Research Article

Complete Continuity of Composition-Differentiation Operators on the Hardy Space $H^1$

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We study composition-differentiation operators on the Hardy space $H^1$ on the unit disk. We prove that if $\phi$ is an analytic self-map of the unit disk such that the composition-differentiation operator induced by $\phi$ is bounded on the Hardy space $H^1$, then it is completely continuous. This result is stronger than the similar result for composition operators which says that the composition operator induced by $\phi$ is completely continuous if and only if $|\phi(e^{i\theta})| < 1$ almost everywhere on the unit circle.

1. Introduction

Let $D$ denote the open unit disk in the complex plane, and let $\phi : D \rightarrow D$ denote an analytic self-map of $D$. Let $X$ be a Banach space of analytic functions on the unit disk such that for each $f \in X$, the function $f \circ \phi \in X$. The composition operator $C_\phi : X \rightarrow X$ is defined by

$$C_\phi(f) = f \circ \phi. \quad (1)$$

The composition operator was first introduced, in the setting of Hardy-Hilbert space $H^2$, by E. Nordgren in 1968 (see [1]). This line of investigation was then followed by several people in the Hardy space $H^p$ as well as in the Bergman space $A^p$, for $0 < p < \infty$.

In this paper, the Banach space $X$ is either the Hardy space $H^p$ or the Bergman space $A^p$. Let us recall the definitions of these spaces. Assume that $p$ is a positive number. An analytic function $f$ on the unit disk is said to belong to the Hardy space $H^p = H^p(D)$ if

$$\|f\|_{H^p}^p = \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta < \infty. \quad (2)$$

In general, the Hardy space $H^p$ for $1 \leq p < \infty$ is a Banach space of analytic functions, and for $p = 2$, it is a Hilbert space with the following inner product:

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f^*(e^{i\theta}) g^*(e^{i\theta}) \, d\theta, \quad (3)$$

where

$$f^*(e^{i\theta}) = \lim_{r \to 1^-} f(re^{i\theta}) \quad (4)$$

is the boundary function of $f$; we recall that for each $f \in H^p$, the boundary function of $f$ exists almost everywhere on the unit circle (see [2] or [3]). It is easy to see that for $f \in H^2$ with Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad (5)$$

is the boundary function of $f$. We study composition-differentiation operators on the Hardy space $H^1$ on the unit disk. We prove that if $\phi$ is an analytic self-map of the unit disk such that the composition-differentiation operator induced by $\phi$ is bounded on the Hardy space $H^1$, then it is completely continuous. This result is stronger than the similar result for composition operators which says that the composition operator induced by $\phi$ is completely continuous if and only if $|\phi(e^{i\theta})| < 1$ almost everywhere on the unit circle.

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the norm of \( f \) is given by
\[
\|f\|_{H^p} = \left( \sum_{n=0}^{\infty} |a_n|^2 \right)^{1/2}.
\] (6)

Another functional Hilbert space on the unit disk is the Bergman space \( A^p = A^p(D) \) consisting of all analytic functions \( f \) in the unit disk for which the area integral
\[
\int_D |f(z)|^p dA(z)
\] (7)
is finite; here, \( dA(z) = \pi^{-1} dx dy \) is the normalized area measure in the unit disk. The norm of \( f \) in the Bergman space is defined by
\[
\|f\|_{A^p} = \left( \int_D |f(z)|^p dA(z) \right)^{1/p}.
\] (8)

It is well-known that \( A^p \), when \( 1 \leq p < \infty \), is a Banach space. Moreover, for \( f \in A^2 \) with Taylor coefficients \( \{a_n\}_{n=0}^{\infty} \) we have
\[
\|f\|_{A^2} = \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1}.
\] (9)

Note that the inner product in \( A^2 \) is defined by
\[
\langle f, g \rangle = \int_D f(z)g(z)dA(z).
\] (10)

For a detailed account on the theory of Bergman spaces, we refer the reader to [4, 5] and [6].

