Hindawi
Journal of Function Spaces
Volume 2023, Article ID 8662375, 12 pages
https://doi.org/10.1155/2023/8662375

Research Article

Existence of Mild Solutions for Nonlocal Evolution Equations with the Hilfer Derivatives

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Received 22 November 2022; Revised 18 January 2023; Accepted 11 March 2023; Published 20 April 2023

Academic Editor: Shrideh K. Q. Al-Omari

The existence of mild solutions for Hilfer fractional evolution equations with nonlocal conditions in a Banach space is investigated in this manuscript. No assumptions about the compactness of a function or the Lipschitz continuity of a nonlinear function are imposed on the nonlocal item and the nonlinear function, respectively. However, we assumed that the nonlocal item is continuous, the nonlinear term is continuous and satisfies some specified assumptions, and the associated semigroup is compact. Our theorems are proved by means of approximate techniques, semigroup methods, and fixed point theorem. These methods are useful for fixing the noncompactness of operators caused by some specified given assumptions on this paper. The results obtained here improve some known results. Finlay, two examples are presented for illustration of our main results.

1. Introduction

In 1659, G.A. de L'Hospital and G.W. Leibniz presented the concept of a fractional derivative for the first time. Some mathematicians have recently made significant developments in this topic (see, e.g., [1–4]). Agarwal et al. [5] presented the concept of the solution of a fractional differential equation with uncertainty. Two global existence conclusions for an initial value problem involving a class of fractional differential equations were obtained by Băleanu and Mustafa [6]. Recently, fractional calculus and its applications have been shown to be a valuable tool in a wide range of scientific fields. In fact, there are numerous applications in viscoelasticity, electrochemistry, electromagnetics, etc. (see, e.g., [7, 8]). Ma et al. [9] investigated fractional impulsive neutral stochastic differential equations via infinite delay to obtain existence of almost periodic solutions. Krasnoselskii's fixed point theorem, operator semigroup, and fractional calculus were used in obtaining the results.

In the past twenty years, Hilfer introduced a fractional derivative named a generalized Riemann-Liouville derivative (the Hilfer derivative for short), which is the interpolation of the Caputo derivative and the Riemann-Liouville (RL for short) derivative [10, 11].

Recently, the study of fractional differential equations with the Hilfer derivatives has received a considerable amount of attention. Furati et al. [12] proved the nonexistence of a global solution to the Cauchy problem with the Hilfer fractional derivative and a polynomial nonlinearity. In [13], Hilfer et al. constructed an operational calculus for the Mikusinski type and applied it to solve linear fractional differential equations with the Hilfer derivatives. In [14], Furati et al. obtained existence results for an IVP for a class of nonlinear Hilfer fractional differential equations. Abbas et al. [15] presented the existence data dependence and

Ulam stability of the solutions for some Hilfer fractional differential inclusion. In [16], Zhang investigated IVPs for two types of impulsive fractional differential equations with the Hilfer derivatives.

$$\begin{cases} {}_{\varrho_0}D_{\varrho}^{\varsigma,\kappa}\nu(\varrho) = f(\varrho, \nu(\varrho)), & \varrho \in (\varrho_0, b], \varrho \neq \varrho_\xi (\xi = 1, 2, \cdots, n), \\ {}^{H}{}_{\varrho_0}I_{\varrho}^{(1-\varsigma)(1-\kappa)}\nu(\varrho)\Big|_{\varrho=\varrho_\xi^+} - {}^{H}{}_{\varrho_0}I_{\varrho}^{(1-\varsigma)(1-\kappa)}\nu(\varrho)\Big|_{\varrho=\varrho_k^-} = J_\xi\left(\nu\left(\varrho_\xi^-\right)\right), & \xi = 1, 2, \cdots, n, \\ {}^{H}{}_{\varrho_0}I_{\varrho}^{(1-\varsigma)(1-\kappa)}\nu(\varrho)\Big|_{\varrho\longrightarrow\varrho_0^+} = \nu_0, \end{cases}$$
$$\begin{cases} {}_{\varrho_0}D_{\varrho}^{\varsigma,\kappa}\nu(\varrho) = f(\varrho, \nu(\varrho)), & \varrho \in (\varrho_0, b], \varrho \neq \varrho_\xi (\xi = 1, 2, \cdots, n), \\ {}^{H}{}_{\varrho_0}I_{\varrho}^{(1-\varsigma)(1-\kappa)}\nu(\varrho)\Big|_{\varrho=\varrho_\xi^+} = \Phi_\xi\left(\nu\left(\varrho_\xi^-\right)\right), & \xi = 1, 2, \cdots, n, \\ {}^{H}{}_{\varrho_0}I_{\varrho}^{(1-\varsigma)(1-\kappa)}\nu(\varrho)\Big|_{\varrho\longrightarrow\varrho_0^+} = \nu_0, \end{cases} \tag{1}$$

where ${}_{\varrho_0}D_{\varrho}^{\varsigma,\kappa}$ denotes the Hilfer derivative of order ς, $0 < \varsigma < 1$, and type κ, $0 \leq \kappa \leq 1$, and ${}^{H}{}_{\varrho_0}I_{\varrho}^{(1-\varsigma)(1-\kappa)}$ represents the Hadamard fractional integral of order $(1-\varsigma)(1-\kappa)$. The functions f and Φ are some appropriate continuous functions. Metpattarahiran et al. [17] established the existence of impulsive fractional differential equations with the Hilfer derivative using sectorial operators.

Nonlocal fractional differential equations have recently inspired many mathematicians and physicists [18]. On the other hand, Byszewski [19–21] investigated abstract nonlocal semilinear initial value problems for the first time. Physical models with nonlocal conditions have been considered in the literature due to their significance and applications, because they yield better results than those with initial conditions (see, e.g., [22–25]). In [26], Fan investigated the existence for nondensely defined evolution equations with nonlocal conditions; he assumed that the nonlocal item is Lipschitz continuous or continuous. The approximate solutions and fixed point theory techniques were used. In [27], Fan and Li used the semigroup theory to investigate the existence of nonlocal impulsive semilinear differential equations.

The concept of mild solutions has recently been developed with the implementation of fixed point theorems in the proof of the existence of solutions to differential equations in Banach spaces. The mild solution to a differential equation is the fixed point of an associated compact operator that is to be introduced on a Banach space [28]. To show the compactness of the operator, some theorems and assumptions are used in the proof.

Recently, fixed point theory has been shown to be an effective tool in nonlinear analysis. There is an extremely rich literature of using the fixed point theory together with some theories in investigating the existence of mild solutions to linear and nonlinear fractional evolution differential equations. The existence of mild solutions for a Hilfer fractional evolution system is investigated in [29]. Sivasankar and Udhayakumar [30] investigated the existence of neutral stochastic Volterra integrodifferential inclusions with the Hilfer derivative and with almost sectorial operators. Basic fractional calculus theory, Bohnenblust-Karlin's fixed point theorem, and stochastic analysis were utilized. Sivasankar et al. [31] utilized basic fractional calculus theory, semigroup method, and Mönch fixed point theorem with the measure of noncompactness to investigate the existence of fractional stochastic differential systems with nonlocal conditions and delay with almost sectorial operators involving the Hilfer derivatives. In [32], Bedi et al. investigated the controllability and stability of fractional evolution equations with the Hilfer derivatives and the results obtained by means of propagation family theory, noncompactness calculation methods, and fixed point theory. Ravichandran et al. obtained results on controllability and the existence of a mild solution of the Hilfer fractional derivative [33]. Zhou et al. [34] obtained results on mild solutions for nonlocal Cauchy problems of fractional evolution equations with the Riemann-Liouville fractional derivative. In [35], Wang and Zhang discussed the existence of solutions to a nonlocal IVP for differential equations involving the Hilfer derivative. Tuan et al. [36] established the existence and uniqueness of a mild solution to a time-fractional semilinear differential equation with a final nonlocal condition.

Gui et al. [37] studied the existence and Hyers-Ulam stability of the almost periodic solution to quadratic mean almost periodic nonlocal fractional differential equations with impulse and fractional Brownian motion; the main results were obtained by utilizing the semigroups of operator method, Mönch fixed point theorem, and Hyers-Ulam stability theory. Du et al. [38] proved the exact controllability results for nonlocal fractional differential inclusions with the Hilfer derivatives. In [39], Gu and Trujillo used the noncompact measure method and Arzela-Ascoli theorem to investigate the existence of a mild solution for the Hilfer fractional evolution equations. They obtained sufficient conditions to ensure the existence of the mild solution to the following system:

$$\begin{cases} D_{0+}^{\varsigma,\kappa}\nu(\varrho) = A\nu(\varrho) + f(\varrho, \nu(\varrho)), & \varrho \in \acute{\mathfrak{J}} = (0, b], \\ I_{0+}^{(1-\varsigma)(1-\kappa)}\nu(0) = \nu_0, \end{cases} \tag{2}$$

where $D_{0+}^{\varsigma,\kappa}$ notates the Hilfer fractional derivative of order $0 < \varsigma < 1$ type $0 \leq \kappa \leq 1$, $\nu(\cdot)$ takes value in a Banach space $\mathscr{V}$, A is considered to be an infinitesimal generator of a C_0-semigroup being strongly continuous and of uniformly bounded linear operator on Banach space $\mathscr{V}$, and for $\mathfrak{J} = [0, b]$, $f : \mathfrak{J} \times \mathscr{V} \longrightarrow \mathscr{V}$ is the given function satisfying some assumptions, $\nu_0 \in \mathscr{V}$.

In the following system [40], Yang and Wang utilized the basic properties of the Hilfer fractional calculus and fixed point methods to obtain existence and uniqueness of mild solutions for a class of the Hilfer evolution equations. The nonlocal item is considered to be continuous and compact; they verified two cases for the corresponding C_0-semigroup, compact and noncompact:

$$\begin{cases} D_{0+}^{\varsigma,\kappa}\nu(\varrho) = A\nu(\varrho) + f(\varrho, \nu(\varrho)), & \varrho \in \acute{\mathfrak{J}} = (0, b], \\ I_{0+}^{(1-\varsigma)(1-\kappa)}[\nu(0) - h(0, \nu(0))] - g(\nu) = \nu_0, \end{cases} \tag{3}$$

where the operator A is considered to be an infinitesimal generator, which generates an analytic semigroup of uniformly bounded linear operator on a Banach space.

