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Research Article

Existence of Solutions for Inclusion Problems in Musielak-Orlicz-Sobolev Space Setting

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In this paper, we mainly prove the existence of (weak) solutions of an inclusion problem with the Dirichlet boundary condition of the following form: $L \in A(x, u, Du) + F(x, u, Du)$, in Ω, and $u = 0$, on $\partial\Omega$, in Musielak-Orlicz-Sobolev spaces $W_0^1 L_\Phi(\Omega)$ by using the surjective theorem, where $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain, L belongs to the dual space $(W_0^1 L_\Phi(\Omega))^*$ of $W_0^1 L_\Phi(\Omega)$, A is a multivalued maximal monotone operator, and F is a multivalued convection term. Some examples for A and F are given in the paper. Then, we give some properties of the solution set of the inclusion problem. We also show the existence of (weak) solutions of the inclusion problem with an obstacle effect.

1. Introduction

In [1], Liu and Motreanu established in Sobolev spaces the existence and location of solutions for an elliptic inclusion problem driven by a (p, q)-Laplacian operator with Dirichlet boundary and a multivalued term that has gradient dependence (the so-called multivalued convection term). In [2], Zeng et al. proved the existence of positive solutions for an elliptic inclusion problem driven by an abstract nonhomogeneous operator with Dirichlet boundary and a multivalued convection term in Sobolev spaces. In [3], we proved the existence of positive solutions for an elliptic inclusion problem driven by a nonlinear operator with a Dirichlet boundary and a multivalued convection term in Orlicz-Sobolev spaces, where the nonlinear operator depends on the gradient of the solution. In [1–3], the conditions which ensured the existence of subsolutions and supersolutions were given, and the proof of the existence of solutions depended on the subsolutions and supersolutions.

Precup and Rodríguez-López [4] proved the existence of solutions for an inclusion problem driven by a ϕ-Laplacian operator depending on the differential of the solution. Chen and Tang [5] discussed periodic solutions for a differential inclusion problem involving the $p(t)$-Laplacian. In [6], Zeng et al. proved the existence and boundedness of the weak solutions to inclusion problems driven by double-phase partial differential operators, obstacle effects, and multivalued convection terms in Sobolev spaces. In [7], Zeng et al. introduced a family of the approximating inclusion problems corresponding to an elliptic obstacle problem with double-phase phenomena and a multivalued reaction convection term in Sobolev spaces. Liu and Papageorgiou [8] proved the existence of a nontrivial bounded solution for a double-phase Dirichlet problem with unilateral constraints in Sobolev spaces. Crespo-Blanco et al. [9] showed the existence and uniqueness of a quasilinear elliptic equation driven by a double-phase operator with variable exponents. In the above results, the differential operators are single-valued functions.

The differential inclusions governed by maximal monotone operators have many applications in heat equations, obstacle problems, mechanics, electricity, and management, for example, [10, 11]. Oppezzi and Rossi [12] proved the existence of solutions of a variational inequality with a maximal monotone operator in Sobolev spaces, where the maximal monotone operator depends on the solution and its gradient. Le [13] proved the existence of the solution of variational inequalities with maximal monotone operators by using a subsupersolution method in variable exponent Sobolev spaces, where the maximal monotone operator depends on the gradient of the solution. The conditions which ensure the existence of subsolutions or supersolutions are not given. Le [14] studied the inclusion problems containing maximal monotone and generalized pseudomonotone mappings in Sobolev spaces. Papageorgiou et al. [15] proved the existence of extremal solutions for nonlinear multivalued systems with maximal monotone terms in Sobolev spaces. Avci and Pankov [16] proved the existence of the (weak) solution of a Dirichlet boundary value problem driven by a maximal monotone differential operator in Musielak-Orlicz-Sobolev spaces, where the maximal monotone operator depends on the gradient of the solution.

Variational inequalities and differential equations, as is known to all, correspond to different function spaces. The variable exponent Sobolev spaces and Orlicz-Sobolev spaces are two special kinds of Musielak-Orlicz-Sobolev spaces. They are distinct extensions of the classical Sobolev spaces. In recent years, there are many results of differential equations in Musielak-Orlicz-Sobolev spaces. For example, Fan [17] and we [18] proved the existence of weak solutions of a class of differential equations of divergence form by using a subsupersolution method in reflexive Musielak-Orlicz-Sobolev spaces and nonreflexive Musielak-Orlicz-Sobolev spaces, respectively; Li et al. [19] proved the existence and uniqueness of entropy solutions and the uniqueness of renormalized solutions to the nonlinear elliptic equations in Musielak-Orlicz-Sobolev spaces; we [20] proved the existence of barrier solutions of elliptic differential equations in Musielak-Orlicz-Sobolev spaces; and Baasandorj et al. [21] established optimal regularity estimates for the gradient of solutions to nonuniformly elliptic equations of the Orlicz double phase with variable exponent types. Musielak-Orlicz functions (described in Section 2) have many applications such as non-Newtonian fluids (see, e.g., [22]), thermistor problem (see, e.g., [23]), and image restoration (see, e.g., [24]).

In this paper, we are interested to find a (weak) solution u of the following inclusion problem with the Dirichlet boundary condition and a convection multivalued term

$$\begin{cases} L \in A(x,u,Du) + F(x,u,Du), & \text{in}\,\Omega \\ u = 0, & \text{on}\,\partial\Omega, \end{cases} \tag{1}$$

in Musielak-Orlicz-Sobolev spaces $W_0^1 L_\Phi(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain, L belongs to the dualspace $(W_0^1 L_\Phi(\Omega))^*$ of $W_0^1 L_\Phi(\Omega)$, $A : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow 2^{\mathbb{R}^N}$ is maximal monotone with respect to the last variable, and $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow 2^{\mathbb{R}}$ is a multivalued mapping.

When A is maximal monotone with respect to the last variable, some particular cases of this problem in variable exponent Sobolev spaces and Musielak-Orlicz-Sobolev spaces were obtained, for example, by Le in [13] and by Avci and Pankov in [16], respectively. There are some other particular cases of problem (1). When A is an Orlicz double phase with variable exponents, a case of this problem was established by Baasandorj et al. in [21]. When $L = 0$, F is a single-valued function in $L^1(\Omega)$ and A is a multivalued and maximal monotone; a particular case of this problem in Sobolev spaces was developed, for example, by Oppezzi and Rossi in [12]. When $L = 0$, F is a multivalued convection term and A is a single-valued elliptic differential operator; the existence results of this problem in Orlicz-Sobolev spaces were provided, for example, by us in [3]. When $L = 0$, A is a double-phase operator and F is a multivalued convection term in Sobolev spaces; a particular case of this problem was developed, for example, by Zeng et al. in [6]. When $L = 0$, A is a double-phase operator with variable exponents and F is a single-valued term in variable exponent Sobolev spaces; a particular case of this problem was obtained, for example, by Crespo-Blanco et al. in [9].

The paper is organized as follows: Section 2 contains some preliminaries and some technical lemmas which will be needed in Section 3. In Section 3, we first define the multivalued differential operator $\mathscr{A}$ defined on $W_0^1 L_\Phi(\Omega)$ with values in $(W_0^1 L_\Phi(\Omega))^*$ by using the multivalued function A. Then, we show that $\mathscr{A}$ is a pseudomonotone operator. Some examples for A are given. Next, we deal with the multivalued convection term. We also give an example for the multivalued convection term. Then, we prove the existence of solutions for problem (1) by using the surjective theorem. Then, some properties of the solution set of problem (1) are given. Finally, we show the existence of the solutions for the inclusion problem with an obstacle effect and give some properties of the solution set of the problem.

In this paper, we always assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary and denote by $L^0(\Omega)$ the set of all real measurable functions defined on Ω.

2. Preliminaries

Now, we first list briefly some definitions and facts about Musielak-Orlicz-Sobolev spaces; for more details, see Musielak [25] or Harjulehto and Hästö [26].

Let $\mathbb{R}_+ = [0,+\infty)$. A real function $\Phi : \Omega \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ will be called a Musielak-Orlicz function, denoted by $\Phi \in N(\Omega)$, if it satisfies the following conditions:

(i) $\Phi(x,u)$ is an N-function of the variable $u \geq 0$ for every $x \in \Omega$; i.e., it is a convex, nondecreasing, and continuous function of u such that $\Phi(x,0) = 0$, $\Phi(x,u) > 0$ for $u > 0$, and there hold the conditions

$$\lim_{u \longrightarrow 0^+} \sup_{x \in \Omega} \frac{\Phi(x,u)}{u} = 0, \lim_{u \longrightarrow \infty} \inf_{x \in \Omega} \frac{\Phi(x,u)}{u} = +\infty, \tag{2}$$

for every $x \in \Omega$.

(ii) $\Phi(x, u)$ is a measurable function of x for all $u \geq 0$.

Equivalently, for all $x \in \Omega$ and all $u \in \mathbb{R}$, Φ admits the representation $\Phi(x, u) = \int_0^{|u|} \varphi(x, \tau) d\tau$, where $\varphi(x, t)$ is the right-hand derivative of $\Phi(x, \cdot)$ at t, for a fixed $x \in \Omega$ and all $t \geq 0$. Then, for every $x \in \Omega$, $\varphi(x, \tau)$ is a right-continuous and nondecreasing function of $\tau \geq 0$, $\varphi(x, 0) = 0$, $\varphi(x, \tau) > 0$ for $\tau > 0$, and $\lim_{u \longrightarrow +\infty} \inf_{x \in \Omega} \varphi(x, \tau) = +\infty$.

The complementary function $\bar{\Phi}$ to a Musielak-Orlicz function Φ is defined by $\bar{\Phi}(x, v) = \sup_{u \geq 0}\{uv - \Phi(x, u)\}$ for $v \geq 0, x \in \Omega$. Then, $\bar{\Phi}$ is a Musielak-Orlicz function and Φ is also the complementary function to $\bar{\Phi}$.

Equivalently, $\bar{\Phi}$ admits the representation $\bar{\Phi}(x, v) = \int_0^v \phi(x, \sigma) d\sigma$, where ϕ is given by $\phi(x, \sigma) = \sup\{\tau : \varphi(x, \tau) \leq \sigma\}$, for all $x \in \Omega$.

For $\Phi \in N(\Omega)$, the inequality $uv \leq \Phi(x, u) + \bar{\Phi}(x, v)$ for all $u, v \geq 0$, $x \in \Omega$, is called the Young inequality.

Let $\Phi \in N(\Omega)$. Φ is said to satisfy the Δ_2 condition ($\Phi \in \Delta_2$, for short), if, for $\ell > 0$, there exist a constant $K > 0$ and $h \in L^1(\Omega)$ with $h \geq 0$ such that

$$\Phi(x, \ell u) \leq K\Phi(x, u) + h(x), \tag{3}$$

for all $u \geq 0$ and a.e. $x \in \Omega$.

For each $x \in \Omega$, the inverse function of $\Phi(x, \cdot)$ is denoted by $\Phi^{-1}(x, \cdot)$; i.e., $\Phi^{-1}(x, \Phi(x, u)) = \Phi(x, \Phi^{-1}(x, u)) = u$, for $u \geq 0$.

Let $\Phi_1, \Phi_2 \in N(\Omega)$. $\Phi_1 \preccurlyeq \Phi_2$ means that Φ_1 is weaker than Φ_2; that is, there exist positive constants K_1, K_2 and a nonnegative function $h \in L^1(\Omega)$ such that $\Phi_1(x, u) \leq K_1 \Phi_2(x, K_2 u) + h(x)$ for all $u \geq 0$ and a.e. $x \in \Omega$.

Φ is called locally integrable, if $\int_\Omega \Phi(x, u) dx < +\infty$ for every $u > 0$.

The following assumptions will be used.

(Φ_1) $\inf_{x \in \Omega} \Phi(x, 1) = c_1 > 0$.

(Φ_2) For every $u_0 > 0$, there exists $c = c(u_0) > 0$ such that $\inf_{x \in \Omega}(\Phi(x, u)/u) \geq c$ and $\inf_{x \in \Omega}(\bar{\Phi}(x, u)/u) \geq c$, for $u \geq u_0$.

Surely, $(\Phi_2) \Rightarrow (\Phi_1)$.

$\Phi \in N(\Omega)$ is said to satisfy the condition (Φ) if $\Phi, \bar{\Phi} \in \Delta_2$, and both Φ and $\bar{\Phi}$ are locally integrable and satisfy (Φ_2).

