \varepsilon \leq \rho\left(x_{m_{(k)}}, x_{n_{(k)}}\right) &\leq \rho\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right) \\ &+ \rho\left(x_{m_{(k)+1}}, x_{m_{(k)-1}}\right) + \rho\left(x_{m_{(k)-1}}, x_{n_{(k)}}\right) \\ &\leq \rho\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right) + \rho\left(x_{m_{(k)+1}}, x_{m_{(k)-1}}\right) + \varepsilon. \end{aligned} \tag{63}$$

Then,

$$\lim_{k\longrightarrow\infty} \rho\left(m_{(k)}, n_{(k)}\right) = \varepsilon. \tag{64}$$

By quadrilateral inequality, we have

$$\begin{aligned} \rho\left(x_{m_{(k)+1}}, x_{n_{(k)}}\right) &\leq \rho\left(x_{m_{(k)+1}}, x_{m_{(k)-1}}\right) + \rho\left(x_{m_{(k)-1}}, x_{m_{(k)}}\right) + \rho\left(x_{m_{(k)}}, x_{n_{(k)}}\right), \\ \rho\left(x_{m_{(k)}}, x_{n_{(k)}}\right) &\leq \rho\left(x_{m_{(k)}}, x_{m_{(k)+2}}\right) + \rho\left(x_{m_{(k)+2}}, x_{m_{(k)+1}}\right) + \rho\left(x_{m_{(k)+1}}, x_{n_{(k)}}\right). \end{aligned} \tag{65}$$

Letting $k \longrightarrow \infty$ in the above inequalities, we obtain

$$\lim_{k\longrightarrow\infty} \rho\left(x_{m_{(k)+1}}, x_{n_{(k)}}\right) = \varepsilon. \tag{66}$$

Now, by quadrilateral inequality, we have

$$\begin{aligned} \rho\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) &\leq \rho\left(x_{m_{(k)+1}}, x_{m_{(k)}}\right) + \rho\left(x_{m_{(k)}}, x_{n_{(k)}}\right) + \rho\left(x_{n_{(k)}}, x_{n_{(k)+1}}\right), \\ \rho\left(x_{m_{(k)}}, x_{n_{(k)}}\right) &\leq \rho\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right) + \rho\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) + \rho\left(x_{n_{(k)+1}}, x_{n_{(k)}}\right). \end{aligned} \tag{67}$$

Letting $k \longrightarrow \infty$ in the above inequalities, we obtain

$$\lim_{k\longrightarrow\infty} \rho\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) = \varepsilon. \tag{68}$$

By quadrilateral inequality, we have

$$\begin{aligned} \rho\left(x_{m_{(k)+2}}, x_{n_{(k)}}\right) &\leq \rho\left(x_{m_{(k)+2}}, x_{m_{(k)}}\right) + \rho\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right) + \rho\left(x_{m_{(k)+1}}, x_{n_{(k)}}\right), \\ \rho\left(x_{m_{(k)}}, x_{n_{(k)}}\right) &\leq \rho\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right) + \rho\left(x_{m_{(k)+1}}, x_{m_{(k)+2}}\right) + \rho\left(x_{m_{(k)+2}}, x_{n_{(k)}}\right). \end{aligned} \tag{69}$$

Letting $k \longrightarrow \infty$ in the above inequalities, we obtain

$$\lim_{k\longrightarrow\infty} \rho\left(x_{m_{(k)+2}}, x_{n_{(k)}}\right) = \varepsilon. \tag{70}$$

By the quadrilateral inequality, we find

$$\begin{aligned} \rho\left(x_{m_{(k)+2}}, x_{n_{(k)+1}}\right) &\leq \rho\left(x_{m_{(k)+2}}, x_{m_{(k)}}\right) + \rho\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right) + \rho\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right), \\ \rho\left(x_{m_{(k)}}, x_{n_{(k)+1}}\right) &\leq \rho\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right) + \rho\left(x_{m_{(k)+1}}, x_{m_{(k)+2}}\right) + \rho\left(x_{m_{(k)+2}}, x_{n_{(k)+1}}\right). \end{aligned} \tag{71}$$

Letting $k \longrightarrow \infty$ in the above inequalities, we obtain

$$\lim_{k\longrightarrow\infty} \rho\left(x_{m_{(k)+2}}, x_{n_{(k)}}\right) = \varepsilon. \tag{72}$$

From (11) and by setting $x = x_{m_{(k)}}$ and $y = x_{n_{(k)}}$, we have

$$\begin{aligned} M\left(x_{m_{(k)}}, x_{n_{(k)}}\right) = \max\Big\{&\rho\left(x_{m_{(k)}}, x_{n_{(k)}}\right), \rho\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right), \\ &\times \rho\left(x_{n_{(k)}}, x_{n_{(k)+1}}\right), \rho\left(x_{m_{(k)+2}}, x_{m_{(k)+1}}\right), \\ &\times \rho\left(x_{m_{(k)+2}}, x_{n_{(k)}}\right), \rho\left(x_{m_{(k)+2}}, x_{n_{(k)+1}}\right), \\ &\times \rho\left(x_{m_{(k)+2}}, x_{m_{(k)+1}}\right)\Big\}.. \end{aligned} \tag{73}$$

Taking the limit as $k \longrightarrow \infty$, we have

$$\lim_{k\longrightarrow\infty} M\left(x_{m_{(k)}}, x_{n_{(k)}}\right) = \max\{\varepsilon, 0, 0, \varepsilon, \varepsilon, \varepsilon, \varepsilon\} = \varepsilon. \tag{74}$$

Applying (11) with $x = x_{m_{(k)}}$ and $y = x_{n_{(k)}}$, we obtain

$\theta[\rho(x_{m_{(k)+1}}, x_{n_{(k)+1}})] \leq [\theta(M(x_{m_{(k)}}, x_{n_{(k)}}))]^{\omega}$ $[\rho(x_{m_{(k)}}, x_{n_{(k)}}), \rho(x_{m_{(k)}}, Tx_{m_{(k)}}), \rho(x_{n_{(k)}}, Tx_{n_{(k)}}), \rho(Tx_{m_{(k)}}, x_{n_{(k)}}), \rho(T^2 x_{m_{(k)}}, x_{n_{(k)}})] = [\theta(M(x_{m_{(k)}}, x_{n_{(k)}}))]^{\omega[\rho(x_{m_{(k)}}, x_{n_{(k)}}), \rho(x_{m_{(k)}}, x_{m_{(k)+1}}), \rho(x_{n_{(k)}}, x_{n_{(k)+1}}), \rho(x_{m_{(k)+1}}, x_{n_{(k)}}), \rho(x_{m_{(k)+2}}, x_{n_{(k)}})]}$. As ω is a continuous function