In [41], Shu and Shi presented the correct form of mild solutions to a linear fractional impulsive evolution equation. Varun Bose and Udhayakumar [42] utilized the fractional calculus theory, the semigroup of operator method, and the Martelli fixed point theorem to investigate the existence of a mild solution fractional neutral integrodifferential inclusion with almost sectorial operator with the Hilfer derivative. Varun Bose et al. studied the approximate controllability of neutral Volterra integrodifferential inclusions via the Hilfer fractional derivative and with almost sectorial operators. The results were proven by making use of multivalued maps and the Leray-Schauder fixed point theorem [43].

Inspired by the above discussion, we extend the study of the existence of mild solutions for nonlocal Hilfer fractional evolution problems. In this paper, no assumptions about the compactness of a function or the Lipschitz continuity are imposed on the nonlocal item and the nonlinear term, respectively. However, we assume that the nonlocal item is a continuous function, the nonlinear term is continuous, and the associated semigroup is compact. Approximate technique, the theory of semigroup, and fixed point theorem are used to obtain our results. We can indicate that the Schauder fixed point is used here because it yields only the existence of a fixed point without its uniqueness and reliability on the compactness of operators and can be used on infinite Banach spaces. We study the following nonlocal Hilfer fractional evolution problem:

$$\begin{cases} D_{0+}^{\varsigma,\kappa}\nu(\varrho) = A\nu(\varrho) + f(\varrho, \nu(\varrho)), & \varrho \in \acute{\mathfrak{J}} = (0, b], \\ I_{0+}^{(1-\varsigma)(1-\kappa)}\nu(0) = \nu_0 - g(\nu), \end{cases} \quad (4)$$

where $D_{0+}^{\varsigma,\kappa}$ denotes the Hilfer fractional derivative of order ς, $0 < \varsigma < 1$, and type κ, $0 \le \kappa \le 1$, the state $\nu(\cdot)$ has a value in a Banach space $\mathscr{V}$ associated with a norm $\|\cdot\|$, and A is considered to be an infinitesimal generator of a strongly C_0 -semigroup $\{Q(\varrho)\}_{\varrho\ge 0}$ of uniformly bounded linear operator on Banach space $\mathscr{V}$; for $\mathfrak{J} = [0, b]$, the $f : \mathfrak{J} \times \mathscr{V} \longrightarrow \mathscr{V}$ is the given function that meets some assumptions which will be stated in Section 2, $\nu_0 \in \mathscr{V}$, and g is the nonlocal item that satisfies an assumption that will be specified later.

In this paper, we mainly use the approximate solutions, the semigroup of operator method, and the fixed point theory techniques to investigate the existence of mild solutions to the nonlocal evolution (4). These methods are useful for fixing the noncompactness of the operators. Therefore, we make no assumptions such as compactness or Lipschitz continuity on g. However, we only assume that g is regarded continuous. We first study the existence of mild solutions for the corresponding nonlocal approximate problem (5) of (4).

We now introduce the corresponding approximate problem of the nonlocal problem (4), as follows:

$$\begin{cases} D_{0+}^{\varsigma,\kappa}\nu(\varrho) = A\nu(\varrho) + f(\varrho, \nu(\varrho)), & \varrho \in \acute{\mathfrak{J}} = (0, b], \\ I_{0+}^{(1-\varsigma)(1-\kappa)}\nu(0) = Q\left(\frac{1}{n}\right)[\nu_0 - g(\nu)], & n \ge 1, \end{cases} \quad (5)$$

where $D_{0+}^{\varsigma,\kappa}$, ς, κ, A, f, and g are specified in (4) and $Q(\varrho)$ is a compact semigroup. We will prove some theorems and lemmas utilizing some specified assumptions, the Schauder fixed point theorem, and the Arzela-Ascoli theorem to get the fixed points of (5). Some sufficient conditions will be needed for proving the relatively compactness of mild solutions. Consequently, by the approximate approach, (4) has at least one mild solution on $\acute{\mathfrak{J}}$.

The structure of this paper is arranged as follows. In Section 2, some preliminaries such as notations, definitions, remarks, and lemmas are recalled. Section 3 is reserved for our main discussion about the existence of mild solutions for the nonlocal evolution (4). In Section 4, we give two examples to illustrate our results. Some conclusions are given in Section 5.

2. Preliminaries

In this section, firstly, the notations that we shall utilize in this manuscript will be introduced. Following that, basic definitions such as the RL-fractional integral, RL-fractional derivative, Caputo fractional derivative, and Hilfer fractional derivative will be presented. In the end, essential theorems supporting the paper will be given.

Let $\mathscr{V}$ be a Banach Space. We denote by $C(\mathfrak{J}\,;\mathscr{V})$ as the space of all continuous functions from $\mathfrak{J}$ to $\mathscr{V}$ and $C(\acute{\mathfrak{J}}\,;\mathscr{V})$ as the space of all continuous functions from $\acute{\mathfrak{J}}$ to $\mathscr{V}$. Let $\mu = \kappa + \varsigma - \kappa\varsigma$, where $0 < \varsigma < 1$ and $0 \le \kappa \le 1$, and then, $1 - \mu = (1 - \varsigma)(1 - \kappa)$. We define the space $C_{1-\mu}(\mathfrak{J}\,;\mathscr{V})$ and the set $\mathfrak{B}'_r$ as follows: $C_{1-\mu}(\mathfrak{J}\,;\mathscr{V}) = \{\nu : \varrho^{1-\mu}\nu(\varrho) \in C(\mathfrak{J}\,;\mathscr{V})\}$ associated with the norm $\|\cdot\|_\mu$, which is given by $\|\nu\|_\mu = \sup\{\varrho^{1-\mu}\|\nu(\varrho)\|,\ \varrho \in \acute{\mathfrak{J}}$, where $\varrho^{1-\mu}\nu(\varrho)|_{\varrho=0} = \lim_{\varrho\longrightarrow 0^+}\varrho^{1-\mu}\nu(\varrho)$; one can easily see that $C_{1-\mu}(\mathfrak{J}\,;\mathscr{V})$ is a Banach space. The subset $\acute{\mathfrak{B}}_r = \{\nu \in C_{1-\mu}(\mathfrak{J}\,;\mathscr{V}), \|\nu\|_\mu \le r\}$ of the space $C_{1-\mu}(\mathfrak{J}\,;\mathscr{V})$ is bounded, closed, and convex.

We now present the following definitions, which are concerned with fractional derivative and fractional integral.

Definition 1 (see [1]). The RL-fractional integral via lower limit a and of the order ϱ for a function $f : [a,+\infty) \longrightarrow \mathscr{V}$ regarding that the right side is pointwise defined on $[a, +\infty)$ is given by

$$I_{a^+}^{\rho}f(\varrho) = \frac{1}{\Gamma(\rho)}\int_a^{\varrho}(\varrho - s)^{\rho-1}f(s)ds, \quad \varrho > a, \rho > 0. \quad (6)$$

Definition 2 (see [4]). The RL-fractional derivative of order $m - 1 < \rho < m$, $m \in \mathbb{Z}^+$ for a function $f : [a,+\infty) \longrightarrow \mathscr{V}$ is given by

$$^{RL}D_{a^+}^{\rho}f(\varrho) = \frac{1}{\Gamma(m-\rho)}\frac{d^m}{dt^m}\int_a^{\varrho}(\varrho - s)^{m-\varrho-1}f(s)ds, \quad \varrho > a, m - 1 \le \rho < m. \quad (7)$$

Definition 3 (see [40]). The Caputo fractional derivative of order $m - 1 < \rho < m$, $m \in \mathbb{Z}^+$ for a function $f : [a,+\infty) \longrightarrow \mathscr{V}$ is introduce by

$$^{C}D_{a^+}^{\rho}f(\varrho) = \frac{1}{\Gamma(\rho)}\int_a^{\varrho}(\varrho - s)^{\rho-1}f^{(m)}(s)ds, \quad \varrho > a, m - 1 < \rho < m. \quad (8)$$

Using the above definitions, we can introduce the Hilfer fractional derivative. Differentiating fractional integrals and interpolating the Caputo fractional derivative and the RL-fractional derivative lead to the following definition.

Definition 4 (see [10]). The left-sided Hilfer fractional derivative of order $0 < \rho < 1$ type $0 \le \gamma \le 1$ for a function $f(\varrho)$ is defined as

$$D_{a^+}^{\rho,\gamma} f(\varrho) = \left(I_{a^+}^{\gamma(1-\rho)} D\left(I_{a^+}^{(1-\gamma)(1-\rho)} f\right)\right)(\varrho), \tag{9}$$

where $D := \mathrm{d}/\mathrm{d}\varrho$.

Theorem 5 (Arzela-Ascoli's theorem) [2]. *Let H be a set of functions in Banach space $C([a,b];\mathscr{V})$ satisfying the following:*

(i) For any $\tau \in [a,b]$, $\{h(\tau): h \in H\}$ is relatively compact in $\mathscr{V}$

(ii) For any $\varepsilon > 0$, there exists a $\delta > 0$, such that $\|h(\tau) - h(\tau')\| < \varepsilon$, for all $h \in H$, and for any $\tau, \tau' \in [a,b]$ implies $|\tau - \tau'| < \delta$. Thus, H is equicontinuous on $[a,b]$

As a result, H is relatively compact.

Theorem 6 (equicontinuous sets of linear mappings) [44]. *Let $\mathscr{V}$ and $\mathscr{U}$ be the two topological vector spaces. A set F of linear maps of $\mathscr{V}$ into $\mathscr{U}$ is said to be equicontinuous if, to every neighborhood of zero R in $\mathscr{U}$, there is a neighborhood of zero Q in $\mathscr{V}$, such that, for all mappings $q \in F$.*

$$t \in Q \text{ implies } q(t) \in R. \tag{10}$$

Theorem 7 (Schauder fixed point theorem) [45]. *If H is is a closed, bounded, and convex subset of a Banach space $\mathscr{V}$, and $F : H \longrightarrow H$ is completely continuous. Then, F has a fixed point in H.*

Theorem 5 and Theorem 7 are essential in proving of the main results of this manuscript.

3. Main Results

This section is mainly focusing on investigating the main results of the paper. To begin, some assumptions and remarks will be presented. The definition of the mild solution of the nonlocal problem (4) will be introduced. However, to obtain such a result, we will present some theorems and lemmas. Fixed point theorem and semigroups method as well as the technique of approximation are involved in the proof of our theorems. Finally, we will give some sufficient conditions to guarantee the existence of mild solutions of (4).

We consider the assumptions below, which are very useful for the main discussion:

(HQ) The operator $Q(\varrho)$ is a C_0-semigroup generated by the infinitesimal operator A. $Q(\varrho)$ is compact for $\varrho > 0$, and $(Q(\varrho))_{\varrho \ge 0}$ is uniformly bounded; that is, there exists $\tilde{M} > 1$ such that $\sup_{\varrho \in [0,+\infty)} \|Q(\varrho)\| < \tilde{M}$.