Let $\Phi \in N(\Omega)$. The Musielak-Orlicz space (i.e., the generalized Orlicz space) $L_\Phi(\Omega)$ is defined by

$$L_\Phi(\Omega) = \left\{u \in L^0(\Omega) : \int_\Omega \Phi\left(x, \frac{|u(x)|}{\lambda}\right) dx < +\infty, \text{for some } \lambda > 0\right\}, \tag{4}$$

with the (Luxemburg) norm

$$\|u\|_{(\Phi)} = \inf\left\{\lambda > 0 : \int_\Omega \Phi\left(x, \frac{|u(x)|}{\lambda}\right) dx \leq 1\right\}. \tag{5}$$

Moreover, the set $K_\Phi(\Omega) = \{u \in L^0(\Omega) : \int_\Omega \Phi(x, |u(x)|) dx < +\infty\}$ will be called the Musielak-Orlicz class (i.e., the generalized Orlicz class). A function $u \in L^0(\Omega)$ is called a finite element of $L_\Phi(\Omega)$, if $\lambda u \in K_\Phi(\Omega)$ for every $\lambda > 0$. The space of all finite elements of $L^0(\Omega)$ is denoted by $E_\Phi(\Omega)$. Then, $K_\Phi(\Omega)$ is a convex subset of $L_\Phi(\Omega)$, $L_\Phi(\Omega)$ is the smallest vector subspace of $L^0(\Omega)$ containing $K_\Phi(\Omega)$, and $E_\Phi(\Omega)$ is the largest vector subspace of $L^0(\Omega)$ contained in $K_\Phi(\Omega)$.

If Φ is locally integrable, then $E_\Phi(\Omega)$ is a separable space, and $E_\Phi(\Omega) = K_\Phi(\Omega) = L_\Phi(\Omega)$ if and only if $\Phi \in \Delta_2$. Then, $L_\Phi(\Omega)$ is reflexive if $\Phi \in N(\Omega)$ satisfies the condition (Φ).

The Musielak-Orlicz-Sobolev space $W^1 L_\Phi(\Omega)$ is defined by

$$W^1 L_\Phi(\Omega) = \{u \in L_\Phi(\Omega) : \forall |\alpha| \leq 1, D^\alpha u \in L_\Phi(\Omega)\}, \tag{6}$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_N)$ is a multi-index with nonnegative integers α_i, $i = 1, 2, \cdots, N$, which has $|\alpha| = |\alpha_1| + |\alpha_2| + \cdots + |\alpha_N|$, and $D^\alpha u$ denotes the distributional derivatives.

Let $\|u\|_{\Phi,\Omega} = \inf\{\lambda > 0 : \sum_{|\alpha| \leq 1} \int_\Omega \Phi(x, (|D^\alpha u(x)|/\lambda)) dx \leq 1\}$ for $u \in W^1 L_\Phi(\Omega)$. Then, $\|\cdot\|_{\Phi,\Omega}$ is a norm on $W^1 L_\Phi(\Omega)$, and the pair $(W^1 L_\Phi(\Omega), \|\cdot\|_{\Phi,\Omega})$ is a Banach space if Φ is locally integrable and satisfies (Φ_1). Taking $\Phi(x, u) = \Phi(u)$, $W^1 L_\Phi(\Omega)$ is an Orlicz-Sobolev space. Taking $\Phi(x, u) = |u|^{p(x)}$, $W^1 L_\Phi(\Omega)$ is the variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$.

It is easy to see that

$$W^1 L_\Phi(\Omega) = \{u \in L_\Phi(\Omega) : |Du| \in L_\Phi(\Omega)\}. \tag{7}$$

Denote $\|Du\|_{(\Phi)} = \||Du|\|_{(\Phi)}$ and $\|u\|_{1,\Phi} = \|u\|_{(\Phi)} + \|Du\|_{(\Phi)}$. Then, $\|u\|_{1,\Phi}$ and $\|u\|_{\Phi,\Omega}$ are two equivalent norms.

The space $W^1 L_\Phi(\Omega)$ will always be identified with a subspace of the product $\prod_{|\alpha| \leq 1} L_\Phi(\Omega) = \Pi L_\Phi$; this subspace is $\sigma(\Pi L_\Phi, \Pi E_{\bar{\Phi}})$ closed. Let $W_0^1 L_\Phi(\Omega)$ be the $\sigma(\Pi L_\Phi, \Pi E_{\bar{\Phi}})$ closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1 L_\Phi(\Omega)$.

Let $\Phi \in N(\Omega)$ be locally integrable and satisfy (Φ_1). Then, the space $W^1 L_\Phi(\Omega)$ is reflexive if $L_\Phi(\Omega)$ is reflexive.

Next, we recall the following notations and lemmas which will be used later.

We denote by 2^X all subsets of a set X.

Let X be a Banach space and X^* be its dual. We denote the dual-pairing (X^*, X) by $\langle \cdot, \cdot \rangle$. Given a multivalued mapping $T : X \longrightarrow 2^{X^*}$, let us fix some notations associated with it.

(i) The domain of T is the set $D(T) = \{u \in X : T(u) \neq \varnothing\}$.

(ii) The graph of T is the set $Gr(T) = \{(u, u^*) \in X \times X^* : u^* \in T(u)\}$.

(iii) $\mathcal{P}_{f(c)}(X) = \{E \subset X : E \text{ is nonempty, closed, (convex)}\}$.

Definition 1 (see [27], Definition 3.2.2). Let X be a Banach space. Given a multivalued mapping $T : X \longrightarrow 2^{X^*}$, we say that

(i) T is monotone if for any $u, v \in D(T)$ and $u^* \in T(u)$, $v^* \in T(v)$, we have $\langle u^* - v^*, u - v \rangle \geq 0$;

(ii) T is maximal monotone if it is monotone and, for $(u, u^*) \in X \times X^*$, the inequalities $\langle u^* - v^*, u - v \rangle \geq 0$ for all $(v, v^*) \in Gr(T)$ imply $(u, u^*) \in Gr(T)$;

(iii) T is bounded if it maps bounded sets in X to bounded sets in X^*.

Definition 2 (see [28], Definition 3.7). Let X and Y be Hausdorff topological spaces and $T : X \longrightarrow 2^Y$ be a multivalued operator. Then, T is called upper semicontinuous at $u_0 \in X$, if for every open subset $V \subset Y$ with $T(u_0) \subset V$, there exists a neighborhood $U(u_0)$ of u_0 such that $T(U(u_0)) \subset V$. T is called upper semicontinuous in X, if T is upper semicontinuous at every $u_0 \in X$.

Definition 3 (see [27], Definition 6.1.29). Let (X, d) be a metric space and $A, B \subset X$. Set

$$h^*(A, B) = \sup \{d(a, B): a \in A\}. \tag{8}$$

$h^*(A, B)$ is called the excess of A over B.

Definition 4 (see [27], Definition 6.1.33, or [29], Definition 2.60). Let X be a Hausdorff topological space and (Y, d) a metric space. A multifunction $F : X \longrightarrow 2^Y \setminus \{\varnothing\}$ is said to be h-upper semicontinuous at $x_0 \in X$ if $x \longrightarrow h^*(F(x), F(x_0))$ is continuous at x_0; i.e., for every $\varepsilon > 0$, there is $U_\varepsilon \in \mathcal{N}(x_0)$ such that for all $x \in U_\varepsilon$, $h^*(F(x), F(x_0)) < \varepsilon$, where $\mathcal{N}(x_0)$ is the filter of neighborhoods of x_0. If F is h-upper semicontinuous at every $x_0 \in X$, then we say that F is h-upper semicontinuous.

Lemma 5 (see [27], Proposition 6.1.35). *If $F : X \longrightarrow 2^Y \setminus \{\varnothing\}$ is upper semicontinuous, then F is h-upper semicontinuous.*

Lemma 6 (see [30], Theorem 1.1.1). *Let X be a reflexive Banach space, X^* be its dual, and T be a multivalued map from X into X^*. Suppose that for each $u \in X$, $T(u)$ is a nonempty closed convex subset of X^*, and for each line interval $[u_1, u_2] := \{u \in X : u = \lambda u_1 + (1 - \lambda) u_2, \lambda \in [0, 1]\}$ in X, T is an upper semicontinuous map from the line interval into X^* endowed with its weak topology. Then, T is maximal monotone.*

Definition 7 (see [28], Definition 3.57). Let X be a reflexive Banach space and $T : X \longrightarrow 2^{X^*}$ be a multivalued operator. T is said to be pseudomonotone, if the following conditions hold:

(i) The set $T(u)$ is nonempty, bounded, closed, and convex for all $u \in X$;

(ii) T is upper semicontinuous from each finite-dimensional subspace of X to X^* endowed with the weak topology;

(iii) If $\{u_n\} \subset X$ with $u_n \rightharpoonup u$ weakly in X, and $u_n^* \in T(u_n)$ is such that

$$\limsup_{n \longrightarrow \infty} \langle u_n^*, u_n - u \rangle \leq 0, \tag{9}$$

then for every $v \in X$, there exists $u^*(v) \in T(u)$ such that

$$\liminf_{n \longrightarrow \infty} \langle u_n^*, u_n - v \rangle \geq \langle u^*(v), u - v \rangle. \tag{10}$$

Lemma 8 (see [27], Proposition 3.2.55). *Let X be a real reflexive Banach space and assume that $T : X \longrightarrow 2^{X^*}$ satisfies the following conditions:*

(i) For each $u \in X$, we have that $T(u)$ is a nonempty, closed, and convex subset of X^*;

(ii) T is bounded;

(iii) If $u_n \rightharpoonup u$ weakly in X and $u_n^* \rightharpoonup u^*$ weakly in X^* with $u_n^* \in T(u_n)$ and if $\limsup_{n \longrightarrow \infty} \langle u_n^*, u_n - u \rangle \leq 0$, then $u^* \in T(u)$ and $\langle u_n^*, u_n \rangle \longrightarrow \langle u^*, u \rangle$.

Then, T is pseudomonotone.

Lemma 9 (see [28], Proposition 3.59). *Let X be a reflexive Banach space.*

(i) If $T : X \longrightarrow 2^{X^*}$ is a maximal monotone operator with $D(T) = X$, then T is pseudomonotone.

(ii) If $T_1, T_2 : X \longrightarrow 2^{X^*}$ are pseudomonotone operators, then $T_1 + T_2$ is pseudomonotone.

Lemma 10 (see [28], Definition 3.61). *Let X be a Banach space and $T : X \longrightarrow 2^{X^*}$ be an operator. We say that T is coercive if either $D(T)$ is bounded or $D(T)$ is unbounded and*

$$\lim_{\|u\|_X \longrightarrow +\infty, u \in D(T)} \frac{\inf \{\langle u^*, u \rangle : u^* \in T(u)\}}{\|u\|_X} = +\infty. \tag{11}$$

Lemma 11 (see [28], Theorem 3.61). *Let X be a reflexive Banach space and $T : X \longrightarrow 2^{X^*}$ be pseudomonotone and coercive. Then, T is surjective, i.e., $R(T) = X^*$.*

Lemma 12 (see [31], Lemma 2.1). *Suppose that M is an N-function. Then,*

$$\int_\Omega M(|u(x)|) dx \leq \int_\Omega M(d |Du(x)|) dx, \tag{12}$$

for any $u \in W_0^1 L_M(\Omega)$, where d is twice the diameter of Ω.

3. Main Results

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with the Lipschitz boundary and $\Phi \in N(\Omega)$ satisfy the condition (Φ). Then, $L_\Phi(\Omega)$, $W_0^1 L_\Phi(\Omega)$, and $W^1 L_\Phi(\Omega)$ are all reflexive. Hence, the weak topology of $W^1 L_\Phi(\Omega)$ (resp. $W_0^1 L_\Phi(\Omega)$) is equivalent to the weak-$*$ topology of $W^1 L_\Phi(\Omega)$ (resp. $W_0^1 L_\Phi(\Omega)$). We assume that there exists $\Psi \in N(\Omega)$ satisfying the condition (Φ) such that $\Phi \preccurlyeq \Psi$ and the embedding $W_0^1 L_\Phi(\Omega) \hookrightarrow L_\Psi(\Omega)$ is compact.