$$\begin{aligned}
&\lim_{k\longrightarrow\infty} \omega\Big[\rho\left(x_{m_{(k)}}, x_{n_{(k)}}\right), \rho\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right), \rho\left(x_{n_{(k)}}, x_{n_{(k)+1}}\right), \\
&\quad \times \rho\left(x_{m_{(k)+1}}, x_{n_{(k)}}\right), \rho\left(x_{m_{(k)+2}}, x_{n_{(k)}}\right)\Big] \\
&= \omega\Big[\lim_{k\longrightarrow\infty}\Big(\rho\left(x_{m_{(k)}}, x_{n_{(k)}}\right), \rho\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right), \rho\left(x_{n_{(k)}}, x_{n_{(k)+1}}\right), \\
&\quad \times \rho\left(x_{m_{(k)+1}}, x_{n_{(k)}}\right), \rho\left(x_{m_{(k)+2}}, x_{n_{(k)}}\right)\Big)\Big] = \omega[\varepsilon, 0, 0, \varepsilon, \varepsilon]
\end{aligned} \tag{75}$$

So, there exist $\delta \in]0, 1[$ such that $\omega[\varepsilon, 0, 0, \varepsilon, \varepsilon] = \delta$. Then,

$$\theta\left(\rho\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right)\right) \leq \left[\theta\left(M\left(x_{m_{(k)}}, x_{n_{(k)}}\right)\right)\right]^{\delta} \tag{76}$$

Letting $k \longrightarrow \infty$ in the above inequality, applying the continuity of θ, we have

$$\theta\left[\lim_{k\longrightarrow\infty}\left(\rho\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right)\right)\right] \leq \left[\theta\left(\lim_{k\longrightarrow\infty} M\left(x_{m_{(k)}}, x_{n_{(k)}}\right)\right)\right]^{\delta} \tag{77}$$

Therefore,

$$\theta(\varepsilon) \leq [\theta(\varepsilon)]^{\delta} < \theta(\varepsilon), \tag{78}$$

which is a contradiction. Then,

$$\lim_{n,m\longrightarrow\infty} \rho(x_m, x_n) = 0 \tag{79}$$

Hence, $\{x_n\}$ is a Cauchy sequence in C. By completeness of (C, ρ), there exists $z \in C$ such that

$$\lim_{n\longrightarrow\infty} \rho(x_n, z) = 0 \tag{80}$$

Now, we show that $\rho(Tz, z) = 0$. Arguing by contradiction, we assume that

$$\rho(Tz, z) > 0 \tag{81}$$

Now, by quadrilateral inequality we get,

$$\rho(Tx_n, Tz) \leq \rho(Tx_n, x_n) + \rho(x_n, z) + \rho(z, Tz) \tag{82}$$

$$\rho(z, Tz) \leq \rho(z, x_n) + \rho(x_n, Tx_n) + \rho(Tx_n, Tz). \tag{83}$$

By letting $n \longrightarrow \infty$ in inequality (82) and (83), we obtain

$$\rho(z, Tz) \leq \lim_{n\longrightarrow\infty} \rho(Tx_n, Tz) \leq d(z, Tz) \tag{84}$$

Therefore,

$$\lim_{n\longrightarrow\infty} \rho(Tx_n, Tz) = \rho(z, Tz). \tag{85}$$

Since $x_n \longrightarrow z$ as $n \longrightarrow \infty$ for all $n \in \mathbb{N}$ and since T is an (λ, μ)-continuous, we conclude that $\lim_{n\longrightarrow\infty} Tx_n = Tz$. Then,

$$\lim_{n\longrightarrow\infty} \rho(Tx_n, Tz) = \rho(z, Tz) = 0. \tag{86}$$

So $z = Tz$

For uniqueness, now, suppose that $z, u \in C$ are two fixed points of T such that $u \neq z$. Therefore, we have

$$\rho(z, u) = \rho(Tz, Tu) > 0 \tag{87}$$

Applying (11) with $x = z$ and $y = u$, we have

$$\begin{aligned}
\theta(\rho(z, u)) &= \theta(\rho(Tu, Tz)) \\
&\leq [\theta(M(z, u))]^{\omega\left(\rho(z,u), \rho(z,Tz), \rho(u,Tu), \rho(u,Tz), \rho\left(T^2 z, u\right)\right)} \\
&= [\theta(M(z, u))]^{\omega(\rho(z,u), \rho(z,z), d(u,u), \rho(u,z), \rho(z,u))} \\
&= [\theta(M(z, u))]^{\omega(d(z,u), 0, 0, \rho(u,z), \rho(z,u))} \\
&= [\theta(M(z, u))]^{\delta}
\end{aligned} \tag{88}$$

where

$$\begin{aligned}
M(z, u) &= \max\{\rho(z, u), \rho(z, Tz), \rho(u, Tu), \rho(Tz, u), \\
&\quad \times \rho\left(T^2 z, Tz\right), \rho\left(T^2 z, u\right), \rho\left(T^2 z, Tu\right)\} \\
&= \max\{\rho(z, u), \rho(z, z), \rho(u, u), \rho(z, u), \rho(z, z), \\
&\quad \times \rho(z, u), \rho(z, u)\} = d(z, u).
\end{aligned} \tag{89}$$

Therefore, we have

$$\theta(\rho(z, u)) \leq [\theta(\rho(z, u))]^{\delta} < \theta(\rho(z, u)), \tag{90}$$

which implies that

$$\rho(z, u) < \rho(z, u), \tag{91}$$

which is a contradiction. Therefore, $u = z$, and hence, the proof is complete. □

Consequently, we have the following:

Corollary 20. *Let (C, ρ) be a $(\lambda, \mu)-$complete generalized metric space, and let $\lambda, \mu : C \times C \longrightarrow [0,+\infty[$ be two functions. Let $T : C \longrightarrow C$ be a self-mapping satisfying the following conditions:*

(i) $\theta[\rho(Tx, Ty)] \leq [\theta(M(x, y))]^k, k \in]0, 1[\, \theta \in \Theta_C$

(ii) T is continuous. Then, T has a unique fixed point

Proof. Define a function $\omega : \mathbf{R}_+^5 \longrightarrow \mathbb{R}_+$ by

$$\omega(t_1, t_2, t_3, t_4, t_5) = k \text{ for } all\, t_1, t_2, t_3, t_4, t_5 \in \mathbb{R}_+. \quad (92)$$

Clearly, $\omega \in \Omega$
Taking

$$\lambda(x, y) = 1. \\ \mu(x, y) = 1 \quad (93)$$

Thus, T is an $(\lambda, \mu) - \omega - \theta$-contraction, and T is a triangular (λ, μ) – admissible mapping. As in the proof of Theorem 19, T has a unique fixed point $x \in C$.

