Research Article

Ground State Solutions of Schrödinger-Kirchhoff Equations with Potentials Vanishing at Infinity

Dongdong Sun

School of Mathematics, Qilu Normal University, Jinan 250013, China

Correspondence should be addressed to Dongdong Sun; sundd@amss.ac.cn

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1. Introduction

In this paper, the following Schrödinger-Kirchhoff equations with potentials vanishing at infinity are studied:

\[
\begin{align*}
& -\left(\varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^3} |\nabla u|^2 \right) \Delta u + V(x)u = K(x)|u|^{p-1}u \text{ in } \mathbb{R}^3, \\
& u > 0, \ u \in H^1(\mathbb{R}^3),
\end{align*}
\]

(1)

where \(a, b > 0\) are constants, and \(\varepsilon > 0\) is a small parameter. The potentials \(V(x)\) and \(K(x)\) satisfy

(V) \(V : \mathbb{R}^3 \to \mathbb{R}\) is Hölder continuous and \(\bar{a}/(1 + |x|^\alpha) \leq V(x) \leq A\) for some \(\bar{a}, A > 0\), and \(0 < \alpha < 2\)

(K) \(K : \mathbb{R}^3 \to \mathbb{R}\) is Hölder continuous and \(0 < K(x) \leq k/(1 + |x|^\beta)\) for some \(\beta, k > 0\)

Problem (1) is related to the stationary analogue of the equation

\[
\frac{u_{\varepsilon}}{\varepsilon} - \left( a + b \int_{\Omega} |\nabla u|^2 \right) \Delta u = g(x, \varepsilon),
\]

proposed by Kirchhoff [1] as an extension of the classical D’Alembert’s wave equation for free vibrations of elastic strings. Early studies on Kirchhoff equations (2) were Bernstein [2] and Pohozaev [3]. In recent years, lots of interesting results on the elliptic Kirchhoff equations have been obtained. Here, we only refer to [4–12] and references therein.

Let \(b = 0\) in (1), then equation (1) becomes the well-known Schrödinger equation

\[
-\varepsilon^2 \Delta u + V(x)u = f(u) \text{ in } \mathbb{R}^N.
\]

(3)

There have been enormous results on (3). Since we cannot give a complete list of references here, we only refer to [13–22].

Recently, many authors considered the existence and concentration of positive solutions for Schrödinger-Kirchhoff equations. In [23], He and Zou studied the Schrödinger-Kirchhoff equation with subcritical nonlinearity. In [24], Wang et al. studied the Schrödinger-Kirchhoff equation with critical nonlinearity. In [25], Figueiredo et al. treated the Schrödinger-Kirchhoff equation with the almost optimal Berestycki-Lions type nonlinearity. In [26], Sun and Zhang investigated the existence and concentration of ground state solutions for Schrödinger-Kirchhoff equations with competing potentials. In [27], Sun and Zhang studied the ground state solutions for Schrödinger-Kirchhoff...
equations with a critical frequency. Further results can be seen in [28–36] etc.

In [37], Ambrosetti et al. studied a class of nonlinear Schrödinger equations with potentials vanishing at infinity. Inspired by [37], we consider the Schrödinger–Kirchhoff equation (1) which contains a nonlocal term \( \int_{\mathbb{R}^d} |\nabla u|^2 \) in it. Because of the nonlocal term, several special difficulties will occur in our arguments. For instance, the estimate (34) in our paper would not be obtained if we follow the proof of Lemma 17 in [37] directly. As a result, we can only prove Lemma 6 for small \( b > 0 \) in (1). To overcome this difficulty, new definition of \( R_{\varepsilon, \omega} \) is given, and the desired inequality (34) is finally proved by delicate analysis.

Let
\[
\sigma := \sigma_{\alpha, \beta} = \begin{cases} \frac{4\beta}{\alpha^2} \quad & \text{if } 0 < \beta < \frac{\alpha}{2}, \\ 3, & \text{otherwise}. \end{cases}
\]

Our main results are as follows.

**Theorem 1.** Assume that (V) and (K) are satisfied. Let
\[
\sigma < p < 5.
\]

Then, for every \( \varepsilon > 0 \), there exists a positive ground state solution \( u_{\varepsilon} \in H^1(\mathbb{R}^d) \) of (1).

Concerning the concentration behaviour of the ground state solutions \( u_{\varepsilon} \) obtained in Theorem 1, we have the following theorem.

