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Research Article

Properties of Univalence for a New General Integral Operator Defined as a Joint Extension of Two Known Integral Operators

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In this paper, we consider a new general integral operator, defined as a joint extension of two already known integral operators, and prove some univalence properties for this operator. Some other well-known operators are mentioned as particular cases of our general operator, and known results are outlined also as particular cases of our results.

1. Introduction

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. Also, let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of univalent functions in $\mathcal{U}$.

One of the topics in geometric function theory is the study of geometric properties of the integral operators. Such properties like univalence, starlikeness, and convexity of different integral operators on various classes of analytic functions have been studied by many authors in their works, among which we mention the papers [1–13].

In our habilitation thesis [2], we introduced (as one of further research directions), the general integral operator $N_{\alpha,\beta}(f, g)$, with $\alpha = (\alpha_1, \alpha_2, \cdots \alpha_n)$ and $(f, g) = (f_1, f_2, \cdots f_n, g_1, g_2, \cdots g_n)$ as follows:

$$N_{\alpha,\beta}(f, g)(z) = \left[\int_0^z \beta t^{\beta-1} \exp\left(\int_0^t \prod_{i=1}^{n} \left(\frac{(f_i * g_i)(u)}{u}\right)^{\alpha_i} du\right) dt\right]^{1/\beta}, \tag{2}$$

where α_i are positive real numbers, β is a complex number with $\operatorname{Re} \beta > 0$, $f_i, g_i \in \mathcal{A}$, $i = \overline{1, n}$, exp is exponential function, and $(f_i * g_i)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k$ is the Hadamard product.

This operator is designed as a joint extension of two already known operators given by Attiya [1] and Frasin [8], but it also covers other known integral operators.

Remark 1.

(i) For $n = 1, \alpha_1 = 1, g_1 = z/(1 - z)$, we obtain the integral operator given by Attiya in [1]

(ii) For $\prod_{i=1}^{n}((f_i * g_i)(z)/z)^{\alpha_i} = \sum_{i=1}^{n} \alpha_i((f_i * g_i)'(z)/(f_i * g_i)(z) - 1/z)$, we get the integral operator defined by Frasin in [8], with α_i positive real numbers, for $i = \overline{1, n}$

(iii) For $\prod_{i=1}^{n}((f_i * g_i)(z)/z)^{\alpha_i} = \sum_{i=1}^{n} \alpha_i((f_i * g_i)'(z)/(f_i * g_i)(z) - 1/z)$, and $g_i(z) = z/(1 - z)$, $i = \overline{1, n}$, we get the integral operator, $I_\beta(f_1, f_2, \cdots, f_n)(z) = \left(\int_0^z \beta t^{\beta-1} \prod_{i=1}^{n}(f_i(t)/t)^{\alpha_i} dt\right)^{1/\beta}$ defined by Breaz D. and Breaz N. in [3], with α_i positive real numbers, for $i = \overline{1, n}$

(iv) For $\prod_{i=1}^{n}((f_i * g_i)(z)/z)^{\alpha_i} = \sum_{i=1}^{n} \alpha_i((f_i * g_i)'(z)/(f_i * g_i)(z) - 1/z)$, $g_i(z) = z/(1 - z)$, $i = \overline{1, n}$, and

$\beta = 1$, we get the integral operator, $I(f_1, f_2, \cdots, f_n)(z) = \int_0^z \prod_{i=1}^n (f_i(t)/t)^{\alpha_i} dt$ defined by Breaz D. and Breaz N. in [3], with α_i positive real numbers, for $i = \overline{1, n}$

(v) For $\prod_{i=1}^n ((f_i * g_i)(z)/z)^{\alpha_i} = \sum_{i=1}^n \alpha_i ((f_i * g_i)'(z)/(f_i * g_i)(z) - 1/z)$, $g_i(z) = z/(1-z)^2$, $i = \overline{1, n}$, and $\beta = 1$, we get the integral operator, $J(z) = \int_0^z \prod_{i=1}^n (f_i'(t))^{\alpha_i} dt$ defined by Breaz et al. in [4], with α_i positive real numbers, for $i = \overline{1, n}$

