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Research Article

On Characterizations of Weighted Harmonic Bloch Mappings and Its Carleson Measure Criteria

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For $\alpha > 0$, several characterizations of the α-Bloch spaces of harmonic mappings are given. We also obtain several similar characterizations for the closed separable subspace. As an application, we give relations between $\mathscr{B}_H^\alpha$ and Carleson's measure.

1. Introduction

Let Ω denote a simply connected region in the complex plane $\mathbb{C}$; a *harmonic mapping* is a complex-valued function h defined on Ω such that the Laplace equation satisfied

$$\Delta h \coloneqq 4h_{w\bar{w}} \equiv 0, \tag{1}$$

where $h_{w\bar{w}}$ is the complex second partial derivative of the harmonic mapping h.

It is known that in the literature, a harmonic mapping h can be written in the form $f + \bar{g}$, where f and g are analytic functions. This form is unique if we fix w_0 such that $g(w_0) = 0$.

Let $\mathbb{D} = \{w \in \mathbb{C} : |w| < 1\}$ be the well-known open unit disk in $\mathbb{C}$ and $\mathcal{H}ol(\mathbb{D})$ and $\mathcal{H}ar(\mathbb{D})$ denote the class of analytic functions and harmonic mappings on $\mathbb{D}$, respectively.

In the last few decades, the Banach spaces of analytic functions on $\mathbb{D}$ have been gaining a great deal of attention, but for the harmonic extensions of analytic spaces, it is still limited. Besides [1] by F. Colonna, papers such as [2] for the study of the operator theory on some spaces of harmonic mappings, [3] for characterizations of Bloch-type spaces of harmonic mappings, [4] for composition operators on some Banach spaces of harmonic mappings, [5] for the study of harmonic Bloch and Besov spaces, [6] for the study harmonic Zygmund spaces, [7] for the study of harmonic ν-Bloch mappings and [8] for the study of harmonic Lipschitz-type spaces. For $\alpha > 0$, α-Bloch space for harmonic mapping is defined such that

$$\mathscr{B}_H^\alpha = \{h \in \mathcal{H}ar(\mathbb{D}) \colon \mathscr{B}_h^\alpha < \infty\}, \tag{2}$$

where

$$\mathscr{B}_h^\alpha \coloneqq \sup_{w\in\mathbb{D}} \left(1 - |w|^2\right)^\alpha \left(|h_w(w)| + |h_{\bar{w}}(w)|\right). \tag{3}$$

The mapping $h \mapsto \|h\|_{\mathscr{B}_H^\alpha} \coloneqq |h(0)| + \mathscr{B}_h^\alpha$ defines a norm which yields a Banach space structure on $\mathscr{B}_H^\alpha$. This space is an extension to harmonic mappings of the classical α-Bloch space $\mathscr{B}^\alpha$ introduced by Zhu in [9], see also [10]. We recall that $f \in \mathcal{H}ol(\mathbb{D})$ belongs to $\mathscr{B}^\alpha$ if and only if

$$\mathscr{B}_f^\alpha \coloneqq \sup_{w\in\mathbb{D}} \left(1 - |w|^2\right)^\alpha \left|f'(w)\right| < \infty, \tag{4}$$

with norm $\|f\|_{\mathscr{B}^\alpha} = |f(0)| + \mathscr{B}_f^\alpha$. Thus, representing $h \in Har(\mathbb{D})$ as $f + \bar{g}$ with $f, g \in \mathcal{H}ol(\mathbb{D})$ and $g(0) = 0$, we see that $h_w = f'$ and $h_{\bar{w}} = \overline{g'}$. Therefore,

$$\mathscr{B}_h^\alpha = \sup_{w\in\mathbb{D}} \left(1 - |w|^2\right)^\alpha \left(\left|f'(w)\right| + \left|g'(w)\right|\right),$$
$$\frac{1}{2}\left(\mathscr{B}_f^\alpha + \mathscr{B}_g^\alpha\right) \le \max\left\{\mathscr{B}_f^\alpha, \mathscr{B}_g^\alpha\right\} \le \mathscr{B}_h^\alpha \le \mathscr{B}_f^\alpha + \mathscr{B}_g^\alpha. \tag{5}$$

Consequently, $h \in \mathscr{B}_H^\alpha$ if and only if the functions f, $g \in \mathcal{H}ol(\mathbb{D})$ such that $h = f + \bar{g}$ with $g(0) = 0$ are in the classical α-Bloch space. When $\alpha = 1$, the space $\mathscr{B}^\alpha$ is the (analytic) Bloch space $\mathscr{B}$ and the corresponding harmonic extension denoted by $\mathscr{B}_H$. The elements of $\mathscr{B}_H$ were first introduced in [1].

The little harmonic α-Bloch space $\mathscr{B}_{H,0}^\alpha$ is defined such that

$$\mathscr{B}_{H,0}^\alpha = \left\{h \in \mathscr{B}_H^\alpha : \lim_{|w|\longrightarrow 1} \left(1 - |w|^2\right)^\alpha (|h_w(w)| + |h_{\bar{w}}(w)|) = 0\right\}. \tag{6}$$

$\mathscr{B}_{H,0}^\alpha$ is a separable closed subspace of $\mathscr{B}_H^\alpha$ (see [2]); for more information about $\mathscr{B}_H^\alpha$ and $\mathscr{B}_{H,0}^\alpha$, see [2, 3, 11] and [1].

For $b \in \mathbb{D}$, the conformal automorphism is given by

$$\varphi_b(w) = \frac{b - w}{1 - \bar{b}w}, \quad \text{for all } w \in \mathbb{D}, \tag{7}$$

and Green's function with logarithmic singularity at the fixed point b is defined by

$$g(w, b) = \log \frac{1}{|\varphi_b(w)|}. \tag{8}$$

For $b \in \mathbb{D}$ and $0 < \delta < 1$, the pseudohyperbolic disk $D(b, \delta)$ with the pseudohyperbolic center b and pseudohyperbolic radius δ is given by

$$D(b, \delta) = \{w \in \mathbb{D} : |\varphi_b(w)| < \delta\}. \tag{9}$$

The pseudohyperbolic disk $D(b, \delta)$ is a Euclidean disk with Euclidean center $(1 - |b|^2)\delta/(1 - \delta^2|b|^2)$ and Euclidean radius $(1 - \delta^2)b/(1 - \delta^2|b|^2)$ (see [12]). Now, we let λ denote the normalized Lebesgue area measure on $\mathbb{D}$, since $D(b, \delta) \subset \mathbb{D}$ is a Lebesgue measureable set; then, the Euclidean area of $D(b, \delta)$ is given by

$$|D(b, \delta)| = \frac{\pi b^2 \left(1 - \delta^2\right)^2}{\left(1 - \delta^2 |b|^2\right)^2}. \tag{10}$$

Thus, by directed computation, we have the following fact:

Fact 1. Let $\delta \in (0, 1)$; then, for all $w \in D(b, \delta)$,

$$1 - |w|^2 \approx |D(b, \delta)|. \tag{11}$$

For any $w, b \in \mathbb{D}$, the hyperbolic distance between w and b is given by

$$\varrho(b, w) = \frac{1}{2} \log \frac{1 + |\varphi_b(w)|}{1 - |\varphi_b(w)|}. \tag{12}$$

Meanwhile, for $R \in (0,\infty)$, the hyperbolic disk is given by

$$\Delta(b, R) = \{w \in \mathbb{D} : \varrho(b, w) < R\}. \tag{13}$$

Throughout this paper, we say that two quantities $Q_1(h)$ and $Q_2(h)$, depending on the harmonic mappings h, are equivalent denoted by $Q_1(h) \approx Q_2(h)$, if there exists a constant $C > 0$ such that

$$\frac{1}{C} Q_2(h) \le Q_1(h) \le C Q_2(h). \tag{14}$$

Moreover, if $Q_1(h) \approx Q_2(h)$, we have $Q_2(h) < \infty \Leftrightarrow Q_1(h) < \infty$.

