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Research Article

Complete Continuity of Composition-Differentiation Operators on the Hardy Space H^1

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We study composition-differentiation operators on the Hardy space H^1 on the unit disk. We prove that if φ is an analytic self-map of the unit disk such that the composition-differentiation operator induced by φ is bounded on the Hardy space H^1, then it is completely continuous. This result is stronger than the similar result for composition operators which says that the composition operator induced by φ is completely continuous if and only if $|\varphi(e^{i\theta})| < 1$ almost everywhere on the unit circle.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane, and let $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ denote an analytic self-map of $\mathbb{D}$. Let X be a Banach space of analytic functions on the unit disk such that for each $f \in X$, the function $f \circ \varphi \in X$. The composition operator $C_\varphi : X \longrightarrow X$ is defined by

$$C_\varphi(f) = f \circ \varphi. \tag{1}$$

The composition operator was first introduced, in the setting of Hardy-Hilbert space H^2, by E. Nordgren in 1968 (see [1]). This line of investigation was then followed by several people in the Hardy space H^p as well as in the Bergman space A^p, for $0 < p < \infty$.

In this paper, the Banach space X is either the Hardy space H^p or the Bergman space A^p. Let us recall the definitions of these spaces. Assume that p is a positive number. An analytic function f on the unit disk is said to belong to the Hardy space $H^p = H^p(\mathbb{D})$ if

$$\|f\|_{H^p}^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\theta}\right) \right|^p d\theta < \infty. \tag{2}$$

In general, the Hardy space H^p for $1 \le p < \infty$ is a Banach space of analytic functions, and for $p = 2$, it is a Hilbert space with the following inner product:

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f^*\left(e^{i\theta}\right) \overline{g^*\left(e^{i\theta}\right)} d\theta, \tag{3}$$

where

$$f^*\left(e^{i\theta}\right) \coloneqq \lim_{r\longrightarrow 1^-} f\left(re^{i\theta}\right) \tag{4}$$

is the boundary function of f; we recall that for each $f \in H^p$, the boundary function of f exists almost everywhere on the unit circle (see [2] or [3]). It is easy to see that for $f \in H^2$ with Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \tag{5}$$

the norm of f is given by

$$\|f\|_{H^2} = \left(\sum_{n=0}^{\infty} |a_n|^2\right)^{1/2}. \tag{6}$$

Another functional Hilbert space on the unit disk is the Bergman space $A^p = A^p(\mathbb{D})$ consisting of all analytic functions f in the unit disk for which the area integral

$$\int_{\mathbb{D}} |f(z)|^p dA(z) \tag{7}$$

is finite; here, $dA(z) = \pi^{-1}dxdy$ is the normalized area measure in the unit disk. The norm of f in the Bergman space is defined by

$$\|f\|_{A^p} = \left(\int_{\mathbb{D}} |f(z)|^p dA(z)\right)^{1/p}. \tag{8}$$

It is well-known that A^p, when $1 \le p < \infty$, is a Banach space. Moreover, for $f \in A^2$ with Taylor coefficients $\{a_n\}_{n=0}^{\infty}$ we have

$$\|f\|_{A^2}^2 = \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1}. \tag{9}$$

Note that the inner product in A^2 is defined by

$$\langle f, g\rangle = \int_{\mathbb{D}} f(z)\bar{g(z)}dA(z). \tag{10}$$

For a detailed account on the theory of Bergman spaces, we refer the reader to [4, 5] and [6].