(vi) For $n = 1$, $((f * g)(z)/z)^\alpha = \alpha((f * g)'(z)/(f * g)(z) - 1/z)$, $g(z) = z/(1-z)$, and $\beta = \alpha = 1$, we get the Alexander integral operator, $I(f)(z) = \int_0^z (f(t)/t) dt$

(vii) For $n = 1$, $((f * g)(z)/z)^\alpha = \alpha((f * g)'(z)/(f * g)(z) - 1/z)$, $g(z) = z/(1-z)$, and $\beta = 1$, we get the Miller-Mocanu integral operator, $I_\alpha(f)(z) = \int_0^z (f(t)/t)^\alpha dt$, with α positive real number

In that follows, we study the univalence of this general integral operator. The following known results will be used in order to prove our main results.

Lemma 2 (see [14]). *Let β be a complex number, $\operatorname{Re} \beta > 0$, and $f \in \mathscr{A}$. If*

$$\frac{1 - |z|^{2 \operatorname{Re} \beta}}{\operatorname{Re} \beta} \left| \frac{z f''(z)}{f'(z)} \right| \leq 1, \tag{3}$$

for all $z \in \mathscr{U}$, then

$$F_\beta(z) = \left[\beta \int_0^z t^{\beta - 1} f'(t) dt \right]^{1/\beta} \in \mathscr{S}. \tag{4}$$

Lemma 3 (see [15]). *Let γ be a complex number, $\operatorname{Re} \gamma > 0$, and $f \in \mathscr{A}$. If*

$$\frac{1 - |z|^{2 \operatorname{Re} \gamma}}{\operatorname{Re} \gamma} \left| \frac{z f''(z)}{f'(z)} \right| \leq 1, \tag{5}$$

for all $z \in \mathscr{U}$, then for any complex number β, with $\operatorname{Re} \beta \geq \operatorname{Re} \gamma$, we have

$$F_\beta(z) = \left[\beta \int_0^z t^{\beta - 1} f'(t) dt \right]^{1/\beta} \in \mathscr{S}. \tag{6}$$

Lemma 4 (see [16]). *Let β be a complex number, $\operatorname{Re} \beta > 0$, $c \in \mathbb{C}$ with $|c| \leq 1$, $c \neq 1$. If $f \in \mathscr{A}$ satisfies*

$$\left| c|z|^{2\beta} + \left(1 - |z|^{2\beta}\right) \cdot \frac{z f''(z)}{\beta f'(z)} \right| \leq 1, \tag{7}$$

for all $z \in \mathscr{U}$, then

$$F_\beta(z) = \left[\beta \int_0^z t^{\beta - 1} f'(t) dt \right]^{1/\beta} \in \mathscr{S}. \tag{8}$$

Lemma 5 (see [10]). *Let γ and λ be complex numbers, $\operatorname{Re} \gamma > 0$, $\lambda \neq 0$, and $f \in \mathscr{A}$. If*

$$\operatorname{Re} \left(e^{i\theta} \frac{z f''(z)}{f'(z)} \right) \leq \frac{\operatorname{Re} \gamma}{4|\lambda|}, (0 < \operatorname{Re} \gamma < 1), \tag{9}$$

$$\text{or } \operatorname{Re} \left(e^{i\theta} \frac{z f''(z)}{f'(z)} \right) \leq \frac{1}{4|\lambda|}, (\operatorname{Re} \gamma \geq 1), \tag{10}$$

for all $z \in \mathscr{U}$ and for some θ, $0 \leq \theta \leq 2\pi$, then for any complex number β, with $\operatorname{Re} \beta \geq \operatorname{Re} \gamma$, we have

$$F_\beta(z) = \left[\beta \int_0^z t^{\beta - 1} \left(f'(t)\right)^\lambda dt \right]^{1/\beta} \in \mathscr{S}. \tag{11}$$

Lemma 6 (Schwarz) (see [17]). *Let f be a regular function in the open disk, $\mathscr{U}_R = \{z \in \mathbb{C} : |z| < R\}$ with $|f(z)| < M$, M fixed. If f has in $z = 0$ one zero with multiplicity $\geq m$, then*

$$|f(z)| \leq \frac{M}{R^m} |z|^m, (z \in \mathscr{U}_R), \tag{12}$$

the equality for $(z \neq 0)$ can hold only if

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m, \tag{13}$$

where θ is constant.