In this work, we expand the study carried out in [13, 14] and [15] for harmonic mappings.

2. Some Integral Criteria for Harmonic α-Bloch Mappings

The following lemma needed in the prove of the main theorem of this section (see Lemma 3 in [15]).

Lemma 1. *Assume that $q, \alpha \in [0,\infty)$ and $q - \alpha > -1$; then,*

$$J(q, \alpha) = \iint_{\mathbb{D}} \left(\log \frac{1}{|w|}\right)^q \frac{d\lambda(w)}{\left(1 - |w|^2\right)^\alpha} < \infty, \tag{15}$$

where λ is the normalized Lebesgue area measure on $\mathbb{D}$.

The following theorem is the main theorem of this section.

Theorem 2. *Assume that $\alpha \in [0,\infty)$, $\delta \in (0, 1)$ and let $p, q \in (1,\infty)$. Then for $h \in \mathcal{H}ar(\mathbb{D})$, the following quantities are equivalent:*

(A) $\|h\|_{\mathscr{B}_H^\alpha}^p$

(B) $\sup_{b\in\mathbb{D}} (1/|D(b, \delta)|^{1-(\alpha p/2)}) \iint_{D(b,\delta)} (|h_w(w)| + |h_{\bar{w}}(w)|)^p \, d\lambda(w)$

(C) $\sup_{b\in\mathbb{D}} \iint_{D(b,\delta)} (|h_w(w)| + |h_{\bar{w}}(w)|)^p (1 - |w|^2)^{\alpha p - 2} d\lambda(w)$

(D) $\sup_{b\in\mathbb{D}} \iint_{\mathbb{D}} (|h_w(w)| + |h_{\bar{w}}(w)|)^p (1-|w|^2)^{\alpha p-2} (1-|\varphi_b(w)|^2)^q \, d\lambda(w)$

(E) $\sup_{b\in\mathbb{D}} \iint_{\mathbb{D}} (|h_w(w)| + |h_{\bar{w}}(w)|)^p g^q(w,b) (1-|w|^2)^{\alpha p-2} \, d\lambda(w)$

Proof. Let $F \in \mathscr{H}ol(\mathbb{D})$. Then, $|F|^p$ is a subharmonic function. Thus,

$$|F(0)|^p \le \frac{1}{\pi\delta^2} \iint_{D(0,\delta)} |F(w)|^p d\lambda(w). \tag{16}$$

Now, assume that $F = h_w \circ \varphi_b$; then,

$$\begin{aligned} |h_w(b)|^p &\le \frac{1}{\pi\delta^2} \iint_{D(0,\delta)} |(h_w(\varphi_b(w))|^p d\lambda(w) \\ &= \frac{1}{\pi\delta^2} \iint_{D(b,\delta)} |h_w(w)|^p \left|\varphi_b'(w)\right|^2 d\lambda(w). \end{aligned} \tag{17}$$

Also, suppose that $F = h_{\bar{w}} \circ \varphi_b$; then, we obtain

$$|h_{\bar{w}}(b)|^p \le \frac{1}{\pi\delta^2} \iint_{D(b,\delta)} |h_{\bar{w}}(w)|^p \left|\varphi_b'(w)\right|^2 d\lambda(w). \tag{18}$$

From [16], we know that

$$\left|\varphi_b'(\zeta)\right| \le \frac{4}{1-|b|^2} \quad b,\zeta \in \mathbb{D}. \tag{19}$$

Also, from [2], we know that $|h_w(\zeta)|^p + |h_{\bar{w}}(\zeta)|^p \le (|h_w(\zeta)| + |h_{\bar{w}}(\zeta)|)^p$.

Using the inequality,

$$(|h_w(\zeta)| + |h_{\bar{w}}(\zeta)|)^p \le 2^p \left(|h_w(\zeta)|^p + |h_{\bar{w}}(\zeta)|^p\right). \tag{20}$$

Therefore,

$$\begin{aligned} &(|h_w(b)| + |h_{\bar{w}}(b)|)^p \\ &\quad \le 2^p \left(|h_w(b)|^p + |h_{\bar{w}}(b)|^p\right) \\ &\quad \le \frac{16(2^p)}{\pi\delta^2 \left(1-|b|^2\right)^2} \iint_{D(b,\delta)} \left(|h_w(w)|^p + |h_{\bar{w}}(w)|^p\right) d\lambda(w). \end{aligned} \tag{21}$$

Thus, we have

$$\begin{aligned} &\sup_{b\in\mathbb{D}} \left(1-|b|^2\right)^{\alpha p} (|h_w(b)| + |h_{\bar{w}}(b)|)^p \\ &\quad \le \sup_{b\in\mathbb{D}} \frac{16(2^p)}{\pi\delta^2 \left(1-|b|^2\right)^{2-\alpha p}} \iint_{D(b,\delta)} \left(|h_w(w)|^p + |h_{\bar{w}}(w)|^p\right) d\lambda(w) \\ &\quad \le \sup_{b\in\mathbb{D}} \frac{M(\delta,p,\alpha)}{|D(b,\delta)|^{1-(\alpha p/2)}} \iint_{D(b,\delta)} (|h_w(w)| + |h_{\bar{w}}(w)|)^p d\lambda(w) \\ &\quad \le \sup_{b\in\mathbb{D}} \frac{1}{|D(b,\delta)|^{1-(\alpha p/2)}} \iint_{D(b,\delta)} (|h_w(w)| + |h_{\bar{w}}(w)|)^p d\lambda(w), \end{aligned} \tag{22}$$

where $M(\delta,p,\alpha) = 16(2^p)/(\sqrt{\pi}\delta)^{\alpha p}(1-\delta^2)^{2-\alpha p}$ is a constant. Therefore, $(A) \lesssim (B)$.

The equivalence $(B) \approx (C)$ follows directly from Fact 1, for all $w \in D(b,\delta)$ and $\delta \in (0,1)$.