**Theorem 2.** Under the same assumptions as in Theorem 1, the positive ground state solutions \( \{u_{\varepsilon}\} \) concentrate at a global minimum point \( x^* \) of \( s'(x) = \sqrt{V(2(p-1)-1/2)(x)/K(2(p-1))(x)} \). That is, \( u_{\varepsilon} \) has a unique maximum point \( x_{\varepsilon} \) and \( x_{\varepsilon} \to x^* \) as \( \varepsilon \to 0^+ \), and
\[
\lim_{\varepsilon \to 0^+} u_{\varepsilon}(x) = U^*(\frac{x - x_{\varepsilon}}{\varepsilon}) + \omega_{x}(x) \quad \text{as } \varepsilon \to 0^+,
\]

where \( \omega_\varepsilon \to 0 \) in \( L^\infty(\mathbb{R}^d) \) and in \( C^{2}_{\text{loc}}(\mathbb{R}^d) \) as \( \varepsilon \to 0^+ \) and \( U^* \) is the unique positive ground state solution of
\[
- \left( a + b \int_{\mathbb{R}^d} |\nabla U^*|^2 \right) \Delta U^* + V(x^*) U^* = K(x^*)(U^*)^{p-1}.
\]

**Remark 3.** The uniqueness of ground state solution of (7) can be seen in [26].

### 2. Existence of Positive Ground State Solutions

#### 2.1. Preliminaries

To prove our results, we work in the following weighted Sobolev spaces:
\[
\mathcal{H}_{\varepsilon} := \left\{ u \in \mathcal{D}^{1,2}(\mathbb{R}^d): \int_{\mathbb{R}^d} \left( \varepsilon^2 a |\nabla u|^2 + V(x) u^2 \right) < +\infty \right\}.
\]

\( \mathcal{H}_{\varepsilon} \) is a Hilbert space, and the scalar product and norm are as follows:
\[
\|v\|_{\mathcal{H}_{\varepsilon}}^2 = \int_{\mathbb{R}^d} \left( \varepsilon^2 a |\nabla v|^2 + V(x) v^2 \right),
\]
\[
(u, v)_{\mathcal{H}_{\varepsilon}} = \int_{\mathbb{R}^d} \left( \varepsilon^2 a \nabla u \nabla v + V(x) u v \right).
\]

Set \( \mathcal{H} = \mathcal{H}_1 \) with norm \( \|\cdot\|_{\mathcal{H}} \). Let \( L^p_{\varepsilon} \) be the weighted space of measurable functions \( u : \mathbb{R}^d \to \mathbb{R} \) such that
\[
\|u\|_{L^p_{\varepsilon}} = \left( \int_{\mathbb{R}^d} K(x) |u|^p dx \right)^{1/p} < \infty.
\]

The following result can be seen in [38].

**Theorem 4.** Assume that (V) and (K) are satisfied. Then \( \mathcal{H}_{\varepsilon} \subset L^{p+1}_{\varepsilon} \) provided \( \sigma \leq p \leq 5 \), and
\[
\|v\|_{L^{p+1}_{\varepsilon}} \leq C_\varepsilon \|v\|_{\mathcal{H}} \quad \forall v \in \mathcal{H}_{\varepsilon},
\]

where \( C_\varepsilon > 0 \) is a constant. Furthermore, the embedding is compact if (5) holds.

**2.2. Proof of Theorem 1.** In this section, we will prove Theorem 1.

Define \( I_{\varepsilon} : \mathcal{H}_{\varepsilon} \to \mathbb{R} \) by
\[
I_{\varepsilon}(u) = \frac{1}{2} \int_{\mathbb{R}^d} \left( \varepsilon^2 a |\nabla u|^2 + V(x) u^2 \right) + \frac{b}{4} \left( \int_{\mathbb{R}^d} |u|^2 \right)^{2} - \frac{1}{p+1} \int_{\mathbb{R}^d} K(x) |u|^{p+1}.
\]

Then, \( I_{\varepsilon} \in C^1(\mathcal{H}_{\varepsilon}, \mathbb{R}) \) and the critical point of the energy functional \( I_{\varepsilon} \) are just a weak solution of problem (1).

We have the following theorem.

**Theorem 5.** Assume that (V) and (K) are satisfied, and let (5) hold. Then
\[
c_\varepsilon = \inf_{v \in \mathcal{N}_{\varepsilon}} \max_{t \geq 0} I_{\varepsilon}(tv)
\]

is the ground energy level of \( I_{\varepsilon} \), and equation (1) has a positive ground state solution \( u_{\varepsilon} \in \mathcal{H}_{\varepsilon} \). Furthermore,
\[
\|u_{\varepsilon}\|_{\mathcal{H}_{\varepsilon}}^2 \leq C c_\varepsilon,
\]

where \( C > 0 \) is a constant.