3.1. The Maximal Monotone Operator. Let $A : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathscr{P}_{fc}(\mathbb{R}^N)$ satisfy the following conditions:

(A1) A is $\mathscr{L}(\Omega) \times \mathscr{B}(\mathbb{R}) \times \mathscr{B}(\mathbb{R}^N)$-measurable, where $\mathscr{L}(\Omega)$ is the family of Lebesgue measurable subsets of Ω, $\mathscr{B}(\mathbb{R})$ is the σ-algebra of Borel sets in $\mathbb{R}$, and $\mathscr{B}(\mathbb{R}^N)$ is the σ-algebra of Borel sets in $\mathbb{R}^N$; i.e., for every open set $U \subset \mathbb{R}^N$, $A^{-1}(U) := \{(x, s, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N : A(x, s, \xi) \cap U \neq \emptyset\} \in \mathscr{L}(\Omega) \times \mathscr{B}(\mathbb{R}) \times \mathscr{B}(\mathbb{R}^N)$.

(A2) For a.e. $x \in \Omega$, every $s \in \mathbb{R}$, $A(x, s, \cdot)$: $\mathbb{R}^N \longrightarrow 2^{\mathbb{R}^N}$ is maximal monotone.

(A3) If ξ, u, $\{u_n\}$, η are given, being $\xi \in \mathbb{R}^N$, $u \in W_0^1 L_\Phi(\Omega)$, $\{u_n\}$ in $W_0^1 L_\Phi(\Omega)$ such that $u_n \longrightarrow u$ a.e. in Ω, and η an $\mathscr{L}(\Omega)$-measurable selection of the map $x \in \Omega \longrightarrow A(x, u(x), \xi) \subset \mathbb{R}^N$, then there exists a sequence $\{\eta_n\}$ converging a.e. to η in Ω, such that, for every $n \in \mathbb{N}$, η_n is an $\mathscr{L}(\Omega)$-measurable selection of the map $x \in \Omega \longrightarrow A(x, u_n(x), \xi) \subset \mathbb{R}^N$.

(A4) There exist $a_1 \in L_{\bar{\Phi}}(\Omega)$, $a_2 \in L^1(\Omega)$, and $b_1 \geq 0$, $b_2 > 0$ such that

$$|\zeta| \leq a_1(x) + b_1 \bar{\Phi}^{-1}(x, \Phi(x, |s|)) + b_1 \bar{\Phi}^{-1}(x, \Phi(x, |\xi|)), \tag{13}$$

$$\zeta\xi \geq a_2(x) + b_2 \Phi(x, |\xi|), \tag{14}$$

for a.e. $x \in \Omega$, all $s \in \mathbb{R}$, all $\xi \in \mathbb{R}^N$, and all $\zeta \in A(x, s, \xi)$.

By Definition 6.2.1 (a) and Proposition 6.2.3 in [27], we can see that the condition (A1) is weaker than the condition Definition 3.3 (ii) in [16].

If $u : \Omega \longrightarrow \mathbb{R}$ and $w : \Omega \longrightarrow \mathbb{R}^N$ are $\mathscr{L}(\Omega)$-measurable, then $x \in \Omega \longrightarrow A(\cdot, u, w)(x) = A(x, u(x), w(x)) \in 2^{\mathbb{R}^N}$ turns out to be measurable as well (see [12], Remark 1.2). If $\eta : \Omega \longrightarrow 2^{\mathbb{R}^N}$ is a measurable selection of $A(\cdot, u, Dv)$, u, $v \in W_0^1 L_\Phi(\Omega)$, then, by (13), $\eta \in (L_{\bar{\Phi}}(\Omega))^N$.

Let $\tilde{A} : W_0^1 L_\Phi(\Omega) \longrightarrow 2^{(L_{\bar{\Phi}}(\Omega))^N}$ be defined by

$$\tilde{A}(u) = \left\{\zeta \in \left(L_{\bar{\Phi}}(\Omega)\right)^N : \zeta(x) \in A(x, u(x), Du(x)) \text{ for a.e. } x \in \Omega\right\}, \tag{15}$$

and $\mathscr{A} : W_0^1 L_\Phi(\Omega) \longrightarrow 2^{(W_0^1 L_\Phi(\Omega))^*}$ by

$$\mathscr{A}(u) = \left\{-\operatorname{div} \zeta : \zeta \in \tilde{A}(u)\right\}. \tag{16}$$

It follows from (A1) and (13) that $\tilde{A}(u) \neq \emptyset$, for any $u \in W_0^1 L_\Phi(\Omega)$. It is easy to see that $D(\mathscr{A}) = W_0^1 L_\Phi(\Omega)$.

Example 1. Let $\Phi(x, t) = t^p$, for $t \geq 0$, all $x \in \bar{\Omega}$, where $1 < q < p < +\infty$ are given. For all $x \in \bar{\Omega}$, $s \in \mathbb{R}$, and $\xi \in \mathbb{R}^N$, we put $A(x, s, \xi) = (\Phi(x, |\xi|)/|\xi|^2)\xi$. Then, $\mathscr{A}(u)$ becomes the p-Laplace operator $-\operatorname{div}(|Du|^{p-2}Du)$, e.g., [1].

Example 2. Let $\Phi(x, t) = t^p + \mu t^q$, for $t \geq 0$, all $x \in \bar{\Omega}$, where the numbers $\mu \geq 0$ and $1 < q < p < +\infty$ are given. For all $x \in \bar{\Omega}$, $s \in \mathbb{R}$, and $\xi \in \mathbb{R}^N$, we put $A(x, s, \xi) = (\Phi(x, |\xi|)/|\xi|^2)\xi$. Then, $\mathscr{A}(u)$ becomes the double-phase operator $-\operatorname{div}(|Du|^{p-2}Du + \mu|Du|^{q-2}Du)$, e.g., [6].

Example 3. Let $\Phi(x, t) = t^{p(x)}$, for $x \in \bar{\Omega}$ and $t \geq 0$, where $p \in C(\bar{\Omega})$ is given such that $1 < p_- := \inf_{x \in \bar{\Omega}} p(x) \leq p(x) \leq p_+ := \sup_{x \in \bar{\Omega}} p(x) < +\infty$, for all $x \in \bar{\Omega}$. For all $x \in \bar{\Omega}$, $s \in \mathbb{R}$, and $\xi \in \mathbb{R}^N$, we put $A(x, s, \xi) = (\Phi(x, |\xi|)/|\xi|^2)\xi$. Then, $\mathscr{A}(u)$ becomes the $p(x)$-Laplace differential operator $-\operatorname{div}(|Du|^{p(x)-2}Du)$, e.g., [21].

Example 4. Let $\Phi(x, t) = t^{p(x)} + \mu(x)t^{q(x)}$, for $x \in \bar{\Omega}$ and $t \geq 0$, where $\mu \in L^1(\Omega)$ is a nonnegative function and $p, q \in C(\bar{\Omega})$ are given such that $1 < p(x) < N$, $p(x) < q(x)$, for all $x \in \bar{\Omega}$. For all $x \in \bar{\Omega}$, $s \in \mathbb{R}$, and $\xi \in \mathbb{R}^N$, we put $A(x, s, \xi) = (\Phi(x, |\xi|)/|\xi|^2)\xi$. Then, $\mathscr{A}(u)$ becomes the double-phase operator with variable exponents $-\operatorname{div}(|Du|^{p(x)-2}Du + \mu(x)|Du|^{q(x)-2}Du)$, e.g., [9].

Example 5. Let $\varphi(x, t) = t^{p(x)-2}t \log(1 + t^{r(x)})$, for $x \in \bar{\Omega}$ and $t \geq 0$, where $p \in L^\infty(\Omega)$ with $p_- > 1$, and $r \in L^\infty(\Omega)$ with $r_- \geq 0$, and let $\Phi(x, t) = \int_0^t \varphi(x, \tau)d\tau$ for $x \in \bar{\Omega}$ and $t \in \mathbb{R}$. Then, Φ satisfies (Φ) (see [17]). For all $x \in \Omega$, $s \in \mathbb{R}$, and $\xi \in \mathbb{R}^N$, we put $A_1(x, s, \xi) = a_1(x, s)(\Phi(x, |\xi|)/|\xi|^2)\xi$ and $A_2(x, s, \xi) = a_2(x, s)(\Phi(x, |\xi|)/|\xi|^2)\xi$, where a_1 and a_2 are Carathéodory functions (i.e., for each $s \in \mathbb{R}$, $x \longrightarrow a_1(x, s)$ and $x \longrightarrow a_2(x, s)$ are measurable; for a.e. $x \in \Omega$, $s \longrightarrow a_1(x, s)$ and $s \longrightarrow a_2(x, s)$ are continuous) on $\Omega \times \mathbb{R}$ satisfying $0 < \alpha \leq a_1(x, s) \leq a_2(x, s) \leq \beta$ for positive constants α and β, for a.e. $x \in \Omega$. As in [12], we define $A(x, s, \xi) = \{\lambda A_1(x, s, \xi) + (1 - \lambda)A_2(x, s, \xi): \lambda \in [0, 1]\}$, for all $x \in \Omega$, $s \in \mathbb{R}$, and $\xi \in \mathbb{R}^N$. Then, A satisfies (A1)–(A4).

Lemma 13. *Under assumptions (A1)–(A4), we have that*

(i) *$\mathscr{A}$ is bounded;*

(ii) *$\mathscr{A}(u) \in \mathscr{P}_{fc}((W_0^1 L_\Phi(\Omega))^*)$, $\forall u \in X$;*

(iii) *$Gr(\mathscr{A})$ is (sequentially) closed in $W_0^1 L_\Phi(\Omega) \times (W_0^1 L_\Phi(\Omega))^*$ with respect to the strong-weak topology (i.e., with respect to the product of the norm topology of $W_0^1 L_\Phi(\Omega)$ and the weak topology of $(W_0^1 L_\Phi(\Omega))^*$);*

(iv) *$\mathscr{A}$ is upper semicontinuous from $W_0^1 L_\Phi(\Omega)$ with the strong topology to $2^{(W_0^1 L_\Phi(\Omega))^*}$ with the weak topology on $(W_0^1 L_\Phi(\Omega))^*$.*

Proof.

(i) For any $u \in W_0^1 L_\Phi(\Omega)$, by (13), we can obtain that

$$\bar{\Phi}\left(x, \frac{1}{\beta}|\zeta(x)|\right) \le \bar{\Phi}(x, |a_1(x)|) + \Phi(x, |u(x)|)) + \Phi(x, |Du(x)|), \tag{17}$$

for a.e. $x \in \Omega$, all $\zeta \in \tilde{A}(u)$, where $\beta = \max\{3, 3b_1\}$. Therefore, $\mathscr{A}$ is bounded.

(ii) A is closed and convex valued, so that the same is true for $\mathscr{A}$.

(iii) Assume that $\{u_n\}$ and $\{u_n^*\}$ are sequences in $W_0^1 L_\Phi(\Omega)$ and $(W_0^1 L_\Phi(\Omega))^*$, respectively, such that $u_n^* \in \mathscr{A}(u_n)$, $\forall n \in \mathbb{N}$,

$$u_n \longrightarrow u \text{ strongly in } W_0^1 L_\Phi(\Omega), \tag{18}$$

$$u_n^* \rightharpoonup u^* \text{ weakly in } \left(W_0^1 L_\Phi(\Omega)\right)^* \quad \text{for } \sigma\left(\prod L_\Phi, \prod E_{\bar{\Phi}}\right). \tag{19}$$

We will show that $u^* \in \mathscr{A}(u)$. From (16), for every $n \in \mathbb{N}$, there exists some $\zeta_n \in \tilde{A}(u_n)$ such that $u_n^* = -\text{div}\,\zeta_n$. Hence,

$$\zeta_n(x) \in A(x, u_n(x), Du_n(x)) \quad \text{for a.e.} x \in \Omega. \tag{20}$$

In view of (18), we see that $\{u_n\}$ is bounded in $W_0^1 L_\Phi(\Omega)$ and

$$\int_\Omega \Phi(x, 2|u_n(x) - u(x)|)dx \longrightarrow 0, \int_\Omega \Phi(x, |Du_n(x) - Du(x)|)dx \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{21}$$

Based on [32], taking to a subsequence if necessary, we have

$$u_n(x) \longrightarrow u(x), Du_n(x) \longrightarrow Du(x) \quad \text{for a.e.} x \in \Omega, \tag{22}$$

and there exist nonnegative functions $h_1, h_2 \in L^1(\Omega)$ such that $\Phi(x, 2|u_n(x) - u(x)|) \le h_1(x)$ for a.e. $x \in \Omega$. Since Φ is convex with respect to the second variable,

$$\begin{aligned}\Phi(x, |u_n(x)|) &\le \frac{1}{2}\Phi(x, 2|u_n(x) - u(x)|) + \frac{1}{2}\Phi(x, 2|u(x)|) \\ &\le \frac{1}{2}h_1(x) + \frac{1}{2}\Phi(x, 2|u(x)|).\end{aligned} \tag{23}$$

It follows from (13) that $\{\zeta_n\}$ is a bounded sequence in $(L_{\bar{\Phi}}(\Omega))^N$, and also, by passing to a subsequence if necessary, there exists $\zeta \in (L_{\bar{\Phi}}(\Omega))^N$ such that

$$\zeta_n \rightharpoonup \zeta \text{ weakly in } \left(L_{\bar{\Phi}}(\Omega)\right)^N \text{ for } \sigma\left(\left(L_{\bar{\Phi}}(\Omega)\right)^N, \left(E_\Phi(\Omega)\right)^N\right). \tag{24}$$

Letting $n \longrightarrow \infty$, from (24), we have

$$\begin{aligned}\langle u_n^*, v\rangle = \langle -\text{div}\,\zeta_n, v\rangle &= \int_\Omega \zeta_n Dv dx \longrightarrow \int_\Omega \zeta Dv dx \\ &= \langle -\text{div}\,\zeta, v\rangle, \quad \forall v \in W_0^1 L_\Phi(\Omega).\end{aligned} \tag{25}$$

Thanks to (19), we get $-\mathrm{d}iv\zeta = u^*$.