$(C) \Rightarrow (D)$. Since $1-\delta^2 \le 1-|\varphi_b(w)|^2$, so for every $w \in D(b,\delta)$, we obtain

$$\begin{aligned} &\sup_{b\in\mathbb{D}} \iint_{D(b,\delta)} (|h_w(w)| + |h_{\bar{w}}(w)|)^p \left(1-|w|^2\right)^{\alpha p-2} d\lambda(w) \\ &\quad \le \sup_{b\in\mathbb{D}} \frac{1}{\left(1-\delta^2\right)^q} \iint_{\mathbb{D}} (|h_w(w)| + |h_{\bar{w}}(w)|)^p \left(1-|w|^2\right)^{\alpha p-2} \\ &\qquad \cdot \left(1-|\varphi_b(w)|^2\right)^q d\lambda(w) \\ &\quad \lesssim \sup_{b\in\mathbb{D}} \iint_{\mathbb{D}} (|h_w(w)| + |h_{\bar{w}}(w)|)^p \left(1-|w|^2\right)^{\alpha p-2} \\ &\qquad \cdot \left(1-|\varphi_b(w)|^2\right)^q d\lambda(w). \end{aligned} \tag{23}$$

Hence, $(C) \lesssim (D)$.

It is clear that $1-|\varphi_b(w)|^2 \lesssim g(w,b)$ for all $w, b \in \mathbb{D}$; thus, we can infer that $(D) \lesssim (E)$.

Now, making a change of variables $w = \varphi_b(w)$ in quantity (E), we have

$$\begin{aligned} &\iint_{\mathbb{D}} (|h_w(w)| + |h_{\bar{w}}(w)|)^p g^q(w,b) \left(1-|w|^2\right)^{\alpha p-2} d\lambda(w) \\ &\quad = \iint_{\mathbb{D}} (|h_w(\varphi_b(w))| + |h_{\bar{w}}(\varphi_b(w))|)^p \left(\log\frac{1}{|w|}\right)^q \\ &\qquad \cdot \left(1-|\varphi_b(w)|^2\right)^{\alpha p-2} \left|\varphi_b'(w)\right|^2 d\lambda(w) \\ &\quad = \iint_{\mathbb{D}} (|h_w(\varphi_b(w))| + |h_{\bar{w}}(\varphi_b(w))|)^p \left(1-|\varphi_b(w)|^2\right)^{\alpha p} \\ &\qquad \cdot \left(\log\frac{1}{|w|}\right)^q \frac{d\lambda(w)}{\left(1-|w|^2\right)^2} \\ &\quad \le \|h\|^p_{\mathscr{B}^\alpha_H} \iint_{\mathbb{D}} \left(\log\frac{1}{|w|}\right)^q \frac{d\lambda(w)}{\left(1-|w|^2\right)^2} \\ &\quad \lesssim \|h\|^p_{\mathscr{B}^\alpha_H}. \end{aligned} \tag{24}$$

From Lemma 1, we know that

$$J(q,2)=\iint_{\mathbb{D}}\left(\log\frac{1}{|w|}\right)^{q}\frac{d\lambda(w)}{\left(1-|w|^{2}\right)^{2}}<\infty. \quad (25)$$

Thus, we deduce that $(E)\lesssim(A)$.

Suppose $D_{1/4}=\{w:|w|<1/4\}$, for $w\in D_{1/4}$; we have

$$\begin{aligned}&\left|\varphi_b'(w)\right|^{2}=\frac{1-\left|b^{2}\right|}{\left|1-\bar{b}w\right|^{4}}\le\frac{1}{\left|1-|w|\right|^{4}}\le\left(\frac{4}{3}\right)^{4},\\&\frac{1}{1-|w|^{2}}\le\frac{16}{15}.\end{aligned} \quad (26)$$

Therefore,

$$\begin{aligned}J_{D_{1/4}}&=\iint_{D_{1/4}}\left(|h_w(w)|+|h_{\bar{w}}(w)|\right)^{p}\left(\log\frac{1}{|w|}\right)^{\alpha p}\left|\varphi_b'(w)\right|^{2}d\lambda(w)\\&\le\|h\|_{\mathscr{B}_H^{\alpha}}^{p}\iint_{D_{1/4}}\left(\log\frac{1}{|w|}\right)^{\alpha p}\frac{\left|\varphi_b'(w)\right|^{2}}{\left(1-|w|^{2}\right)^{\alpha p}}d\lambda(w)\\&\le\left(\frac{4}{3}\right)^{4}\left(\frac{16}{15}\right)^{\alpha p}J(\alpha p,0)\|h\|_{\mathscr{B}_H^{\alpha}}^{p}.\end{aligned} \quad (27)$$

Since $J(\alpha p,0)$ is finite by Lemma 1, we have

$$J_{D_{1/4}}\lesssim\|h\|_{\mathscr{B}_H^{\alpha}}^{p}. \quad (28)$$

Next, since $\log 1/s\le 1/\delta(1-s^{2})$ for any $0<\delta\le s\le 1$, so

$$\log\frac{1}{|w|}\le 4(1-|w|). \quad (29)$$

Then, for $w\in\mathbb{D}\setminus D_{1/4}$, we obtain

$$\begin{aligned}J_{\mathbb{D}\setminus D_{1/4}}&=\iint_{\mathbb{D}\setminus D_{1/4}}\left(|h_w(w)|+|h_{\bar{w}}(w)|\right)^{p}\\&\quad\cdot\left(\log\frac{1}{|w|}\right)^{\alpha p}\left|\varphi_b'(w)\right|^{2}d\lambda(w)\\&\le 4^{\alpha p}\iint_{\mathbb{D}\setminus D_{1/4}}\left(|h_w(w)|+|h_{\bar{w}}(w)|\right)^{p}\\&\quad\cdot\left(1-|w|^{2}\right)^{\alpha p}\left|\varphi_b'(w)\right|^{2}d\lambda(w)\\&\le\pi 4^{\alpha p}\|h\|_{\mathscr{B}_H^{\alpha}}^{p}.\end{aligned} \quad (30)$$

Thus,

$$J_{\mathbb{D}\setminus D_{1/4}}\lesssim\|h\|_{\mathscr{B}_H^{\alpha}}^{p}. \quad (31)$$

Combining (28) and (31), we have $(F)\lesssim(A)$. This completes the proof. □

Remark 3. The above characterizations of harmonic α-Bloch functions is an extension of Theorem 1 proved by Zhao in [15]. We extend the known characterizations of α-Bloch space for analytic function to the harmonic setting using the conditions of the analytic functions f and g and their subharmonicity on the unit disk. Therefore, we used the proof technique as in the proof of Theorem 1 in [15].

3. Some Integral Criteria for Little Harmonic α-Bloch Mappings

We will give some concerned integral-type criteria for the little harmonic α-Bloch space $\mathscr{B}_{H,0}^{\alpha}$ and the little harmonic Besov space.