(HF) For any $\varrho \in \mathfrak{J}$, $f(\varrho,)$: $\mathscr{V} \longrightarrow \mathscr{V}$ is a continuous function, and for each $\nu \in \mathscr{V}$, $f(,\nu)$: $\mathfrak{J}' \longrightarrow \mathscr{V}$ is a strongly measurable function. Besides that, for $p > 1$, there exists a function $\psi \in L^p(\mathfrak{J};\mathbb{R}^+)$ and a positive constant l such that

$$\|f(\varrho,\nu(\varrho))\| \le \psi(\varrho) + l\varrho^{1-\mu}\|\nu(\varrho)\|, \tag{11}$$

for each $\varrho \in \mathfrak{J}$ and any $\nu \in C_{1-\mu}(\mathfrak{J};\mathscr{V})$.

We also assume

$$I_{0^+}^{\varsigma}\psi \in C_{1-\mu}(\mathfrak{J};\mathbb{R}^+), \quad \lim_{\varrho \longrightarrow 0+} I_{0^+}^{\varsigma}\varrho^{(1-\kappa)(1-\varsigma)}\psi(\varrho) = 0. \tag{12}$$

(Hg) $g : C_{1-\mu}(\mathfrak{J};\mathscr{V}) \longrightarrow \mathscr{V}$ is a continuous function that maps $\mathfrak{B}'_r$ into a bounded set, and there is a $\lambda = \lambda(r) \in (0,b)$ in a manner that $g(\nu) = g(\nu')$ for any $\nu, \nu' \in \mathfrak{B}'_r$ with $\nu(\tau) = \nu'(\tau)$, $\tau \in [\delta, b]$.

Before we propose the definition of mild solution to (4), we first state the following lemmas.

Lemma 8. *The nonlocal problem (4) is equivalent to the following integral equation:*

$$\begin{aligned}\nu(\varrho) = {} & \frac{[\nu_0 - g(\nu)]}{\Gamma(\kappa(1-\varsigma)+\varsigma)}\varrho^{\mu-1} + \frac{1}{\Gamma(\varsigma)}\int_0^{\varrho}(\varrho - s)^{\varsigma-1} \\ & \cdot [A\nu(s) + f(s,\nu(s))]ds, \quad \varrho \in \mathfrak{J}'.\end{aligned} \tag{13}$$

Proof. To avoid the repetition, we omit the proof because it is contained in [14]. □

Lemma 9. *If the given integral equation (13) holds, then we obtain*

$$\nu(\varrho) = S_{\varsigma,\kappa}(\varrho)[\nu_0 - g(\nu)] + \int_0^{\varrho} T_{\varsigma}(\varrho - s)f(s,\nu(s))ds, \quad \varrho \in \mathfrak{J}', \tag{14}$$

where $T_{\varsigma}(\varrho) = \varrho^{\varsigma-1}P_{\varsigma}(\varrho)$, $P_{\varsigma}(\varrho) = \int_0^{\infty}\varsigma\theta M_{\varsigma}(\theta)Q(\varrho^{\varsigma}\theta)d\theta$, and $S_{\varsigma,\kappa}(\varrho) = I_{0+}^{\kappa(1-\varsigma)}T_{\varsigma}(\varrho)$.

The function $M_{\varsigma}(\theta) = \sum_{n=1}^{\infty}(-\theta)^{n-1}/(n-1)\Gamma(1-\rho n)$, $0 < \rho < 1$, $\theta \in C$, is the wright function, which satisfies the following equality: $\int_0^{\infty}\theta^{\iota}M_{\varsigma}(\theta)d\theta = \Gamma(1+\iota)/\Gamma(1+\varsigma\iota)$ for $\iota \ge 0$, $\theta \ge 0$.

Proof. The proof can be given in a similar way to that in Lemma 2.11 in [39]. So we omit it. □

We now introduce mild solutions of the nonlocal problem (4) by the following definition.

Definition 10. If a function $\nu \in C_{1-\mu}(\mathfrak{J};\mathscr{V})$ satisfies

$$\nu(\varrho) = S_{\varsigma,\kappa}(\varrho)[\nu_0 - g(\nu)] + \int_0^\varrho T_\varsigma(\varrho - s) f(s, \nu(s)) ds, \varrho \in \mathfrak{J}', \tag{15}$$

where $T_\varsigma(\varrho)$, $P_\varsigma(\varrho)$, and $S_{\varsigma,\kappa}(\varrho)$ are notated in Lemma 9; then, it is said to be a mild solution of (4).

Next, we recall some remarks from some cited papers and present some lemmas to be utilized to prove Theorem 15 and the associated theorems and lemmas.

Remark 11 (see [40]). Under assumption (HQ), one can see that $P_\varsigma(\varrho)$ is a continuous mapping in the uniform operator for $\varrho > 0$.

Remark 12 (see [39]). By the assumption (HQ), we get $\|T_\varsigma(\varrho)\nu\| \le \tilde{M}\varrho^{\varsigma-1}/\Gamma(\varsigma)\|\nu\|$, $\|S_{\varsigma,\kappa}(\varrho)\nu\| \le \tilde{M}\varrho^{(\kappa-1)(1-\varsigma)}/\Gamma(\kappa(1-\varsigma)+\varsigma)\|\nu\|$. In addition, $T_\varsigma(\varrho)$ and $S_{\varsigma,\kappa}(\varrho)$ are linear for any $\nu \in \mathscr{V}$ and any fixed $\varrho > 0$.

Remark 13 (see [39]). By Remark 12 and (Hf), we have

$$\lim_{t\longrightarrow 0+} \varrho^{(1-\kappa)(1-\varsigma)} S_{\varsigma,\kappa}(\varrho)\nu_0 = \frac{\nu_0}{\Gamma(\kappa(1-\varsigma)+\varsigma)}. \tag{16}$$

Remark 14 (see [40]). Under assumption (HQ), we have $\{T_\varsigma(\varrho)\}_{\rho>0}$ and $\{S_{\varsigma,\kappa}(\varrho)\}_{\varrho>0}$ that are continuous in the uniform operator. That means, for any $\nu \in \mathscr{V}$ and $0 < \varrho < s \le b$, we have

$$\begin{aligned} \|T_\varsigma(\varrho)\nu - T_\varsigma(s)\nu\| &\longrightarrow 0, \\ \|S_{\varsigma,\kappa}(\varrho)\nu - S_{\varsigma,\kappa}(s)\nu\| &\longrightarrow 0, \end{aligned} \tag{17}$$

as $\varrho \longrightarrow s$.

Theorem 15. *Suppose that (HQ)-(Hg) are fulfilled. The nonlocal Hilfer evolution problem (4) thus has at least one mild solution on $\mathfrak{J}' = (0, b]$ given that*

$$\left[\frac{\tilde{M}^2}{\Gamma(\kappa(1-\varsigma)+\varsigma)} \sup_{\nu\in\mathfrak{B}'_r}\|\nu_0 - g(\nu)\| + \frac{\tilde{M}}{\Gamma(\varsigma)} \frac{b^{\kappa(\varsigma-1)+1/q}}{[q(\varsigma-1)+1]^{1/q}} \times \|\psi\|_p + \frac{\tilde{M}lr}{\Gamma(\varsigma+1)} b^{1+\kappa(\varsigma-1)}\right] \le r, \tag{18}$$

where $p > 1$, $q > 1$, $(1/p) + (1/q) = 1$, and $q(\varsigma - 1) + 1 > 0$.

Now, we use the approximate technique to prove Theorem 15. We consider the following corresponding approximate problem

$$\begin{cases} D_{0+}^{\varsigma,\kappa}\nu(\varrho) = A\nu(\varrho) + f(\varrho, \nu(t)), & \varrho \in \mathfrak{J}' = (0, b], \\ I_{0+}^{(1-\varsigma)(1-\kappa)}\nu(0) = Q\left(\frac{1}{n}\right)[\nu_0 - g(\nu)], \end{cases} \tag{19}$$

where $n \ge 1$ and $D_{0+}^{\varsigma,\kappa}$, ς, and κ are notated the same as of system (4).

Lemma 16. *If all the assumptions of Theorem 15 are fulfilled, the nonlocal problem (19) then has at least one mild solution $\nu_n \in C_{1-\mu}(\mathfrak{J};\mathscr{V})$, where $n \ge 1$.*

Proof. Let us introduce an operator $G_n : C_{1-\mu}(\mathfrak{J};\mathscr{V}) \longrightarrow C_{1-\mu}(\mathfrak{J};\mathscr{V})$, $n \ge 1$, as follows:

$$\begin{aligned} (G_n\nu)(\varrho) = {} & S_{\varsigma,\kappa}(\varrho)Q\left(\frac{1}{n}\right)[\nu_0 - g(\nu)] \\ & + \int_0^\varrho T_\varsigma(\varrho - s) f(s, \nu(s)) \mathrm{d}s, \quad \varrho \in \mathfrak{J}', \end{aligned} \tag{20}$$

$$(G_n\nu)(\varrho) = (G_{n1}\nu)(\varrho) + (G_{n2}\nu)(\varrho), \tag{21}$$

where

$$\begin{aligned} (G_{n1}\nu)(\varrho) &= S_{\varsigma,\kappa}(\varrho)Q\left(\frac{1}{n}\right)[\nu_0 - g(\nu)], \quad \varrho \in \mathfrak{J}', \\ (G_{n2}\nu)(\varrho) &= \int_0^\varrho T_\varsigma(\varrho - s) f(s, \nu(s)) ds, \quad \varrho \in \mathfrak{J}'. \end{aligned} \tag{22}$$

G_n is the mild solution of system (19). □

Next, we will show that G_n has a fixed point by utilizing Theorem 7. The proof contains the subsequent steps below.