In order to make the statement for the main results, we need to define the following class (a different class but using the same idea was defined by Frasin in [8]):

$$\mathscr{A}(g, M) = \{f \in \mathscr{A} : |(f * g)(z)| \leq M\}, g \in \mathscr{A}, M \geq 1. \tag{14}$$

2. Main Results

Theorem 7. *Let be $\alpha_i, M_i, i = \overline{1, n}$, positive real numbers, $M_i \geq 1, i = \overline{1, n}$, β complex number, $\operatorname{Re} \beta > 0$, and $\prod_{i=1}^n M_i^{\alpha_i} \leq (1 + 2 \operatorname{Re} \beta)^{1 + (1/(2 \operatorname{Re} \beta))}/2$. If $f_i \in \mathscr{A}(g_i, M_i)$, $i = \overline{1, n}$, then $N_{\alpha,\beta}(f, g) \in \mathscr{S}$.*

Proof. For the beginning, let consider the function:

$$l(z) = \int_0^z \exp \left(\int_0^t \prod_{i=1}^n \left(\frac{(f_i * g_i)(u)}{u} \right)^{\alpha_i} du \right) dt. \tag{15}$$

We have $l(0) = l'(0) - 1 = 0$, where

$$l'(z) = \exp \int_0^z \prod_{i=1}^{n} \left(\frac{(f_i * g_i)(u)}{u} \right)^{\alpha_i} du. \tag{16}$$

By logarithmical differentiation, we get

$$\frac{zl''(z)}{l'(z)} = z \cdot \prod_{i=1}^{n} \left(\frac{(f_i * g_i)(z)}{z} \right)^{\alpha_i}, \tag{17}$$

and further

$$\frac{1-|z|^{2 \operatorname{Re} \beta}}{\operatorname{Re} \beta} \cdot \left| \frac{zl''(z)}{l'(z)} \right| = |z| \cdot \frac{1-|z|^{2 \operatorname{Re} \beta}}{\operatorname{Re} \beta} \cdot \prod_{i=1}^{n} \frac{|(f_i * g_i)(z)|^{\alpha_i}}{|z|^{\alpha_i}}. \tag{18}$$

Since $f_i \in \mathscr{A}(g_i, M_i), i = \overline{1, n}$, according to Schwarz lemma, we obtain

$$|(f_i * g_i)(z)| < M_i |z|. \tag{19}$$

Further, since $\alpha_i > 0, i = \overline{1, n}$, if we put (19) in (18), we have

$$\frac{1-|z|^{2 \operatorname{Re} \beta}}{\operatorname{Re} \beta} \cdot \left| \frac{zl''(z)}{l'(z)} \right| \leq |z| \cdot \frac{1-|z|^{2 \operatorname{Re} \beta}}{\operatorname{Re} \beta} \cdot \prod_{i=1}^{n} M_i^{\alpha_i}. \tag{20}$$

On the other hand, the maximum value of the function

$$H(r) = \frac{1}{\operatorname{Re} \beta} \cdot r \cdot \left(1 - r^{2 \operatorname{Re} \beta}\right) \tag{21}$$

is $2/(1+2 \operatorname{Re} \beta)^{1+(1/(2 \operatorname{Re} \beta))}$, obtained for $r = (1+2 \operatorname{Re} \beta)^{-1/(2 \operatorname{Re} \beta)}$. Hence, if we use this information in (20), we get

$$\frac{1-|z|^{2 \operatorname{Re} \beta}}{\operatorname{Re} \beta} \cdot \left| \frac{zl''(z)}{l'(z)} \right| \leq \left(\frac{2}{(1+2 \operatorname{Re} \beta)^{1+(1/(2 \operatorname{Re} \beta))}} \right) \prod_{i=1}^{n} M_i^{\alpha_i}, \tag{22}$$

and from the hypothesis on $\prod_{i=1}^{n} M_i^{\alpha_i}$, further we have

$$\frac{1-|z|^{2 \operatorname{Re} \beta}}{\operatorname{Re} \beta} \cdot \left| \frac{zl''(z)}{l'(z)} \right| \leq 1. \tag{23}$$