It is clear that, if x is a fixed point of T, then x is also a fixed point of T^n for every $n \in \mathbb{N}$. The notion of property P introduced first by Jeong and Rhoades [31] that if a mapping T satisfies $Fix(T) = Fix(T^n)$ for each $n \in \mathbb{N}$, then it is said that T has property P or that T has no periodic points. □

Theorem 21. *Let λ, μ: $C \times C \longrightarrow \mathbb{R}^+$ be two functions, and let (C, ρ) be an (λ, μ) -generalized complete metric space. Let $T : C \longrightarrow C$ be a mapping satisfying the following conditions:*

(i) T is a triangular (λ, μ) -admissible mapping

(ii) T is an $(\lambda, \mu) - \omega - \theta$ -contraction

(iii) $\lambda(z, Tz) \geq 1$ and $\mu(z, Tz) \leq 1$, for all $z \in Fix\,(T)$. Then, T has the property P,$(T^n x = Tx)$.

Proof. Let $z \in \text{Fix}\,(T^n)$ for some fixed $n > 1$. As $\lambda(z, Tz) \geq 1$ and $\mu(z, Tz) \leq 1$ and T is a triangular (λ, μ)-admissible mapping, then

$$\lambda(Tz, T^2 z) \geq 1 \text{ and } \mu(Tz, T^2 z) \leq 1. \quad (94)$$

Continuing this process, we have

$$\lambda(T^n z, T^{n+1} z) \geq 1 \text{ and } \mu(T^n z, T^{n+1} z) \leq 1 \quad (95)$$

for all $n \in \mathbb{N}$. By (T_3) and (T_4), we get

$$\lambda(T^m z, T^n z) \geq 1 \text{ and } \mu(T^m z, T^n z) \leq 1, \forall m, n \in \mathbb{N}, n \neq m. \quad (96)$$

Assume that $z \notin$ Fix (T), i.e., $\rho(z, Tz) > 0$.
Applying (11) with $x = T^{n-1} z$ and $y = z$, we get

$$\rho(z, Tz) = \rho(T^n z, Tz) = \rho(TT^{n-1} z, Tz), \quad (97)$$

which implies that

$$\begin{aligned} &\theta(\rho(TT^{n-1} z, Tz) \\ &\leq [\theta(M(T^{n-1} z, z))]^{\omega(\rho(T^{n-1} z, z), \rho(T^{n-1} z, TT^{n-1} z), \rho(z, Tz), \rho(TT^{n-1} z, z), \rho(z, T^2 T^{n-1} z))} \\ &= [\theta(M(T^{n-1} z, z))]^{\omega(\rho(T^{n-1} z, z), \rho(T^{n-1} z, T^n z), \rho(z, Tz), \rho(T^n z, z), \rho(z, T^{n+1} z))} \\ &= [\theta(M(T^{n-1} z, z))]^{\omega(\rho(T^{n-1} z, z), \rho(T^{n-1} z, T^n z), \rho(z, Tz), \rho(T^n z, z), 0)}. \end{aligned} \quad (98)$$

Thus, there exists $\delta \in (0, 1)$ such that

$$\omega(\rho(T^{n-1} z, z), \rho(T^{n-1} z, T^n z), \rho(z, Tz), d(T^n z, z), 0) = \delta. \quad (99)$$

Then,

$$\rho(z, Tz) = \rho(T^n z, Tz) = \rho(TT^{n-1} z, Tz)\, [\theta(M(T^{n-1} z, z))]^{\delta}, \quad (100)$$

where

$$\begin{aligned} M(z, T^{n-1} z) &= \max \{\rho(T^{n-1} z, z), \rho(T^{n-1} z, TT^{n-1} z), \\ &\quad \times \rho(z, Tz), \rho(T^{n-1} z, z), \rho(T^2 T^{n-1} z, z), \\ &\quad \times \rho(T^2 T^{n-1} z, Tz), \rho(T^2 T^{n-1} z, T^{n-1} z)\} \\ &= \max \{\rho(T^{n-1} z, z), \rho(T^{n-1} z, T^n z), \rho(z, Tz), \\ &\quad \times \rho(T^{n-1} z, z), \rho(TT^n z, z), \rho(TT^n z, Tz), \\ &\quad \times \rho(TT^n z, T^{n-1} z)\} \\ &= \max \{\rho(T^{n-1} z, z), \rho(T^{n-1} z, z), \rho(z, Tz), \\ &\quad \times \rho(T^{n-1} z, z), \rho(Tz, z), \rho(Tz, Tz), \\ &\quad \times \rho(Tz, T^{n-1} z)\}. \end{aligned} \quad (101)$$

As $\rho(T^{n-1} z, T^n z) \longrightarrow 0$, taking the limit as $n \longrightarrow \infty$

$$\lim_{n \longrightarrow +\infty} M(z, T^{n-1} z) = \rho(z, Tz). \quad (102)$$

Since θ is an increasing and contentious function, therefore,

$$\theta(\rho(z, Tz)) \leq [\theta(\rho(z, Tz))]^{\delta} < \theta(\rho(z, Tz)), \quad (103)$$

which is a contradiction. So, $\rho(z, Tz) > 0$. Then, Fix $(T^n) = \text{Fix}(T)$. Therefore, T has the property (P). □

Assuming the following conditions, we prove that Theorem 19 still holds for T not necessarily continuous.

Theorem 22. *Let $\lambda, \mu : C \times C \longrightarrow \mathbb{R}^+$ be two functions, and let (C, ρ) be an (λ, μ) – complete generalized metric space.*

Let $T : C \longrightarrow C$ be a mapping, satisfying the following assertions:

(i) T is triangular $(\lambda, \mu) - $admissible

(ii) T is $(\lambda, \mu) - \theta - \omega - $contraction

(iii) There exists $x_0 \in X$ such that $\lambda(x_0, Tx_0) \geq 1$ and $\mu(x_0, Tx_0) \leq 1$

(iv) (X, d) is (λ, μ)-regular

Then, T has a fixed point. Moreover, T has a unique fixed point whenever $\lambda(z, u) \geq 1$ and $\mu(z, u) \leq 1$ for all $z, u \in Fix(T)$.

Proof. Let $x_0 \in X$ such that $\lambda(x_0, Tx_0) \geq 1$ and $\mu(x_0, Tx_0) \leq 1$. Similar to the proof of Theorem 19, we can conclude that

$$\begin{aligned}(\lambda(x_n, x_{n+1}) \geq 1 \text{ and } \mu(x_n, x_{n+1}) \leq 1),\\ x_n \longrightarrow z \text{as} n \longrightarrow \infty,\end{aligned} \tag{104}$$

where $x_{n+1} = Tx_n$. From $(iv)\lambda(x_{n+1}, z) \geq 1$ and $\mu(x_{n+1}, z) \leq 1$ hold for all $n \in \mathbb{N}$.