Step 1. We prove that the operator G_n maps the set $\mathfrak{B}'_r$ into itself. Recalling the conditions (HQ)-(Hg) and Hölder inequality, for any $\nu \in \mathfrak{B}'_r$, we have

$$\begin{aligned} \|G_n\nu\|_\mu &= \sup_{\varrho\in\mathfrak{J}'} \rho^{1-\mu}\left\| S_{\varsigma,\kappa}(t)Q\left(\frac{1}{n}\right)[\nu_0 - g(\nu)] + \int_0^\varrho T_\varsigma(\varrho - s) f(s, \nu(s)) ds \right\| \\ &\le \frac{\tilde{M}^2}{\Gamma(\kappa(1-\varsigma)+\varsigma)} \sup_{\nu\in\mathfrak{B}'_r}\|\nu_0 - g(\nu)\| + \sup_{\varrho\in\mathfrak{J}'}\varrho^{1-\mu} \\ &\quad \cdot \left\| \int_0^\varrho (\varrho - s)^{\varsigma-1} P_\varsigma(\varrho - s) f(s, \nu(s)) ds \right\| \\ &\le \frac{\tilde{M}^2}{\Gamma(\kappa(1-\varsigma)+\varsigma)} \sup_{\nu\in\mathfrak{B}'_r}\|\nu_0 - g(\nu)\| + \frac{\tilde{M}}{\Gamma(\varsigma)} \sup_{\rho\in\mathfrak{J}'}\varrho^{1-\mu} \\ &\quad \cdot \left\| \int_0^\varrho (\varrho - s)^{\varsigma-1} f(s, \nu(s)) ds \right\| \end{aligned}$$

$$\begin{aligned}
&\le \frac{\tilde{M}^2}{\Gamma(\kappa(1-\varsigma)+\varsigma)} \sup_{\nu\in\mathfrak{B}_r'} \|\nu_0 - g(\nu)\| \\
&\quad + \frac{\tilde{M}}{\Gamma(\varsigma)} \sup_{\rho\in\mathfrak{J}'} \varrho^{1-\mu} \int_0^{\varrho} (\varrho-s)^{\varsigma-1} \left(\psi(s) + l s^{1-\mu}\|\nu(s)\|\right) ds \\
&\le \frac{\tilde{M}^2}{\Gamma(\kappa(1-\varsigma)+\varsigma)} \sup_{\nu\in\mathfrak{B}_r'} \|\nu_0 - g(\nu)\| + \frac{\tilde{M}}{\Gamma(\varsigma)} \sup_{\rho\in\mathfrak{J}'} \rho^{1-\mu} \\
&\quad \cdot \left[\int_0^{\varrho} (\varrho-s)^{\varsigma-1}\psi(s) ds + l\int_0^{\varrho} (\varrho-s)^{\varsigma-1} s^{1-\mu}\|\nu(s)\| ds\right] \\
&\le \frac{\tilde{M}^2}{\Gamma(\kappa(1-\varsigma)+\varsigma)} \sup_{\nu\in\mathfrak{B}_r'} \|\nu_0 - g(\nu)\| + \frac{\tilde{M}}{\Gamma(\varsigma)} \sup_{\rho\in\mathfrak{J}'} t^{1-\mu} \\
&\quad \cdot \left(\int_0^{\varrho} (\varrho-s)^{q(\varsigma-1)} ds\right)^{1/q} \|\psi\|_p + \frac{\tilde{M}l}{\Gamma(\varsigma)} b^{1-\mu} \\
&\quad \cdot \int_0^b (\varrho-s)^{\varsigma-1} \sup_{s\in\mathfrak{J}'} \left(s^{1-\mu}\|\nu(s)\|\right) ds \\
&\le \left[\frac{\tilde{M}^2}{\Gamma(\kappa(1-\varsigma)+\varsigma)} \sup_{\nu\in\mathfrak{B}_r'} \|\nu_0 - g(\nu)\| + \frac{\tilde{M}}{\Gamma(\varsigma)}\right. \\
&\quad \left. \cdot \frac{b^{\kappa(\varsigma-1)+1/q}}{[q(\varsigma-1)+1]^{1/q}} \|\psi\|_p + \frac{\tilde{M}lr}{\Gamma(\varsigma+1)} b^{1+\kappa(\varsigma-1)}\right] \le r.
\end{aligned} \tag{23}$$

Hence, $\|G_n\nu\|_\mu \le r$ for any $\nu \in \mathfrak{B}_r' \subseteq C_{1-\mu}(\mathfrak{J};\mathscr{V})$. Therefore, G_n is a bounded operator in $C_{1-\mu}(\mathfrak{J};\mathscr{V})$. Consequently, the operator G_n is a self-mapping, that is, $G_n : {\mathfrak{B}'}_r \longrightarrow {\mathfrak{B}'}_r$.

Next, we verify the continuity of the operator G_n on ${\mathfrak{B}'}_r$.

Step 2. The mapping G_n is continuous on ${\mathfrak{B}'}_r$.

$$\begin{aligned}
&\|G_n\nu_1 - G_n\nu_2\|_\mu \\
&\quad \le \sup_{\varrho\in\mathfrak{J}'} \varrho^{1-\mu} \left[\left\| S_{\varsigma,\kappa}(\varrho) Q\left(\frac{1}{n}\right)[\nu_0 - g(\nu_1)] - S_{\varsigma,\kappa}(\varrho) Q\left(\frac{1}{n}\right)\right.\right. \\
&\qquad \left. \times [\nu_0 - g(\nu_2)]\right\| + \left\| \int_0^{\varrho} T_\varsigma(\varrho-s) f(s,\nu_1(s)) ds \right. \\
&\qquad \left.\left. - \int_0^{\varrho} T_\varsigma(\varrho-s) f(s,\nu_2(s)) ds \right\|\right] \\
&\quad \le \sup_{\varrho\in\mathfrak{J}'} \varrho^{1-\mu} \left\| S_{\varsigma,\kappa}(\varrho) Q\left(\frac{1}{n}\right)[g(\nu_2) - g(\nu_1)]\right\| + \sup_{\varrho\in\mathfrak{J}'} \varrho^{1-\mu} \\
&\qquad \times \left\| \int_0^{\varrho} T_\varsigma(\varrho-s)[f(s,\nu_1(s)) - f(s,\nu_2(s))] ds \right\| \\
&\quad \le \frac{\tilde{M}^2}{\Gamma(\kappa(1-\varsigma)+\varsigma)} \sup_{\nu\in\mathfrak{B}_r'} \|g(\nu_2) - g(\nu_1)\| + \frac{\tilde{M}}{\Gamma(\varsigma)} \sup_{\varrho\in\mathfrak{J}'} \varrho^{1-\mu} \\
&\qquad \times \int_0^{\varrho} (\varrho-s)^{\varsigma-1} \|f(s,\nu_1(s)) - f(s,\nu_2(s))\| ds \\
&\quad \le \frac{\tilde{M}^2}{\Gamma(\kappa(1-\varsigma)+\varsigma)} \sup_{\nu\in\mathfrak{B}_r'} \|g(\nu_2) - g(\nu_1)\| + \frac{\tilde{M}}{\Gamma(\varsigma)} \frac{b^{\kappa(\varsigma-1)+1/q}}{[q(\varsigma-1)+1]^{1/q}} \\
&\qquad \times \left(\int_0^b \|f(s,\nu_1(s)) - f(s,\nu_2(s))\|^p ds\right)^{1/p} \longrightarrow 0 \text{ as } \nu_1 \longrightarrow \nu_2 \text{ in } {\mathfrak{B}'}_r,
\end{aligned} \tag{24}$$

because g is continuous according to (Hg). Then, the first term of the last inequality is continuous. By (Hf) and Lebesgue dominated theorem, one can easily verify that the second term is continuous. As a result, G_n is continuous on ${\mathfrak{B}'}_r$.

Step 3. $G_n{\mathfrak{B}'}_r$ is equicontinuous. The following two subsections are devoted to show that $G_n{\mathfrak{B}'}_r$ is equicontinuous.

We prove that $G_{n1}{\mathfrak{B}'}_r$ is equicontinuous. For any $\nu \in {\mathfrak{B}'}_r$, $\varrho_1 = 0$, $0 < \varrho_2 \le b$, by Remark 13, we have

$$\begin{aligned}
&\left\| \varrho_2^{1-\mu}(G_{n1}\nu)(\varrho_2) - \varrho_1^{1-\mu}(G_{n1}\nu)(\varrho_1)\Big|_{\varrho_1=0} \right\| \\
&\quad \le \left\| \varrho_2^{1-\mu} S_{\varsigma,\kappa}(\varrho_2) Q\left(\frac{1}{n}\right)[\nu_0 - g(\nu)] \right. \\
&\qquad \left. - \frac{1}{\Gamma(\kappa(1-\varsigma))} Q\left(\frac{1}{n}\right)[\nu_0 - g(\nu)] \right\| \longrightarrow 0,
\end{aligned} \tag{25}$$

since $Q(1/n)$ is compact.

And for any $\nu \in {\mathfrak{B}'}_r$, $0 < \varrho_1 < \varrho_2 \le b$, we have

$$\begin{aligned}
&\left\| \varrho_2^{1-\mu}(G_{n1}\nu)(\varrho_2) - \varrho_1^{1-\mu}(G_{n1}\nu)(\varrho_1) \right\| \\
&\quad \le \left\| \varrho_2^{1-\mu} S_{\varsigma,\kappa}(\varrho_2) Q\left(\frac{1}{n}\right)[\nu_0 - g(\nu)] \right. \\
&\qquad \left. - \varrho_1^{1-\mu} S_{\varsigma,\kappa}(\varrho_1) Q\left(\frac{1}{n}\right)[\nu_0 - g(\nu)] \right\| \\
&\quad \le \tilde{M} \left\| \varrho_2^{1-\mu} S_{\varsigma,\kappa}(\varrho_2) - \varrho_1^{1-\mu} S_{\varsigma,\kappa}(\varrho_1) \right\| \|\nu_0 - g(\nu)\| \longrightarrow 0,
\end{aligned} \tag{26}$$

as $\varrho_2 \longrightarrow \varrho_1$, since $\varrho^{1-\mu} S_{\varsigma,\kappa}(\varrho)$ is uniformly continuous in uniform topology by Remark 14.

Therefore, $G_{n1}{\mathfrak{B}'}_r$ is equicontinuous.