Assume now that \( \varphi : D \rightarrow D \) is an analytic self-map of the unit disk. We know that the composition operator is bounded on the Hardy space \( H^p \) as well as on the Bergman space \( A^p \) (see [7]). Indeed, for \( 1 \leq p < \infty \), the composition operator \( C_\varphi : H^p \rightarrow H^p \) is bounded and
\[
\|C_\varphi\|_{H^p} \leq \left( \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} \right)^{1/p},
\] (11)
where \( \|C_\varphi\|_{H^p} \) denotes the norm of \( C_\varphi \) on \( H^p \) (see [6]). Similar statement holds for \( C_\varphi : A^p \rightarrow A^p \), that is,
\[
\|C_\varphi\|_{A^p} \leq \left( \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} \right)^{2/p}.
\] (12)

In contrast, the compactness of \( C_\varphi \) depends on the behavior of the function \( \varphi \). For instance, it is proved in ([6], Theorem 10.3.5) that \( C_\varphi \) is compact on \( A^2 \) if and only if
\[
\lim_{|z| \rightarrow 1} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0.
\] (13)

The equivalent condition for the compactness of \( C_\varphi \) on the Hardy space \( H^2 \) can be stated in terms of the Nevanlinna counting function (see [6], Theorem 10.4.10).

For a function \( \psi \in X \), the weighted composition operator \( C_{\psi, \varphi} : X \rightarrow X \) is given by
\[
C_{\psi, \varphi}(f) = \psi \cdot (f \circ \varphi).
\] (14)

This operator was studied by several authors from different aspects, see the papers [8–15] and the references therein.

Another operator related to the composition operator is the composition-differentiation operator defined by
\[
D_{\varphi}(f) = f' \circ \varphi.
\] (15)

Some authors consider \( D_{\varphi} \) as the composition of two successive operators \( C_{\varphi} \) and \( D \) where \( D(f) = f' \) is the differentiation operator. For this reason, they use the notation \( C_{\varphi} D \) for what we denoted by \( D_{\varphi} \). The weighted composition-differentiation operator is defined by
\[
D_{\psi, \varphi}(f) = \psi \cdot (f' \circ \varphi).
\] (16)

This operator was studied by many authors; see for instance [16–18] and the references therein. Compared to the composition operator, the behavior of composition-differentiation operator is more subtle. This is due to the presence of differentiation operator which is known to be unbounded even on the Hardy space \( H^p \). It should be emphasized that if \( \psi \) is a bounded analytic function on the open unit disk, and if \( \varphi \) is a nonconstant self-map of the unit disk such that \( \|\psi\|_{\infty} \leq r < 1 \), then \( D_{\psi, \varphi} \) is bounded on \( H^2 \); indeed, Fatehi and Hammond ([17], Proposition 4) proved that
\[
\|D_{\psi, \varphi}\| \leq \left( \frac{r + |\varphi(0)|}{r - |\varphi(0)|} \right)^{1/2} \frac{1}{|1-r|^{1/(1-r)-1}},
\] (17)
where \( \lfloor . \rfloor \) denotes the greatest integer function.

In this paper, we study the complete continuity of the composition-differentiation operator \( D_{\psi, \varphi} \) and its higher-order variants
\[
D_{\psi, \varphi}^{(k)}(f) = \psi \cdot (f^{(k)} \circ \varphi),
\] (18)
where \( k \) is a positive integer. When \( k = 1 \), we suppress the superscript\(^{(1)}\) and just write \( D_{\psi, \varphi} \).

We intend to study the complete continuity of these operators on the nonreflexive Banach space \( H^1 \). We recall that an operator \( T : X \rightarrow X \) is said to be completely continuous if \( x_n \rightarrow x \) weakly in \( X \) implies \( \|Tx_n - Tx\| \rightarrow 0 \). It is well-known that on a Banach space \( X \), every compact operator is completely continuous. On the other hand, if the Banach space \( X \) is reflexive, these two notions coincide. In this paper, we shall focus on the nonreflexive Hardy space.
$H^1$ and try to find conditions under which the weighted composition-differentiation operator $D_{\psi, \varphi}$ is completely continuous. We prove that if $D_{\psi, \varphi}^{(k)}$ is bounded, then $\varphi = 0$ almost everywhere in

$$\{ e^{i \theta} : \varphi(e^{i \theta}) = 1 \}. \quad (19)$$

This implies that $D_{\psi, \varphi}^{(k)}$ is completely continuous on $H^1$. The results we obtain are in line with similar results already obtained by Cima and Matheson [8] for composition operators and then by Contreras and Hernández-Díaz [9] for weighted composition operators.