Theorem 4. *Suppose that $\alpha\in[0,\infty)$, $\delta\in(0,1)$ and let $p,q\in(1,\infty)$. For any harmonic function $h\in\mathcal{H}ar(\mathbb{D})$, the following statements are equivalent:*

$(\tilde{A})$ $h\in\mathscr{B}_{H,0}^{\alpha}$

$(\tilde{B})$ $\lim_{|b|\longrightarrow 1^{-}}(1/|D(b,\delta)|^{1-(\alpha p/2)})\iint_{D(b,\delta)}(|h_w(w)|+|h_{\bar{w}}(w)|)^{p}\, d\lambda(w)=0$

$(\tilde{C})$ $\lim_{|b|\longrightarrow 1^{-}}\iint_{D(b,\delta)}(|h_w(w)|+|h_{\bar{w}}(w)|)^{p}(1-|w|^{2})^{\alpha p-2}d\lambda(w)=0$

$(\tilde{D})$ $\lim_{|b|\longrightarrow 1^{-}}\iint_{\mathbb{D}}(|h_w(w)|+|h_{\bar{w}}(w)|)^{p}(1-|w|^{2})^{\alpha p-2}(1-|\varphi_b(w)|^{2})^{q}\quad d\lambda(w)=0$

$(\tilde{E})$ $\lim_{|b|\longrightarrow 1^{-}}\iint_{\mathbb{D}}(|h_w(w)|+|h_{\bar{w}}(w)|)^{p}g^{q}(w,b)(1-|w|^{2})^{\alpha p-2}\, d\lambda(w)=0$

Proof. Suppose that $h\in\mathscr{B}_{H,0}^{\alpha}$; then,

$$\lim_{|w|\longrightarrow 1}\left(|h_w(w)|+|h_{\bar{w}}(w)|\right)\left(1-|w|^{2}\right)^{\alpha}=0. \quad (32)$$

By making the change of variables $\zeta=\varphi_b(w)$, we have

$$\begin{aligned}I_{\mathbb{D}}(b)&=\iint_{\mathbb{D}}\left(|h_w(w)|+|h_{\bar{w}}(w)|\right)^{p}g^{q}(w,b)\left(1-|w|^{2}\right)^{\alpha p-2}d\lambda(w)\\&=\iint_{\mathbb{D}}\left(|h_w(\varphi_b(w))|+|h_{\bar{w}}(\varphi_b(w))|\right)^{p}\left(1-|\varphi_b(w)|^{2}\right)^{\alpha p}\\&\quad\cdot\left(\log\frac{1}{|w|}\right)^{q}\frac{d\lambda(w)}{\left(1-|w|^{2}\right)^{2}}.\end{aligned} \quad (33)$$

Let $\delta\in(0,1)$ such that $\pi(1-\delta^{2})^{q-1}/\delta^{q}(q-1)=\varepsilon$, for $\varepsilon>0$. Set $D_{\delta}=\{w:|w|<\delta\}$; for any $w\in\mathbb{D}\setminus D_{\delta}$, we have $\log 1/|w|\le(1-|w|^{2})/\delta$. Then,

$$\begin{aligned}I_{\mathbb{D}\setminus D_{\delta}}(b)&=\iint_{\mathbb{D}\setminus D_{\delta}}\left(|h_w(\varphi_b(w))|+|h_{\bar{w}}(\varphi_b(w))|\right)^{p}\\&\quad\cdot\left(1-|\varphi_b(w)|^{2}\right)^{\alpha p}\left(\log\frac{1}{|w|}\right)^{q}\frac{d\lambda(w)}{\left(1-|w|^{2}\right)^{2}}\\&\le\|h\|_{\mathscr{B}_H^{\alpha}}^{p}\iint_{\mathbb{D}\setminus D_{\delta}}\left(\log\frac{1}{|w|}\right)^{q}\frac{d\lambda(w)}{\left(1-|w|^{2}\right)^{2}}\\&\le\frac{1}{\delta^{q}}\|h\|_{\mathscr{B}_H^{\alpha}}^{p}\iint_{\mathbb{D}\setminus D_{\delta}}\frac{d\lambda(w)}{\left(1-|w|^{2}\right)^{2-q}}\\&=\frac{\pi\left(1-\delta^{2}\right)^{q-1}}{\delta^{q}(q-1)}\|h\|_{\mathscr{B}_H^{\alpha}}^{p}.\end{aligned} \quad (34)$$

This means that

$$I_{\mathbb{D}\setminus D_\delta}(b) \le \varepsilon \|h\|^p_{\mathscr{B}^\alpha_H}. \tag{35}$$

Since $h \in \mathscr{B}^\alpha_{H,0}$, then $\lim_{|b|\longrightarrow 1}(|h_w(\varphi_b(w))| + |h_{\bar{w}}(\varphi_b(w))|)(1-|\varphi_b(w)|^2)^\alpha = 0$, and the convergence is uniform for any $w \in D_\delta$. So for the given $\varepsilon > 0$, there is $\delta_0 \in (0,1)$ such that

$$(|h_w(\varphi_b(w))| + |h_{\bar{w}}(\varphi_b(w))|)^p (1-|\varphi_b(w)|^2)^{\alpha p} < \varepsilon, \forall b \in \mathbb{D}\setminus D_{\delta_0}. \tag{36}$$

Thus, we obtain

$$\begin{aligned} I_{\mathbb{D}\setminus D_{\delta_0}}(b) &= \iint_{\mathbb{D}\setminus D_{\delta_0}} (|h_w(\varphi_b(w))| + |h_{\bar{w}}(\varphi_b(w))|)^p \\ &\quad \cdot (1-|\varphi_b(w)|^2)^{\alpha p}\left(\log\frac{1}{|w|}\right)^q \frac{d\lambda(w)}{(1-|w|^2)^2} \\ &\le \varepsilon \iint_{\mathbb{D}\setminus D_{\delta_0}} \left(\log\frac{1}{|w|}\right)^q \frac{d\lambda(w)}{(1-|w|^2)^2}. \end{aligned} \tag{37}$$

Then, we have

$$I_{\mathbb{D}\setminus D_{\delta_0}}(b) \le \varepsilon J(q,2). \tag{38}$$

By Lemma 1, for all $q \in (1,\infty)$, we know that $J(q,2) < \infty$. Now combining (35) and (38), we have

$$I_{\mathbb{D}}(b) \le \varepsilon A(h,p,\alpha), \quad \forall b \in \mathbb{D}\setminus D_{\delta_0}, \tag{39}$$

where $A(h,p,\alpha)$ is a constant depending on p,α and the harmonic function h. That is, $\lim_{|b|\longrightarrow 1} I(b) = 0$. Thus, we deduce that $(\tilde{A}) \Rightarrow (\tilde{E})$.

Since $1-|\varphi_b(w)|^2 \lesssim g(w,b)$ for all $w,b \in \mathbb{D}$, we have $(\tilde{E}) \Rightarrow (\tilde{D})$.

Next, $(\tilde{D}) \Rightarrow (\tilde{C})$ follows immediately from the inequality

$$\begin{aligned} &\iint_{D(b,\delta)} (|h_w(w)| + |h_{\bar{w}}(w)|)^p (1-|w|^2)^{\alpha p-2} d\lambda(w) \\ &\quad \lesssim \iint_{\mathbb{D}} (|h_w(w)| + |h_{\bar{w}}(w)|)^p (1-|w|^2)^{\alpha p-2} \\ &\quad \cdot (1-|\varphi_b(w)|^2)^q \, d\lambda(w). \end{aligned} \tag{40}$$

For $w \in D(b,\delta)$ and $\delta \in (0,1)$, we have that $1-|w|^2 \approx |D(b,\delta)|$; then, we get $(\tilde{C}) \Leftrightarrow (\tilde{B})$.