Suppose that $Tz = x_{n_{0+1}} = Tx_{n_0}$ for some $n_0 \in \mathbb{N}$. From Theorem 19, we know that the members of the sequence $\{x_n\}$ are distinct. Hence, we have $Tz \neq Tx_n$, i.e., $\rho(Tz, Tx_n) > 0$ for all $n > n_0$. Thus, we can apply (11), to x_n and z for all $n > n_0$ to get

$$\begin{aligned}\theta(\rho(Tx_n, Tz)) &\leq [\theta(M(x_n, z))]^{\omega\left(\rho(x_n,z),\rho(x_n,Tx_n),\rho(z,Tz),\rho(x_n,Tz),\rho\left(T^2x_n,z\right)\right)}\\ &= [\theta(M(x_n, z))]^{\omega(\rho(x_n,z),\rho(x_n,x_{n+1}),\rho(z,Tz),\rho(x_n,Tz),\rho(x_{n+2},z))}.\end{aligned} \tag{105}$$

Therefore,

$$\theta(\rho(Tx_n, Tz)) \leq [\theta(M(x_n, z))]^{\omega(\rho(x_n,z),\rho(x_n,x_{n+1}),\rho(z,Tz),\rho(x_n,Tz),\rho(x_{n+2},z))}, \tag{106}$$

where

$$\begin{aligned}M(x_n, z) &= \max\{\rho(x_n, z), \rho(x_n, Tx_n), \rho(z, Tz), \rho(Tx_n, z),\\ &\quad \times \rho\left(T^2x_n, Tz\right), \rho\left(T^2x_n, z\right), \rho\left(T^2x_n, Tx_n\right)\\ &= \max\{\rho(x_n, z), \rho(x_n, x_{n+1}), \rho(z, Tz), \rho(x_{n+1}, z),\\ &\quad \times \rho(x_{n+2}, Tz), \rho(x_{n+2}, z), \rho(x_{n+2}, x_{n+1}).\end{aligned} \tag{107}$$

Thus,

$$\begin{aligned}\lim_{n\longrightarrow\infty} M(x_n, z) &= \max\left\{\lim_{n\longrightarrow\infty}[\rho(x_n, z), \rho(x_n, x_{n+1}), \rho(z, Tz),\right.\\ &\quad \times \rho(x_{n+1}, z), \rho(x_{n+2}, Tz), \rho(x_{n+2}, z),\\ &\quad \left.\times \rho(x_{n+2}, x_{n+1})]\right\}\\ &= \max\left\{0, 0, \rho(z, Tz), 0, \lim_{n\longrightarrow\infty} d(x_{n+2}, Tz), 0, 0\right\}.\end{aligned} \tag{108}$$

Since

$$0 \leq \rho(x_{n+2}, Tz) \leq \rho(x_{n+2}, x_n) + \rho(x_n, z) + \rho(x_n, Tz), \tag{109}$$

$$\lim_{n\longrightarrow\infty} \rho(x_{n+2}, Tz) \leq \rho(z, Tz). \tag{110}$$

Thus,

$$\lim_{n\longrightarrow\infty} M(x_n, z) \leq \rho(z, Tz), \tag{111}$$

and there exist $\delta \in]0, 1[$ such that

$$\omega(\rho(x_n, z), \rho(x_n, x_{n+1}), \rho(z, Tz), \rho(x_n, Tz), \rho(x_{n+2}, z)) = \delta. \tag{112}$$

If $\rho(z, Tz) > 0$, then by (110) and the fact that θ and ω are continuous and by taking the limit as $n \longrightarrow \infty$ in (106), we obtain

$$\begin{aligned}\theta\left(\lim_{n\longrightarrow\infty} \rho(Tx_n, Tz)\right) &\leq \left[\theta\left(\lim_{n\longrightarrow+\infty} M(x_n, z)\right)\right]^{\delta}\\ &\leq [\theta(\rho(z, Tz))]^{\delta} < \theta[\rho(z, Tz)].\end{aligned} \tag{113}$$

Using (85), we get

$$\theta\rho(z, Tz) < \theta[\rho(z, Tz)]. \tag{114}$$

It is a contradiction. Therefore, $\rho(z, Tz) = 0$, that is, z is a fixed point of T, and so $z = Tz$. Thus, z is a fixed point of T. The proof of the uniqueness is similar to that of Theorem 19. □

Definition 23. Let $\rho(C, d)$ be a $(\lambda - \mu)$ -generalized metric space, and let T be a self-mapping on C. Suppose that $\lambda, \mu : X \times X \longrightarrow [0,+\infty[$ are two functions. We say that T is an $(\lambda, \mu) - \omega - \theta_G$ -contraction, if for all $x, y \in C$ with $(\lambda(x, y) \geq 1$ and $\mu(x, y) \leq 1)$ and $\rho(Tx, Ty) > 0$, we have

$$\theta(\rho(Tx, Ty)) \leq [\theta(M(x, y))]^{\omega\left(\rho(x,y),\rho(x,Tx),\rho(y,Ty),\rho(Tx,y),\rho\left(T^2x,y\right)\right)}, \tag{115}$$

where $\theta \in \Theta_G$, $\omega \in \Omega$ and

$$\begin{aligned}M(x, y) &= \max\{\rho(x, y), \rho(x, Tx), \rho(y, Ty), \rho(Tx, y),\\ &\quad \times \rho\left(T^2x, y\right), \rho\left(T^2x, Ty\right), \rho\left(T^2x, Tx\right)\}.\end{aligned} \tag{116}$$

Theorem 24. *Let (C, ρ) be a (λ, μ) -complete generalized metric space, and let $\lambda, \mu : C \times C \longrightarrow [0,+\infty)$ be two functions. Let $T : C \longrightarrow C$ be a self-mapping satisfying the following conditions:*

(i) T is a triangular $(\lambda, \mu) - $admissible mapping

(ii) T is an $(\alpha, \eta) - \theta - \omega$ -contraction

(iii) There exists $x_0 \in C$ such that $\lambda(x_0, Tx_0) \geq 1$ and $\mu(x_0, Tx_0) \leq 1$

(iv) T is a (λ, μ) -continuous

Then, T has a fixed point. Moreover, T has a unique fixed point when $\lambda(x, y) \geq 1$ and $\mu(x, y) \leq 1$ for all $x, y \in C$.