2. Boundedness and Compactness

Given $\psi \in H^p$ and a self-map $\varphi$ on the unit disk, we define

$$\mu(E) = \mu_{\psi, \varphi}(E) = \int_{\varphi^{-1}(E) \cap \mathbb{T}} |\psi|^p\,dm, \quad E \subseteq \overline{D}, \quad (20)$$

where

$$\mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \} \quad (21)$$

is the boundary of the unit disk, and $dm$ is the normalized Lebesgue measure on $\mathbb{T}$.

**Theorem 1.** Let $\psi \in H^p$ and let $\varphi$ be a self-map on $\mathbb{D}$. For $1 \leq p, q < \infty$, the operator $D_{\psi, \varphi} : H^p \to H^q$ is bounded (compact, resp.) if and only if $f \to D(f) = f' \circ \varphi$ maps $H^p$ (boundedly (compactly, resp.) into $L^q(\overline{D}, dm)$.

**Proof.** Let $g = \sum_{i=1}^{n} c_i \chi_{E_i}$ be a nonnegative simple function. Since $\chi_{E} \circ \varphi = \chi_{\varphi^{-1}(E)}$, it follows that

$$\int_{\overline{D}} g\,dm = \sum_{i=1}^{n} c_i \mu(E_i) = \sum_{i=1}^{n} c_i \int_{\varphi^{-1}(E_i) \cap \mathbb{T}} |\psi|^p\,dm$$

$$= \int_{\mathbb{T}} |\psi|^p \sum_{i=1}^{n} c_i \chi_{\varphi^{-1}(E_i) \cap \mathbb{T}}\,dm = \int_{\mathbb{T}} |\psi|^p (g \circ \varphi)\,dm. \quad (22)$$

This is indeed true for each nonnegative measurable function $g$ on $\overline{D}$ (for details, see [9], Lemma 2.1). Now letting $g = |f'|^p$, we obtain

$$\int_{\overline{D}} |f'|^p\,dm = \int_{\mathbb{T}} |\psi|^p |f' \circ \varphi|^p\,dm = ||D_{\psi, \varphi}(f)||_{L^p}^p. \quad (23)$$

Assume that $p, q \in [1, \infty)$. It follows from (23) that $D_{\psi, \varphi} : H^p \to H^q$ is bounded (compact, resp.) if and only if $f \to f'$ maps $H^p$ boundedly (compactly, resp.) into $L^q(\overline{D}, dm)$.

We now consider the weighted composition-differentiation operator $D_{\psi, \varphi}$ on the Bergman space $A^p$. Given $\psi \in A^p$ and a self-map $\varphi$ on the unit disk, we define

$$v(E) = \mu_{\psi, \varphi}(E) = \int_{\varphi^{-1}(E)} |\psi|^p\,d\nu, \quad E \subseteq \overline{D}. \quad (24)$$

It is easy to see that for every non-negative measurable function $g$ on the unit disk,

$$\int_{\overline{D}} g\,dv = \int_{\overline{D}} |\psi|^p (g \circ \varphi)\,d\nu. \quad (25)$$

**Theorem 2.** Let $\psi \in A^p$ and let $\varphi, \varphi'$ be self-maps on $\mathbb{D}$. For $1 \leq p, q < \infty$, the operator $D_{\psi, \varphi} : A^p \to A^q$ is bounded (compact, resp.) if and only if $f \to D(f) = f' \circ \varphi$ maps $A^p$ boundedly (compactly, resp.) into $L^q(\overline{D}, d\nu)$.