**Proof.** As Lemma 3.2 in [26] (or Lemma 2.2 in [27]), we know that \( c_\varepsilon \) is the ground energy level of \( I_{\varepsilon} \), i.e.,
\[
c_\varepsilon = \inf_{v \in \mathcal{N}_{\varepsilon}} I_{\varepsilon}(v),
\]

where \( \mathcal{N}_{\varepsilon} \) is the Nehari manifold of \( I_{\varepsilon} : \mathcal{H}_{\varepsilon} \to \mathbb{R} \) defined
by

\[ \mathcal{A}_\epsilon = \left\{ v \in \mathcal{H}_\epsilon \setminus \{0\} : \int_{\mathbb{R}^3} (\epsilon^2 a |\nabla v|^2 + V(x)v^2) + \epsilon b \left( \int_{\mathbb{R}^3} |\nabla v|^2 \right)^2 = \int_{\mathbb{R}^3} K(x) |v|^{p-1} \right\}. \]  

(16)

It is easy to verify the Mountain-Pass geometry of \( I_\epsilon : \mathcal{H}_\epsilon \to \mathbb{R} \) (see [26] or [27] for example). Now, we prove that \( I_\epsilon \) satisfies the Palais-Smale condition.

Let \( \{v_n\} \subset \mathcal{H}_\epsilon \) be such that \( I_\epsilon(v_n) \to c \) for some \( c > 0 \) and \( I_\epsilon'(v_n) \rightharpoonup 0 \). First standard arguments show that \( \{v_n\} \) is bounded in \( \mathcal{H}_\epsilon \). Thus, there exists \( v \in \mathcal{H}_\epsilon \) and if necessary a subsequence of \( \{v_n\} \) such that \( v_n \rightharpoonup v \) in \( \mathcal{H}_\epsilon \). Since the embedding of \( \mathcal{H}_\epsilon \) into \( L^{p+1}_K(\mathbb{R}^3) \) is compact (see Theorem 4), we can get a subsequence of \( \{v_n\} \) (also denoted by \( \{v_n\} \)) such that \( v_n \to v \) in \( L^{p+1}_K \). Note that

\[ \langle I_\epsilon'(v_n) - I_\epsilon'(v), v_n - v \rangle = \left( \epsilon^2 a + \epsilon b \right) \int_{\mathbb{R}^3} v_n \cdot \nabla v_n - \int_{\mathbb{R}^3} V(x)(v_n - v)^2 \]

\[ - \left( \epsilon^2 a + \epsilon b \right) \int_{\mathbb{R}^3} v_n \cdot \nabla v_n - \int_{\mathbb{R}^3} K(x)(|v_n|^{p-1}v_n - |v|^{p-1}v) \]

\[ \cdot (v_n - v) - \left( \epsilon^2 a + \epsilon b \right) \int_{\mathbb{R}^3} \nabla v_n \cdot \nabla v_n - \int_{\mathbb{R}^3} V(x)(v_n - v)^2 \]

\[ + \epsilon b \left( \int_{\mathbb{R}^3} |\nabla v_n|^2 - \int_{\mathbb{R}^3} |\nabla v|^2 \right) \cdot \nabla v_n \cdot \nabla (v_n - v) \]

\[ - \int_{\mathbb{R}^3} K(x)(|v_n|^{p-1}v_n - |v|^{p-1}v)(v_n - v) \geq ||v_n - v||^2 \]

\[ - \epsilon b \left( \int_{\mathbb{R}^3} |\nabla v_n|^2 - \int_{\mathbb{R}^3} |\nabla v|^2 \right) \cdot \nabla v_n \cdot \nabla (v_n - v) \]

\[ - \int_{\mathbb{R}^3} K(x)(|v_n|^{p-1}v_n - |v|^{p-1}v)(v_n - v). \]

(17)

which implies that

\[ ||v_n - v||^2 \leq \langle I_\epsilon'(v_n) - I_\epsilon'(v), v_n - v \rangle \]

\[ + \epsilon b \left( \int_{\mathbb{R}^3} |\nabla v|^2 - \int_{\mathbb{R}^3} |\nabla v_n|^2 \right) \cdot \int_{\mathbb{R}^3} \nabla v \cdot \nabla (v_n - v) \]

\[ + \int_{\mathbb{R}^3} K(x)(|v_n|^{p-1}v_n - |v|^{p-1}v)(v_n - v). \]

(18)

Since \( \{v_n\} \) is bounded in \( \mathcal{H}_\epsilon \), \( I_\epsilon'(v_n) \rightharpoonup 0 \) and \( v_n \to v \) in \( \mathcal{H}_\epsilon \), we have that \( \langle I_\epsilon'(v_n) - I_\epsilon'(v), v_n - v \rangle \to 0 \) and \( \epsilon b \left( \int_{\mathbb{R}^3} |\nabla v|^2 - \int_{\mathbb{R}^3} |\nabla v_n|^2 \right) \cdot \int_{\mathbb{R}^3} \nabla v \cdot \nabla (v_n - v) \to 0 \) as \( n \to \infty \). Furthermore,

\[ \int_{\mathbb{R}^3} K(x)(|v_n|^{p-1}v_n - |v|^{p-1}v)(v_n - v) \]

\[ = \int_{\mathbb{R}^3} K^{(p+1)}(x)(|v_n|^{p-1}v_n - |v|^{p-1}v)K^{(l)(p+1)}(x)(v_n - v) \]

\[ \leq \left( \int_{\mathbb{R}^3} K(x)|v_n|^{p-1}v_n - |v|^{p-1}v \right)^{p/(p+1)} \cdot \left( \int_{\mathbb{R}^3} K(x)|v_n - v|^{p+1} \right)^{1/(p+1)}. \]