By Definition 6.2.1 (a) and (b) and Proposition 6.2.10 in [27], A is graph measurable. Hence, based on Theorem 2.1.4 of [30], for every $\xi \in \mathbb{R}^N$, there exists a measurable selection η of $A(\cdot, u, \xi)$. Based on (A3), there exists a sequence $\{\eta_n\}$ such that, for every $n \in \mathbb{N}$, $\{\eta_n\}$ is an $\mathscr{L}(\Omega)$-measurable selection of the map $x \in \Omega \longrightarrow A(x, u_n(x), \xi)$ and

$$\eta_n(x) \longrightarrow \eta(x), \quad \text{for a.e.} x \in \Omega. \tag{26}$$

Now, we show that

$$\eta_n \longrightarrow \eta \text{ strongly in} \left(L_{\bar{\Phi}}(\Omega)\right)^N, \quad \text{as } n \longrightarrow \infty. \tag{27}$$

Indeed, suppose that there exists $\varepsilon_0 > 0$ such that $\|\eta_n - \eta\|_{(\bar{\Phi})} \ge \varepsilon_0$, for any $n \in \mathbb{N}$, where $\|\cdot\|_{(\bar{\Phi})} = |||\cdot|||_{(\bar{\Phi})}$ is the Luxemburg norm in $(L_{\bar{\Phi}}(\Omega))^N$. Then,

$$\int_\Omega \bar{\Phi}\left(x, \frac{|\eta_n(x) - \eta(x)|}{\varepsilon_0}\right) dx \ge 1, \quad \text{for any } n \in \mathbb{N}. \tag{28}$$

On the one hand, based on (26), for a.e. $x \in \Omega$, we have

$$\bar{\Phi}\left(x, \frac{|\eta_n(x) - \eta(x)|}{\varepsilon_0}\right) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{29}$$

On the other hand, via (13), $\bar{\Phi} \in \Delta_2$, and the convexity of $\bar{\Phi}$, we can deduce that

$$\begin{aligned}\bar{\Phi}\left(x, \frac{|\eta_n(x) - \eta(x)|}{\varepsilon_0}\right) \le \tilde{K}\big[&\bar{\Phi}(x, |a_1(x)|) + \Phi(x, |u_n(x)|) \\ &+ \Phi(x, |u(x)|) + \Phi(x, |\xi(x)|)\big] \\ &+ h(x),\end{aligned} \tag{30}$$

for any $n \in \mathbb{N}$ and a.e. $x \in \Omega$, where $\tilde{K}$ is a positive constant and $h \in L^1(\Omega)$. Since $\Phi \preccurlyeq \Psi$ and $\Psi \in \Delta_2$, combing (26), (30), and (23), we obtain, by Lebesgue's theorem, that

$$\int_\Omega \bar{\Phi}\left(x, \frac{|\eta_n(x) - \eta(x)|}{\varepsilon_0}\right) dx \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{31}$$

It is a contradiction.

For every nonnegative function $v \in \mathscr{D}(\Omega)$, in view of (A2), we have

$$\int_{\Omega} (\zeta_n(x) - \eta_n(x))(Du_n(x) - \xi)v(x)dx \geq 0. \tag{32}$$

Letting $n \longrightarrow \infty$, and combining (18), (24), and (27), we have

$$\begin{aligned} &\int_{\Omega} (\zeta_n(x) - \eta_n(x))(Du_n(x) - \xi)v(x)dx \\ &\quad \longrightarrow \int_{\Omega} (\zeta(x) - \eta(x))(Du(x) - \xi)v(x)dx. \end{aligned} \tag{33}$$

Thanks to (32), we get

$$\int_{\Omega} (\zeta(x) - \eta(x))(Du(x) - \xi)v(x)dx \geq 0. \tag{34}$$

Since v is arbitrary, we get $(\zeta(x) - \eta(x))(Du(x) - \xi) \geq 0$ for a.e. $x \in \Omega$. Based on (A1) and Theorem III.9 of [33], there exists a sequence of measurable selections $\{\sigma_n\}$ of $A(\cdot, u, \xi)$, such that $\{\sigma_n(x)\colon n \in \mathbb{N}\}$ is dense in $A(x, u(x), \xi)$. It implies that $(\zeta(x) - \vartheta)(Du(x) - \xi) \geq 0$ for a.e. $x \in \Omega$ and for every $\vartheta \in A(x, u(x), \xi)$. Thanks to (A2), $\zeta(x) \in A(x, u(x), Du(x))$ for a.e. $x \in \Omega$. Consequently, $u^* \in \mathscr{A}(u)$.

(iv) To prove (iv), we need to show that if $u \in W_0^1 L_\Phi(\Omega)$, $\{u_n\} \subset W_0^1 L_\Phi(\Omega)$ with

$$u_n \longrightarrow u \text{ strongly in } W_0^1 L_\Phi(\Omega), \quad \text{as } n \longrightarrow \infty, \tag{35}$$

and $V \subset (W_0^1 L_\Phi(\Omega))^*$ is a weakly open set such that $\mathscr{A}(u) \subset V$; then, we can find $n_0 \in \mathbb{N}$ such that $\mathscr{A}(u_n) \subset V$ for all $n \geq n_0$. If we suppose not, then there exists a subsequence $\{u_k\}$ of $\{u_n\}$ such that $\mathscr{A}(u_k) \cap V^C \neq \varnothing$. Let $w_k \in \mathscr{A}(u_k) \cap V^C$. Based on (35), $\{u_n\}$ is bounded in $W_0^1 L_\Phi(\Omega)$. In view of (13), $\{\mathscr{A}(u_n)\}$ is bounded in $(W_0^1 L_\Phi(\Omega))^*$. It follows that $\{w_k\}$ is bounded in $(W_0^1 L_\Phi(\Omega))^*$. Consequently, by passing to a subsequence if necessary, there exists $w \in (W_0^1 L_\Phi(\Omega))^*$ such that $w_k \rightharpoonup w$ weakly in $(W_0^1 L_\Phi(\Omega))^*$ for $\sigma(\Pi L_{\bar{\Phi}}, \Pi E_\Phi)$, as $k \longrightarrow +\infty$. Based on (iii), $w \in \mathscr{A}(u)$. Since $W_0^1 L_\Phi(\Omega)$ is reflexive, $w \in V^C$. It is a contradiction to the choice of V. □

Theorem 14. *Under assumptions (A1)–(A4), the mapping $\mathscr{A}$ is pseudomonotone.*

Proof. Let sequences $\{u_n\} \subset W_0^1 L_\Phi(\Omega)$ and $\{u_n^*\} \subset (W_0^1 L_\Phi(\Omega))^*$ satisfy

$$u_n \rightharpoonup u_0 \text{ weakly in } W_0^1 L_\Phi(\Omega) \quad \text{for } \sigma\left(\prod L_\Phi, \prod E_{\bar{\Phi}}\right), \tag{36}$$

$$u_n^* \rightharpoonup u_0^* \text{ weakly in } \left(W_0^1 L_\Phi(\Omega)\right)^* \quad \text{for } \sigma\left(\prod L_{\bar{\Phi}}, \prod E_\Phi\right), \tag{37}$$

$$\limsup_{n \longrightarrow \infty} \langle u_n^*, u_n - u_0 \rangle \leq 0. \tag{38}$$

We will show that $u_0^* \in \mathscr{A}(u_0)$ and

$$\lim_{n \longrightarrow \infty} \langle u_n^*, u_n \rangle = \langle u_0^*, u_0 \rangle. \tag{39}$$

For each $u \in W_0^1 L_\Phi(\Omega)$, we define the multivalued operator $\mathscr{B}_u : W_0^1 L_\Phi(\Omega) \longrightarrow 2^{(W_0^1 L_\Phi(\Omega))^*}$, by

$$\mathscr{B}_u(v) := \{-\operatorname{div} \eta : \eta(x) \in A(x, u(x), Dv(x)), \text{for a.e.} x \in \Omega\}, \quad \forall v \in W_0^1 L_\Phi(\Omega). \tag{40}$$

Using the similar and more simple proof ((A3) is not needed in this case), we can deduce that Lemma 13 holds if $\mathscr{A}$ is replaced by $\mathscr{B}_u$. Moreover, based on (A2), $\mathscr{B}_u$ is monotone. Therefore, it is maximal monotone. Clearly, $\mathscr{B}_u(u) = \mathscr{A}(u)$, for any $u \in W_0^1 L_\Phi(\Omega)$.

For each $n \in \mathbb{N}$, there exists $\zeta_n \in \tilde{A}(u_0)$ such that $u_n^* = -\operatorname{div}\zeta_n$. From (36), $\{u_n\}$ is bounded in $W_0^1 L_\Phi(\Omega)$. Then, by (13), $\{\zeta_n\}$ is bounded in $(L_{\bar{\Phi}}(\Omega))^N$. Therefore, there exists $\zeta \in (L_{\bar{\Phi}}(\Omega))^N$ such that, by extracting a subsequence if necessary, $\zeta_n \rightharpoonup \zeta$ in $(L_{\bar{\Phi}}(\Omega))^N$ weakly for $\sigma((L_{\bar{\Phi}}(\Omega))^N, (E_\Phi(\Omega))^N)$, as $n \longrightarrow \infty$. Analogous with the proof in Lemma 13, we have $-\operatorname{div}\zeta = u_0^*$.

Let $v \in W_0^1 L_\Phi(\Omega)$ and $\eta \in (L_{\bar{\Phi}}(\Omega))^N$ be such that $\eta(x) \in A(x, u_0(x), Dv(x))$ for a.e. $x \in \Omega$. Since $W_0^1 L_\Phi(\Omega) \hookrightarrow L_\Psi(\Omega)$ is compact, in view of (36), there exists a subsequence of $\{u_n\}$ still denoted by $\{u_n\}$ such that

$$u_n \longrightarrow u_0 \text{ strongly in } L_\Psi(\Omega), \quad \text{as } n \longrightarrow \infty. \tag{41}$$

By passing to a further subsequence if necessary, we can suppose

$$u_n \longrightarrow u_0 \text{ a.e. in } \Omega, \quad \text{as } n \longrightarrow \infty, \tag{42}$$

$$\Psi(x, 2|u_n(x) - u_0(x)|) \leq h_0(x), \quad \text{for a.e.} x \in \Omega, \tag{43}$$

for some $h_0 \in L^1(\Omega)$ (see [32]). Since $\Psi \in \Delta_2$ and Ψ is convex with respect to the second variable,

$$\begin{aligned} \Psi(x, |u_n(x)|) &\leq \frac{1}{2}\Psi(x, 2|u_n(x) - u_0(x)|) + \frac{1}{2}\Psi(x, 2|u_0(x)|) \\ &\leq \frac{1}{2}h_0(x) + \frac{1}{2}\Psi(x, 2|u_0(x)|), \end{aligned} \tag{44}$$

for a.e. $x \in \Omega$. According to (A3), there exists a sequence $\{\eta_n\}$ such that

$$\eta_n \longrightarrow \eta \text{ a.e. in } \Omega, \quad \text{as } n \longrightarrow \infty, \tag{45}$$

and η_n is an $\mathscr{L}(\Omega)$-measurable selection of $x \in \Omega \longrightarrow A(x, u_n(x), Dv(x)) \subset \mathbb{R}^N$, for every $n \in \mathbb{N}$. Since $\Phi \preccurlyeq \Psi$ and $\Psi \in \Delta_2$, similar to the proof of (27), we have

$$\eta_n \longrightarrow \eta \text{ strongly in } \left(L_{\bar{\Phi}}(\Omega)\right)^N, \quad \text{as } n \longrightarrow \infty. \tag{46}$$

For each $n \in \mathbb{N}$, from the monotonicity of $A(x, u_n(x), \cdot)$ for a.e. $x \in \Omega$, we obtain

$$\begin{aligned} 0 &\le \limsup_{n\longrightarrow\infty} \int_\Omega (\zeta_n(x) - \eta_n(x))(Du_n(x) - Dv(x))dx \\ &= \limsup_{n\longrightarrow\infty} \left(\langle u_n^*, u_n - u_0 \rangle + \int_\Omega \zeta_n(x)(Du_0(x) - Dv(x))dx \right. \\ &\quad \left. - \int_\Omega \eta_n(x)(Du_n(x) - Dv(x))dx \right) \\ &= \limsup_{n\longrightarrow\infty} \langle u_n^*, u_n - u_0 \rangle + \int_\Omega (\zeta(x) - \eta(x))(Du_0(x) - Dv(x))dx \\ &\le \langle -\text{div}\, \zeta - (-\text{div}\, \eta), u_0 - v \rangle. \end{aligned} \tag{47}$$

Since $\mathscr{B}_{u_0}$ is maximal monotone, we get that $u_0^* = -\text{div}\, \zeta \in \mathscr{B}_{u_0}(u_0) = \mathscr{A}(u_0)$.