Now, from Lemma 2, we obtain $N_{\alpha,\beta}(f, g) \in \mathscr{S}$. □

Theorem 8. *Let be $\alpha_i, M_i, i = \overline{1, n}$, positive real numbers, $M_i \geq 1, i = \overline{1, n}$, γ complex number, $\operatorname{Re} \gamma > 0$, and $\prod_{i=1}^{n} M_i^{\alpha_i} \leq (1+2 \operatorname{Re} \gamma)^{1+(1/(2 Re \gamma))}/2$. If $f_i \in \mathscr{A}(g_i, M_i)$, $i = \overline{1, n}$, then for any complex number β, with $\operatorname{Re} \beta \geq \operatorname{Re} \gamma$, we have $N_{\alpha,\beta}(f, g) \in \mathscr{S}$.*

Proof. The proof is similar with Theorem 7 if we use Lemma 3, instead of Lemma 2. □

Theorem 9. *Let be $\alpha_i, M_i, i = \overline{1, n}$, positive real numbers, $M_i \geq 1, i = \overline{1, n}$, β complex number, $\operatorname{Re} \beta \geq 2 \cdot \prod_{i=1}^{n} M_i^{\alpha_i}$ and c complex number, $c \neq -1$, such that*

$$|c| \leq 1 - \frac{2 \cdot \prod_{i=1}^{n} M_i^{\alpha_i}}{\operatorname{Re} \beta}. \tag{24}$$

If $f_i \in \mathscr{A}(g_i, M_i)$, $i = \overline{1, n}$, then $N_{\alpha,\beta}(f, g) \in \mathscr{S}$.

Proof. We consider again the function

$$l(z) = \int_0^z \exp \left(\int_0^t \prod_{i=1}^{n} \left(\frac{(f_i * g_i)(u)}{u} \right)^{\alpha_i} du \right) dt, \tag{25}$$

and after some calculations, we get

$$\frac{zl''(z)}{l'(z)} = z \cdot \prod_{i=1}^{n} \left(\frac{(f_i * g_i)(z)}{z} \right)^{\alpha_i}. \tag{26}$$

In order to apply Lemma 4, we evaluate the expression $|c|z|^{2\beta} + (1-|z|^{2\beta})(zl''(z)/\beta l'(z))|$ and from (26), we get

$$\left| c|z|^{2\beta} + \left(1-|z|^{2\beta}\right) \frac{zl''(z)}{\beta l'(z)} \right| \leq |c| + \left| \frac{1-|z|^{2\beta}}{\beta} \right| \cdot |z| \cdot \frac{\prod_{i=1}^{n} |(f_i * g_i)(z)|^{\alpha_i}}{\prod_{i=1}^{n} |z|^{\alpha_i}}. \tag{27}$$

Since $f_i \in \mathscr{A}(g_i, M_i), i = \overline{1, n}$, according to Schwarz lemma, we obtain

$$|(f_i * g_i)(z)| < M_i |z|. \tag{28}$$

Further, since $\alpha_i > 0, i = \overline{1, n}$, if we put (28) in (27), we have

$$\left| c|z|^{2\beta} + \left(1-|z|^{2\beta}\right) \frac{zl''(z)}{\beta l'(z)} \right| \leq |c| + \left| \frac{1-|z|^{2\beta}}{\beta} \right| \cdot |z| \cdot \prod_{i=1}^{n} M_i^{\alpha_i}. \tag{29}$$