(19)

Since \( \{v_n\} \) is bounded in \( L^{p+1}_K(\mathbb{R}^3) \) and \( v_n \to v \) in \( L^{p+1}_K(\mathbb{R}^3) \), we have

\[ \int_{\mathbb{R}^3} K(x)(|v_n|^{p-1}v_n - |v|^{p-1}v)(v_n - v) \to 0, \]

as \( n \to \infty \). Thus, we have \( ||v_n - v|| \to 0 \) as \( n \to \infty \), i.e., \( v_n \to v \) in \( \mathcal{H}_\epsilon \).

From the above arguments, we know that \( I_\epsilon : \mathcal{H}_\epsilon \to \mathbb{R} \) satisfies the Mountain-Pass geometry and the Palais-Smale condition; hence, by the Mountain-Pass theorem, we can get a critical point \( u_\epsilon \in \mathcal{H}_\epsilon \) of \( I_\epsilon \) with \( c_\epsilon = I_\epsilon(u_\epsilon) \). As in [26], we can also know that \( u_\epsilon > 0 \). From

\[ - \left( \epsilon^2 a + \epsilon b \right) \int_{\mathbb{R}^3} |\nabla u_\epsilon|^2 \Delta u_\epsilon + V(x)u_\epsilon = K(x)u_\epsilon^p, \]

we have

\[ \int_{\mathbb{R}^3} \left( \epsilon^2 a |\nabla u_\epsilon|^2 + V(x)u_\epsilon^2 \right) + \epsilon b \left( \int_{\mathbb{R}^3} |\nabla u_\epsilon|^2 \right)^2 = \int_{\mathbb{R}^3} K(x)|u_\epsilon|^{p+1}. \]

(22)

Thus,

\[ c_\epsilon = I_\epsilon(u_\epsilon) = \left( \frac{1}{2} - \frac{1}{p+1} \right) \int_{\mathbb{R}^3} \left( \epsilon^2 a |\nabla u_\epsilon|^2 + V(x)u_\epsilon^2 \right) + \frac{1}{4} \left( \frac{1}{p+1} \right) \epsilon b \left( \int_{\mathbb{R}^3} |\nabla u_\epsilon|^2 \right)^2 \geq \frac{1 - \frac{1}{p+1} \epsilon b}{4} \|u_\epsilon\|_L^2, \]

(23)

which implies (14).

Next, we will show that the positive ground state solution \( u_\epsilon \) of (1) found in Theorem 5 belongs indeed to \( H^1(\mathbb{R}^3) \). We first prove

Lemma 6. Let \( u_\epsilon \) be solutions of (1) found in Theorem 5 and suppose there exists \( \Gamma > 0 \) such that

\[ \|u_\epsilon\|_L^2 \leq \Gamma \epsilon^3. \]

(24)

Then, there exist \( K_\Gamma > 0 \) and \( R_\Gamma > 0 \) such that for \( R \geq R_\Gamma \) and \( \Omega_n \subset \mathbb{R}^3 \setminus B_R \),
\[
\int_{\Omega_{n+\varepsilon}} (\varepsilon^2 a|\nabla u_\varepsilon|^2 + V(x)u_\varepsilon^2) \leq \frac{3}{4} \int_{\Omega_{\varepsilon}} (\varepsilon^2 a|\nabla u_\varepsilon|^2 + V(x)u_\varepsilon^2),
\]

where \(\Omega_{n+\varepsilon} = \mathbb{R}^3 \setminus B_{R_{n+\varepsilon}}\) and \(R_{n+\varepsilon} = \varepsilon K_T n^{2(\gamma - 1)}\), where \(\bar{C}_T > 0\) is a constant.