Thanks to (A3), there exists $\{v_n\}$ such that $v_n \longrightarrow \zeta$ a.e. in Ω, as $n \longrightarrow \infty$, and v_n is an $\mathscr{L}(\Omega)$-measurable selection of $x \in \Omega \longrightarrow A(x, u_n(x), Du(x)) \subset \mathbb{R}^N$, for each $n \in \mathbb{N}$. Then, by using the monotonicity of $A(x, u_n(x), \cdot)$ for a.e. $x \in \Omega$, we have

$$\begin{aligned} \langle u_n^*, u_n \rangle \ge &\int_\Omega \eta_n(x)(Du_n(x) - Du_0(x))dx \\ &+ \int_\Omega \zeta_n(x) Du_0(x)dx. \end{aligned} \tag{48}$$

It yields that $\liminf_{n\longrightarrow\infty} \langle u_n^*, u_n \rangle \ge \langle u_0^*, u_0 \rangle$. Consequently, based on (38), we can deduce that (39) holds. Based on Lemma 8, $\mathscr{A}$ is pseudomonotone. □

3.2. The Multivalued Convection Term. We define $\Phi^-(t) := ess \inf_{x\in\Omega} \Phi(x, t)$, for any $t \ge 0$ (see [26]). By the condition (Φ_1), we obtain that $\Phi^-|_{(0,+\infty)} \not\equiv 0$ and $\Phi^-|_{(0,+\infty)} \not\equiv +\infty$. According to Lemma 2.5.16 of [26], Φ^- is an N-function. Hence, $\Phi^-(t) > 0$ for $t > 0$. Moreover, analogous to the proof of Lemma 2.5.24 of [26], we can obtain that $\Phi^- \in \Delta_2$ whenever $\Phi \in \Delta_2$. Then, there exists a constant $K_d > 0$ such that

$$\Phi^-(ds) \le K_d \Phi^-(s), \quad \forall s \ge s_0, \tag{49}$$

for some $s_0 \ge 0$, where d is twice the diameter of Ω.

Let $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathscr{P}_{fc}(\mathbb{R})$ satisfy the following conditions:

(F1) F is graph measurable.

(F2) For a.e. $x \in \Omega$, the multivalued mapping $F(x, \cdot, \cdot)$: $\mathbb{R} \times \mathbb{R}^N \longrightarrow 2^\mathbb{R}$ is upper semicontinuous.

(F3) There exist a function $a_3 \in L_{\bar{\Psi}}(\Omega)$ and a constant $b_3 > 0$ such that

$$\begin{aligned} \sup \{|f|: f \in F(x, s, \xi)\} \le a_3(x) + b_3 \bar{\Psi}^{-1}(x, \Psi(x, |s|)) \\ + b_3 \bar{\Psi}^{-1}(x, \Phi(x, |\xi|)), \end{aligned} \tag{50}$$

for a.e. $x \in \Omega$, all $s \in \mathbb{R}$ and all $\xi \in \mathbb{R}^N$.

(F4) There exist a function $\omega \in L^1(\Omega)$ and two constants $b_4, b_5 \ge 0$ such that

$$fs \le \omega(x) + b_4 \Phi^-(|s|) + b_5 \Phi(x, |\xi|), \tag{51}$$

for a.e. $x \in \Omega$, all $s \in \mathbb{R}$ and all $\xi \in \mathbb{R}^N$, and for all $f \in F(x, s, \xi)$. Moreover,

$$b_2 - \tilde{\lambda}^{-1} b_4 - b_5 > 0, \tag{52}$$

where

$$\tilde{\lambda} = \inf_{u\in D_\hbar} \frac{\int_\Omega \Phi^-(|Du(x)|)dx}{\int_\Omega \Phi^-(|u(x)|)dx}, \tag{53}$$

with $D_\hbar = \{u \in W_0^1 L_{\Phi^-}(\Omega) \setminus \{0\}: \int_\Omega \Phi^-(|Du(x)|)dx \ge \hbar\}$ for some $\hbar > 0$.

Remark 15. When $Du(x) \not\equiv 0$ for a.e $x \in \Omega$, we have $\int_\Omega \Phi^-(|Du(x)|)dx > 0$.

Remark 16. From Lemma 12 and (49),

$$\int_\Omega \Phi^-(|u(x)|)dx \le K_d \int_\Omega \Phi^-(|Du(x)|)dx + \text{meas}(\Omega) \cdot \Phi^-(ds_0), \tag{54}$$

which yields that

$$\frac{\int_\Omega \Phi^-(|Du(x)|)dx}{\int_\Omega \Phi^-(|u(x)|)dx} \ge \frac{\int_\Omega \Phi^-(|Du(x)|)dx}{K_d \int_\Omega \Phi^-(|Du(x)|)dx + \text{meas}(\Omega) \cdot \Phi^-(ds_0)}. \tag{55}$$

Hence, $\tilde{\lambda} \ge \hbar/(K_d \hbar + \text{meas}(\Omega) \cdot \Phi^-(ds_0)) > 0$.

Example 17. Assume that $a, b, c, d \in L^\infty(\Omega)$, $e, k \in L_{\bar{\Psi}}(\Omega)$, $a(x) \le b(x)$, $c(x) \le d(x)$, and $e(x) \le k(x)$, for a.e. $x \in \Omega$, and

$$b_2 - \tilde{\lambda}^{-1} \max\left\{\|a\|_{L^\infty(\Omega)}, \|b\|_{L^\infty(\Omega)}\right\} - \max\left\{\|c\|_{L^\infty(\Omega)}, \|d\|_{L^\infty(\Omega)}\right\} > 0. \tag{56}$$

Put $F(x, s, \xi) = \{-d_1(\Phi^-(|s|)/|s|^2)s + d_2\varsigma + v : d_1 \in [a(x), b(x)],\ d_2 \in [c(x),\ d(x)],\ \varsigma \in [\bar{\Phi}^{-1}(x, \Phi(x,\ |\xi|)),\ \bar{\Psi}^{-1}(x, \Phi(x, |\xi|))], v \in [e(x), k(x)]\}$, for $x \in \Omega$, $s \in \mathbb{R}$, and $\xi \in \mathbb{R}^N$. Then, F satisfies (F1)–(F4).

For each $u \in W_0^1 L_\Phi(\Omega)$, let

$$\tilde{F}(u) = \left\{ f \in L^0(\Omega) : f(x) \in F(x, u(x), Du(x)), \text{for a.e.} x \in \Omega \right\}. \tag{57}$$

It follows from (F1) that $\tilde{F}(u) \neq \emptyset$ whenever $u \in W_0^1 L_\Phi(\Omega)$. Moreover, from (F3), we can see that $\tilde{F}(u) \subset L_{\bar{\Psi}}(\Omega)$ whenever $u \in W_0^1 L_\Phi(\Omega)$.

Let $\mathscr{F} : W_0^1 L_\Phi(\Omega) \longrightarrow 2^{(W_0^1 L_\Phi(\Omega))^*}$ be defined by $\mathscr{F}(u) = \{\widehat{f} \in (W_0^1 L_\Phi(\Omega))^* : f \in \tilde{F}(u)\}$, where $\widehat{f}$ is defined for each $f \in L_{\bar{\Psi}}(\Omega)$ by

$$\left\langle \widehat{f}, v \right\rangle = \int_\Omega f(x) v(x) dx, \quad \forall v \in W_0^1 L_\Phi(\Omega). \tag{58}$$

Analogous to the proof of Theorem 6.4.16 of [27], we have the following lemma, whose proof is omitted.

Lemma 18. *Let $w : \Omega \times \mathbb{R} \longrightarrow \bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ be a $\mathscr{L}(\Omega) \times \mathscr{B}(\mathbb{R})$-measurable function, $\mathscr{T} : \Omega \longrightarrow 2^{\mathbb{R}} \setminus \{\emptyset\}$ be graph-measurable, and the integral functional*

$$I_w(v) = \int_\Omega w(x, v(x)) dx, v \in L_\Phi(\Omega) \tag{59}$$

be defined for all $v \in S_\Phi(\mathscr{T}) = \{v \in L_\Phi(\Omega): v(x) \in \mathscr{T}(x) \text{ for a.e.} x \in \Omega\}$, where $\Phi \in N(\Omega)$. If there exists $v_0 \in L_\Phi(\Omega)$ such that $I_w(v_0) > -\infty$, then

$$\sup_{v \in S_\Phi(\mathscr{T})} I_w(v) = \int_\Omega \sup_{\eta \in \mathscr{T}(x)} w(x, \eta) dx. \tag{60}$$

If there exists $v_0 \in L_\Phi(\Omega)$ such that $I_w(v_0) < +\infty$, then $x \longrightarrow \inf_{\eta \in \mathscr{T}(x)} w(x, \eta)$ is (Lebesgue) measurable and

$$\inf_{v \in S_\Phi(\mathscr{T})} I_w(v) = \int_\Omega \inf_{\eta \in \mathscr{T}(x)} w(x, \eta) dx. \tag{61}$$

Now, we give the following lemma about h-upper semicontinuous of multivalued mappings.

Lemma 19. *Assume that (F1)–(F3) hold. Then,*

(i) $\tilde{F}(u) \in \mathscr{P}_{fc}(L_{\bar{\Psi}}(\Omega))$ for all $u \in W_0^1 L_\Phi(\Omega)$;

(ii) $\tilde{F}$ is bounded;

(iii) $\tilde{F} : W_0^1 L_\Phi(\Omega) \longrightarrow \mathscr{P}_{fc}(L_{\bar{\Psi}}(\Omega))$ is h-upper semicontinuous; that is, if $u_n \longrightarrow u_0$ strongly in $W_0^1 L_\Phi(\Omega)$, then

$$h^*\left(\tilde{F}(u_n), \tilde{F}(u_0)\right) = \sup_{v' \in \tilde{F}(u_n)} \inf_{v \in \tilde{F}(u_0)} \left\| v' - v \right\|_{(\bar{\Psi})} \longrightarrow 0, \tag{62}$$

as $n \longrightarrow \infty$.

Proof.

(i) The proof of the convexity and closedness of $\tilde{F}(u)$ is a direct consequence of the fact that $F(x, s, \xi)$ is convex, closed, and bounded in $\mathbb{R}$ for a.e. $x \in \Omega$, all $s \in \mathbb{R}$, and all $\xi \in \mathbb{R}^N$.

(ii) The boundedness of $\tilde{F}$ follows directly from (F3).