By the inequality

$$\begin{aligned} &(1-|b|^2)^{\alpha p}(|h_w(b)| + |h_{\bar{w}}(b)|)^p \\ &\quad \lesssim \frac{1}{|D(b,\delta)|^{1-(p/2)}} \iint_{D(b,\delta)} (|h_w(w)| + |h_{\bar{w}}(w)|)^p d\lambda(w), \end{aligned} \tag{41}$$

we obtain $(\tilde{B}) \Rightarrow (\tilde{A})$.

Let $h \in \mathscr{B}^\alpha_{H,0}$; that is, $\lim_{|w|\longrightarrow 1^-}(|h_w(w)| + |h_{\bar{w}}(w)|)(1-|w|^2)^\alpha = 0$. Then, for any $\varepsilon > 0$, there is $\delta \in (1/4, 1)$ such that $(|h_w(w)| + |h_{\bar{w}}(w)|)^p(1-|w|^2)^{\alpha p} < \varepsilon$, for every $w \in \mathbb{D}\setminus D_\delta$.

Now, we set

$$\tilde{J}_{\mathbb{D}}(b) = \iint_{\mathbb{D}} (|h_w(w)| + |h_{\bar{w}}(w)|)^p \left(\log\frac{1}{|w|}\right)^{\alpha p} |\varphi_b'(w)|^2 \, d\lambda(w). \tag{42}$$

For each $w \in \mathbb{D}\setminus D_\delta$, we know $\log 1/|w| \le 4(1-|w|^2)$. Thus,

$$\begin{aligned} \tilde{J}_{\mathbb{D}\setminus D_\delta} &= \iint_{\mathbb{D}\setminus D_\delta} (|h_w(w)| + |h_{\bar{w}}(w)|)^p \\ &\quad \cdot \left(\log\frac{1}{|w|}\right)^{\alpha p} |\varphi_b'(w)|^2 \, d\lambda(w) \\ &\le 2^{2\alpha p}\varepsilon\pi. \end{aligned} \tag{43}$$

Since $\lim_{|b|\longrightarrow 1^-}|\varphi_b'(w)|^2 = 0$ for every $w \in \mathbb{D}$, and the uniformly convergence for $w \in D_\delta$. Then, for any $\varepsilon > 0$ and $\delta \in (1/4, 1)$, there is $\delta_0 \in (0,1)$ such that $|\varphi_b'(w)|^2 < \varepsilon$, for every $b \in \mathbb{D}\setminus D_{\delta_0}$.

Hence,

$$\begin{aligned} \tilde{J}_{\mathbb{D}\setminus D_{\delta_0}} &= \iint_{\mathbb{D}\setminus D_{\delta_0}} (|h_w(w)| + |h_{\bar{w}}(w)|)^p \\ &\quad \cdot \left(\log\frac{1}{|w|}\right)^{\alpha p} |\varphi_b'(w)| d\lambda(w) \\ &\le \|h\|^p_{\mathscr{B}^\alpha_H} \iint_{\mathbb{D}\setminus D_{\delta_0}} \left(\log\frac{1}{|w|}\right)^{\alpha p} \frac{|\varphi_b'(w)|}{(1-|w|^2)^{\alpha p}} d\lambda(w) \\ &\le \varepsilon J(\alpha p, \alpha p)\|h\|^p_{\mathscr{B}^\alpha_H}. \end{aligned} \tag{44}$$

Again by Lemma 1, for all $p \in (1,\infty)$ and $\alpha \in [0,\infty)$ we know that $J(\alpha p, \alpha p) < \infty$. This means that

$$\tilde{J}_{\mathbb{D}\setminus D_{\delta_0}} \lesssim \|h\|^p_{\mathscr{B}^\alpha_H}. \tag{45}$$

By combining equations (43) and (45), we have

$$\tilde{J}_{\mathbb{D}} \le \varepsilon A(h,p,\alpha), \forall b \in \mathbb{D}\setminus D_{\delta_0}, \tag{46}$$

where $A(h,p,\alpha)$ is a constant depending on p,α and the harmonic function h. That is, $\lim_{|b|\longrightarrow 1^-}\tilde{J}_{\mathbb{D}}(b) = 0$. Thus, we deduce that $(\tilde{A}) \Rightarrow (\tilde{F})$. That ends the proof of Theorem 4. □

Remark 5. When h is analytic function, $\mathscr{B}^\alpha_{H,0}$ is the little α-Bloch space $\mathscr{B}^\alpha_0$, and Theorem 4 is proved by Zhao in [15].

4. Carleson's Measures and Harmonic α-Bloch Mappings

Let μ be a positive measure on $\mathbb{D}$. For a subarc $J \subseteq \partial\mathbb{D}$, we let $S(J)$ be the Carleson box based on J; that is,

$$S(J) := \{\zeta \in \mathbb{D} : 1 - |J| \le |\zeta| < 1, \zeta/|\zeta| \in J\}. \quad (47)$$

For $J = \partial\mathbb{D}$, we let $S(J) = \mathbb{D}$.

Let $s > 0$; then, a positive Borel measure μ on $\mathbb{D}$ is called a s-Carleson measure if

$$\|\mu\|^s = \sup_{J\subseteq\partial\mathbb{D}} \frac{\mu(S(J))}{|J|^s} < \infty. \quad (48)$$

Note that $s = 1$ gives the classical Carleson measure. We say that μ is a vanishing s-Carleson measure if

$$\lim_{|J|\longrightarrow 1} \frac{\mu(S(J))}{|J|^s} = 0. \quad (49)$$

As is well known, the Berezin transform of a positive Borel measure μ on $\mathbb{D}$ is bounded if and only if μ is a Carleson measure (see, for example, [13, 14]). Then, for any $\delta \in (0, 1)$, we say $d\mu$ is a Carleson measure if

$$\sup_{b\in\mathbb{D}} \frac{\mu(D(b,\delta))}{\lambda(D(b,\delta))} < \infty, \quad (50)$$

where λ is the normalized Lebesgue area measure on $\mathbb{D}$. Moreover, we say $d\mu$ is a compact Carleson measure if

$$\lim_{|b|\longrightarrow 1} \frac{\mu(D(b,\delta))}{\lambda(D(b,\delta))} = 0. \quad (51)$$

For all $\alpha \in (0,\infty)$, a positive measure μ on $\mathbb{D}$ is a bounded α-Carleson measure if and only if

$$\sup_{b\in\mathbb{D}} \iint_{\mathbb{D}} \left(\frac{1-|b|^2}{|1-\bar{b}w|^2}\right)^{\alpha} d\mu(\zeta) < \infty. \quad (52)$$

Moreover, μ is a compact α-Carleson measure if and only if

$$\lim_{|b|\longrightarrow 1} \iint_{\mathbb{D}} \left(\frac{1-|b|^2}{|1-\bar{b}w|^2}\right)^{\alpha} d\mu(\zeta) = 0. \quad (53)$$

For all $\alpha \in (-1,\infty)$, $p \in [0,\infty)$, we denote by $A_H^{p,\alpha}(\mathbb{D})$ the weighted harmonic Bergman space, where $A_H^{p,\alpha}(\mathbb{D})$ is the set of all $h \in \mathscr{H}ar(\mathbb{D})$ for which

$$\|h\|^p_{A_H^{p,\alpha}} = \iint_{\mathbb{D}} |h(w)|^p d\lambda_\alpha(w) < \infty, \quad (54)$$

where $d\lambda_\alpha(w) = (1-|w|^2)^\alpha d\lambda(w)$.