**Proof.** By (24), we have

\[
\int_{\mathbb{R}^3} a|\nabla u_\varepsilon|^2 \leq \Gamma \varepsilon.
\]

Then

\[
\varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 \leq \varepsilon^2 \left(a + b \frac{\Gamma}{a}\right). \tag{27}
\]

Let \(M_1 := a + b(\Gamma/a)\), and choose \(K_T > 0\) large enough such that

\[
\frac{M_1^2}{a} K_T^{\alpha - 2} \leq \frac{\bar{a}}{2}, \tag{28}
\]

where \(\bar{a}\) is defined in (V). Let \(R_{n+\varepsilon} := \varepsilon K_T n^{2(\gamma - 1)}\), and let \(\chi_{n+\varepsilon}(r)\) be piecewise affine functions such that

\[
\begin{align*}
\chi_{n+\varepsilon}(r) &= 0 \quad \text{for } r \leq R_{n+\varepsilon}, \\
\chi_{n+\varepsilon}(r) &= 1 \quad \text{for } r \geq R_{n+1,\varepsilon}.
\end{align*} \tag{29}
\]

Then

\[
\begin{align*}
|R_{n+1,\varepsilon} - R_{n+\varepsilon}| &= \varepsilon K_T \left| (n+1)^{2/(\gamma - 1)} - n^{2/(\gamma - 1)} \right| \\
&= \varepsilon K_T \left[(n+1)^{a/(\gamma - 1)} - n^{a/(\gamma - 1)} \right] \\
&= \varepsilon^{(\gamma - 2)/2} K_T^{(\gamma - 2)/2} \left( (n+1)^{2/(\gamma - 1)} - n^{2/(\gamma - 1)} \right) a^{2/\gamma} \\
&\quad \cdot \left| (n+1) - n \frac{n+1}{n+1} \right| \\
&\geq \varepsilon^{(\gamma - 2)/2} K_T^{(\gamma - 2)/2} \frac{a^{2/\gamma}}{R_{n+1,\varepsilon}}, \tag{30}
\end{align*}
\]

which yields

\[
|R_{n+1,\varepsilon} - R_{n+\varepsilon}| \leq \varepsilon^{-2} K_T^{\gamma - 2} R_{n+1,\varepsilon}^\alpha. \tag{31}
\]

By (28) and (31), we have

\[
\frac{M_1^2}{a} \varepsilon^2 |\nabla \chi_{n+\varepsilon}(x)|^2 \leq \frac{M_1^2}{a} \varepsilon^2 |R_{n+1,\varepsilon} - R_{n+\varepsilon}|^{-2} \leq \frac{M_1^2}{a} \varepsilon^2 K_T^{\gamma - 2} R_{n+1,\varepsilon}^\alpha \\
\leq \frac{M_1^2}{a} K_T^{\gamma - 2} R_{n+1,\varepsilon}^\alpha \leq \frac{\bar{a}}{2} R_{n+1,\varepsilon}^\alpha \leq V(x), x \in \mathbb{R}^3. \tag{32}
\]

Now, we test (1) on \(\chi_{n+\varepsilon} u_\varepsilon\). We get

\[
\begin{align*}
\left(\varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 \right) \int_{\Omega_{n+\varepsilon}} \chi_{n+\varepsilon} |u_\varepsilon|^2 + \int_{\Omega_{n+\varepsilon}} \nabla u_\varepsilon \nabla \chi_{n+\varepsilon} u_\varepsilon \\
&= \int_{\Omega_{n+\varepsilon}} \chi_{n+\varepsilon} K_{n+1} u_\varepsilon^{p+1} - \left(\varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 \right) \int_{\Omega_{n+\varepsilon}} \nabla u_\varepsilon \nabla \chi_{n+\varepsilon} u_\varepsilon. \tag{33}
\end{align*}
\]

Now, by (32), we have

\[
\begin{align*}
- \left(\varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 \right) \int_{\Omega_{n+\varepsilon}} \nabla u_\varepsilon \nabla \chi_{n+\varepsilon} u_\varepsilon \\
&\leq \varepsilon^2 M_1 \int_{\Omega_{n+\varepsilon}} |\nabla u_\varepsilon||\nabla \chi_{n+\varepsilon}||u_\varepsilon| \\
&= \int_{\Omega_{n+\varepsilon}} (\varepsilon \sqrt{a|\nabla u_\varepsilon|}) \left( \frac{1}{\sqrt{a}} M_1 |\nabla \chi_{n+\varepsilon}||u_\varepsilon| \right) \\
&\leq \frac{1}{2} \int_{\Omega_{n+\varepsilon}} \varepsilon^2 a |\nabla u_\varepsilon|^2 + \frac{M_1^2}{a} \varepsilon^2 |\nabla \chi_{n+\varepsilon}(x)|^2 u_\varepsilon^2 \\
&\leq \frac{1}{2} \int_{\Omega_{n+\varepsilon}} \varepsilon^2 a |\nabla u_\varepsilon|^2 + V(x) u_\varepsilon^2.
\end{align*} \tag{34}
\]