(iii) The following proof is referred to Theorem 2.7 of [13]. Assume that $\{u_n\} \subset W_0^1 L_\Phi(\Omega)$ is a sequence such that

$$u_n \longrightarrow u_0 \text{ strongly in } W_0^1 L_\Phi(\Omega), \quad \text{as } n \longrightarrow \infty, \tag{63}$$

for some $u_0 \in W_0^1 L_\Phi(\Omega)$. We will show that

$$\sup_{v' \in \tilde{F}(u_n)} \inf_{v \in \tilde{F}(u_0)} \left\| v' - v \right\|_{(\bar{\Psi})} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{64}$$

Since $W_0^1 L_\Phi(\Omega) \hookrightarrow L_\Psi(\Omega)$ is compact,

$$u_n \longrightarrow u_0 \text{ strongly in } L_\Psi(\Omega), \quad \text{as } n \longrightarrow \infty. \tag{65}$$

It follows from (65) and (63) that there exists a subsequence of $\{u_n\}$ still denoted by $\{u_n\}$ and $h_1, h_2 \in L^1(\Omega)$ such that

$$u_n(x) \longrightarrow u_0(x), Du_n(x) \longrightarrow Du_0(x), \quad \text{for a.e.} x \in \Omega, \text{as } n \longrightarrow \infty, \tag{66}$$

and $\Psi(x, 2|u_n(x) - u_0(x)|) \leq h_1(x)$, $\Phi(x, 2|Du_n(x) - Du_0(x)|) \leq h_2(x)$, for a.e. $x \in \Omega$. Hence,

$$\begin{aligned} \Psi(x, |u_n(x)|) &\leq \frac{1}{2} h_1(x) + \frac{1}{2} \Psi(x, 2|u_0(x)|), \Phi(x, |Du_n(x)|) \\ &\leq \frac{1}{2} h_2(x) + \frac{1}{2} \Phi(x, 2|Du_0(x)|), \end{aligned} \tag{67}$$

for a.e. $x \in \Omega$. Based on (F2) and Lemma 5, the multivalued mapping $F(x, \cdot, \cdot): \mathbb{R} \times \mathbb{R}^N \longrightarrow 2^{\mathbb{R}}$ is h-upper semicontinuous, for a.e. $x \in \Omega$. It follows that $\sup_{\xi \in F(x, u_n(x), Du_n(x))} \inf_{\eta \in F(x, u_0(x), Du_0(x))} |\xi - \eta| \longrightarrow 0$ as $n \longrightarrow \infty$, for a.e. $x \in \Omega$, and thus,

$$\sup_{\xi \in F(x, u_n(x), Du_n(x))} \inf_{\eta \in F(x, u_0(x), Du_0(x))} \bar{\Psi}(x, |\xi - \eta|) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{68}$$

for a.e. $x \in \Omega$.

On the other hand, since $\Psi, \bar{\Psi} \in \Delta_2$, based on (F3) and (67), there exist some constants $K_1, K_2 > 0$ and a function

$h \in L^1(\Omega)$ such that

$$\begin{aligned}\bar{\Psi}(x,|\xi-\eta|) &\le \frac{K_1}{2}\left[\bar{\Psi}(x,|\xi|)+\bar{\Psi}(x,|\eta|)\right]+h(x)\\ &\le \frac{K_1K_2}{6}\left[2\bar{\Psi}(x,|a_3(x)|)+\Psi(x,|u_n(x)|)\right.\\ &\quad+\Phi(x,|Du_n(x)|)+\Psi(x,|u_0(x)|)\\ &\quad\left.+\Phi(x,|Du_0(x)|)\right]+(K_1+1)h(x)\\ &\le \frac{K_1K_2}{6}\left[2\bar{\Psi}(x,|a_3(x)|)+\frac{1}{2}h_1(x)\right.\\ &\quad+\frac{1}{2}\Psi(x,2|u_0(x)|)+\frac{1}{2}h_2(x)\\ &\quad+\frac{1}{2}\Phi(x,2|Du_0(x)|)+\Psi(x,|u_0(x)|)\\ &\quad\left.+\Phi(x,|Du_0(x)|)\right]+(K_1+1)h(x)\coloneqq H(x),\end{aligned} \tag{69}$$

for all $n\in\mathbb{N}$, a.e. $x\in\Omega$, and all $\xi\in F(x,u_n(x),Du_n(x))$, $\eta\in F(x,u_0(x),Du_0(x))$. Thus,

$$\sup_{\xi\in F(x,u_n(x),Du_n(x))}\ \inf_{\eta\in F(x,u_0(x),Du_0(x))}\bar{\Psi}(x,|\xi-\eta|)\le H(x), \tag{70}$$

for all $n\in\mathbb{N}$, a.e. $x\in\Omega$. Evidently, the function H belongs to $L^1(\Omega)$. Using Lebesgue's dominated convergence theorem, we deduce that

$$\int_\Omega \sup_{\xi\in F(x,u_n(x),Du_n(x))}\ \inf_{\eta\in F(x,u_0(x),Du_0(x))}\bar{\Psi}(x,|\xi-\eta|)dx\longrightarrow 0 \quad \text{as } n\longrightarrow\infty. \tag{71}$$

Let $n\in\mathbb{N}$. Based on (F1), F is graph measurable. For $v'\in\tilde{F}(u_n)$, using Lemma 18, we can obtain that

$$\begin{aligned}&\inf_{v\in\tilde{F}(u_0)}\int_\Omega\bar{\Psi}\left(x,\left|v'(x)-v(x)\right|\right)dx\\ &\quad=\int_\Omega\inf_{\eta\in F(x,u_0(x),Du_0(x))}\bar{\Psi}\left(x,\left|v'(x)-\eta\right|\right)dx.\end{aligned} \tag{72}$$

Using Lemma 18 again, we get

$$\begin{aligned}&\sup_{v'\in\tilde{F}(u_n)}\int_\Omega\inf_{\eta\in F(x,u_0(x),Du_0(x))}\bar{\Psi}\left(x,\left|v'(x)-\eta\right|\right)dx\\ &\quad=\int_\Omega\sup_{\xi\in F(x,u_n(x),Du_n(x))}\ \inf_{\eta\in F(x,u_0(x),Du_0(x))}\bar{\Psi}(x,|\xi-\eta|)dx.\end{aligned} \tag{73}$$

Letting $n\longrightarrow\infty$, and combining (72), (73), and (71), we have

$$\sup_{v'\in\tilde{F}(u_n)}\ \inf_{v\in\tilde{F}(u_0)}\int_\Omega\bar{\Psi}\left(x,\left|v'(x)-v(x)\right|\right)dx\longrightarrow 0. \tag{74}$$

Since $\bar{\Psi}\in\Delta_2$, we can deduce (64), which completes our proof. □

Then, we have the following theorem.

Theorem 20. *If F satisfies (F1)–(F3), then*

(i) $\mathscr{F}$ is bounded;

(ii) $Gr(\mathscr{F})$ is (sequentially) closed in $W_0^1L_\Phi(\Omega)\times(W_0^1L_\Phi(\Omega))^*$ with respect to the strong-weak topology.

Proof.

(i) It is easy from (F3) to see that $\mathscr{F}$ is bounded.

(ii) We show that $Gr(\mathscr{F})$ is (sequentially) closed in $W_0^1 L_\Phi(\Omega)\times(W_0^1L_\Phi(\Omega))^*$ with respect to the strong-weak topology. Assume that $\{u_n\}$ and $\{\hat{f}_n\}$ are sequences in $W_0^1L_\Phi(\Omega)$ and $(W_0^1L_\Phi(\Omega))^*$, respectively, such that

$$u_n\longrightarrow u \text{ strongly in } W_0^1L_\Phi(\Omega), \tag{75}$$

$$\hat{f}_n\rightharpoonup\hat{f} \text{ weakly in } \left(W_0^1L_\Phi(\Omega)\right)^* \quad \text{for } \sigma\left(\Pi L_{\bar{\Phi}}(\Omega),\Pi E_{\Phi}(\Omega)\right), \tag{76}$$

$$\hat{f}_n\in\mathscr{F}(u_n),\quad \forall n\in\mathbb{N}. \tag{77}$$

Let us prove that

$$\hat{f}\in\mathscr{F}(u). \tag{78}$$

Thanks to (77), there exists $f_n\in\tilde{F}(u_n)$ such that, for every $n\in\mathbb{N}$,

$$\left\langle\hat{f}_n,v\right\rangle=\int_\Omega f_n(x)v(x)dx,\quad\forall v\in W_0^1L_\Phi(\Omega). \tag{79}$$

From (75) and Lemma 19 (iii), we obtain that $h^*(\tilde{F}(u_n),\tilde{F}(u))\longrightarrow 0$, and thus,

$$\inf_{w\in\tilde{F}(u)}\|f_n-w\|_{(\bar{\Psi})}\longrightarrow 0,\quad \text{as } n\longrightarrow\infty. \tag{80}$$

Consequently, there is a sequence $\{w_n\}\subset\tilde{F}(u)$ such that

$$\|f_n-w_n\|_{(\bar{\Psi})}\longrightarrow 0,\quad \text{as } n\longrightarrow\infty. \tag{81}$$

Since $\tilde{F}(u)$ is bounded in $L_{\bar{\Psi}}(\Omega)$, by passing to a subsequence if necessary, there exists $w\in L_{\bar{\Psi}}(\Omega)$ such that

$$w_n\rightharpoonup w \text{ weakly in } L_{\bar{\Psi}}(\Omega) \text{ for } \sigma\left(L_{\bar{\Psi}}(\Omega),E_{\Psi}(\Omega)\right),\quad \text{as } n\longrightarrow\infty. \tag{82}$$

According to Lemma 19 (i), $w \in \tilde{F}(u)$. Letting $n \longrightarrow \infty$, and combing (81) and (82), we get that

$$f_n \rightharpoonup w \text{ weakly in } L_{\bar{\Psi}}(\Omega) \quad \text{for } \sigma\left(L_{\bar{\Psi}}(\Omega), E_{\Psi}(\Omega)\right). \tag{83}$$

It implies that $\langle \widehat{f}, v\rangle = \int_\Omega w(x)v(x)dx, \forall v \in W_0^1 L_\Phi(\Omega)$. Therefore, $\widehat{f} \in \mathscr{F}(u)$. □

3.3. Existence Theorem for Inclusion Problems. A function $u \in W_0^1 L_\Phi(\Omega)$ is called a (weak) solution of problem (1), if there exist $\zeta \in (L_{\bar{\Phi}}(\Omega))^N$ and $f \in L_{\bar{\Psi}}(\Omega)$ such that

$$\begin{gathered}\zeta(x) \in A(x, u(x), Du(x)), f(x) \in F(x, u(x), Du(x)), \quad \text{for a.e.} x \in \Omega, \\ \int_\Omega \zeta(x) Dv(x)dx + \int_\Omega f(x)v(x)dx = \int_\Omega L(x)v(x)dx, \quad \forall v \in W_0^1 L_\Phi(\Omega).\end{gathered} \tag{84}$$

In this section, we are going to prove an existence result for our problem (1).

Theorem 21. *Let A and f satisfy (A1)–(A4) and (F1)–(F4); then, there exists a solution for problem (1).*

Proof. We define the multivalued mapping $\mathscr{T} : W_0^1 L_\Phi(\Omega) \longrightarrow 2^{(W^1 L_\Phi(\Omega))^*}$ by

$$\mathscr{T}(u) = \mathscr{A}(u) + \mathscr{F}(u). \tag{85}$$

Then, $\mathscr{T}$ is well-defined.

Step 1. We are going to show that $\mathscr{T}$ is pseudomonotone. Let sequences $\{u_n\} \subset W_0^1 L_\Phi(\Omega)$ and $\{\omega_n\} \subset (W_0^1 L_\Phi(\Omega))^*$ have the properties

$$u_n \rightharpoonup u \text{ weakly in } W_0^1 L_\Phi(\Omega) \text{ for } \sigma\left(\prod L_\Phi, \prod E_{\bar{\Phi}}\right), \quad \text{as } n \longrightarrow \infty, \tag{86}$$

$$\widehat{\omega}_n \rightharpoonup \widehat{\omega} \text{ weakly in } \left(W_0^1 L_\Phi(\Omega)\right)^* \text{ for } \sigma\left(\prod L_{\bar{\Phi}}, \prod E_\Phi\right), \quad \text{as } n \longrightarrow \infty, \tag{87}$$

with $\widehat{\omega}_n \in \mathscr{T}(u_n)$ and $\limsup_{n\longrightarrow\infty}\langle \widehat{\omega}_n, u_n - u\rangle \le 0$. From (85), there exist $\widehat{\zeta}_n \in \mathscr{A}(u_n)$, $\widehat{f}_n \in \mathscr{F}(u_n)$ such that $\widehat{\omega}_n = \widehat{\zeta}_n + \widehat{f}_n$, for every $n \in \mathbb{N}$, where $\widehat{\zeta}_n = -div\zeta_n$ with $\zeta_n \in \tilde{A}(u_n)$ and $\widehat{f}_n$ satisfies $\langle \widehat{f}_n, v\rangle = \int_\Omega f_n(x)v(x)dx$, $\forall v \in W_0^1 L_\Phi(\Omega)$, for some $f_n \in \tilde{F}(u_n)$.