Theorem 6. *Assume that $h \in \mathcal{H}ar(\mathbb{D})$, and let $\alpha \in (0,\infty)$. Then, the following are equivalent:*

(a) $h \in \mathscr{B}_H^\alpha$

(b) $\sup_{b,w\in\mathbb{D}} |h(w) - h(b)|/\rho(b,w) < \infty$

(c) $\sup_{b\in\mathbb{D}} \iint_{\mathbb{D}} \exp\,[C|h(\varphi_b(w)) - h(b)|]\,d\lambda_\alpha(w) < \infty$,

where C is a positive constant

Proof. Firstly, $(a) \Rightarrow (b)$. Suppose that $h \in \mathscr{B}_H^\alpha$, and let $F_b \in \mathcal{H}\text{ar}(\mathbb{D})$ defined as

$$F_b(\zeta) = h(\varphi_b(\zeta)) - h(b), \forall b, \zeta \in \mathbb{D}. \quad (55)$$

Then, $F_b(0) = 0$ and $\|F_b\|_{\mathscr{B}_H^\alpha} \le \|h\|_{\mathscr{B}_H^\alpha} < \infty$. By Theorem 3.6 in [2], we obtain

$$|F_b(\zeta)| \le \frac{1}{2}\|h\|_{\mathscr{B}_H^\alpha} \log \frac{1+|\zeta|}{1-|\zeta|}. \quad (56)$$

Now, setting $w = \varphi_b(\zeta)$, we have

$$|h(w) - h(b)| \le \frac{1}{2}\|h\|_{\mathscr{B}_H^\alpha} \log \frac{1+|\varphi_b(w)|}{1-|\varphi_b(w)|}, \quad (57)$$

which means that

$$\sup_{b,w\in\mathbb{D}} \frac{|h(w)-h(b)|}{\rho(b,w)} \le \|h\|_{\mathscr{B}_H^\alpha} < \infty. \quad (58)$$

That is, (b) holds.

Secondly, $(b) \Rightarrow (c)$. Suppose that

$$0 < \|h\|^* = \sup_{b,w\in\mathbb{D}} \frac{|h(w)-h(b)|}{\rho(b,w)} < \infty. \quad (59)$$

Then for any $r \ge 0$,

$$\{w : w \in \mathbb{D}, |F_b(w)| > r\} \subset \left\{w : |w| > \frac{\exp\,[2r/\|h\|^* - 1]}{\exp\,[(2r/\|h\|^*) + 1]}\right\}, \quad (60)$$

when $0 < C < 2\alpha + 2/\|h\|^*$, then

$$\begin{aligned}
&\iint_{\mathbb{D}} \exp\,[C|F_b(w)|]\,d\lambda_\alpha(w) \\
&= C\int_0^\infty \exp\,[Cr] \cdot \lambda_\alpha(\{w : w \in \mathbb{D}, |F_b(w)| > r\})\,dr \\
&\le \frac{4\pi C}{\alpha+1}\int_0^\infty \exp\,[Cr] \exp\left[-\frac{2r(\alpha+1)}{\|h\|^*}\right] dr \\
&= \frac{4\pi C}{\alpha+1}\left[\frac{\|h\|^*}{2(\alpha+1) - C\|h\|^*}\right] \\
&= \frac{4\pi C\|h\|^*}{2(\alpha+1)^2 - C(\alpha+1)\|h\|^*}.
\end{aligned} \quad (61)$$

Finally, $(c) \Rightarrow (a)$. For some $C > 0$, let

$$\begin{aligned}\|h\|^{**} &= \sup_{b\in\mathbb{D}} \iint_{\mathbb{D}} \exp\left[C|h(\varphi_b(w)) - h(b)|\right] d\lambda_\alpha(w) \\ &= \sup_{b\in\mathbb{D}} \iint_{\mathbb{D}} \exp\left[C|F_b(w)|\right] d\lambda_\alpha(w) < \infty.\end{aligned} \tag{62}$$

Then,

$$\|F_b(w)\|^p_{A_H^{p,\alpha}} \le \frac{\|h\|^{**}}{C} < \infty. \tag{63}$$

For any $h \in \mathcal{H}\text{ar}(\mathbb{D})$, since $f, g \in \mathcal{H}\text{ol}(\mathbb{D})$ such that $h = f + \bar{g}$ with $g(0) = 0$, h has the Taylor series

$$\sum_{m=0}^{\infty} a_m w^m + \sum_{m=1}^{\infty} c_m \bar{w}^m. \tag{64}$$

Hence, by a simple calculation for a Taylor series of $F_b(w)$, which converges uniformly on $D_\delta = \{w : |w| < \delta\}$, $\delta \in (0, 1)$, we have

$$\begin{aligned}a_1 &= (F_b)_w(0) \\ &= \frac{(2\alpha+2)(\alpha+1)}{1 - \left[\delta^2(\alpha+1)+1\right]\left(1-\delta^2\right)^{\alpha+1}} \iint_{D_\delta} F_b(w)\bar{w}\, d\lambda_\alpha(w).\end{aligned} \tag{65}$$

Now, we let $\delta \longrightarrow 1$; then,

$$\left|(F_b)_w(0) + (F_b)_{\bar{w}}(0)\right| \le \left(\alpha^2 + 3\alpha + 2\right) \iint_{\mathbb{D}} |F_b(w)|\, d\lambda_\alpha(w). \tag{66}$$

Similarly, for $\delta \longrightarrow 1$, we see that

$$\left|(F_b)_{\bar{w}}(0)\right| \le \left(\alpha^2 + 3\alpha + 2\right) \iint_{\mathbb{D}} |F_b(w)|\, d\lambda_\alpha(w), \tag{67}$$

which means that

$$\left(1 - |w|^2\right)^\alpha \left(|h_w(w) + h_{\bar{w}}(w)|\right) \le \frac{2}{C}\left(\alpha^2 + 3\alpha + 2\right)\|h\|^{**}. \tag{68}$$

So, we have $h \in \mathscr{B}_H^\alpha$. □

Theorem 7. *Assume that $h \in \mathcal{H}ar(\mathbb{D})$, and let $\alpha \in [0,\infty)$ and $p \in (0,\infty)$. Then, the following are equivalent:*

($\tilde{a}$) $h \in \mathscr{B}_H^\alpha$

($\tilde{b}$) $d\mu_1 = (|h_w(w)| + |h_{\bar{w}}(w)|)^p (\log 1/|w|)^{\alpha p} d\lambda(w)$ is p-Carleson measure

($\tilde{c}$) $d\mu_2 = (|h_w(w)| + |h_{\bar{w}}(w)|)^p (1 - |w|^2)^{\alpha p} d\lambda(w)$ is p-Carleson measure