Proof. Let $x_0 \in C$ such that $\lambda(x_0, Tx_0) \geq 1$ and $\mu(x_0, Tx_0) \leq 1$. Similar to the proof of Theorem 19, we can conclude that

$$\begin{gathered}(\lambda(x_n, x_{n+1}) \geq 1 \text{ and } \mu(x_n, x_{n+1}) \leq 1),\\ \lim_{n\longrightarrow\infty} \rho(x_n, x_{n+1}) = 0, \lim_{n\longrightarrow\infty} \rho(x_n, x_{n+2}) = 0.\end{gathered} \tag{117}$$

By (θ_3), there exist $r \in]0,1[$ and $l \in (0,+\infty]$ such that $\lim_{n\longrightarrow\infty}(\theta(\rho(x_n, x_{n+1})) - 1)/\rho(x_n, x_{n+1})^r = l$. Suppose that $l < \infty$. So, there exists $n_1 \in \mathbb{N}$ such that

$$\left|\frac{\theta(\rho(x_n, x_{n+1})) - 1}{\rho(x_n, x_{n+1})^r}\right| < \frac{l}{2}, \text{for all } n \geq n_1. \tag{118}$$

Taking $M = 2/l$, we have

$$n[\rho(x_n, x_{n+1})^r] < M.n[\theta(\rho(x_n, x_{n+1})) - 1], \text{for all } n \geq n_1. \tag{119}$$

Suppose now that $l = \infty$. Let $N > 0$ be an arbitrary positive number. So, there exists $n_2 \in \mathbb{N}$ such that

$$\left|\frac{\theta(\rho(x_n, x_{n+1})) - 1}{\rho(x_n, x_{n+1})^r}\right| > N, \text{for } \mathit{all}\ n \geq n_2. \tag{120}$$

Taking $M = 1/N$, we have

$$n[\rho(x_n, x_{n+1})^r] < n.M[\theta(\rho(x_n, x_{n+1})) - 1], \text{for all } n \geq n_2. \tag{121}$$

Thus, in all cases, there exist $M > 0$ and $q \in \mathbb{N}(q = \max(n_1, n_2))$ such that

$$n[\rho(x_n, x_{n+1})^r] < M.n[\theta(\rho(x_n, x_{n+1})) - 1], \forall n \geq n_q. \tag{122}$$

By induction, we obtain

$$\begin{aligned}n[\rho(x_n, x_{n+1})^r] &< n.M[\theta(\rho(x_n, x_{n+1})) - 1]\\ &< \cdots < Mn\left[(\theta(\rho(x_0, x_1)))^{r^n} - 1\right].\end{aligned} \tag{123}$$

Letting $n \longrightarrow \infty$ in the above inequality, we obtain

$$\lim_{n\longrightarrow\infty} n[\rho(x_n, x_{n+1})^r] = 0. \tag{124}$$

So, there exists $n_3 \in \mathbb{N}$ such that

$$\rho(x_n, x_{n+1}) \leq \frac{1}{n^{1/r}}, \text{for all } n \geq n_3. \tag{125}$$

By (θ_3), there exist $r \in (0, 1)$ and $h \in]0,+\infty[$ such that $\lim_{n\longrightarrow\infty}(\theta(\rho(x_n, x_{n+2})) - 1)/\rho(x_n, x_{n+2})^r = h$.
Suppose that $h < \infty$. So, there exists $n_4 \in \mathbb{N}$ such that

$$\left|\frac{\theta(\rho(x_n, x_{n+2})) - 1}{\rho(x_n, x_{n+2})^r}\right| < \frac{h}{2}, \text{for all } n \geq n_1. \tag{126}$$

Taking $p = 2/h$, we have

$$n[\rho(x_n, x_{n+2})^r] < P.n[\theta(\rho(x_n, x_{n+2})) - 1], \text{for all } n \geq n_4. \tag{127}$$

Suppose now that $h = \infty$. Let $Q > 0$ be an arbitrary positive number. So, there exists $n_5 \in \mathbb{N}$ such that

$$\left|\frac{\theta(\rho(x_n, x_{n+2})) - 1}{\rho(x_n, x_{n+2})^r}\right| > Q, \text{for all } n \geq n_5. \tag{128}$$

So by taking $P = 1/Q$, we have

$$n[\rho(x_n, x_{n+2})^r] < n.P[\theta(\rho(x_n, x_{n+2})) - 1], \text{for all } n \geq n_5. \tag{129}$$

Thus, in all cases, there exist $P > 0$ and $w \in \mathbb{N}(w = \max(n_4, n_5))$ such that

$$n[\rho(x_n, x_{n+2})^r] < P.n[\theta(\rho(x_n, x_{n+2})) - 1], \text{for all } n \geq w. \tag{130}$$

By induction, we obtain

$$\begin{aligned}n[\rho(x_n, x_{n+2})^r] &< n.P[\theta(\rho(x_n, x_{n+2})) - 1]\\ &< \cdots < n.P\left[(\theta(\rho(x_0, x_2)))^{r^n} - 1\right].\end{aligned} \tag{131}$$

Letting $n \longrightarrow \infty$ in the above inequality, we obtain

$$\lim_{n\longrightarrow\infty} n[\rho(x_n, x_{n+2})^r] = 0. \tag{132}$$

So, there exists $n_6 \in \mathbb{N}$ such that

$$\rho(x_n, x_{n+2}) \leq \frac{1}{n^{1/r}}, \text{for all } n \geq n_6. \tag{133}$$

If $m > n$ and $m = n + 2k + 1$ with $k \in \mathbb{N}$, then

$$\rho(x_n, x_m) \leq \rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_{n+2}) + \cdots + \rho(x_{m-1}, x_m) \leq \sum_{i=n}^{m-1} \frac{1}{i^{1/r}}. \tag{134}$$

If $m > n$ and $m = n + 2k$ with $k \in \mathbb{N}$, then

$$\begin{aligned}\rho(x_n, x_m) &\leq \rho(x_n, x_{n+2}) + \rho(x_{n+2}, x_{n+3}) + \cdots + \rho(x_{m-1}, x_m)\\ &\leq \rho(x_n, x_{n+2}) + \sum_{i=n+2}^{m-1} \frac{1}{i^{1/r}} \leq \sum_{i=n}^{m-1} \frac{1}{i^{1/r}}.\end{aligned} \tag{135}$$

Therefore,

$$\rho(x_n, x_m) \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/r}}. \tag{136}$$

As $0 < r < 1$, the series $\sum_{i=n}^{\infty} 1/i^{1/r}$ converges. Therefore, by taking the limit as $n, m \longrightarrow \infty$ in (136), we get

$$\lim_{n\longrightarrow\infty} \rho(x_n, x_m) = 0. \quad (137)$$

Hence, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since C is complete, there exists $z \in C$ such that

$$\lim_{n\longrightarrow\infty} \rho(x_n, z) = 0. \quad (138)$$

Since T is (λ, μ)-continuous,

$$\lim_{n\longrightarrow\infty} \rho(Tx_n, Tz) = 0. \quad (139)$$

Then,

$$z = \lim_{n\longrightarrow\infty} x_{n+1} = \lim_{n\longrightarrow\infty} Tx_n = Tz. \quad (140)$$

This proves that z is a fixed point of T. □

Corollary 25. *Let (C, ρ) be a (λ, μ)-complete generalized metric space. Let $\lambda, \mu : C \times C \longrightarrow [0,+\infty[$ be two functions. Let $T : C \longrightarrow C$ be a self-mapping satisfying the following conditions:*

(i) $\theta[\rho(Tx, Ty)] \leq [\theta(M(x, y))]^r$, $r \in]0, 1[$ $\theta \in \Theta_G$

(ii) T is a triangular (λ, μ) -admissible mapping

(iii) There exists $x_0 \in C$ such that $\lambda(x_0, Tx_0) \geq 1$ and $\mu(x_0, Tx_0) \leq 1$

(iv) T is a (λ, μ) – continuous

Then, T has a fixed point. Moreover, T has a unique fixed point when $\lambda(x, y) \geq 1$ and $\mu(x, y) \leq 1$ for all $x, y \in C$.