Now, we prove that

$$u_n \longrightarrow u \text{ strongly in } W_0^1 L_\Phi(\Omega), \quad \text{as } n \longrightarrow \infty. \tag{88}$$

To show (88), based on Proposition 1.9 of [17], it needs to be proven that

$$Du_n \longrightarrow Du \text{ strongly in } (L_\Phi(\Omega))^N, \quad \text{as } n \longrightarrow \infty. \tag{89}$$

We suppose that there exits $\varepsilon_0 > 0$ such that

$$\|Du_n - Du\|_{(\Phi)} \ge \varepsilon_0, \quad \text{for any } n \in \mathbb{N}. \tag{90}$$

Then,

$$\int_\Omega \Phi\left(x, \frac{|Du_n(x) - Du(x)|}{\varepsilon_0}\right)dx \ge 1, \quad \text{for any } n \in \mathbb{N}. \tag{91}$$

From (86), the sequence $\{u_n\}$ is bounded in $W_0^1 L_\Phi(\Omega)$, and by passing to a subsequence if necessary,

$$u_n \longrightarrow u \text{ strongly in } L_\Psi(\Omega), \tag{92}$$

$$u_n \longrightarrow u \text{ a.e.in } \Omega, \tag{93}$$

as $n \longrightarrow \infty$. By passing to a further subsequence if necessary, there exists $h_0 \in L^1(\Omega)$ such that

$$\Psi(x, |u_n(x)|) \le \frac{1}{2}h_0(x) + \frac{1}{2}\Psi(x, 2|u(x)|), \tag{94}$$

for a.e. $x \in \Omega$. Thanks to (F3), $\{f_n\}$ is bounded in $L_{\bar{\Psi}}(\Omega)$. It follows that

$$\left\langle \widehat{f}_n, u_n - u\right\rangle = \int_\Omega f_n(x)(u_n(x) - u(x))dx \le 2\|f_n\|_{(\bar{\Psi})}\|u_n - u\|_{(\Psi)} \longrightarrow 0, \tag{95}$$

as $n \longrightarrow \infty$. Consequently, $\limsup_{n\longrightarrow\infty}\langle \widehat{\zeta}_n, u_n - u\rangle \le 0$. By (13), $\widehat{\zeta}_n$ is bounded in $(W_0^1 L_\Phi(\Omega))^*$. Hence, by extracting a subsequence if necessary, there exists $\widehat{\zeta} \in (W_0^1 L_\Phi(\Omega))^*$ such that

$$\widehat{\zeta}_n \rightharpoonup \widehat{\zeta} \text{ weakly in } \left(W_0^1 L_\Phi(\Omega)\right)^* \text{ for } \sigma\left(\prod L_{\bar{\Phi}}, \prod E_\Phi\right), \quad \text{as } n \longrightarrow \infty. \tag{96}$$

From Theorem 14, $\mathscr{A}$ is pseudomonotone; then, we get $\widehat{\zeta} \in \mathscr{A}(u)$ and

$$\lim_{n\longrightarrow\infty}\left\langle \widehat{\zeta}_n, u_n\right\rangle = \left\langle \widehat{\zeta}, u\right\rangle. \tag{97}$$

Let η be a measurable selection of $A(\cdot, u, Du)$. Based on (93) and (A3), there exists a sequence $\{\eta_n\}$ converging a.e. to η in Ω, as $n \longrightarrow \infty$, such that η_n is an $\mathscr{L}(\Omega)$-measurable selection of the map $x \in \Omega \longrightarrow A(x, u_n(x), Du(x)) \in 2^{\mathbb{R}^N}$, for every $n \in \mathbb{N}$. Same as the proof of (46), we can obtain that

$$\eta_n \longrightarrow \eta \text{ strongly in } \left(L_{\bar{\Phi}}(\Omega)\right)^N, \quad \text{as } n \longrightarrow \infty. \tag{98}$$

Denote $\widehat{\eta}_n = -\text{div}\eta_n$ and $\widehat{\eta} = -\text{div}\eta$.

On the one hand, combing (86), (96), (97), and (98), we can obtain that

$$\left\langle \widehat{\zeta}_n - \widehat{\eta}_n, u_n - u \right\rangle = \left\langle \widehat{\zeta}_n, u_n \right\rangle - \left\langle \widehat{\zeta}_n, u \right\rangle - \langle \widehat{\eta}_n - \widehat{\eta}, u_n \rangle + \langle \widehat{\eta}_n, u \rangle - \langle \widehat{\eta}, u_n \rangle \longrightarrow 0, \tag{99}$$

as $n \longrightarrow \infty$. On the other hand, for a.e. $x \in \Omega$, $d_n(x) \coloneqq (\zeta_n(x) - \eta_n(x))(Du_n(x) - Du(x)) \geq 0$. Hence, $d_n \longrightarrow 0$ in $L^1(\Omega)$. By passing to a subsequence if necessary,

$$d_n(x) \longrightarrow 0 \text{ for a.e.} x \in \Omega, \quad \text{as } n \longrightarrow \infty. \tag{100}$$

Since $\Phi \in \Delta_2$ and $\Phi \preccurlyeq \Psi$, by using (13), (14), (94), (100), and the Young inequality, one has $\Phi(x, |Du_n(x)|) \leq \ell(x)$, for some function $\ell \in L^1(\Omega)$, for all $n \in \mathbb{N}$, for a.e. $x \in \Omega$. It implies that $\{Du_n(x)\}$ is bounded in $\mathbb{R}^N$, for a.e. $x \in \Omega$. Taking a subsequence if necessary, there exists $\vartheta \in \mathbb{R}^N$ such that $Du_n(x) \longrightarrow \vartheta$ as $n \longrightarrow \infty$, for a.e. $x \in \Omega$. Similar to Lemma 1.4 of [34], we can show that $\vartheta \in (L_\Phi(\Omega))^N$ and $Du_n \longrightarrow \vartheta$ for $\sigma((L_\Phi(\Omega))^N, (L_{\bar{\Phi}}(\Omega))^N)$, as $n \longrightarrow \infty$. Therefore, $\vartheta = Du$, and thus, $Du_n(x) \longrightarrow Du(x)$ as $n \longrightarrow \infty$, for a.e. $x \in \Omega$. Based on Lebesgue's Theorem, we can show that

$$\int_\Omega \Phi\left(x, \frac{|Du_n(x) - Du(x)|}{\varepsilon_0}\right) dx \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{101}$$

It is a contradiction with (89).

Denote $\widehat{f} = \widehat{\omega} - \widehat{\zeta}$. Combing (87) and (96),

$$\widehat{f}_n \rightharpoonup \widehat{f} \text{ weakly in } \left(W_0^1 L_\Phi(\Omega)\right)^*, \text{for } \sigma\left(\prod L_{\bar{\Phi}}, \prod E_\Phi\right), \quad \text{as } n \longrightarrow \infty. \tag{102}$$

Invoking Theorem 20, we obtain that $\widehat{f} \in \mathscr{F}(u)$. Thus, $\widehat{\omega} \in \mathscr{T}(u)$. In view of (88), and (87), we get

$$\langle \widehat{\omega}_n, u_n \rangle = \langle \widehat{\omega}_n, u_n - u \rangle + \langle \widehat{\omega}_n, u \rangle \longrightarrow \langle \widehat{\omega}, u \rangle, \quad \text{as } n \longrightarrow \infty. \tag{103}$$

Step 2. We will see that $\mathscr{T}$ is coercive. Let $u \in W_0^1 L_\Phi(\Omega)$ and $\widehat{\omega} \in \mathscr{T}(u)$. Then, there exist $\widehat{\zeta} \in \mathscr{A}(u)$ and $\widehat{f} \in \mathscr{F}(u)$ such that $\widehat{\omega} = \widehat{\zeta} + \widehat{f}$. From (A4), for any $\widehat{\zeta} \in \mathscr{A}(u)$, we have that

$$\left\langle \widehat{\zeta}, u \right\rangle \geq b_2 \int_\Omega \Phi(x, |Du(x)|) dx - \|a_2\|_{L^1(\Omega)}. \tag{104}$$

For any $\widehat{f} \in \mathscr{F}(u)$, by (51) and (53), we deduce that

$$\begin{aligned}\left\langle \widehat{f}, u \right\rangle &\leq \|\omega\|_{L^1(\Omega)} + b_4 \int_\Omega \Phi^-(|u(x)|) dx + b_5 \int_\Omega \Phi(x, |Du(x)|) dx \\ &\leq \|\omega\|_{L^1(\Omega)} + \frac{b_4}{\tilde{\lambda}} \int_\Omega \Phi^-(|Du(x)|) dx + b_5 \int_\Omega \Phi(x, |Du(x)|) dx,\end{aligned} \tag{105}$$

for all $u \in D_\hbar$.

Hence, we obtain that

$$\langle \widehat{\omega}, u \rangle \geq \left(b_2 - \frac{b_4}{\tilde{\lambda}} - b_5\right) \int_\Omega \Phi(x, |Du(x)|) dx - \|a_2\|_{L^1(\Omega)} - \|\omega\|_{L^1(\Omega)}, \tag{106}$$

for all $u \in D_\hbar$. Based on Proposition 1.9 of [17], there exists a constant $C^* > 0$ such that

$$\|u\|_{1,\Omega} \leq (1 + C^*) \|Du\|_{(\Phi)}. \tag{107}$$

Then, we get that

$$\begin{aligned}\frac{\langle \widehat{\omega}, u \rangle}{\|u\|_{1,\Omega}} \geq \left(b_2 - \frac{b_4}{\tilde{\lambda}} - b_5\right) \frac{\int_\Omega \Phi(x, |Du(x)|) dx}{(1 + C^*) \|Du\|_{(\Phi)}} \\ - \left(\|a_2\|_{L^1(\Omega)} + \|\omega\|_{L^1(\Omega)}\right) \frac{1}{\|u\|_{1,\Omega}},\end{aligned} \tag{108}$$

for all $u \in D_\hbar$.

Based on Proposition 3.1 of [18], there are functions $h \in L^1(\Omega)$ with $h \geq 0$ and $G : [k, +\infty) \longrightarrow \mathbb{R}$ for some $k > 2$ such that $G(s) \longrightarrow +\infty$ as $s \longrightarrow +\infty$ and

$$\Phi(x, su) \geq sG(s)\Phi(x, u) - sh(x), \quad \text{for all } s \geq k, u \geq 0, \text{a.e.} x \in \Omega. \tag{109}$$

It follows that

$$\begin{aligned}\int_\Omega \Phi(x, |Du(x)|) dx &\geq \left(\|Du\|_{(\Phi)} - \varepsilon\right) \left[G\left(\|Du\|_{(\Phi)} - \varepsilon\right) \int_\Omega \Phi \cdot \left(x, \frac{|Du(x)|}{\|Du\|_{(\Phi)} - \varepsilon}\right) dx - \|h\|_{L^1(\Omega)}\right] \\ &\geq \left(\|Du\|_{(\Phi)} - \varepsilon\right) \left[G\left(\|Du\|_{(\Phi)} - \varepsilon\right) - \|h\|_{L^1(\Omega)}\right],\end{aligned} \tag{110}$$

if $\|Du\|_{(\Phi)}$ is large enough, for any $\varepsilon \in (0, 1)$.

Combining (108) and (110), we can deduce that

$$\lim_{\|u\|_{1,\Omega} \longrightarrow +\infty, u \in W_0^1 L_\Phi(\Omega)} \frac{\inf \{\langle \widehat{\omega}, u \rangle : \widehat{\omega} \in \mathscr{T}(u)\}}{\|u\|_{1,\Omega}} = +\infty. \tag{111}$$

Step 3. According to Lemma 11, there exists $u \in W_0^1 L_\Phi(\Omega)$ such that

$$\langle \widehat{\omega}, u \rangle = \langle L, u \rangle, \tag{112}$$

for some $\widehat{\omega} \in \mathcal{T}(u)$, where $\langle L, u\rangle = \int_\Omega L(x)u(x)dx$. Therefore, there exist $\widehat{\zeta} \in \mathcal{A}(u)$ and $\widehat{f} \in \mathcal{F}(u)$ such that $\widehat{\omega} = \widehat{\zeta} + \widehat{f}$. This completes the proof. □

The set of solutions of problem (1) is denoted by $\mathcal{S}$. Then, we have the following results for set $\mathcal{S}$.