Assume now that $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ is an analytic self-map of the unit disk. We know that the composition operator is bounded on the Hardy space H^p as well as on the Bergman space A^p (see [7]). Indeed, for $1 \le p < \infty$, the composition operator $C_\varphi : H^p \longrightarrow H^p$ is bounded and

$$\|C_\varphi\|_{H^p} \le \left(\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}\right)^{1/p}, \tag{11}$$

where $\|C_\varphi\|_{H^p}$ denotes the norm of C_φ on H^p (see [6]). Similar statement holds for $C_\varphi : A^p \longrightarrow A^p$, that is,

$$\|C_\varphi\|_{A^p} \le \left(\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}\right)^{2/p}. \tag{12}$$

In contrast, the compactness of C_φ depends on the behavior of the function φ. For instance, it is proved in ([6], Theorem 10.3.5) that C_φ is compact on A^2 if and only if

$$\lim_{|z|\longrightarrow 1^-} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0. \tag{13}$$

The equivalent condition for the compactness of C_φ on the Hardy space H^2 can be stated in terms of the Nevanlinna counting function (see [6], Theorem 10.4.10).

For a function $\psi \in X$, the weighted composition operator $C_{\psi,\varphi} : X \longrightarrow X$ is given by

$$C_{\psi,\varphi}(f) = \psi \cdot (f \circ \varphi). \tag{14}$$

This operator was studied by several authors from different aspects, see the papers [8–15] and the references therein. Another operator related to the composition operator is the composition-differentiation operator defined by

$$D_\varphi(f) = f' \circ \varphi. \tag{15}$$

Some authors consider D_φ as the composition of two successive operators C_φ and D where $D(f) = f'$ is the differentiation operator. For this reason, they use the notation $C_\varphi D$ for what we denoted by D_φ. The weighted composition-differentiation operator is defined by

$$D_{\psi,\varphi}(f) = \psi \cdot \left(f' \circ \varphi\right). \tag{16}$$

This operator was studied by many authors; see for instance [16–18] and the references therein. Compared to the composition operator, the behavior of composition-differentiation operator is more subtle. This is due to the presence of differentiation operator which is known to be unbounded even on the Hardy space H^p. It should be emphasized that if ψ is a bounded analytic function on the open unit disk, and if φ is a nonconstant self-map of the unit disk such that $\|\varphi\|_\infty \le r < 1$, then $D_{\psi,\varphi}$ is bounded on H^2; indeed, Fatehi and Hammond ([17], Proposition 4) proved that

$$\|D_\varphi\| \le \left(\frac{r + |\varphi(0)|}{r - |\varphi(0)|}\right)^{1/2} \left\lfloor \frac{1}{1-r} \right\rfloor r^{\lfloor 1/(1-r) \rfloor - 1}, \tag{17}$$

where $\lfloor\cdot\rfloor$ denotes the greatest integer function.

In this paper, we study the complete continuity of the composition-differentiation operator $D_{\psi,\varphi}$ and its higher-order variants

$$D_{\psi,\varphi}^{(k)}(f) = \psi \cdot \left(f^{(k)} \circ \varphi\right), \tag{18}$$

where k is a positive integer. When $k = 1$, we suppress the superscript$^{(1)}$ and just write $D_{\psi,\varphi}$.

We intend to study the complete continuity of these operators on the nonreflexive Banach space H^1. We recall that an operator $T : X \longrightarrow X$ is said to be completely continuous if $x_n \longrightarrow x$ weakly in X implies $\|Tx_n - Tx\| \longrightarrow 0$. It is well-known that on a Banach space X, every compact operator is completely continuous. On the other hand, if the Banach space X is reflexive, these two notions coincide. In this paper, we shall focus on the nonreflexive Hardy space

H^1 and try to find conditions under which the weighted composition-differentiation operator $D_{\psi,\varphi}$ is completely continuous. We prove that if $D^{(k)}_{\psi,\varphi}$ is bounded, then $\psi = 0$ almost everywhere in

$$\left\{e^{i\theta} : \left|\varphi\left(e^{i\theta}\right)\right| = 1\right\}. \tag{19}$$

This implies that $D^{(k)}_{\psi,\varphi}$ is completely continuous on H^1. The results we obtain are in line with similar results already obtained by Cima and Matheson [8] for composition operators and then by Contreras and Hernández-Diaz [9] for weighted composition operators.