Proof. Define a function $\omega : \mathbf{R}_+^5 \longrightarrow \mathbb{R}_+$ by

$$\omega(a_1, a_2, a_3, a_4, a_5) = r \text{ for all } a_1, a_2, a_3, a_4, a_5 \in \mathbb{R}_+. \quad (141)$$

So, $\omega \in \Omega$, and T is an $(\lambda, \mu) - \omega - \theta$-contraction. As in the proof of Theorem 24, T has a unique fixed point $x \in X$. □

Example 26. Let $C = [1,+\infty)$ and $a \in (0, 1)$. Define $\rho : C \times C \longrightarrow [0,+\infty)$ by

$$\rho(x, y) = (|x - y|). \quad (142)$$

Then, (C, ρ) is a complete generalized metric space. Define $T : C \longrightarrow C$ by

$$\begin{aligned} &T(t) = a\sqrt{t} \text{ for all } t \in [1,+\infty), \\ &\lambda(x, y) = \frac{\max\{x, y\} + a}{\min\{x, y\} + a}, \text{ for all } x, y \in \mathbb{R}_+, \\ &\mu(x, y) = \frac{\min\{x, y\} + a}{\max\{x, y\} + a}, \text{ for all } x, y \in \mathbb{R}_+, \\ &\omega(t_1, t_2, t_3, t_4, t_5) = \sqrt{a} \text{ for all } t_1, t_2, t_3, t_4, t_5 \in \mathbb{R}_+. \end{aligned} \quad (143)$$

Then, T is an (λ, μ)-continuous triangular (λ, μ) – admissible mapping.

Case 1. $0 \leq x \leq y$

$$\begin{aligned} &\rho(Tx, Ty) = \left(a\sqrt{y} - a\sqrt{x}\right), \\ &M(\rho(x, y)) = \max\left\{\rho(x, y), \rho\left(x, a\sqrt{x}\right), \rho\left(y, a\sqrt{y}\right), \rho\left(y, a\sqrt{x}\right),\right. \\ &\qquad \times \rho\left(a^2\sqrt{\sqrt{x}}, y\sqrt{y}\right), \rho\left(a^2\sqrt{\sqrt{x}}, a\sqrt{y}\right), \\ &\qquad \left.\times \rho\left(a^2\sqrt{\sqrt{x}}, a\sqrt{x}\right)\right\}. \end{aligned} \quad (144)$$

Since $x \leq y$ and $a \in (0, 1)$, we get

$$\begin{aligned} M(x, y) = \max &\left\{(y - x), \left(x - a\sqrt{x}\right), \left(y - a\sqrt{y}\right), \left(y - a\sqrt{x}\right),\right. \\ &\times \left(y\sqrt{y} - a^2\sqrt{\sqrt{x}}\right), \left(a\sqrt{y} - a^2\sqrt{\sqrt{x}},\right) \\ &\left.\times \left(a\sqrt{x} - a^2\sqrt{\sqrt{x}},\right)\right\}. \end{aligned} \quad (145)$$

Thus,

$$M(x, y) \geq y - x \geq a(y - x). \quad (146)$$

On the other hand,

$$a(y - x) = \sqrt{a}\sqrt{a}(y - x), \quad (147)$$

which implies that

$$\theta(M(x, y)) \geq \theta(\rho(x, y)) = e^{(y-x)}. \quad (148)$$

Thus,

$$\begin{aligned} &\theta(\rho(x, y))^{\omega\left(\rho(x,Tx),\rho(y,Ty),\rho(x,Ty),\rho(y,Tx),\rho\left(T^2x,y\right)\right)} = e^{\sqrt{a}(y-x)} = e^{\sqrt{a}\left(\sqrt{y}-\sqrt{x}\right)\left(\sqrt{y}+\sqrt{x}\right)}, \\ &\theta(\rho(Tx, Ty)) = e^{a\left(\sqrt{y}-\sqrt{x}\right)}. \end{aligned} \quad (149)$$

As $x, y \in [1,\infty[$

$$e^{a\left(\sqrt{y}-\sqrt{x}\right)} \leq e^{\sqrt{a}\left(\sqrt{y}-\sqrt{x}\right)\left(\sqrt{y}+\sqrt{x}\right)}. \quad (150)$$

Thus,

$$\theta(\rho(Tx, Ty)) \leq \theta(\rho(x, y))^{\omega\left(\rho(x,y),\rho(x,Tx),\rho(y,Ty),\rho(y,Tx),\rho\left(T^2x,y\right)\right)}. \quad (151)$$

Case 2. $x > y > 0$

$$\begin{aligned} \rho(Tx, Ty) &= \left(a\sqrt{x} - a\sqrt{y}\right), \\ M(x, y) &= \max\left\{\rho(x, y), \rho\left(x, a\sqrt{x}\right), \rho\left(y, a\sqrt{y}\right), \rho\left(y, a\sqrt{x}\right),\right. \\ &\quad \times \rho\left(a^2\sqrt{\sqrt{x}}, y\sqrt{y}\right), \rho\left(a^2\sqrt{\sqrt{x}}, a\sqrt{y}\right), \\ &\quad \left.\times \rho\left(a^2\sqrt{\sqrt{x}}, a\sqrt{x}\right)\right\}. \end{aligned} \quad (152)$$

As $x > y$ and $a \in]0, 1[$,

$$\begin{aligned} M(x, y) = \max &\left\{(x - y), \left(x - a\sqrt{x}\right), \left(y - a\sqrt{y}\right), \left(\left|y - a\sqrt{x}\right|\right),\right. \\ &\times \left(\left|y\sqrt{y} - a^2\sqrt{\sqrt{x}}\right|\right), \left(a\left|\sqrt{y} - a^2\sqrt{\sqrt{x}}\right|\right), \\ &\left.\times \left(a\sqrt{x} - a^2\sqrt{\sqrt{x}},\right)\right\}. \end{aligned} \quad (153)$$