**Proof.** Letting $g = |f'|^p$ in (25), we obtain

$$\int_{\overline{D}} |f'|^p\,dv = \int_{\overline{D}} |\psi|^p |f' \circ \varphi|^p\,d\nu = ||D_{\psi, \varphi}(f)||_{A^p}^p, \quad f \in A^p. \quad (26)$$

On the other hand, assume there is a constant $C > 0$ such that for each $f \in A^p$ we have

$$||f'||_{L^1(\overline{D}, d\nu)} \leq C ||f||_{A^p}. \quad (27)$$

This together with (26) implies that

$$||D_{\psi, \varphi}(f)||_{A^q} = ||f'||_{L^1(\overline{D}, d\nu)} \leq C ||f||_{A^p}, \quad (28)$$

which means that $D_{\psi, \varphi}$ is bounded. Conversely, assume that

$$||D_{\psi, \varphi}(f)||_{A^q} \leq C' ||f||_{A^p}, \quad (29)$$

for some $C' > 0$. It then follows from (26) that

$$||f'||_{L^1(\overline{D}, d\nu)} \leq C' ||f||_{A^p}. \quad (30)$$

Therefore, $D$ is bounded. The equivalence of compactness of two operators is a consequence of (28).

3. Complete Continuity

In this section, we study the complete continuity of composition-differentiation operators on the Hardy space $H^1$. We begin by recalling two statements; one in measure theory that characterizes weak convergence in $L^1$ and the other from general functional analysis on the weak-to-weak continuity of bounded operators. Let $(X, \mathcal{S}, \mu)$ be a measure space. A sequence $f_n : X \to \mathbb{R}$ of measurable
functions is said to converge in measure to a measurable function \( f : X \rightarrow \mathbb{R} \) if for every \( \eta > 0 \),
\[
\lim_{n \to \infty} \mu(\{ x \in X : |f_n(x) - f(x)| > \eta \}) = 0. \tag{31}
\]

It is well-known that if \( \mu(X) < \infty \), and if \( f_n \rightharpoonup f \) almost everywhere, then \( f_n \rightharpoonup f \) in measure (see [19], page 100).

**Lemma 3** (see [20], page 295). Let \( (f_n) \) be a sequence in \( L^1(X, \mathcal{S}, \mu) \) that converges weakly to \( f \). Then, \( f_n \) converges strongly to \( f \) if and only if \( f_n \) converges to \( f \) in measure on every measurable subset of finite measure.

**Lemma 4.** Let \( X \) be a Banach space and \( T \) be a bounded operator on \( X \). Assume that \( \{x_n\} \) is a sequence in \( X \) such that \( x_n \rightharpoonup 0 \) weakly. Then, \( T x_n \rightharpoonup 0 \) weakly.

**Proof.** Let \( f \) be an arbitrary element in the dual space \( X^* \). Since \( f \circ T \) is continuous, it follows that \((f \circ T)(x_n) \rightarrow (f \circ T)(0) = 0\) which is the desired result.

**Theorem 5.** Let \( \varphi \) be an analytic self-map of the open unit disk \( \mathbb{D} \) such that \( D_{\varphi} \) is bounded on \( H^1 \). Then, \( D_{\varphi} \) is completely continuous.

**Proof.** We first consider the following sequence of functions in \( H^1 \):
\[
w_n(z) = \frac{z^{n+k}}{n+1}, \quad n \geq 0. \tag{32}
\]

It is clear that
\[
\|w_n\|_{H^1} = \frac{1}{n + 1} \rightarrow 0, \quad n \rightarrow \infty. \tag{33}
\]

Since \( D_{\varphi} \) is bounded, it follows that
\[
\|\varphi^n\|_{H^1} = \|D_{\varphi}(w_n)\|_{H^1} \rightarrow 0, \quad n \rightarrow \infty, \tag{34}
\]
which is not possible if \( |\varphi(e^{i\theta})| = 1 \) on a set of positive measure. Therefore,
\[
|\varphi(e^{i\theta})| < 1, \quad a.e. \text{ on } \mathbb{T}. \tag{35}
\]