We prove that $G_{n2}{\mathfrak{B}'}_r$ is equicontinuous. For any $\nu \in {\mathfrak{B}'}_r$, $\varrho_1 = 0$, $0 < \varrho_2 \le b$, we get

$$\begin{aligned}
&\left\| \varrho_2^{1-\mu}(G_{n2}\nu)(\varrho_2) - \varrho_1^{1-\mu}(G_{n2}\nu)(\varrho_1)\Big|_{\varrho_1=0} \right\| \\
&\quad \le \left\| \varrho_2^{1-\mu} \int_0^{\varrho_2} T_\varsigma(\varrho_2-s) f(s,\nu(s)) ds \right\| \\
&\quad \le \frac{\tilde{M}}{\Gamma(\varsigma)} \frac{\varrho_2^{\kappa(\varsigma-1)+1/q}}{[q(\varsigma-1)+1]^{1/q}} \|\psi\|_p + \frac{\tilde{M}lr}{\Gamma(\varsigma+1)} \varrho_2^{1+\kappa(\varsigma-1)} r \\
&\quad \longrightarrow 0, \varrho_2 \longrightarrow 0.
\end{aligned} \tag{27}$$

For any $\nu \in {\mathfrak{B}'}_r$, $0 < \varrho_1 < \varrho_2 \le b$, we have

$$\begin{aligned}
&\left\| \varrho_2^{1-\mu}(G_{n2}\nu)(\varrho_2) - \varrho_1^{1-\mu}(G_{n2}\nu)(\varrho_1) \right\| \\
&\quad \le \left\| \varrho_2^{1-\mu} \int_0^{\varrho_2} T_\varsigma(\varrho_2-s) f(s,\nu(s)) ds \right. \\
&\qquad \left. - \varrho_1^{1-\mu} \int_0^{\varrho_1} T_\varsigma(\varrho_1-s) f(s,\nu(s)) ds \right\| \\
&\quad \le \left\| \varrho_2^{1-\mu} \int_{\varrho_1}^{\varrho_2} (\varrho_2-s)^{\varsigma-1} P_\varsigma(\varrho_2-s) f(s,\nu(s)) ds \right\|
\end{aligned}$$

$$+\left\|\int_0^{\varrho_1}\left(\varrho_2^{1-\mu}(\varrho_2-s)^{\varsigma-1}-\varrho_1^{1-\mu}(\varrho_1-s)^{\varsigma-1}\right)\times P_\varsigma(\varrho_2-s)f(s,\nu(s))ds\right\|+\left\|\int_0^{\varrho_1}\varrho_1^{1-\mu}(\varrho_1-s)^{\varsigma-1}\times\left(P_\varsigma(\varrho_2-s)-P_\varsigma(\varrho_1-s)\right)f(s,\nu(s))ds\right\|$$
$$\le\varrho_2^{1-\mu}\frac{\tilde{M}}{\Gamma(\varsigma)}\frac{(\varrho_2-\varrho_1)^{(\varsigma-1)+1/q}}{[q(\varsigma-1)+1]^{1/q}}\|\psi\|_p+\varrho_2^{1-\mu}\times\frac{\tilde{M}lr}{\Gamma(\varsigma+1)}(\varrho_2-\varrho_1)^{\varsigma}+\frac{\tilde{M}}{\Gamma(\varsigma)}\times\left[\int_0^{\varrho_1}\left(\varrho_2^{1-\mu}(\varrho_2-s)^{\varsigma-1}-\varrho_1^{1-\mu}(\varrho_1-s)^{\varsigma-1}\right)^q ds\right]^{1/q}\times\|\psi\|_p+\frac{\tilde{M}lr}{\Gamma(\varsigma)}\int_0^{\varrho_1}\left(\varrho_2^{1-\mu}(\varrho_2-s)^{\varsigma-1}-\varrho_1^{1-\mu}(\varrho_1-s)^{\varsigma-1}\right)\mathrm{d}s+\left\|\int_0^{\varrho_1-\tau}\varrho_1^{1-\mu}(\varrho_1-s)^{\varsigma-1}\left(P_\varsigma(\varrho_2-s)-P_\varsigma(\varrho_1-s)\right)f(s,\nu(s))\mathrm{d}s\right\|+\left\|\int_{\varrho_1-\tau}^{\varrho_1}\varrho_1^{1-\mu}(\varrho_1-s)^{\varsigma-1}\left(P_\varsigma(\varrho_2-s)-P_\varsigma(\varrho_1-s)\right)f(s,\nu(s))ds\right\|$$
$$\le\varrho_2^{1-\mu}\frac{\tilde{M}}{\Gamma(\varsigma)}\frac{(\varrho_2-\varrho_1)^{(\varsigma-1)+1/q}}{[q(\varsigma-1)+1]^{1/q}}\|\psi\|_p+\varrho_2^{1-\mu}\frac{\tilde{M}lr}{\Gamma(\varsigma+1)}\times(\varrho_2-\varrho_1)^{\varsigma}+\frac{\tilde{M}}{\Gamma(\varsigma)}\left[\int_0^{\varrho_1}\varrho_2^{1-\mu}(\varrho_2-s)^{q(\varsigma-1)}ds\right]^{1/q}\|\psi\|_p-\frac{\tilde{M}}{\Gamma(\varsigma)}\left[\varrho_1^{1-\mu}\int_0^{\varrho_1}(\varrho_1-s)^{q(\varsigma-1)}\mathrm{d}s\right]^{1/q}\|\psi\|_p+\frac{\tilde{M}lr}{\Gamma(\varsigma)}\times\left[\varrho_2^{1-\mu}\int_0^{\varrho_1}(\varrho_2-s)^{(\varsigma-1)}ds-\varrho_1^{1-\mu}\int_0^{\varrho_1}(\varrho_1-s)^{(\varsigma-1)}ds\right]+\left\|\int_0^{\varrho_1-\tau}\varrho_1^{1-\mu}(\varrho_1-s)^{\varsigma-1}\left(P_\varsigma(\varrho_2-s)-P_\varsigma(\varrho_1-s)\right)f(s,\nu(s))ds\right\|+\left\|\int_{\varrho_1-\tau}^{\varrho_1}\varrho_1^{1-\mu}(\varrho_1-s)^{\varsigma-1}\left(P_\varsigma(\varrho_2-s)-P_\varsigma(\varrho_1-s)\right)f(s,\nu(s))ds\right\|$$
$$\le\varrho_2^{1-\mu}\frac{\tilde{M}}{\Gamma(\varsigma)}\frac{(\varrho_2-\varrho_1)^{(\varsigma-1)+1/q}}{[q(\varsigma-1)+1]^{1/q}}\|\psi\|_p+\varrho_2^{1-\mu}\frac{\tilde{M}lr}{\Gamma(\varsigma+1)}(\varrho_2-\varrho_1)^{\varsigma}+\frac{\tilde{M}\varrho_2^{1-\mu}}{(\varsigma-1+(1/q))\Gamma(\varsigma)}\left(\varrho_2^{q(\varsigma-1)+1}-(\varrho_2-\varrho_1)^{q(\varsigma-1)+1}\right)^{1/q}\|\psi\|_p-\frac{\tilde{M}}{\Gamma(\varsigma)}\left[\frac{\varrho_1^{\varsigma-\mu+1/q}}{\varsigma-1+1/q}\right]\|\psi\|_p+\frac{\tilde{M}lr}{\Gamma(\varsigma+1)}\times\left[\varrho_2^{1-\mu+\varsigma}-\varrho_2^{1-\mu}(\varrho_2-\varrho_1)^{\varsigma}-\varrho_1^{1-\mu+\varsigma}\right]+\frac{\varrho_1^{(1-\mu)}}{\varsigma-1+1/q}\times\left(\varrho_1^{q(\varsigma-1)+1}-\tau^{q(\varsigma-1)+1}\right)^{1/q}\|\psi\|_p\sup_{s\in(0,\varrho_1-\tau)}\|P_\varsigma(\varrho_2-s)-P_\varsigma(\varrho_1-s)\|+\frac{lrb^{1-\mu}}{\varsigma}\varrho_1^{1-\mu}(\varrho_1^{\varsigma}-\tau^{\varsigma})\sup_{s\in(0,\varrho_1-\tau)}\|P_\varsigma(\varrho_2-s)-P_\varsigma(\varrho_1-s)\|+\left\|\int_{\varrho_1-\tau}^{\varrho_1}\varrho_1^{1-\mu}(\varrho_1-s)^{\varsigma-1}\left(P_\varsigma(\varrho_2-s)-P_\varsigma\times(\varrho_1-s)\right)f(s,\nu(s))ds\right\|\longrightarrow 0, \tag{28}$$

as $\varrho_2\longrightarrow\varrho_1$ and $\tau\longrightarrow 0$. The above inequality tends to zero regardless of $\nu\in\mathfrak{B}'_r$. Therefore, $G_{n2}\mathfrak{B}'_r$ is equicontinuous. As a result, $G_n\mathfrak{B}'_r$ is equicontinuous.

Step 4. For any $\varrho\in\mathfrak{J}'$, we will prove that $Z(\varrho)=\{G_n\nu(\varrho),\nu\in\mathfrak{B}'_r\}$ is relatively compact in $\mathscr{V}$.

Recall equation (20) and $T_\varsigma(\varrho)=\varrho^{\varsigma-1}P_\varsigma(\varrho)$, $P_\varsigma(\varrho)=\int_0^\infty\varsigma\theta M_\varsigma(\theta)Q(\varrho^\varsigma\theta)\mathrm{d}\theta$, and $S_{\varsigma,\kappa}(\varrho)=I_{0+}^{\kappa(1-\varsigma)}T_\varsigma(\varrho)$ from Lemma 9, and then, we have

$$(G_n\nu)(\varrho)=\frac{1}{\Gamma(\kappa(1-\varsigma))}\int_0^\varrho(\varrho-s)^{\kappa(1-\varsigma)-1}s^{\varsigma-1}\int_0^\infty\varsigma\theta M_\varsigma(\theta)Q\cdot(s^\varsigma\theta)Q\left(\frac{1}{n}\right)[\nu_0-g(\nu)]d\theta ds+\int_0^\varrho\int_0^\infty\varsigma\theta M_\varsigma(\theta)Q\cdot((\varrho-s)^\varsigma\theta)(\varrho-s)^{\varsigma-1}f(s,\nu(s))d\theta ds. \tag{29}$$