2. Boundedness and Compactness

Given $\psi \in H^p$ and a self-map φ on the unit disk, we define

$$\mu(E) = \mu_{\psi,\varphi}(E) = \int_{\varphi^{-1}(E)\cap\mathbb{T}} |\psi|^p \mathrm{d}m, \quad E \subseteq \bar{\mathbb{D}}, \tag{20}$$

where

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\} \tag{21}$$

is the boundary of the unit disk, and dm is the normalized Lebesgue measure on $\mathbb{T}$.

Theorem 1. *Let $\psi \in H^p$ and let φ be a self-map on $\mathbb{D}$. For $1 \le p, q < \infty$, the operator $D_{\psi,\varphi} : H^p \longrightarrow H^q$ is bounded (compact, resp.) if and only if $f \mapsto D(f) = f'$ maps H^p boundedly (compactly, resp.) into $L^q(\mathbb{D}, d\nu)$.*

Proof. Let $g = \sum_{i=1}^n c_i \chi_{E_i}$ be a nonnegative simple function. Since $\chi_E \circ \varphi = \chi_{\varphi^{-1}(E)}$, it follows that

$$\begin{aligned}\int_{\bar{\mathbb{D}}} g \mathrm{d}\mu &= \sum_{i=1}^n c_i \mu(E_i) = \sum_{i=1}^n c_i \int_{\varphi^{-1}(E_i)\cap\mathbb{T}} |\psi|^p \mathrm{d}m \\ &= \int_{\mathbb{T}} |\psi|^p \sum_{i=1}^n c_i \chi_{\varphi^{-1}(E_i)\cap\mathbb{T}} \mathrm{d}m = \int_{\mathbb{T}} |\psi|^p (g \circ \varphi) \mathrm{d}m.\end{aligned} \tag{22}$$

This is indeed true for each nonnegative measurable function g on $\bar{\mathbb{D}}$ (for details, see [9], Lemma 2.1). Now letting $g = |f'|^p$, we obtain

$$\int_{\bar{\mathbb{D}}} |f'|^p \mathrm{d}\mu = \int_{\mathbb{T}} |\psi|^p |f' \circ \varphi|^p \mathrm{d}m = \|D_{\psi,\varphi}(f)\|^p_{H^p}. \tag{23}$$

Assume that $p, q \in [1, \infty)$. It follows from (23) that $D_{\psi,\varphi} : H^p \longrightarrow H^q$ is bounded (compact, resp.) if and only if $f \mapsto f'$ maps H^p boundedly (compactly, resp.) into $L^q(\bar{\mathbb{D}}, \mathrm{d}\mu)$.

We now consider the weighted composition-differentiation operator $D_{\psi,\varphi}$ on the Bergman space A^p. Given $\psi \in A^p$ and a self-map φ on the unit disk, we define

$$\nu(E) = \mu_{\psi,\varphi}(E) = \int_{\varphi^{-1}(E)} |\psi|^p \mathrm{d}A, \quad E \subseteq \mathbb{D}. \tag{24}$$

It is easy to see that for every non-negative measurable function g on the unit disk,

$$\int_{\mathbb{D}} g \mathrm{d}\nu = \int_{\mathbb{D}} |\psi|^p (g \circ \varphi) \mathrm{d}A. \tag{25}$$

□

Theorem 2. *Let $\psi \in A^p$ and let φ, φ' be self-maps on $\mathbb{D}$. For $1 \le p, q < \infty$, the operator $D_{\psi,\varphi} : A^p \longrightarrow A^q$ is bounded (compact, resp.) if and only if $f \mapsto D(f) = f'$ maps A^p boundedly (compactly, resp.) into $L^q(\mathbb{D}, d\nu)$.*

Proof. Letting $g = |f'|^p$ in (25), we obtain

$$\int_{\mathbb{D}} |f'|^p \mathrm{d}\nu = \int_{\mathbb{D}} |\psi|^p |f' \circ \varphi|^p \mathrm{d}A = \|D_{\psi,\varphi}(f)\|^p_{A^p}, \quad f \in A^p. \tag{26}$$