To prove the complete continuity of the composition-differentiation operator \( D_{\varphi} \), we let \((f_n)\) be a sequence that converges weakly to zero in \( H^1 \). It follows that \( f_n \rightharpoonup 0 \) uniformly on compact subsets of \( \mathbb{D} \). This, in turn, implies that \( f_n' \rightharpoonup 0 \) uniformly on compact subsets of the unit disk. Therefore, \( f_n'(\varphi(z)) \rightharpoonup 0 \) pointwise in \( \mathbb{D} \), and hence (due to the fact that \( |\varphi'(e^{i\theta})| < 1 \) a.e. on \( \mathbb{T} \))
\[
D_{\varphi}(f_n)(e^{i\theta}) = f_n'(\varphi(e^{i\theta})) \rightarrow 0, \quad a.e. \text{ on } \mathbb{T}. \tag{36}
\]

We should recall that in finite measure spaces, almost everywhere (pointwise) convergence implies convergence in measure (see [19], page 100). Therefore \( D_{\varphi}(f_n) \) converges in measure to zero in \( L^1(\mathbb{T}, dm) \). Moreover, according to Lemma 4, the boundedness of \( D_{\varphi} \) on \( H^1 \) implies that \( D_{\varphi}(f_n) \rightharpoonup 0 \) in the weak topology of \( H^1 \), and hence in the weak topology of \( L^1(\mathbb{T}) \). Finally, we know from ([20], page 295) that weak convergence of \( D_{\varphi}(f_n) \) together with its convergence in measure implies that \( \|D_{\varphi}(f_n)\|_{H^1} \rightarrow 0 \). Hence, \( D_{\varphi} \) is completely continuous.

**Theorem 6.** Let \( \varphi \) be an analytic self-map of the open unit disk \( \mathbb{D} \) such that \( D_{\varphi}^{(k)} \) is bounded on \( H^1 \). Then, \( D_{\varphi}^{(k)} \) is completely continuous.

**Proof.** Let us consider the sequence of functions
\[
w_n(z) = \frac{z^{n+k}}{(n + 1) \cdots (n + k)}, \quad n \geq 1, \tag{37}
\]
in \( H^1 \). It follows that \((d^k/dz^k)w_n = z^n\), and
\[
\|w_n\|_{H^1} = \frac{1}{(n + 1) \cdots (n + k)} \rightarrow 0, \quad n \rightarrow \infty. \tag{38}
\]

Therefore, the boundedness of \( D_{\varphi}^{(k)} \) implies that
\[
\|\varphi^n\|_{H^1} = \|D_{\varphi}^{(k)}(w_n)\|_{H^1} \rightarrow 0, \quad n \rightarrow \infty, \tag{39}
\]
which is impossible unless \( |\varphi| < 1 \) almost everywhere on \( \mathbb{T} \).

For the complete continuity of \( D_{\varphi}^{(k)} \), we note that if \( (f_n) \) is a weak null sequence in \( H^1 \), then \( f_n^{(k)} \rightharpoonup 0 \) uniformly on compact subsets of \( \mathbb{D} \), from which it follows that
\[
D_{\varphi}^{(k)}(f_n)(e^{i\theta}) = f_n^{(k)}(\varphi(e^{i\theta})) \rightarrow 0, \quad a.e. \text{ on } \mathbb{T}. \tag{40}
\]

The rest of the argument is routine.

**Theorem 7.** Let \( \psi \in H^1 \) and \( \varphi \) be an analytic self-map of the open unit disk \( \mathbb{D} \) such that \( D_{\psi\varphi} \) is bounded on \( H^1 \). Then, \( D_{\psi\varphi} \) is completely continuous.

**Proof.** Again, we consider the sequence
\[
w_n(z) = \frac{z^{n+1}}{n + 1}, \quad n \geq 1, \tag{41}
\]
which converges to zero in \( H^1 \). Since \( D_{\psi\varphi} \) is bounded, it follows that
\[
\|D_{\psi\varphi}(w_n)\|_{H^1} \rightarrow 0, \quad n \rightarrow \infty. \tag{42}
\]
On the other hand,
\[
\int_{\{e^{\alpha} : |\varphi(e^{\alpha})| = 1\}} |\psi|dm = \int_{\{e^{\alpha} : |\varphi(e^{\alpha})| < 1\}} |\psi||\varphi||n|dm
\leq \int_{T} |\psi||\varphi|^n dm = \|D_{\varphi}(w_n)\|_{H^1} \longrightarrow 0, \quad n \longrightarrow \infty.
\]
(43)