Then, $\forall\eta\in(0,\varrho)$ and $\gamma>0$; we define

$$(G_n^{\eta,\gamma}\nu)(\varrho)=\frac{1}{\Gamma(\kappa(1-\varsigma))}\int_0^{\varrho-\eta}(\varrho-s)^{\kappa(1-\varsigma)-1}s^{\varsigma-1}\int_\gamma^\infty\varsigma\theta M_\varsigma\cdot(\theta)Q(s^\varsigma\theta)Q\left(\frac{1}{n}\right)[\nu_0-g(\nu)]d\theta ds+\int_0^{\varrho-\eta}\int_\gamma^\infty\varsigma\theta M_\varsigma(\theta)Q((t-s)^\varsigma\theta)(\varrho-s)^{1-\varsigma}f\cdot(s,\nu(s))d\theta ds, \tag{30}$$

from the compactness of $Q(\eta^\varsigma\theta)$ and $Q(1/n)$, $(\eta^\varsigma\theta)>0$; we can see that $\forall\eta\in(0,\varrho)$ and $\gamma>0$, and the set $Z^{\eta,\gamma}(\varrho)=\{(G_n^{\eta,\gamma}\nu)(\varrho),\nu\in\mathfrak{B}'_r\}$ is relatively compact in $\mathscr{V}$. Then utilizing the previous fact, we derive

$$\|G_n\nu(\varrho)-G_n^{\eta,\gamma}\nu(\varrho)\|_\mu\le\sup_{\varrho\in\mathfrak{J}'}\varrho^{1-\mu}\left\|\frac{1}{\Gamma(\kappa(1-\varsigma))}\int_0^\varrho(\varrho-s)^{\kappa(1-\varsigma)-1}s^{\varsigma-1}\times\int_0^\infty\varsigma\theta M_\varsigma(\theta)Q(s^\varsigma\theta)Q\left(\frac{1}{n}\right)[\nu_0-g(\nu)]d\theta ds+\int_0^\varrho\int_0^\infty\varsigma\theta M_\varsigma(\theta)Q((\varrho-s)^\varsigma\theta)(\varrho-s)^{\varsigma-1}f(s,\nu(s))d\theta ds-\frac{1}{\Gamma(\kappa(1-\varsigma))}\int_0^{\varrho-\eta}(\varrho-s)^{\kappa(1-\varsigma)-1}s^{\varsigma-1}\int_\gamma^\infty\varsigma\theta M_\varsigma(\theta)Q(s^\varsigma\theta)Q\times\left(\frac{1}{n}\right)[\nu_0-g(\nu)]d\theta ds-\int_0^{\varrho-\eta}\int_\gamma^\infty\varsigma\theta M_\varsigma(\theta)Q((\varrho-s)^\varsigma\theta)\times(\varrho-s)^{\varsigma-1}f(s,\nu(s))d\theta ds\right\|$$
$$\le\sup_{\rho\in\mathfrak{J}'}\varrho^{1-\mu}\left\|\frac{1}{\Gamma(\kappa(1-\varsigma))}\int_0^\varrho(\varrho-s)^{\kappa(1-\varsigma)-1}s^{\varsigma-1}\int_0^\gamma\varsigma\theta M_\varsigma(\theta)Q\times\left(s^\varsigma\theta+\frac{1}{n}\right)[\nu_0-g(\nu)]d\theta ds+\frac{1}{\Gamma(\kappa(1-\varsigma))}\times\int_0^\varrho(\varrho-s)^{\kappa(1-\varsigma)-1}s^{\varsigma-1}\int_\gamma^\infty\varsigma\theta M_\varsigma(\theta)Q\left(s^\varsigma\theta+\frac{1}{n}\right)\times[\nu_0-g(\nu)]d\theta ds-\frac{1}{\Gamma(\kappa(1-\varsigma))}\int_0^{\varrho-\eta}(\varrho-s)^{\kappa(1-\varsigma)-1}s^{\varsigma-1}\right.$$

$$
\begin{aligned}
&\times\int_{\gamma}^{\infty}\varsigma\theta M_{\varsigma}(\theta)Q\left(s^{\varsigma}\theta+\frac{1}{n}\right)[\nu_0-g(\nu)]d\theta ds\\
&+\int_0^{\varrho}\int_0^{\gamma}\varsigma\theta M_{\varsigma}(\theta)Q((\varrho-s)^{\varsigma}\theta)(\varrho-s)^{\varsigma-1}f(s,\nu(s))d\theta ds\\
&+\int_0^{\varrho}\int_{\gamma}^{\infty}\varsigma\theta M_{\varsigma}(\theta)Q((\varrho-s)^{\varsigma}\theta)(\varrho-s)^{\varsigma-1}f(s,\nu(s))d\theta ds\\
&-\int_0^{\varrho-\eta}\int_{\gamma}^{\infty}\varsigma\theta M_{\varsigma}(\theta)Q((\varrho-s)^{\varsigma}\theta)(\varrho-s)^{\varsigma-1}f(s,\nu(s))d\theta ds\|\\
\le{}& \sup_{\varrho\in\mathfrak{J}'}\varrho^{1-\mu}\frac{\varsigma\tilde{M}}{\Gamma(\kappa(1-\varsigma))}\left\|\int_0^{\varrho}(\varrho-s)^{\kappa(1-\varsigma)-1}s^{\varsigma-1}\right.\\
&\times\left.\int_0^{\gamma}\theta M_{\varsigma}(\theta)[\nu_0-g(\nu)]d\theta ds\right\|+\sup_{\varrho\in\mathfrak{J}'}\varrho^{1-\mu}\frac{\varsigma\tilde{M}}{\Gamma(\kappa(1-\varsigma))}\\
&\times\left\|\int_{\varrho-\eta}^{\varrho}(\varrho-s)^{\kappa(1-\varsigma)-1}s^{\varsigma-1}\int_{\gamma}^{\infty}\theta M_{\varsigma}(\theta)\,[\nu_0-g(\nu)]d\theta ds\right\|\\
&+\sup_{\varrho\in\mathfrak{J}'}\varrho^{1-\mu}\left\|\varsigma\int_0^{\varrho}\int_0^{\gamma}\theta M_{\varsigma}(\theta)Q((\varrho-s)^{\varsigma}\theta)\right.\\
&\times\left.(\varrho-s)^{\varsigma-1}f(s,\nu(s))d\theta ds\right\|+\sup_{\varrho\in\mathfrak{J}'}\varrho^{1-\mu}\left\|\varsigma\int_{\varrho-\eta}^{\varrho}\int_{\gamma}^{\infty}\theta M_{\varsigma}\right.\\
&\times\left.(\theta)Q((\varrho-s)^{\varsigma}\theta)(\varrho-s)^{\varsigma-1}f(s,\nu(s))d\theta ds\right\|\\
\le{}&\frac{\varsigma\tilde{M}}{\Gamma(\kappa(1-\varsigma))}\sup_{\nu\in\mathfrak{B}_r'}(\|\nu_0-g(\nu)\|)B(\kappa(1-\varsigma),\varsigma)\\
&\times\int_0^{\gamma}\theta M_{\varsigma}(\theta)d\theta+\frac{\tilde{M}}{\Gamma(\kappa(1-\varsigma))\Gamma(\varsigma)}\sup_{\nu\in\mathfrak{B}_r'}\\
&\times(\|\nu_0-g(\nu)\|)\sup_{\varrho\in\mathfrak{J}'}\varrho^{1-\mu}\int_{\varrho-\eta}^{\varrho}(\varrho-s)^{\kappa(1-\varsigma)-1}s^{\varsigma-1}ds+M\\
&\times\left[\frac{\varsigma}{(q(\varsigma-1)+1)^{1/q}}b^{\kappa(\varsigma-1)+1/q}\|\psi\|_p+rlb^{1-\kappa(1-\varsigma)}\right]\\
&\times\int_0^{\gamma}\theta M_{\varsigma}(\theta)d\theta+\tilde{M}b^{1-\mu}\left[\frac{\eta^{\varsigma-1+1/q}}{(q(\varsigma-1)+1)^{1/q}\Gamma(\varsigma)}\|\psi\|_p\right.\\
&+\left.\frac{rl}{\Gamma(\varsigma+1)}\eta^{\varsigma}\right]\coloneqq\sum_{m=1}^{4}\mathfrak{K}_m.
\end{aligned}
\tag{31}
$$

In fact, one can see that the integrals $\mathfrak{K}_m\longrightarrow 0$, $(m=1, 2, 3, 4)$ as $\eta,\gamma\longrightarrow 0+$.

Thus, there exist relatively compact sets, which are arbitrarily close to the set $Z(\rho)=\{G_n\nu(\varrho),\nu\in\mathfrak{B}_r'\}$; thus, by Arzela-Ascoli theorem, we conclude that the set $\{G_n\nu(\varrho),\nu\in\mathfrak{B}_r'\}$ is also relatively compact in $\mathscr{V}$.

We have proved that the mapping $G_n:\mathfrak{B}'_r\longrightarrow\mathfrak{B}'_r$ is continuous and relatively compact. Consequently, by utilizing Theorem 7, we conclude that the operator G_n has a fixed point in $\mathfrak{B}'_r$, which is said to be the mild solution of the nonlocal problem (19). Hence, the proof of Lemma 16 is completed.

We now construct the solution set:

$$
B=\left\{\nu_n\in C_{1-\mu}(\mathfrak{J};\mathscr{V})\colon\nu_n=G_n\nu_n,n\ge 1\right\}.\tag{32}
$$

Next, we will give some lemmas to prove that the solution set B is precompact in $C_{1-\mu}(\mathfrak{J};\mathscr{V})$.

Lemma 17. *For each $\varrho\in\mathfrak{J}'$, $B(\varrho)$ is relatively compact and B is an equicontinuous set on $\mathfrak{J}'$.*

Proof. We first show that for each $\varrho\in\mathfrak{J}'$, $B(\varrho)$ is relatively compact. Let $\varrho\in\mathfrak{J}'$, $\varepsilon>0$, and $\nu_n\in B$. By semigroup property and condition (HQ) and the definition of mild solution for (5), there exists $h\in(0,\varrho)$ such that

$$
\begin{aligned}
&\|\nu_n(\varrho)-Q(h)\nu_n(\varrho-h)\|\\
&\le\varrho^{1-\mu}\left\|S_{\varsigma,\kappa}(\varrho)Q\left(\frac{1}{n}\right)[\nu_0-g(\nu_n)]-S_{\varsigma,\kappa}(\varrho-h)Q\left(\frac{1}{n}\right)Q(h)\right.\\
&\quad\times\left.[\nu_0-g(\nu_n)]\right\|+\varrho^{1-\mu}\left\|\int_0^{\rho}T_{\varsigma}(\varrho-s)f(s,\nu_n(s))ds\right.\\
&\quad-\left.\int_0^{\varrho-h}Q(h)T_{\varsigma}(\varrho-h-s)f(s,\nu_n(s))ds\right\|\\
&\le\varrho^{1-\mu}\left\|S_{\varsigma,\kappa}(\varrho)Q\left(\frac{1}{n}\right)[\nu_0-g(\nu_n)]-S_{\varsigma,\kappa}(\varrho)Q\left(\frac{1}{n}\right)Q(h)\right.\\
&\quad\times\left.[\nu_0-g(\nu_n)]\right\|+\varrho^{1-\mu}\left\|S_{\varsigma,\kappa}(\varrho)Q\left(\frac{1}{n}\right)Q(h)[\nu_0-g(\nu_n)]\right.\\
&\quad-\left.S_{\varsigma,\kappa}(\varrho-h)Q\left(\frac{1}{n}\right)Q(h)[\nu_0-g(\nu_n)]\right\|+\varrho^{1-\mu}\\
&\quad\times\left\|\int_0^{\varrho-h}\left(T_{\varsigma}(\varrho-s)-Q(h)T_{\varsigma}(\varrho-h-s)\right)f(s,\nu_n(s))ds\right\|\\
&\quad+\varrho^{1-\mu}\left\|\int_{\varrho-h}^{\varrho}\left(T_{\varsigma}(\varrho-s)-T_{\varsigma}(\varrho-s)\right)f(s,\nu_n(s))ds\right\|\\
&\le\tilde{M}\varrho^{1-\mu}\|S_{\varsigma,\kappa}(\varrho)-S_{\varsigma,\kappa}(\varrho)Q(h)\|\|\nu_0-g(\nu_n)\|+\tilde{M}\varrho^{1-\mu}\\
&\quad\times\|S_{\varsigma,\kappa}(\varrho)Q(h)-S_{\varsigma,\kappa}(\varrho-h)Q(h)\|\|\nu_0-g(\nu_n)\|+\varrho^{1-\mu}\\
&\quad\times\left[\int_0^{\varrho-h}\|T_{\varsigma}(\varrho-s)-Q(h)T_{\varsigma}(\varrho-h-s))\|^q ds\right]^{1/q}\|\psi\|_p\\
&\quad+lr\varrho^{2(1-\mu)}\int_0^{\varrho-h}\|T_{\varsigma}(\varrho-s)-Q(h)T_{\varsigma}(\varrho-h-s))\|ds+\varrho^{1-\mu}\\
&\quad\times\left\|\int_{\varrho-h}^{\varrho}\left(T_{\varsigma}(\varrho-s)-T_{\varsigma}(\varrho-s)\right)f(s,\nu_n(s))ds\right\|\le\varepsilon.
\end{aligned}
\tag{33}
$$