Proof. First of all, $(\tilde{a}) \Rightarrow (\tilde{b})$. Suppose that $h \in \mathscr{B}_H^\alpha$, and let the integral below:

$$\begin{aligned}I &= \iint_{\mathbb{D}} \left(\frac{1 - |b|^2}{\left|1 - \bar{b}w\right|^2}\right)^p d\mu_1(w) \\ &= \left\{\iint_{\{|w|>1/4\}} + \iint_{\{|w|\le 1/4\}}\right\} \left(\frac{1 - |b|^2}{\left|1 - \bar{b}w\right|^2}\right)^p d\mu_1(w) = I_1 + I_2.\end{aligned} \tag{69}$$

Since $\log 1/|w| \lesssim 1 - |w|^2$ when $|w| > 1/4$,

$$\begin{aligned}I_1 &= \iint_{\{|w|>1/4\}} (|h_w(w)| + |h_{\bar{w}}(w)|)^p \cdot \left(\frac{1 - |b|^2}{\left|1 - \bar{b}w\right|^2}\right)^p \left(\log\frac{1}{|w|}\right)^{\alpha p} d\lambda(w) \\ &\lesssim \iint_{\{|w|>1/4\}} (|h_w(w)| + |h_{\bar{w}}(w)|)^p \cdot \left(\frac{\left(1 - |b|^2\right)\left(1 - |w|^2\right)^\alpha}{\left|1 - \bar{b}w\right|^2}\right)^p d\lambda(w) \\ &\lesssim \|h\|^p_{\mathscr{B}_H^\alpha} \iint_{\mathbb{D}} \left(\frac{\left(1 - |b|^2\right)}{\left|1 - \bar{b}w\right|^2}\right)^p d\lambda(w) \lesssim \|h\|^p_{\mathscr{B}_H^\alpha}.\end{aligned} \tag{70}$$

At the same time,

$$\begin{aligned}I_2 &= \iint_{\{|w|\le 1/4\}} (|h_w(w)| + |h_{\bar{w}}(w)|)^p \cdot \left(\frac{1 - |b|^2}{\left|1 - \bar{b}w\right|^2}\right)^p \left(\log\frac{1}{|w|}\right)^{\alpha p} d\lambda(w) \\ &\le \left(\frac{16}{15}\right)^{\alpha p} \|h\|^p_{\mathscr{B}_H^\alpha} \iint_{\{|w|\le 1/4\}} \left(\frac{\left(1 - |b|^2\right)}{\left|1 - \bar{b}w\right|^2}\right)^p \left(\log\frac{1}{|w|}\right)^{\alpha p} d\lambda(w) \\ &\le \left(\frac{16}{15}\right)^{\alpha p} \left(\frac{4}{3}\right)^p \|h\|^p_{\mathscr{B}_H^\alpha} \iint_{\{|w|\le 1/4\}} \left(\log\frac{1}{|w|}\right)^{\alpha p} d\lambda(w) \|h\|^p_{\mathscr{B}_H^\alpha}.\end{aligned} \tag{71}$$

Consequently, $I\|h\|^p_{\mathscr{B}_H^\alpha}$. So, we see that $(\tilde{b})$ holds.

Next, $(\tilde{b}) \Rightarrow (\tilde{c})$. This is unmistakable, since $1 - |w|^2 \le 2\log 1/|w|$.

Finally, $(\tilde{c}) \Rightarrow (\tilde{a})$. Assuming that $d\mu_2$ is p-Carleson measure, then

$$\begin{aligned}\infty > I &= \iint_{\mathbb{D}} (|h_w(w)| + |h_{\bar{w}}(w)|)^p \left(\frac{1 - |b|^2}{\left|1 - \bar{b}w\right|^2}\right)^p \cdot \left(1 - |w|^2\right)^{\alpha p} d\lambda(w) \\ &\ge \iint_{\mathbb{D}} (|h_w(w)| + |h_{\bar{w}}(w)|)^p \left(1 - |w|^2\right)^{\alpha p} d\lambda(w).\end{aligned} \tag{72}$$

Moreover,

$$(|h_w(b)| + |h_{\bar{w}}(b)|)^p \le \frac{3}{\pi} \iint_{\mathbb{D}} (|h_w(w)| + |h_{\bar{w}}(w)|)^p \cdot \left(\frac{1-|b|^2}{|1-\bar{b}w|^2}\right)^p d\lambda(w). \quad (73)$$

Hence,

$$\begin{aligned}(|h_w(b)| + |h_{\bar{w}}(b)|)^p (1-|w|^2)^{\alpha p} &\le \frac{3}{\pi} \iint_{\mathbb{D}} (|h_w(w)| + |h_{\bar{w}}(w)|)^p (1-|w|^2)^{\alpha p} \cdot \left(\frac{1-|b|^2}{|1-\bar{b}w|^2}\right)^p d\lambda(w) \\ &\le I < \infty.\end{aligned} \quad (74)$$

So, we have $h \in \mathscr{B}_H^\alpha$. □

Now, we give new characteristics of the little harmonic α-Bloch space $\mathscr{B}_{H,0}^\alpha$.

Theorem 8. *Assume that $h \in \mathscr{H}ar(\mathbb{D})$, and let $\alpha \in (0,\infty)$. Then, the following are equivalent:*

(i) $h \in \mathscr{B}_{H,0}^\alpha$

(ii) $\lim_{|b|\longrightarrow 1} \sup_{w\in\Delta(b,R)} |h(w) - h(b)|/\rho(b,w) = 0$

(iii) $\lim_{|b|\longrightarrow 1} \iint_{\mathbb{D}} \exp\,[C|h(\varphi_b(w)) - h(b)|]d\lambda_\alpha(w) = \pi/(\alpha+1)$,

where C is a positive constant

Proof. Firstly, (i) ⇔ (ii). Suppose that $h \in \mathscr{B}_{H,0}^\alpha$, as in the proof of Theorem 6; let $F_b \in \mathcal{H}\mathrm{ar}(\mathbb{D})$ defined as

$$F_b(\zeta) = h(\varphi_b(\zeta)) - h(b), \forall b, \zeta \in \mathbb{D}. \quad (75)$$

Then, $F_b(0) = 0$ and $\|F_b\|_{\mathscr{B}_H^\alpha} \le \|h\|_{\mathscr{B}_H^\alpha} < \infty$. By Theorem 3.6 in [2], we obtain

$$|F_b(\zeta)| \le \frac{1}{2}\|h\|_{\mathscr{B}_H^\alpha} \log \frac{1+|\zeta|}{1-|\zeta|}. \quad (76)$$

For $R \in (0,\infty)$ and $\delta = (e^R - 1)/(e^R + 1)$, we obtain

$$|h(w) - h(b)| \le \frac{1}{2}\rho(0, \varphi_b(w)) \sup_{z\in D(0,\delta)} (1-|z|^2)^\alpha \cdot \left(|(F_b)_z(z)| + |(F_b)_{\bar{z}}(z)|\right). \quad (77)$$

Meanwhile, since $\lim_{|b|\longrightarrow 1}|\varphi_b(z)| = 1$ for any $z \in D(0,\delta)$ and $h \in \mathscr{B}_{H,0}^\alpha$, we have

$$\lim_{|b|\longrightarrow 1} \sup_{z\in D(0,\delta)} \left(1-|\varphi_b(\zeta)|^2\right)^\alpha \left(|h_\zeta(\zeta)| + \left|h_{\bar{\zeta}}(\zeta)\right|\right) = 0. \quad (78)$$