Then (33) and (34) imply that

\[
\begin{align*}
\int_{\Omega_{n+\varepsilon}} \varepsilon^2 a |\nabla u_\varepsilon|^2 + V(x) u_\varepsilon^2 &\leq \int_{\Omega_{n+\varepsilon}} u_\varepsilon^2 a |\nabla \chi_{n+\varepsilon}(x)|^2 + V(x) u_\varepsilon^2 \\
&\leq \varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 \int_{\Omega_{n+\varepsilon}} \chi_{n+\varepsilon} |u_\varepsilon|^2 \\
&\quad + \int_{\Omega_{n+\varepsilon}} \nabla u_\varepsilon \nabla \chi_{n+\varepsilon} u_\varepsilon \\
&\leq \frac{1}{2} \int_{\Omega_{n+\varepsilon}} (\varepsilon^2 a |\nabla u_\varepsilon|^2 + V(x) u_\varepsilon^2). \tag{35}
\end{align*}
\]

Then, by Proposition 11 in [37], let \(\delta > 0\) be fixed, and for sufficiently large \(R > 0\), we have

\[
\begin{align*}
\int_{\Omega_{n+\varepsilon}} (\varepsilon^2 a |\nabla u_\varepsilon|^2 + V(x) u_\varepsilon^2) \\
&\leq \frac{1}{2} \int_{\Omega_{n+\varepsilon}} (\varepsilon^2 a |\nabla u_\varepsilon|^2 + V(x) u_\varepsilon^2) \tag{36}
\end{align*}
\]

\[
+ \delta \varepsilon^{-3(p - 1)/2} \left( \int_{\Omega_{n+\varepsilon}} \varepsilon^2 a |\nabla u_\varepsilon|^2 + V(x) u_\varepsilon^2 \right)^{(p+1)/2}.
\]

By (24), we have

\[
\begin{align*}
\left( \int_{\Omega_{n+\varepsilon}} \varepsilon^2 a |\nabla u_\varepsilon|^2 + V(x) u_\varepsilon^2 \right)^{(p+1)/2} \\
&\leq I^{(p-1)/2} \varepsilon^{(p-1)/2} \int_{\Omega_{n+\varepsilon}} \varepsilon^2 a |\nabla u_\varepsilon|^2 + V(x) u_\varepsilon^2. \tag{37}
\end{align*}
\]
Thus, by the above two estimates, we know
\[
\int_{\Omega_{\rho,n}} \left( \varepsilon^2 a|\nabla u|^2 + V(x)|u|^2 \right) \leq \left( \frac{1}{2} + \delta r^{(p-1)/2} \right) \int_{\Omega_{\rho,n}} \left( \varepsilon^2 a|\nabla u|^2 + V(x)|u|^2 \right).
\] (38)

Choosing \( \delta > 0 \) sufficiently small, we can get that
\[
\int_{\Omega_{\rho,n}} \left( \varepsilon^2 a|\nabla u|^2 + V(x)|u|^2 \right) \leq \frac{3}{4} \int_{\Omega_{\rho,n}} \left( \varepsilon^2 a|\nabla u|^2 + V(x)|u|^2 \right).
\] (39)

Lemma 7. Let \( u_e \) be solutions of (1), and let \( \rho, R_f, \) and \( K_f \) be as in Lemma 6. Then, for all \( \rho \geq 2R_f, \)
\[
\int_{|z| > \rho} \left( \varepsilon^2 a|\nabla u_e|^2 + V(x)|u_e|^2 \right) \leq C_f \varepsilon^3 \left\{ \frac{1}{2} \log \frac{3}{4} (K_f \varepsilon)^{-2} \right\},
\] (40)
\[
\int_{|z| > \rho} \left( \varepsilon^2 a|\nabla u_e|^2 + V(x)|u_e|^2 \right) \leq C_f \varepsilon^3 \left\{ \frac{1}{2} \log \frac{3}{4} (K_f \varepsilon)^{-2} \right\},
\] (41)
\[
\int_{|z| > \rho} \left( \varepsilon^2 a|\nabla u_e|^2 + V(x)|u_e|^2 \right) \leq C_f \varepsilon^3 \left\{ \frac{1}{2} \log \frac{3}{4} (K_f \varepsilon)^{-2} \right\},
\] (42)
\[
\int_{|z| > \rho} \left( \varepsilon^2 a|\nabla u_e|^2 + V(x)|u_e|^2 \right) \leq C_f \varepsilon^3 \left\{ \frac{1}{2} \log \frac{3}{4} (K_f \varepsilon)^{-2} \right\},
\] (43)

By the choices of \( \tilde{n}, \bar{n}, \)
\[
\rho - \varepsilon K_f \tilde{n}^{2(2-a)},
\] (44)
\[
R_f - \varepsilon K_f \tilde{n}^{2(2-a)},
\]
which implies
\[
\tilde{n} - \bar{n} \geq \frac{1}{2} (K_f \varepsilon)^{-2} \left( \rho^{2-a} - R_f^{2-a/2} \right),
\] (45)

Then, from (43) and the above formula, we can get the estimate (40).