On the other hand, assume there is a constant $C > 0$ such that for each $f \in A^p$ we have

$$\|f'\|_{L^q(\mathbb{D},\mathrm{d}\nu)} \le C\|f\|_{A^p}. \tag{27}$$

This together with (26) implies that

$$\|D_{\psi,\varphi}(f)\|_{A^q} = \|f'\|_{L^q(\mathbb{D},\mathrm{d}\nu)} \le C\|f\|_{A^p}, \tag{28}$$

which means that $D_{\psi,\varphi}$ is bounded. Conversely, assume that

$$\|D_{\psi,\varphi}(f)\|_{A^q} \le C'\|f\|_{A^p} \tag{29}$$

for some $C' > 0$. It then follows from (26) that

$$\|f'\|_{L^q(\mathbb{D},\mathrm{d}\nu)} \le C'\|f\|_{A^p}. \tag{30}$$

Therefore, D is bounded. The equivalence of compactness of two operators is a consequence of (28). □

3. Complete Continuity

In this section, we study the complete continuity of composition-differentiation operators on the Hardy space H^1. We begin by recalling two statements; one in measure theory that characterizes weak convergence in L^1 and the other from general functional analysis on the weak-to-weak continuity of bounded operators. Let $(X, \mathcal{S}, \mu)$ be a measure space. A sequence $f_n : X \longrightarrow \mathbb{R}$ of measurable functions is said to converge in measure to a measurable function $f : X \longrightarrow \mathbb{R}$ if for every $\eta > 0$,

$$\lim_{n\longrightarrow\infty}\mu(\{x \in X: |f_n(x) - f(x)| > \eta) = 0. \quad (31)$$

It is well-known that if $\mu(X) < \infty$, and if $f_n \longrightarrow f$ almost everywhere, then $f_n \longrightarrow f$ in measure (see [19], page 100).

Lemma 3 (see [20], page 295). *Let (f_n) be a sequence in $L^1(X, S, \mu)$ that converges weakly to f. Then, f_n converges strongly to f if and only if f_n converges to f in measure on every measurable subset of finite measure.*

Lemma 4. *Let X be a Banach space and T be a bounded operator on X. Assume that $\{x_n\}$ is a sequence in X such that $x_n \longrightarrow 0$ weakly. Then, $Tx_n \longrightarrow 0$ weakly.*

Proof. Let f be an arbitrary element in the dual spaceX^*. Since $f \circ T$ is continuous, it follows that $(f \circ T)(x_n) \longrightarrow (f \circ T)(0) = 0$ which is the desired result. □

Theorem 5. *Let φ be an analytic self-map of the open unit disk $\mathbb{D}$ such that D_φ is bounded on H^1. Then, D_φ is completely continuous.*

Proof. We first consider the following sequence of functions in H^1:

$$w_n(z) = \frac{z^{n+1}}{n+1}, \quad n \ge 0. \quad (32)$$

It is clear that

$$\|w_n\|_{H^1} = \frac{1}{n+1} \longrightarrow 0, \quad n \longrightarrow \infty. \quad (33)$$

Since D_φ is bounded, it follows that

$$\|\varphi^n\|_{H^1} = \|D_\varphi(w_n)\|_{H^1} \longrightarrow 0, \quad n \longrightarrow \infty, \quad (34)$$

which is not possible if $|\varphi(e^{i\theta})| = 1$ on a set of positive measure. Therefore,

$$\left|\varphi\left(e^{i\theta}\right)\right| < 1, \quad a.e. on\, \mathbb{T}. \quad (35)$$

To prove the complete continuity of the composition-differentiation operator D_φ, we let (f_n) be a sequence that converges weakly to zero in H^1. It follows that $f_n \longrightarrow 0$ uniformly on compact subsets of $\mathbb{D}$. This, in turn, implies that $f'_n \longrightarrow 0$ uniformly on compact subsets of the unit disk. Therefore, $f'_n(\varphi(z)) \longrightarrow 0$ pointwise in $\mathbb{D}$, and hence (due to the fact that $|\varphi(e^{i\theta})| < 1$ a.e. on $\mathbb{T}$)

$$D_\varphi(f_n)\left(e^{i\theta}\right) = f'_n\left(\varphi\left(e^{i\theta}\right)\right) \longrightarrow 0, \quad a.e. on\, \mathbb{T}. \quad (36)$$