By combining equations (77) and (78), we deduce (ii). The other direction comes easily from inequality

$$\left(1-|w|^2\right)^\alpha (|h_w(w)| + |h_{\bar{w}}(w)|) \le \sup_{w\in\Delta(b,R)} \frac{|h(w) - h(b)|}{\rho(b,w)}. \quad (79)$$

Secondly, (i) ⇔ (iii). For any $f \in \mathcal{H}\mathrm{ol}(\mathbb{D})$, since $\mathscr{B}_0^\alpha$ is the closure in $\mathscr{B}^\alpha$ of the polynomials, there exist a polynomial $\mathbf{p}$ such that (see [17])

$$\|f - \mathbf{p}\|_{\mathscr{B}^\alpha} < \frac{2(\alpha+1)}{C}, C \in (0,\infty). \quad (80)$$

Also since $\mathscr{B}_{H,0}^\alpha$ is the closure in $\mathscr{B}_H^\alpha$ of the polynomials, there exist a polynomial $\mathbf{P} = \mathbf{p}_1 + \bar{\mathbf{p}}_2$ where $\mathbf{p}_1, \mathbf{p}_2 \in \mathscr{B}^\alpha$, such that

$$\|h - \mathbf{P}\|_{\mathscr{B}_H^\alpha}^p < \frac{2(\alpha+1)}{C}. \quad (81)$$

Hence,

$$\sup_{b\in\mathbb{D}} \iint_{\mathbb{D}} \exp\,[C|(h-\mathbf{P})(\varphi_b(w)) - (h-\mathbf{P})(b)|]d\lambda_\alpha(w) < \infty. \quad (82)$$

Furthermore,

$$M = \sup_{b\in\mathbb{D}} \iint_{\mathbb{D}} \exp\,[C|h(\varphi_b(w)) - h(b)|]d\lambda_\alpha(w) < \infty. \quad (83)$$

Now, set $\delta = 1 - \varepsilon$, where $\varepsilon \in (0,1)$ in (77); we get $\delta_1 \in (0,1)$ so that for $\mathbb{D} \setminus D(\bar{0}, \delta_1)$,

$$\begin{aligned}\frac{\pi}{\alpha+1} &\le \iint_{\mathbb{D}} \exp\left[\frac{C}{2}|h(\varphi_b(w)) - h(b)|\right] d\lambda_\alpha(w) \\ &\le \iint_{\mathbb{D}\setminus D(0,\delta)} \exp\left[\frac{C}{2}|h(\varphi_b(w)) - h(b)|\right] d\lambda_\alpha(w) \\ &\quad + \iint_{D(0,\delta)} \exp\left[\frac{C}{2}|h(\varphi_b(w)) - h(b)|\right] d\lambda_\alpha(w) \\ &\le \sqrt{2\pi M\varepsilon} + \frac{\pi\left(1-\varepsilon^{\alpha+1}\right)^2}{\alpha+1}\left(\frac{2}{\varepsilon}\right)^{C\varepsilon/2}.\end{aligned} \quad (84)$$

So, we see that (iii) holds.

On the other hand, suppose that (iii) is true. For any $C \in (0,\infty)$, we see that

$$\begin{aligned}&\left(1-|w|^2\right)^{\alpha}\left(|h_w(w)|+|h_{\bar{w}}(w)|\right)\\&\quad\le \frac{\alpha^2+3\alpha+2}{C}\iint_{\mathbb{D}} \exp\left[\frac{C}{2}|h(\varphi_b(w))-h(b)|\right]d\lambda_\alpha(w).\end{aligned} \tag{85}$$

So, we have $h \in \mathscr{B}^{\alpha}_{H,0}$. □

Next, we give relation between $\mathscr{B}^{\alpha}_{H,0}$ and the compact Carleson measure.

Theorem 9. *Assume that $h \in \mathcal{H}ar(\mathbb{D})$, and let $\alpha \in [0,\infty)$ and $p \in (0,\infty)$. Then, the following are equivalent:*

(i) $h \in \mathscr{B}^{\alpha}_{H,0}$

(ii) $d\mu_1 = (|h_w(w)| + |h_{\bar{w}}(w)|)^p (\log 1/|w|)^{\alpha p} d\lambda(w)$ is a compact p-Carleson measure

(iii) $d\mu_2 = (|h_w(w)| + |h_{\bar{w}}(w)|)^p (1-|w|^2)^{\alpha p} d\lambda(w)$ is a compact p-Carleson measure

Proof. First of all, (i) ⇒ (ii). Suppose that $h \in \mathscr{B}^{\alpha}_{H,0}$, and let $\varepsilon \in (0,1)$. Then, there is $\delta \in (1/4, 1)$ such that

$$\left(1-|w|^2\right)^{\alpha}\left(|h_w(w)|+|h_{\bar{w}}(w)|\right) \le \varepsilon, \tag{86}$$

for $w \in \mathbb{D} \setminus D(0,\delta)$, and

$$\iint_{\mathbb{D}\setminus D(0,\delta)}\left(|h_w(w)|+|h_{\bar{w}}(w)|\right)^p\left(\frac{1-|b|^2}{\left|1-\bar{b}w\right|^2}\right)^p\left(\log\frac{1}{|w|}\right)^{\alpha p}d\lambda(w)\le C\varepsilon. \tag{87}$$

Also,

$$\begin{aligned}&\iint_{D(0,\delta)}\left(|h_w(w)|+|h_{\bar{w}}(w)|\right)^p\left(\frac{1-|b|^2}{\left|1-\bar{b}w\right|^2}\right)^p\left(\log\frac{1}{|w|}\right)^{\alpha p}d\lambda(w)\\&\quad\le C\left(1-|b|^2\right)^p\|h\|^p_{\mathscr{B}^{\alpha}_H}.\end{aligned} \tag{88}$$

By combining equations (87) and (88), we see that (ii) holds.

Next, (ii) ⇒ (iii). This is very clear since $1-|w|^2 \le 2\log 1/|w|$.

After that, (iii) ⇒ (i). Assuming that $d\mu_2$ is a compact p-Carleson measure, as in the proof of Theorem 7, we have

$$\begin{aligned}&\left(|h_w(b)|+|h_{\bar{w}}(b)|\right)^p\left(1-|w|^2\right)^{\alpha p}\\&\quad\lesssim\frac{3}{\pi}\iint_{\mathbb{D}}\left(|h_w(w)|+|h_{\bar{w}}(w)|\right)^p\left(1-|w|^2\right)^{\alpha p}\\&\quad\cdot\left(\frac{1-|b|^2}{\left|1-\bar{b}w\right|^2}\right)^p d\lambda(w)\\&\quad<\infty.\end{aligned} \tag{89}$$

So, we have $h \in \mathscr{B}^{\alpha}_{H,0}$. □

Data Availability

The data is not applicable to this concerned article as no concerned data sets were created or used through this concerned study.

Conflicts of Interest

The authors declare no conflict of interest.

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