We combine the aforementioned inequality with the compactness of $Q(h)$ in the uniform topology for $h\in(0,\varrho)$ and uniform continuity of $T_{\varsigma}(\varrho)$ in the uniform topology for $\varrho>0$. Therefore, $B(\varrho)$ is relatively compact. □

Then, we prove that B is equicontinuous on $\mathfrak{J}'$. The argument of the proof is much similar to the proof in

Step 3. Now, for $0 \le \varrho_1 < \varrho_2 \le b$, we proceed the proof as follows:

$$
\begin{aligned}
&\left\| \varrho_2^{1-\mu} \nu_n(\varrho_2) - \varrho_1^{1-\mu} \nu_n(\varrho_1) \right\| \\
&\quad \le \left\| \varrho_2^{1-\mu} S_{\varsigma,\kappa}(\varrho_2) Q\left(\frac{1}{n}\right) [\nu_0 - g(\nu_n)] - \varrho_1^{1-\mu} S_{\varsigma,\kappa}(\varrho_1) Q\left(\frac{1}{n}\right) \right. \\
&\qquad \left. \times [\nu_0 - g(\nu_n)] \right\| + \left\| \int_0^{\varrho_2} \varrho_2^{1-\mu} T_\varsigma(\varrho_2 - s) f(s, \nu_n(t)) ds \right. \\
&\qquad \left. - \int_0^{\varrho_1} \varrho_1^{1-\mu} T_\varsigma(\varrho_1 - s) f(s, \nu_n(\varrho)) ds \right\| \\
&\quad \le \left\| \left[\varrho_2^{1-\mu} S_{\varsigma,\kappa}(\varrho_2) - \varrho_1^{1-\mu} S_{\varsigma,\kappa}(\varrho_1) \right] Q\left(\frac{1}{n}\right) [\nu_0 - g(\nu_n)] \right\| \\
&\qquad + \left\| \varrho_2^{1-\mu} \int_{\varrho_1}^{\varrho_2} T_\varsigma(\varrho_2 - s) f(s, \nu_n(s)) ds \right. \\
&\qquad \left. + \int_0^{\varrho_1} \left(\varrho_2^{1-\mu} T_\varsigma(\varrho_2 - s) - \varrho_1^{1-\mu} T_\varsigma(\varrho_1 - s) \right) f(s, \nu_n(s)) ds \right\| \\
&\quad \le \left\| \left[\varrho_2^{1-\mu} S_{\varsigma,\kappa}(\varrho_2) - \varrho_1^{1-\mu} S_{\varsigma,\kappa}(\varrho_1) \right] Q\left(\frac{1}{n}\right) [\nu_0 - g(\nu_n)] \right\| \\
&\qquad + \varrho_2^{1-\mu} \frac{\tilde{M}}{\Gamma(\varsigma)} \frac{(\varrho_2 - \varrho_1)^{(\varsigma-1)+1/q}}{[q(\varsigma-1)+1]^{1/q}} \|\psi\|_p + \varrho_2^{1-\mu} \frac{\tilde{M} l r}{\Gamma(\varsigma+1)} (\varrho_2 - \varrho_1)^\varsigma \\
&\qquad + \frac{\tilde{M}}{\Gamma(\varsigma)} \left[\int_0^{\varrho_1} \left(\varrho_2^{1-\mu} (\varrho_2 - s)^{\varsigma-1} - \varrho_1^{1-\mu} (\varrho_1 - s)^{\varsigma-1} \right)^q ds \right]^{1/q} \|\psi\|_p \\
&\qquad + \frac{\tilde{M} l r}{\Gamma(\varsigma)} \int_0^{\varrho_1} \left(\varrho_2^{1-\mu} (\varrho_2 - s)^{\varsigma-1} - \varrho_1^{1-\mu} (\varrho_1 - s)^{\varsigma-1} \right) ds \\
&\qquad + \left\| \int_0^{\varrho_1 - \tau} \varrho_1^{1-\mu} (\varrho_1 - s)^{\varsigma-1} \left(P_\varsigma(\varrho_2 - s) - P_\varsigma(\varrho_1 - s) \right) f(s, \nu_n(s)) ds \right\| \\
&\qquad + \left\| \int_{\varrho_1 - \tau}^{\varrho_1} \varrho_1^{1-\mu} (\varrho_1 - s)^{\varsigma-1} \left(P_\varsigma(\varrho_2 - s) - P_\varsigma(\varrho_1 - s) \right) f(s, \nu_n(s)) ds \right\| \\
&\quad \le \left\| \left[\varrho_2^{1-\mu} S_{\varsigma,\kappa}(\varrho_2) - \varrho_1^{1-\mu} S_{\varsigma,\kappa}(\varrho_1) \right] Q\left(\frac{1}{n}\right) [\nu_0 - g(\nu_n)] \right\| + \varrho_2^{1-\mu} \\
&\qquad \times \frac{\tilde{M}}{\Gamma(\varsigma)} \frac{(\varrho_2 - \varrho_1)^{(\varsigma-1)+1/q}}{[q(\varsigma-1)+1]^{1/q}} \|\psi\|_p + \varrho_2^{1-\mu} \frac{\tilde{M} l r}{\Gamma(\varsigma+1)} (\varrho_2 - \varrho_1)^\varsigma \\
&\qquad + \left[\frac{\tilde{M} \varrho_2^{1-\mu}}{(\varsigma - 1 + 1/q)\Gamma(\varsigma)} \left[(\varrho_2 - \varrho_1)^{q(\varsigma-1)+1} - \varrho_2^{q(\varsigma-1)+1} \right]^{1/q} \|\psi\|_p \right. \\
&\qquad - \frac{\tilde{M} \varrho_1^{\varsigma-\mu+1/q}}{(\varsigma - 1 + (1/q))\Gamma(\varsigma)} \|\psi\|_p + \frac{\tilde{M} l r}{\Gamma(\varsigma+1)} \left(\varrho_2^{1-\mu+\varsigma} - \varrho_2^{1-\mu} (\varrho_2 - \varrho_1)^\varsigma \right) \\
&\qquad - \frac{\tilde{M} l r}{(\varsigma+1)} \varrho_1^{1-\mu+\varsigma} + \frac{\rho_1^{1-\mu}}{\varsigma - 1 + 1/q} \left(\varrho_1^{q(\varsigma-1)+1} - \tau^{q(\varsigma-1)+1} \right)^{1/q} \\
&\qquad \times \|\psi\|_p \sup_{s \in (0, \varrho_1 - \tau)} \left\| P_\varsigma(\varrho_2 - s) - P_\varsigma(\varrho_1 - s) \right\| + \frac{l r b^{1-\mu}}{\varsigma} \varrho_1^{1-\mu} \\
&\qquad \times \left(\varrho_1^\varsigma - \tau^\varsigma \right) \sup_{s \in (0, \varrho_1 - \tau)} \left\| P_\varsigma(\varrho_2 - s) - P_\varsigma(\varrho_1 - s) \right\| \\
&\qquad + \left\| \int_{\varrho_1 - \tau}^{\varrho_1} \varrho_1^{1-\mu} (\varrho_1 - s)^{\varsigma-1} \left(P_\varsigma(\varrho_2 - s) - P_\varsigma(\varrho_1 - s) \right) f(s, \nu_n(s)) ds \right\| \coloneqq \sum_{\iota=0}^{7} \tilde{I}_\iota.
\end{aligned}
\tag{34}
$$

We can easily see that $\tilde{I}_1 \longrightarrow 0$ independently of $\nu_n \in \mathfrak{B}'_r$ as $\varrho_2 \longrightarrow \varrho_1$, and $\tilde{I}_7$, $\tilde{I}_7 \longrightarrow 0$ independently of $\nu_n \in \mathfrak{B}'_r$ as $\tau \longrightarrow 0$, $\varrho_2 \longrightarrow \varrho_1$. Since the uniform continuity of $\varrho^{1-\mu} S_{\varsigma,\kappa}(\varrho)$, then we obtain that $(\tilde{I}_\iota, \iota = 2, 3, 4, 5, 6)$ tends to zero as $\varrho_2 \longrightarrow \varrho_1$. Therefore, $B(\varrho)$ is equicontinuous for $\varrho \in [0, b]$.