We should recall that in finite measure spaces, almost everywhere (pointwise) convergence implies convergence in measure (see [19], page 100). Therefore $D_\varphi(f_n)$ converges in measure to zero in $L^1(\mathbb{T}, dm)$. Moreover, according to Lemma 4, the boundedness of D_φ on H^1 implies that $D_\varphi(f_n) \longrightarrow 0$ in the weak topology of H^1, and hence in the weak topology of $L^1(\mathbb{T})$. Finally, we know from ([20], page 295) that weak convergence of $D_\varphi(f_n)$ together with its convergence in measure implies that $\|D_\varphi(f_n)\|_{H^1}$ converges to zero. Hence, D_φ is completely continuous. □

Theorem 6. *Let φ be an analytic self-map of the open unit disk $\mathbb{D}$ such that $D_\varphi^{(k)}$ is bounded on H^1. Then, $D_\varphi^{(k)}$ is completely continuous.*

Proof. Let us consider the sequence of functions

$$w_n(z) = \frac{z^{n+k}}{(n+1)\cdots(n+k)}, \quad n \ge 1, \quad (37)$$

in H^1. It follows that $(d^k/dz^k)w_n = z^n$, and

$$\|w_n\|_{H^1} = \frac{1}{(n+1)\cdots(n+k)} \longrightarrow 0, \quad n \longrightarrow \infty. \quad (38)$$

Therefore, the boundedness of $D_\varphi^{(k)}$ implies that

$$\|\varphi^n\|_{H^1} = \left\|D_\varphi^{(k)}(w_n)\right\|_{H^1} \longrightarrow 0, \quad n \longrightarrow \infty, \quad (39)$$

which is impossible unless $|\varphi| < 1$ almost everywhere on $\mathbb{T}$.

For the complete continuity of $D_\varphi^{(k)}$, we note that if (f_n) is a weak null sequence in H^1, then $f_n^{(k)} \longrightarrow 0$ uniformly on compact subsets of $\mathbb{D}$, from which it follows that

$$D_\varphi^{(k)}(f_n)\left(e^{i\theta}\right) = f_n^{(k)}\left(\varphi\left(e^{i\theta}\right)\right) \longrightarrow 0, \quad a.e. on\, \mathbb{T}. \quad (40)$$

The rest of the argument is routine. □

Theorem 7. *Let $\psi \in H^1$ and φ be an analytic self-map of the open unit disk $\mathbb{D}$ such that $D_{\psi,\varphi}$ is bounded on H^1. Then, $D_{\psi,\varphi}$ is completely continuous.*

Proof. Again, we consider the sequence

$$w_n(z) = \frac{z^{n+1}}{n+1}, \quad n \ge 1, \quad (41)$$

which converges to zero in H^1. Since $D_{\psi,\varphi}$ is bounded, it follows that

$$\|D_{\psi,\varphi}(w_n)\|_{H^1} \longrightarrow 0, \quad n \longrightarrow \infty. \quad (42)$$

On the other hand,

$$\int_{\{e^{i\theta}:|\varphi(e^{i\theta})|=1\}}|\psi|\mathrm{d}m=\int_{\{e^{i\theta}:|\varphi(e^{i\theta})|=1\}}|\psi||\varphi|^{n}\mathrm{d}m \le \int_{\mathbb{T}}|\psi||\varphi|^{n}\mathrm{d}m=\|D_{\psi,\varphi}(w_n)\|_{H^1}\longrightarrow 0,\quad n\longrightarrow\infty. \tag{43}$$