Thus,

$$M(x, y) \geq y - x \geq a(x - y). \quad (154)$$

On the other hand,

$$a(y - x) = \sqrt{a}\sqrt{a}(x - y). \quad (155)$$

which implies that

$$\theta(M(x, y)) \geq \theta(\rho(x, y)) = e^{(x-y)}. \quad (156)$$

Thus,

$$\begin{aligned} \theta(\rho(x, y))^{\omega\left(\rho(x,Tx),\rho(y,Ty),\rho(x,Ty),\rho(y,Tx),\rho\left(T^2x,y\right)\right)} &= e^{\sqrt{a}(x-y)} = e^{\sqrt{a}(\sqrt{x}-\sqrt{y})(\sqrt{y}+\sqrt{x})}, \\ \theta(\rho(Tx, Ty)) &= e^{a(\sqrt{x}-\sqrt{y})}. \end{aligned} \quad (157)$$

As $x, y \in [1,\infty[$

$$e^{a(\sqrt{x}-\sqrt{y})} \leq e^{\sqrt{a}(\sqrt{x}-\sqrt{y})(\sqrt{y}+\sqrt{x})}. \quad (158)$$

Thus,

$$\theta(\rho(Tx, Ty)) \leq \theta(\rho(x, y))^{\omega\left(\rho(x,y),\rho(x,Tx),d\rho(y,Ty),\rho(y,Tx),d\left(T^2x,y\right)\right)}, \quad (159)$$

where $\theta \in \Theta_C \cap \Theta_G$. Hence, conditions (11) and (115) are satisfied. Therefore, T has a unique fixed point $z = 1$.

4. Application to Nonlinear Integral Equations

In this section, we endeavour to apply Theorems 19 and 24 to prove the existence and uniqueness of the integral equation of the Fredholm type.

$$u(t) = v\int_m^n h(t, r, u(r))\mathrm{ds}, \quad (160)$$

where $m, n \in \mathbb{R}$, $u \in C([m, n], \mathbb{R})$, and $h : [m, n]^2 \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions and for some constant v depending on the parameters m and n.

Theorem 27. *Suppose the function* h *is such that* $|h(t, r, u(r)) - h(t, r, v(r))| \leq (|u(t) - v(t)|)\forall t, r \in \mathbb{R}$ *and* $u, v \in C([m, n], \mathbb{R})$. *Then, the equation (160) has a unique solution* $u \in C([m, n], \mathbb{R})$ *and* $|v| \leq m/n$.

Proof. Let $C = C([m, n], \mathbb{R})$ and $T : C \longrightarrow C$ defined by

$$T(u)(t) = v\int_m^n h(t, r, u(r))\mathrm{ds}. \quad (161)$$

$\forall u \in C$. Define $\rho : C \times C \longrightarrow [0,+\infty[$ given by

$$\rho(u, v) = \left(\max_{t\in[m,n]}|u(t) - v(t)|\right). \quad (162)$$

Then, (C, ρ) is a complete generalized metric space. Assume that $u, v \in C$ and $t, r \in [m, n]$. Then, we get

$$\begin{aligned} |Tu(t) - Tv(t)| &= |v|\left(\left|\int_m^n h(t, r, u(r))\mathrm{dr} - \int_m^n h(t, r, v(r))\mathrm{dr}\right|\right) \\ &= |v|\left|\int_m^n h(t, r, u(r)) - h(t, r, v(r))\mathrm{dr}\right| \\ &\leq |v|\int_m^n |h(t, r, u(r)) - h(t, r, v(r))\mathrm{dr}| \\ &\leq |v|\int_m^n (|u(r) - v(r)|\mathrm{dr}) = |v|\left[\int_m^n ((|u(r)| - |v(r)|))\mathrm{dr}\right]. \end{aligned} \quad (163)$$

Thus,

$$\begin{aligned} \max_{t\in[m,n]}(|Tu(t) - Tv(t)|) &= \max_{t\in[m,n]}|v|\int_m^n |h(t, r, u(r)) - h(t, r, v(r))\mathrm{dr}| \\ &\leq \max_{t\in[m,n]}|v|\int_m^n ((|u(r) - v(r)|)\mathrm{dr}) \\ &\leq |v|\int_m^n \left(\left(\max_{r\in[m,n]}|u(r) - v(r)|\right)\mathrm{dr}\right). \end{aligned} \quad (164)$$

As $\rho(Tx, Ty) > 0$ and $\rho(x, y) > 0$ for any $x \neq y$, then we can take natural exponential sides and get

$$e^{[\rho(Tx,Ty)]} = e^{\left[|\nu| \max_{t\in[a,b]} \int_a^b |h(t,r,x(r))-h(t,r,y(r))\mathrm{d}r|\right]} \le e^{\left[|\nu| \int_a^b \left(\left(\max_{r\in[a,b]} |x(r)-y(r)|\right) \mathrm{d}r\right)\right]}. \quad (165)$$

Since $|\nu| \le m/n$, which implies that

$$e^{[\rho(Tu,Tv)]} = e^{\left[|\nu| \max_{t\in[m,n]} \int_m^n |h(t,r,u(r))-h(t,r,v(r))dr|\right]} \le e^{\left[m/n \int_m^n \left(\left(\max_{r\in[m,n]} |u(r)-v(r)|\right) dr\right)\right]}. \quad (166)$$

Hence,

$$\theta(\rho(Tu,Tv)) \le [\theta(M(u,v))]^{\omega}, \quad (167)$$

for all $u, v \in C$ with $\theta(t) = e^{(t)}$ and $\omega(t) = |m/n|$. Then, T satisfies conditions (11) and (115) which are hold. □

5. Concluding Remarks

The paper deals with $\theta - \omega -$ contraction in (λ, μ)-generalized metric spaces, which is an extension of the Banach contraction principle. We prove fixed point theorems of some generalized contractions which are defined on generalized metric spaces that satisfy a (λ, μ)-complete generalized metric space condition. Our generalized results are based on θ-contraction. Finally, we present an application dealing with the existence of solutions for integral equation of the Fredholm type. Further, we also need to illustrate some generalizations of the introduced $\theta - \omega -$ contraction mappings for generalized metric spaces with a graph. Some open problems for the future, for example, fixed circle problem or fixed figure problem of the $\theta - \omega -$ contraction mappings for generalized metric spaces.

Data Availability

No underlying data was collected.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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References

[1] G. Nallaselli, A. J. Gnanaprakasam, G. Mani, and O. Ege, "Solving integral equations via admissible contraction mappings," *Univerzitet u Nišu*, vol. 36, no. 14, pp. 4947–4961, 2022.