Theorem 22. *Assume (A1)–(A4) and (F1)–(F4) hold. Then, $\mathcal{S}$ is nonempty, bounded, and closed.*

Proof.

(1) According to Theorem 21, $\mathcal{S}$ is nonempty.

(2) We are going to prove that $\mathcal{S}$ is bounded. Assume that $\mathcal{S}$ is unbounded. Then, there exists a sequence $\{u_n\}$ in $\mathcal{S}$ such that $\|u_n\|_{1,\Omega} \longrightarrow +\infty$, as $n \longrightarrow \infty$. Since there exists a constant $C > 0$ such that $\langle L, u_n\rangle \le C\|L\|_{(W_0^1 L_\Phi(\Omega))^*}\|u_n\|_{1,\Omega}$, combining (108) and (110), we have that

$$\lim_{\|u_n\|_{1,\Omega}\longrightarrow+\infty, u_n\in W_0^1L_\Phi(\Omega)} \frac{\inf\{\langle\widehat{\omega}_n - L, u_n\rangle : \widehat{\omega}_n \in \mathcal{T}(u_n)\}}{\|u_n\|_{1,\Omega}} = +\infty. \quad (113)$$

However, it follows from (112) that $\langle \widehat{\omega}_n - L, u_n\rangle = 0$, where $\widehat{\omega}_n \in \mathcal{T}(u_n)$, for any $n \in \mathbb{N}$. It is a contradiction.

(3) We will show that $\mathcal{S}$ is closed. Let $\{u_n\}$ be a sequence in $\mathcal{S}$ such that

$$u_n \longrightarrow u_0 \text{ in } W_0^1L_\Phi(\Omega), \quad \text{as } n \longrightarrow \infty, \quad (114)$$

for some $u_0 \in W_0^1L_\Phi(\Omega)$. Then, there exist $\widehat{\zeta}_n \in \mathcal{A}(u_n)$ and $\widehat{f}_n \in \mathcal{F}(u_n)$ such that $\widehat{\omega}_n = \widehat{\zeta}_n + \widehat{f}_n \in \mathcal{T}(u_n)$ and

$$\langle \widehat{\omega}_n, u_n\rangle = \langle L, u_n\rangle, \quad \text{for any } n \in \mathbb{N}. \quad (115)$$

Based on (114), $\{u_n\}$ is bounded in $W_0^1L_\Phi(\Omega)$. In view of (13), and (F3), $\{\widehat{\zeta}_n\}$ and $\{\widehat{f}_n\}$ are bounded in $(L_{\bar{\Phi}}(\Omega))^N$ and $L_{\bar{\Psi}}(\Omega)$, respectively. Extracting a subsequence if necessary, there exist $\widehat{\zeta}_0, \widehat{f}_0 \in (W_0^1L_\Phi(\Omega))^*$ such that

$$\widehat{\zeta}_n \rightharpoonup \widehat{\zeta}_0 \text{ weakly in } \left(W_0^1L_\Phi(\Omega)\right)^* \quad \text{for } \sigma\left(\prod L_{\bar{\Phi}}, \prod E_\Phi\right), \quad (116)$$

$$\widehat{f}_n \rightharpoonup \widehat{f}_0 \text{ weakly in } \left(W_0^1L_\Phi(\Omega)\right)^* \quad \text{for } \sigma\left(\prod L_{\bar{\Phi}}, \prod E_\Phi\right), \quad (117)$$

as $n \longrightarrow \infty$. Using Lemma 13 (iii), we have $\widehat{\zeta}_0 \in \mathcal{A}(u_0)$. Based on Theorem 20 (ii), we get $\widehat{f}_0 \in \mathcal{F}(u_0)$.

Based on (114),

$$\begin{aligned} &\left\langle \widehat{\zeta}_n, u_n - u_0 \right\rangle \longrightarrow 0, \\ &\left\langle \widehat{f}_n, u_n - u_0 \right\rangle \longrightarrow 0, \end{aligned} \quad (118)$$

$$\langle L, u_n - u_0\rangle \longrightarrow 0, \quad (119)$$

as $n \longrightarrow \infty$.

Based on (115)–(119),

$$\langle \widehat{\omega}_n - L, u_n\rangle \longrightarrow \left\langle \widehat{\zeta}_0 + \widehat{f}_0 - L, u_0\right\rangle, \quad \text{as } n \longrightarrow \infty. \quad (120)$$

Denote $\widehat{\omega}_0 = \widehat{\zeta}_0 + \widehat{f}_0$. Then, $\widehat{\omega}_0 \in \mathcal{T}(u_0)$. Consequently, $\langle \widehat{\omega}_0, u_0\rangle = \langle L, u_0\rangle$. It implies that $u_0 \in \mathcal{S}$. □

3.4. Obstacle Problems. Now, we consider the following inclusion problem with an obstacle effect

$$\begin{cases} L \in A(x,u,Du) + F(x,u,Du), & \text{in } \Omega, \\ u(x) \le v(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (121)$$

where $v : \Omega \longrightarrow \mathbb{R}_+ \cup \{+\infty\}$ is a given function. Let $\mathcal{K} = \{u \in W_0^1L_\Phi(\Omega) : u \le v \text{ a.e.in } \Omega\}$. It is easy to see that K is a nonempty, closed, and convex subset of $W_0^1L_\Phi(\Omega)$ and contains 0. The indicator function $I_K : W_0^1L_\Phi(\Omega) \longrightarrow \mathbb{R} \cup \{+\infty\}$ of K is defined by

$$I_K(u) := \begin{cases} 0, & \text{if } u \in K, \\ +\infty, & \text{otherwise}. \end{cases} \quad (122)$$

Let ∂I_K be the subdifferential of I_K in the sense of convex analysis (see Definition 32.11 of [35]).

A function $u \in \mathcal{K}$ is called a (weak) solution of problem (121), if there exist $\zeta \in (L_{\bar{\Phi}}(\Omega))^N$ and $f \in L_{\bar{\Psi}}(\Omega)$ such that

$$\zeta(x) \in A(x,u(x),Du(x)), f(x) \in F(x,u(x),Du(x)), \quad \text{for a.e. } x \in \Omega, \quad (123)$$

$$\int_\Omega \zeta(x)Dv(x)dx + \int_\Omega f(x)v(x)dx = \int_\Omega L(x)v(x)dx, \quad \forall v \in \mathcal{K}. \quad (124)$$

From the definition above, it is not difficult to show that problem (121) is equivalent to the following variational inequality:

Find $u \in W_0^1 L_\Phi(\Omega)$ such that

$$\langle \mathscr{A}(u), v-u\rangle + \langle \mathscr{F}(u), v-u\rangle + I_K(v) - I_K(u) \ge \langle L, v-u\rangle, \quad \forall v \in W_0^1 L_\Phi(\Omega), \tag{125}$$

which is formulated as the following inclusion:

Find $u \in W_0^1 L_\Phi(\Omega)$ such that

$$L \in \mathscr{A}(u) + \mathscr{F}(u) + \partial I_K(u). \tag{126}$$

We have the following theorem.

Theorem 23. *Assume that (A1)–(A4) and (F1)–(F4) hold. Then, there exists a solution u of problem (121) such that $\|u\|_{1,\Omega} < R$ for some $R > 0$.*

Proof. Let $\mathscr{T}$ be defined by (85). Then, based on the proof of Theorem 21, $\mathscr{T}$ is pseudomonotone.

According to Example 32.15 of [35], ∂I_K is maximal monotone.

We are going to check that there exists a positive constant R such that

$$\langle \widehat{\omega} + \widehat{\eta} - L, u\rangle > 0, \tag{127}$$

for all $u \in W_0^1 L_\Phi(\Omega)$ with $\|u\|_{1,\Omega} = R$, for all $\widehat{\omega} \in \mathscr{T}(u)$ and all $\widehat{\eta} \in \partial I_K(u)$.

Let $u \in W_0^1 L_\Phi(\Omega)$ with $\|Du\|_{(\Phi)} > k$, where k is the constant in (109). Based on Remark 15, there exists $\hbar > 0$ such that $u \in D_\hbar$. From (106) and (110), we obtain that

$$\begin{aligned}\langle \widehat{\omega}, u\rangle \ge \left(b_2 - \frac{b_4}{\tilde{\lambda}} - b_5\right)\Big[\left(\|Du\|_{(\Phi)} - \varepsilon\right) G\left(\|Du\|_{(\Phi)} - \varepsilon\right) \\ - \|h\|_{L^1(\Omega)}\Big] - \|a_2\|_{L^1(\Omega)} - \|\omega\|_{L^1(\Omega)},\end{aligned} \tag{128}$$

for any $\varepsilon \in (0,1)$ and for all $\widehat{\omega} \in \mathscr{T}(u)$.

Similar to the proof of (2.50) in [36], there exists $w \in (W_0^1 L_\Phi(\Omega))^*$ and $\alpha_K \in \mathbb{R}$ such that

$$I_K(u) \ge -\tilde{C}\|w\|_{(W_0^1 L_\Phi(\Omega))^*}\|u\|_{1,\Omega} - \alpha_K, \quad \forall u \in W_0^1 L_\Phi(\Omega), \tag{129}$$

for some constant $\tilde{C} > 0$.

Combining (107), (128), and (129), and based on the arbitrariness of ε, we have

$$\begin{aligned}\langle \widehat{\omega} + \widehat{\eta} - L, u\rangle &\ge \left(b_2 - \frac{b_4}{\tilde{\lambda}} - b_5\right)\|Du\|_{(\Phi)} G\left(\|Du\|_{(\Phi)}\right) \\ &\quad - \left(b_2 - \frac{b_4}{\tilde{\lambda}} - b_5\right)\|h\|_{L^1(\Omega)} - \|a_2\|_{L^1(\Omega)} \\ &\quad - \|\omega\|_{L^1(\Omega)} - \tilde{C}\|w\|_{(W_0^1 L_\Phi(\Omega))^*}\|u\|_{1,\Omega} - \alpha_K \\ &\quad - C\|L\|_{(W_0^1 L_\Phi(\Omega))^*}\|u\|_{1,\Omega} \ge \|Du\|_{(\Phi)} \\ &\quad \cdot \Big[\left(b_2 - \frac{b_4}{\tilde{\lambda}} - b_5\right) G\left(\|Du\|_{(\Phi)}\right) \\ &\quad - (1+C^*)\left(\tilde{C}\|w\|_{(W_0^1 L_\Phi(\Omega))^*} + C\|L\|_{(W_0^1 L_\Phi(\Omega))^*}\right)\Big] \\ &\quad - \left(b_2 - \frac{b_4}{\tilde{\lambda}} - b_5\right)\|h\|_{L^1(\Omega)} - \|a_2\|_{L^1(\Omega)} \\ &\quad - \|\omega\|_{L^1(\Omega)} - \alpha_K,\end{aligned} \tag{130}$$

for all $\widehat{\omega} \in \mathscr{T}(u)$ and all $\widehat{\eta} \in \partial I_K(u)$, where C is a positive constant. It is easy to see that we can take $R_0 > k$ to be large enough such that

$$\begin{aligned}R_0 &\Big[\left(b_2 - \frac{b_4}{\tilde{\lambda}} - b_5\right) G(R_0) - (1+C^*) \\ &\cdot \left(\tilde{C}\|w\|_{(W_0^1 L_\Phi(\Omega))^*} + C\|L\|_{(W_0^1 L_\Phi(\Omega))^*}\right)\Big] \\ &- \left(b_2 - \frac{b_4}{\tilde{\lambda}} - b_5\right)\|h\|_{L^1(\Omega)} - \|a_2\|_{L^1(\Omega)} \\ &- \|\omega\|_{L^1(\Omega)} - \alpha_K > 0.\end{aligned} \tag{131}$$

Let $R \ge (1+C^*)R_0$. Then, (127) holds for all $u \in W_0^1 L_\Phi(\Omega)$ with $\|u\|_{1,\Omega} = R$, for all $\widehat{\omega} \in \mathscr{T}(u)$ and all $\widehat{\eta} \in \partial I_K(u)$. Using Theorem 2.2 of [14], inclusion (126) has a solution u such that $\|u\|_{1,\Omega} < R$. □

We denote the set of solutions of inclusion (126) based on $\mathscr{S}_K$. With the analogous proof of Theorem 22, we can deduce the following theorem.

Theorem 24. *Assume that (A1)–(A4) and (F1)–(F4) hold. Then, $\mathscr{S}_K$ is nonempty, bounded, and closed.*

Data Availability

No underlying data was collected or produced in this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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