Therefore, the integral on the left-hand side must be zero, from which it follows that
\[
\psi(e^{\theta}) = 0, \quad a.e. in \{e^{\alpha} : |\varphi(e^{\alpha})| = 1\}. \quad (44)
\]

To prove that \(D_{\varphi, \psi}(f_n)\) is completely continuous, we let \((f_n)\) be a weak null sequence in \(H^1\). It follows that \(f'_n \longrightarrow 0\) uniformly on compact subsets of \(D\). Using this fact together with the assumption that \(\psi = 0\) almost everywhere in \(\{e^{\theta} : |\varphi(e^{\theta})| = 1\}\), we conclude that
\[
D_{\varphi, \psi}(f_n)(e^{\theta}) = \psi(e^{\theta})f'_n(e^{\theta}) \longrightarrow 0, \quad a.e. on T.
\]
(45)

It now follows that \(D_{\varphi, \psi}(f_n)\) converges to zero in measure in \(L^1(T)\) (see [19], page 100). Moreover, the boundedness of \(D_{\varphi, \psi}\) on \(H^1\) implies that \(D_{\varphi, \psi}(f_n) \longrightarrow 0\) in the weak topology of \(H^1\), and hence in the weak topology of \(L^1(T)\) (by Lemma 4). Finally, we invoke the fact that weak convergence of a given sequence together with its convergence in measure implies its norm convergence (see [20], page 295), that is, \(\|D_{\varphi, \psi}(f_n)\|_{H^1} \longrightarrow 0\) as \(n \longrightarrow \infty\).

\[\square\]

In the following theorem, we extend the above result to weighted composition-differentiation operators.

**Theorem 8.** Let \(\psi \in H^1\) and \(\varphi\) be an analytic self-map of the unit disk \(D\) such that \(D_{\varphi}^{(k)}\) is bounded on \(H^1\). Then, \(D_{\varphi, \psi}^{(k)}\) is completely continuous.

**Proof.** Using the sequence introduced in Theorem 6, we have
\[
\int_{\{e^{\alpha} : |\varphi(e^{\alpha})| < 1\}} |\psi|dm = \int_{\{e^{\alpha} : |\varphi(e^{\alpha})| < 1\}} |\psi||\varphi||n|dm
\leq \int_{T} |\psi||\varphi|^n dm = \|D_{\varphi}^{(k)}(w_n)\|_{H^1} \longrightarrow 0, \quad n \longrightarrow \infty.
\]
(46)

The rest of argument is routine. \[\square\]

### 4. Conclusion

In this paper, we studied the composition-differentiation operator \(D_{\varphi}\) and its variants including the weighted composition-differentiation operators \(D_{\varphi, \psi}\) and \(D_{\varphi, \psi}^{(k)}\). We have proved that each of these operators is completely continuous provided that it is bounded. The motivation for the above description comes from the following theorem proved in [18], Proposition 1): the composition operator \(C_{\varphi}\) is completely continuous on \(H^1\) if and only if \(|\varphi(e^{\alpha})| < 1\) almost everywhere on the unit circle. This result was then generalized to weighted composition operators ([19], Theorem 4.1): the weighted composition operator \(C_{\varphi, \psi}\) is completely continuous on \(H^1\) if and only if \(\psi = 0\) almost everywhere on the set \(\{e^{\theta} : |\varphi(e^{\theta})| = 1\}\).

As a matter of fact, given an analytic self-map of the unit disk \(\varphi\), the composition operator \(C_{\varphi}\) is always bounded on \(H^1\) while \(D_{\varphi}\) may not be bounded; so in our results, we have assumed that \(D_{\varphi}\) is bounded on \(H^1\). On the other hand, as we have seen in the proofs presented in this paper, the boundedness assumption on \(D_{\varphi}\) implies that \(|\varphi(e^{\alpha})| < 1\) almost everywhere on the unit circle. In this way, we obtained the stronger result that the composition-differentiation operators are completely continuous provided that they are bounded.

**Data Availability**

Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

**Conflicts of Interest**

This research was done as part of my usual duty as an employee of IKIU.

**References**