But, since for $|z| < 1$, we have $|1-|z|^{2\beta}| \cdot |z| \leq 2$, from the last inequality, we get

$$\left| c|z|^{2\beta} + \left(1-|z|^{2\beta}\right) \frac{zl''(z)}{\beta l'(z)} \right| \leq |c| + \frac{2 \cdot \prod_{i=1}^{n} M_i^{\alpha_i}}{|\beta|} \leq |c| + \frac{2 \cdot \prod_{i=1}^{n} M_i^{\alpha_i}}{Re\beta}. \tag{30}$$

Further, if we use the condition from the hypothesis, related to c, we obtain

$$\left| c|z|^{2\beta} + \left(1 - |z|^{2\beta}\right) \frac{zl''(z)}{\beta l'(z)} \right| \le 1. \tag{31}$$

Now, from Lemma 4, we obtain $N_{\alpha,\beta}(f, g) \in \mathcal{S}$. □

Theorem 10. *Let be $\alpha_i, M_i, i = \overline{1, n}$, positive real numbers, $M_i \ge 1, i = \overline{1, n}$, γ complex number, $\operatorname{Re} \gamma > 0$, and $\prod_{i=1}^{n} M_i^{\alpha_i} \le \min \{\operatorname{Re} \gamma/4, 1/4\}$. If $f_i \in \mathcal{A}(g_i, M_i)$, $i = \overline{1, n}$, then for any complex number β, with $\operatorname{Re} \beta \ge \operatorname{Re} \gamma$, we have $N_{\alpha,\beta}(f, g) \in \mathcal{S}$.*

Proof. We consider the function

$$l(z) = \int_0^z \exp\left(\int_0^t \prod_{i=1}^{n} \left(\frac{(f_i * g_i)(u)}{u} \right)^{\alpha_i} du \right) dt. \tag{32}$$

After some calculations, we obtain $l(0) = l'(0) - 1 = 0$ and

$$\frac{zl''(z)}{l'(z)} = z \cdot \prod_{i=1}^{n} \left(\frac{(f_i * g_i)(z)}{z} \right)^{\alpha_i}, \tag{33}$$

and further

$$\operatorname{Re}\left(e^{i\theta} \cdot \frac{zl''(z)}{l'(z)} \right) \le \left| e^{i\theta} \cdot \frac{zl''(z)}{l'(z)} \right| = |z| \cdot \left| e^{i\theta} \right| \cdot \prod_{i=1}^{n} \frac{|(f_i * g_i)(z)|^{\alpha_i}}{|z|^{\alpha_i}}. \tag{34}$$

Since $f_i \in \mathcal{A}(g_i, M_i), i = \overline{1, n}$, according to Schwarz lemma, we obtain

$$|(f_i * g_i)(z)| < M_i |z|. \tag{35}$$

Further, since $\alpha_i > 0, i = \overline{1, n}$, if we put (35) in (34), we have

$$\operatorname{Re}\left(e^{i\theta} \cdot \frac{zl''(z)}{l'(z)} \right) \le |z| \cdot \left| e^{i\theta} \right| \cdot \prod_{i=1}^{n} M_i^{\alpha_i}. \tag{36}$$

Using further the hypothesis $\prod_{i=1}^{n} M_i^{\alpha_i} \le \min \{\operatorname{Re} \gamma/4, 1/4\}$ and also $|z| < 1$, $|e^{i\theta}| = 1$, from the last inequality, we get

$$\operatorname{Re}\left(e^{i\theta} \cdot \frac{zl''(z)}{l'(z)} \right) \le \min\left\{ \frac{\operatorname{Re} \gamma}{4}, \frac{1}{4} \right\}. \tag{37}$$

Now, if we apply Lemma 5 for $\lambda = 1$, we obtain $N_{\alpha,\beta}(f, g) \in \mathcal{S}$. □

Remark 11. If in Theorems 7–10 we take different particular cases, as for example those from Remarks 1, we get the condition of univalence for different known integral operators. Theorems 7–9 extend the results obtained by Frasin in [8].

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares no conflict of interest.

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