Now, we are in position to prove Theorem 1. To prove \( u_e \in H^1(\mathbb{R}^3) \), we actually need to show that \( u_e \in L^2(\mathbb{R}^3) \). In the following, we follow [37] to give a proof here.

First, for the simplicity of the notation, we can take \( \varepsilon \) to 1. Let \( u \in \mathcal{X} \) be a solution of (1)(with \( \varepsilon = 1 \)), and we choose \( y \in \mathbb{R}^3 \) such that \( |y| > 2 \). Then
\[
\int_{B_{1}(y)} u^2 \leq \frac{1}{V(x)} \int_{B_{1}(y)} V(x) u^2 \leq c_1 |y|^a \int_{B_{1}(y)} V(x) u^2. \tag{46}
\]

For \( R = (1/2)|y| \) we know that
\[
\int_{B_{1}(y)} u^2 \leq \int_{R^3 \setminus B_{2}} V(x) u^2. \tag{47}
\]

From the above two estimates and Lemma 2.8, we have
\[
\int_{B_{1}(y)} u^2 \leq C_1 |y|^a \exp \left\{ -C_2 |y|^{-a/2} \right\} |y| > 1. \tag{48}
\]

Let \( B_{3} \setminus B_{2} \subset \cup_{y \in \mathcal{X}} B_{1}(y) \) where \( m \in \mathbb{N} \) and \( y \in \mathbb{R}^3 \). Define \( y_{1,k} = 2^{-k}y \). Then
\[
\int_{R^3 \setminus B_{2}} u^2 \leq \sum_{k=0}^{\infty} \int_{B_{1}(y_{1,k})} u^2. \tag{49}
\]

By (48) and \( 0 < a < 2 \), we have
\[
\int_{R^3 \setminus B_{2}} u^2 \leq C_1 \sum_{k=0}^{\infty} |y_{1,k}|^a \exp \left\{ -C_2 |y_{1,k}|^{-a/2} \right\} < \infty. \tag{50}
\]

Thus, we have that \( u \in L^2(\mathbb{R}^3) \), whence \( u \in H^1(\mathbb{R}^3) \). Finally, standard elliptic estimate (see Theorem 4.1 in [39] for example) shows that \( \lim_{|x| \to \infty} u(x) = 0 \).

3. Concentration Behaviour of Ground State Solutions

In this section, we will study the concentration behaviour of the positive ground state solutions \{\( u_e \)\} obtained in Theorems 1 as \( \varepsilon \to 0^+ \). Assume that \( (V) \) and \( (K) \) are satisfied, and let (5) hold. By (5), we can know that \( \delta(x) \to 0 \) as \( |x| \to \infty \), where \( \delta(x) = V^{2(p-1)/2(x)} K^{2(p-1)/2(x)} \) is defined in Theorem 2, and therefore, \( \delta(x) \) has a global minimum point \( x_0 \) in \( \mathbb{R}^3 \). Now, let \( u_{0} > 0 \) be the unique positive ground state solution (see [26]) of
\[
- \left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 \right) \Delta u + V(x_0) u = K(x_0) u^p, u \in \mathbb{R}^3, \tag{51}
\]
and let \( c_0 \) be the ground energy level associated to (51),
i.e., \( c_0 = I_0(u_0) \) where

\[
I_0(u) = \frac{1}{2} \int_{\mathbb{R}^3} (a|\nabla u|^2 + V(x_0)|u|^2) + \frac{b}{2} \left( \int_{\mathbb{R}^3} |\nabla u|^2 \right)^{\frac{p+1}{p}} K(x_0)|u|^{p+1}.
\]

(52)

Let \( c_\varepsilon \) be as in Theorem 5, then we have the following lemma.

**Lemma 8.**

\[
\limsup_{\varepsilon \to 0^+} c_\varepsilon \leq c_0.
\]

(53)

**Proof.** Since the proof is just the same as the proof of Lemma 11 in [26], we omit it here. \(\square\)

**Remark 9.** In particular, there exists \( C^* > 0 \) such that \( c_\varepsilon \leq C^* \varepsilon^3 \) for sufficiently small \( \varepsilon > 0 \).

**Corollary 10.** Let \( u_\varepsilon \) be the ground state solutions obtained in Theorem 1. Then by Lemma 8 and (14), we have that there exists \( \Gamma > 0 \) such that for sufficiently small \( \varepsilon > 0 \),

\[
\|u_\varepsilon\|_{\varepsilon}^2 \leq \Gamma \varepsilon^3.
\]

(54)

Now, we give a uniform pointwise decay estimate for the solutions \( u_\varepsilon \):

**Lemma 11.** Let \( \Gamma, R_\Gamma, K_\Gamma \) and \( u_\varepsilon \) be as in Lemma 6. Then, there exist constants \( C > 0 \) and \( d > 0 \), such that for \( |x| \geq 2R_\Gamma + C \),

\[
|u_\varepsilon(x)| \leq C|x|^d e^{-d|x|} \exp \left\{ -\frac{1}{2} \log \frac{3}{4} (K_\Gamma e)^{-(2-a)/2} \left( |x|^{(2-a)/2} - K_\Gamma^{-a+2}/2 \right) \right\},
\]

(55)

where \( C \) depends only on \( \Gamma \) and \( p \) and \( d \) depends on \( p, a, \) and \( \beta \).