Lemma 18. *B is relatively compact in $C_{1-\mu}(\mathfrak{J}; \mathscr{V})$.*

Proof.

$$
\bar{\nu}_n(\varrho) = \begin{cases} \nu_n(\varrho), & \varrho \in [\delta, b], \\ \nu_n(\delta), & \varrho \in [0, \delta]. \end{cases} \tag{35}
$$

By the condition (Hg), $g(\nu_n) = g(\bar{\nu}_n)$. Simultaneously, by utilizing Lemma 17 and without loss of generality, we may assume $\bar{\nu}_n \longrightarrow \tilde{\nu} \in C_{1-\mu}(\mathfrak{J}; \mathscr{V})$. Thus, by the continuity of $Q(\varrho)$ and g, we get

$$
\begin{aligned}
&\left\| S_{\varsigma,\kappa}(\varrho) Q\left(\frac{1}{n}\right) [\nu_0 - g(\nu_n)] - S_{\varsigma,\kappa}(\varrho) Q\left(\frac{1}{n}\right) [\nu_0 - g(\tilde{\nu})] \right\| \\
&\quad \le \left\| S_{\varsigma,\kappa}(\varrho) Q\left(\frac{1}{n}\right) [\nu_0 - g(\bar{\nu}_n)] - S_{\varsigma,\kappa}(\varrho) Q\left(\frac{1}{n}\right) [\nu_0 - g(\tilde{\nu})] \right\| \\
&\quad \longrightarrow 0,
\end{aligned}
\tag{36}
$$

as $n \longrightarrow +\infty$; i.e., $B(0)$ is relatively compact. □

Then again, similar to the proof of Lemma 17, we obtain that the set $B \in C_{1-\mu}(\mathfrak{J}; \mathscr{V})$ is equicontinuous at $\varrho = 0$. The proof is completed.

Since Lemma 17 and Lemma 18 are proved. Then, the set B is precompact in $C_{1-\mu}(\mathfrak{J}; \mathscr{V})$. Without loss of generality, we may suppose that $\nu_n \longrightarrow \nu^* \in C_{1-\mu}(\mathfrak{J}; \mathscr{V})$ as $n \longrightarrow +\infty$. Recall the definition of mild solution for (19), which can be formulated similarly with the mild solutions of (4) as follows:

$$
\nu_n(\varrho) = S_{\varsigma,\kappa}(\varrho) Q\left(\frac{1}{n}\right) [\nu_0 - g(\nu)] + \int_0^\varrho T_\varsigma(\varrho - s) f(s, \nu(s)) ds, \tag{37}
$$

for $\varrho \in \mathfrak{J}$. Taking the limit to the above equation, that is, $n \longrightarrow +\infty$, we get

$$
\nu^*(\varrho) = S_{\varsigma,\kappa}(\varrho) [\nu_0 - g(\nu^*)] + \int_0^\varrho T_\varsigma(\varrho - s) f(s, \nu^*(s)) ds, \tag{38}
$$

for $\varrho \in \mathfrak{J}$. Thus, Theorem 15 is proved, implying that ν^* is the mild solution of the nonlocal evolution problem (4).

4. Applications

This section is mainly concerned with the applications of our results. Nonlocal conditions are well known for their precision and effectiveness in physics and engineering science models. Now, if real-life phenomenon arises in physics or anywhere in which can be formulated same as (4), then our results ensure that there exists a mild solution to the system. We apply our results to the following examples.

Example 1. We apply Theorem 15 on the following nonlocal problem with Hilfer fractional derivative:

$$\begin{cases} D_{0+}^{\varsigma,\kappa} y(\varrho,\xi) = \partial_x^2 y(\varrho,\xi) + \Phi(\varrho, y(\varrho,\xi)), \varrho \in \mathfrak{J}' = (0,1], \xi \in \Omega = [0,\pi], \\ y(\varrho,0) = y(\varrho,\pi) = 0, \\ I_{0+}^{(1-\varsigma)(1-\kappa)} y(0,\xi) = y_0(\nu) - \Xi(y(0,\xi)), \end{cases} \tag{39}$$

where $D_{0+}^{\varsigma,\kappa}$, ς, and κ are specified in (4). We assume $\mathscr{V} = L^2([0,\pi];\mathbb{R})$ is a Banach space, $y_0(\xi) \in X$, and A is characterized by $Au = u''$, such that $A : D(A) \subset L^2([0,\pi];\mathbb{R}) \longrightarrow L^2([0,\pi];\mathbb{R})$ with the domain $D(A)$ presented by

$$D(A) = \left\{ u \in \mathscr{V} : u \text{ and } u' \text{ are absolutely continuous}, u'' \in \mathscr{V}, u(0) = 0 = u(\pi) \right\}. \tag{40}$$

Thus, A generates a compact semigroup $\{Q(\varrho)\}_{\varrho \geq 0}$, which is strongly continuous. We now can reformulate (39) similar to (4) as follows:

$$\begin{cases} D_{0+}^{\varsigma,\kappa} \nu(\varrho) = A\nu(\varrho) + f(\varrho, \nu(\varrho)), \quad \varrho \in \mathfrak{J}' = (0,1], \\ I_{0+}^{(1-\varsigma)(1-\kappa)} \nu(0) = \nu_0 - g(\nu), \end{cases} \tag{41}$$

where $\nu(\varrho) = y(\varrho,\xi)$, $\varrho \in (0,1]$, $\xi \in \Omega$. The continuous function $f : \mathfrak{J}' \times \mathscr{V} \longrightarrow \mathscr{V}$ is given by

$$f(\varrho, \nu(\varrho)) = \Phi(\varrho, y(\varrho,\xi)). \tag{42}$$

According to the condition (Hf), an integrable function $\psi \in L^p(\mathfrak{J};\mathbb{R}^+)$ and a positive constant l exist such that $\|\Phi(\varrho, y(\varrho,\xi))\| = \psi(\varrho) + l\varrho^{1-\mu}\|\nu(\varrho)\|$. The operator $g : C_{1-\mu}(\mathfrak{J};\mathscr{V}) \longrightarrow \mathscr{V}$ is given by $g(\nu) = \Xi(y(\varrho,\xi))$. By the condition (Hg), we have g maps $\mathfrak{B}'_r \in C_{1-\mu}(\mathfrak{J};\mathscr{V})$ into itself. Therefore, f and g satisfy the conditions (HQ)-(Hg), and then,

$$\left[\frac{\tilde{M}^2}{\Gamma(\kappa(1-\varsigma)+\varsigma)} \sup_{\nu \in \mathfrak{B}'_r} \|\nu_0 - g(\nu)\| + \frac{\tilde{M}}{\Gamma(\varsigma)} \frac{1}{[q(\varsigma-1)+1]^{1/q}} \times \|\psi\|_p + \frac{\tilde{M} l r}{\Gamma(\varsigma+1)} \right] \leq r \tag{43}$$

holds. Thus, by Theorem 15, the nonlocal evolution equation (39) has at least one mild solution.

Example 2. This example focuses on the study of the existence of a mild solution for the nonlocal model with nonlocal conditions associated with electrostatic Micro-Electro-Mechanical-Systems control with the Hilfer derivative. Let $\mathfrak{D}_b = (0,b] \times \Omega$.

$$\begin{cases} D_{0+}^{\varsigma,\kappa} z(\varrho,h) = \Delta z(\varrho,h) + \dfrac{\lambda}{(1-z)^2(1+\beta|\Omega|)^2} & \text{in } \mathfrak{D}_b, \\ z = 0 & \text{in } (0,b] \times \partial\Omega, \\ I_{0+}^{(1-\varsigma)(1-\kappa)} z(0,h) = z_0(h) - \Pi(z) & \text{in } \bar{\Omega}. \end{cases} \tag{44}$$

The function z stands for the deformation of an elastic membrane which is part of Micro-Electro-Mechanical-Systems device, where Δ is a Laplace operator, $|\Omega|$ is one-dimensional Lebesgue measure of Ω, $h \in \Omega$, β is a constant, and λ is a positive integer. Let $\mathscr{V} = L(\Omega;\mathbb{R})$ be a Banach space; then, $z_0(h) \in \mathscr{V}$. The operator A is defined by $A\rho = \rho'$ such that $A : D(A) \subset L(\Omega;\mathbb{R}) \longrightarrow L(\Omega;\mathbb{R})$ with the following domain:

$$D(A) = \left\{ \rho \in \mathscr{V} : \rho, \rho' \text{ are absolutely continuous}, \rho' \in \mathscr{V}, \rho(h) = 0 \text{ when } h \in \Omega \right\}. \tag{45}$$

Then, A generates compact semigroup $\{Q(\varrho)\}_{\varrho \geq 0}$ which is a strongly continuous. We now compare (44) with (4); we get $\nu(\varrho) = z(\varrho,h)$, $\varrho \in (0,b]$, $h \in \Omega$. The continuous function $f : \mathfrak{J}' \times \mathscr{V} \longrightarrow \mathscr{V}$ is defined by $f(\varrho, \nu(\varrho)) = \lambda/((1-z)^2(1+\beta|\Omega|)^2)$. It follows from the condition (Hf); an integrable function $\psi \in L^p(J;\mathbb{R}^+)$ and a positive constant l exist such that $\|\lambda/((1-z)^2(1+\beta|\Omega|)^2)\| = \psi(\varrho) + l\varrho^{1-\mu}\|\nu(\varrho)\|$. The operator $g : C_{1-\mu}(\mathfrak{J};\mathscr{V}) \longrightarrow \mathscr{V}$ is represented by $g(\nu) = \Pi(z)$. By the condition (Hg), one can see that g maps $\mathfrak{B}'_r \in C_{1-\mu}(\mathfrak{J};\mathscr{V})$ into itself. Therefore, f and g satisfy the conditions (HQ)-(Hg), and then,

$$\left[\frac{\tilde{M}^2}{\Gamma(\kappa(1-\varsigma)+\varsigma)} \sup_{\nu \in \mathfrak{B}'_r} \|\nu_0 - g(\nu)\| + \frac{\tilde{M}}{\Gamma(\varsigma)} \frac{1}{[q(\varsigma-1)+1]^{1/q}} \times \|\psi\|_p + \frac{\tilde{M} l r}{\Gamma(\varsigma+1)} \right] \leq r \tag{46}$$

holds. The nonlocal evolution equation (44) then has at least one mild solution, according to Theorem 15.

5. Conclusion

In this paper, we have discussed the existence of mild solutions for nonlocal Hilfer evolution equations of the type (4). We assumed that the nonlocal item g is a continuous function with no assumptions such as the compactness or Lipschitz continuity utilized. Moreover, the nonlinear term is continuous with no Lipschitz continuity used, and the associated C_0-semigroup is compact. We obtained our results by utilizing the approximate technique, the semigroup methods, and the fixed point theorem. Furthermore, some sufficient conditions to ensure the existence of mild solutions are obtained. Finally, a theoretical and real-life example is given to illustrate our results. It is of great interest for future research. Using the proposed methods of the paper, one can study the existence and Hyers-Ulam stability of mild solutions for nonlocal differential equations with impulse and fractional Brownian motion in the sense of the Caputo fractional derivative or the Hilfer derivative.

Data Availability

There was no data used in this study.

Conflicts of Interest

The authors declare that the publication of this paper does not involve any conflicts of interest.

Authors' Contributions

In this paper, all authors contributed equally. The paper was read and approved by all authors.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (grant numbers 11871064 and 11571300).

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