Therefore, the integral on the left-hand side must be zero, from which it follows that

$$\psi\left(e^{i\theta}\right)=0,\quad a.e.\text{in}\left\{e^{i\theta}:\left|\varphi\left(e^{i\theta}\right)\right|=1\right\}. \tag{44}$$

To prove that $D_{\psi,\varphi}$ is completely continuous, we let (f_n) be a weak null sequence in H^1. It follows that $f_n'\longrightarrow 0$ uniformly on compact subsets of $\mathbb{D}$. Using this fact together with the assumption that $\psi=0$ almost everywhere in $\{e^{i\theta}:|\varphi(e^{i\theta})|=1\}$, we conclude that

$$D_{\psi,\varphi}(f_n)\left(e^{i\theta}\right)=\psi\left(e^{i\theta}\right)f_n'\left(\varphi\left(e^{i\theta}\right)\right)\longrightarrow 0,\quad a.e.on\,\mathbb{T}. \tag{45}$$

It now follows that $D_{\psi,\varphi}(f_n)$ converges to zero in measure in $L^1(\mathbb{T})$ (see [19], page 100). Moreover, the boundedness of $D_{\psi,\varphi}$ on H^1 implies that $D_{\psi,\varphi}(f_n)\longrightarrow 0$ in the weak topology of H^1, and hence in the weak topology of $L^1(\mathbb{T})$ (by Lemma 4). Finally, we invoke the fact that weak convergence of a given sequence together with its convergence in measure implies its norm convergence (see [20], page 295), that is, $\|D_{\psi,\varphi}(f_n)\|_{H^1}\longrightarrow 0$ as $n\longrightarrow\infty$. □

In the following theorem, we extend the above result to weighted composition-differentiation operators.

Theorem 8. *Let $\psi\in H^1$ and φ be an analytic self-map of the unit disk $\mathbb{D}$ such that $D_{\psi,\varphi}^{(k)}$ is bounded on H^1. Then, $D_{\psi,\varphi}^{(k)}$ is completely continuous.*

Proof. Using the sequence introduced in Theorem 6, we have

$$\int_{\{e^{i\theta}:|\varphi(e^{i\theta})|=1\}}|\psi|\mathrm{d}m=\int_{\{e^{i\theta}:|\varphi(e^{i\theta})|=1\}}|\psi||\varphi|^{n}\mathrm{d}m \le \int_{\mathbb{T}}|\psi||\varphi|^{n}\mathrm{d}m=\left\|D_{\psi,\varphi}^{(k)}(w_n)\right\|_{H^1}\longrightarrow 0, n\longrightarrow\infty. \tag{46}$$

The rest of argument is routine. □

4. Conclusion

In this paper, we studied the composition-differentiation operator D_φ and its variants including the weighted composition-differentiation operators $D_{\psi,\varphi}$ and $D_{\psi,\varphi}^{(k)}$. We have proved that each of these operators is completely continuous provided that is it bounded. The motivation for the above description comes from the following theorem proved in ([8], Proposition 1): the composition operator C_φ is completely continuous on H^1 if and only if $|\varphi(e^{i\theta})|<1$ almost everywhere on the unit circle. This result was then generalized to weighted composition operators ([9], Theorem 4.1): the weighted composition operator $C_{\psi,\varphi}$ is completely continuous on H^1 if and only if $\psi=0$ almost everywhere on the set $\{e^{i\theta}:|\varphi(e^{i\theta})|=1\}$.

As a matter of fact, given an analytic self-map of the unit disk φ, the composition operator C_φ is always bounded on H^1 while D_φ may not be bounded; so in our results, we have assumed that D_φ is bounded on H^1. On the other hand, as we have seen in the proofs presented in this paper, the boundedness assumption on D_φ implies that $|\varphi(e^{i\theta})|<1$ almost everywhere on the unit circle. In this way, we obtained the stronger result that the composition-differentiation operators are completely continuous provided that they are bounded.

Data Availability

Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

Conflicts of Interest

This research was done as part of my usual duty as an employee of IKIU.

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