[2] G. Nallaselli, A. J. Gnanaprakasam, G. Mani, O. Ege, D. Santina, and N. Mlaiki, "A study on fixed-point techniques under the α-ꝼ-convex contraction with an application," *Axioms*, vol. 12, no. 2, p. 139, 2023.

[3] M. Asim, M. Imdad, and S. Radenovic, "Fixed point results in M_v –metric spaces with an application," *Journal of Inequalities and Applications*, vol. 2019, no. 1, p. 50, 2019.

[4] M. Younis, D. Singh, and A. Goyal, "A novel approach of graphical rectangular b-metric spaces with an application to the vibrations of a vertical heavy hanging cable," *Journal of Fixed Point Theory and Applications*, vol. 21, no. 1, 2019.

[5] O. Ege, "Complex valued rectangular b-metric spaces and an application to linear equations," *The Journal of Nonlinear Science and Applications*, vol. 8, no. 6, pp. 1014–1021, 2015.

[6] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Fundamenta Mathematicae*, vol. 3, pp. 133–181, 1922.

[7] B. Mohammadi, M. De la Senb, V. Parvanahc, F. Golkarmaneshd, and H. Aydi, "On Jleli-Samet-Ciric-Presic type contractive mappings," *Univerzitet u Nišu*, vol. 34, no. 14, pp. 4685–4695, 2020.

[8] H. K. Nashine, D. Gopal, D. Jain, and A. Al-Rawashdeh, "Solutions of initial and boundary value problems via F-contraction mappings in metric-like space," *International Journal of Nonlinear Analysis and Applications*, vol. 9, no. 1, pp. 129–145, 2018.

[9] J. R. Roshan, V. Parvaneh, Z. Kadelburg, and N. Hussain, "New fixed point results in b-rectangular metric spaces," *Nonlinear Analysis: Modelling and Control*, vol. 21, no. 5, pp. 614–634, 2016.

[10] M. Abbas, V. Parvaneh, and A. Razani, "Periodic points of T-Ciric generalized contraction mappings in ordered metric spaces," *Georgian Mathematical Journal*, vol. 19, no. 4, pp. 597–610, 2012.

[11] M. Jleli, E. Karapınar, and B. Samet, "Further generalizations of the Banach contraction principle," *Journal of Inequalities and Applications*, vol. 2014, no. 1, 2014.

[12] M. Rossafi, A. Kari, E. M. Marhrani, and M. Aamri, "Fixed point theorems for generalizedθ–φ–expansive mapping in rectangular metric spaces," *Abstract and Applied Analysis*, vol. 2021, Article ID 6642723, 10 pages, 2021.

[13] Z. Mustafa, V. Parvaneh, M. M. M. Jaradat, and Z. Kadelburg, "Extended rectangular b-metric spaces and some fixed point theorems for contractive mappings," *Symmetry*, vol. 11, no. 4, p. 594, 2019.

[14] I. A. Bakhtin, "The contraction mapping principle in quasimetric spaces," *Functional Analysis*, vol. 30, pp. 26–37, 1989.

[15] S. Czerwik, "Contraction mappings in b-metric spaces," *Acta Mathematica et Informatica Universitatis Ostraviensis*, vol. 1, pp. 5–11, 1993.

[16] A. Branciari, "A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces," *Publicationes Mathematicae*, vol. 57, no. 1-2, pp. 31–37, 2000.

[17] B. Samet, "Discussion on fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces by

A. Branciari," *Publicationes Mathematicae*, vol. 76, no. 4, pp. 493-494, 2010.

[18] M. Jlleli and B. Samet, "The KANNANS fixed point theorem in a cone rectangular metric space," *Journal of Nonlinear Sciences and Applications*, vol. 2, no. 3, pp. 161–167, 2009.

[19] M. Nazam, "OnJc-contraction and related fixed-point problem with applications," *Mathematical Methods in the Applied Sciences*, vol. 43, no. 17, pp. 10221–10236, 2020.

[20] M. Nazam and O. Acar, "Fixed points of (α,ψ)-contractions in Hausdorff partial metric spaces," *Mathematical Methods in the Applied Sciences*, vol. 42, no. 16, pp. 5159–5173, 2019.

[21] M. Nazam, E. Ameer, M. Mursaleen, and O. Acar, "Nonlinear inequalities and related fixed-point problems," *Journal of Mathematical Inequalities*, vol. 15, no. 3, pp. 941–967, 2007.

[22] M. Jleli and B. Samet, "A new generalization of the Banach contraction principle," *Journal of Inequalities and Applications*, vol. 2014, no. 1, 2014.

[23] W. Shatanawi, A. Al-Rawashdeh, H. Aydi, and H. K. Nashine, "On a fixed point for generalized contractions in generalized metric spaces," *Abstract and Applied Analysis*, vol. 2012, Article ID 246085, 13 pages, 2012.

[24] L. G. Huang and X. Zhang, "Cone metric spaces and fixed point theorems of contractive mappings," *Journal of Mathematical Analysis and Applications.*, vol. 332, no. 2, pp. 1468–1476, 2007.

[25] N. Mehmood, A. Al Rawashdeh, and S. Radenović, "New fixed point results for E-metric spaces," *Positivity*, vol. 23, no. 5, pp. 1101–1111, 2019.

[26] A. Kari, M. Rossafi, E. M. Marhrani, and M. Aamri, "Fixed-point theorems forθ–ϕ-contraction in generalized asymmetric metric spaces," *International Journal of Mathematics and Mathematical Sciences*, vol. 2020, Article ID 8867020, 19 pages, 2020.

[27] N. Hussain and P. Salimi, "Suzuki-Wardowski type fixed point theorems for $\alpha - G - F$-contractions," *Thai Journal of Mathematics*, vol. 18, pp. 1879–1895, 2014.

[28] N. Hussain, M. A. Kutbi, and P. Salimi, "Fixed point theory in α-complete metric spaces with applications," *Abstract and Applied Analysis*, vol. 2014, Article ID 280817, 11 pages, 2014.

[29] X. D. Liu, S.-S. Chang, Y. Xiao, and L. C. Zhao, "Existence of fixed points for θ-type contraction and θ-type Suzuki contraction in complete metric spaces," *Fixed Point Theory and Applications*, vol. 2016, no. 8, 2016.

[30] N. Hussain, V. Parvaneh, B. A. S. Alamria, and Z. Kadelburg, "F-HR-type contractions on (\alpha,\eta)-complete rectangular b-metric spaces," *Journal of Nonlinear Sciences and Applications*, vol. 10, no. 3, pp. 1030–1043, 2017.

[31] G. S. Jeong and B. E. Rhoades, "Maps for which $F(T) = F(T^{(n)})$," *Fixed Point Theory applications*, vol. 6, pp. 71–105, 2007.