**Proof.** Let \( x_0 \in \mathbb{R}^3 \) be such that \( |x_0| \geq 2R_\Gamma + 2 \), and let \( \eta \) be a smooth cut-off function satisfying \( \eta = 1 \) for \( x \in B_1(x_0) \), \( \eta = 0 \) for \( x \in \mathbb{R}^3 \setminus B_2(x_0) \), and \( |\nabla \eta| \leq 2 \). For simplicity of notation, we let \( v = u_\varepsilon \). For \( \Gamma > 0 \) and \( s \geq 0 \), define \( \phi = \phi_{s, \Gamma} := v \min \left\{ \frac{\varepsilon}{2}, L^2 \right\} \eta^2 \). The function \( v \) satisfies

\[
- \left( \varepsilon^2 a + b \int_{\mathbb{R}^3} |\nabla v|^2 \right) \Delta v + V(x) v = K(x) v^p,
\]

and by (24), we have \( \int_{\mathbb{R}^3} \varepsilon^2 a |\nabla v|^2 \leq \Gamma \varepsilon^3 \), which implies that \( \int_{\mathbb{R}^3} |\nabla v|^2 \leq (\Gamma/a) \varepsilon \). Now, test (56) on \( \phi \) we can obtain that

\[
\varepsilon^2 \int_{\mathbb{R}^3} a|\nabla v|^2 \min \left\{ \frac{\varepsilon}{2}, L^2 \right\} \eta^2 + \frac{\varepsilon^2}{2} \int_{\mathbb{R}^3} a|\nabla \left( \frac{\varepsilon}{2} \right)|^2 \eta^2 + \int_{\mathbb{R}^3} V(x) v^2 \eta^2 \min \left\{ \frac{\varepsilon}{2}, L^2 \right\} \]

\[
\leq -2\varepsilon \int_{\mathbb{R}^3} a \eta \min \left\{ \frac{\varepsilon}{2}, L^2 \right\} |\nabla v| \eta - 2\varepsilon \int_{\mathbb{R}^3} |\nabla v|^2 \]

\[
+ \varepsilon \int_{\mathbb{R}^3} \eta \min \left\{ \frac{\varepsilon}{2}, L^2 \right\} |\nabla v| \eta + K \varepsilon \eta \min \left\{ \frac{\varepsilon}{2}, L^2 \right\}
\]

\[
\leq 2\varepsilon \int_{\mathbb{R}^3} a \eta \min \left\{ \frac{\varepsilon}{2}, L^2 \right\} |\nabla v| \eta + \frac{2\varepsilon^2}{a} \int_{\mathbb{R}^3} \eta \min \left\{ \frac{\varepsilon}{2}, L^2 \right\} |\nabla v| \eta + K \varepsilon \eta \min \left\{ \frac{\varepsilon}{2}, L^2 \right\}
\]

\[
+ C \varepsilon^2 \int_{\mathbb{R}^3} \eta \min \left\{ \frac{\varepsilon}{2}, L^2 \right\} |\nabla v| \eta + K \varepsilon \eta \min \left\{ \frac{\varepsilon}{2}, L^2 \right\}.
\]

(57)

Next, we can follow the proof of Lemma 22 in [37] directly, so we omit the details here.

By Lemma 11, we know that \( \lim_{|x| \to +\infty} u_\varepsilon = 0 \) for any small and fixed \( \varepsilon > 0 \). Thus, there exists a maximum point \( x_\varepsilon \) in \( \mathbb{R}^3 \) for \( u_\varepsilon \). We have the following lemma.

**Lemma 12.** Let \( u_\varepsilon \) be the solution of (1) satisfying (24), and let \( x_\varepsilon \) be any maximum point of \( u_\varepsilon \). Then, for sufficiently small \( \varepsilon > 0 \), \( |x_\varepsilon| \leq C \), where \( C = C(\Gamma) \).

**Proof.** As \( x_\varepsilon \) is a maximum point of \( u_\varepsilon \), we know that \( \Delta u_\varepsilon(x_\varepsilon) \leq 0 \), and furthermore, \(- \varepsilon^2 a + b \int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 \Delta u_\varepsilon(x_\varepsilon) \geq 0 \). Therefore, from (1), we have that

\[
V(x_\varepsilon) K^{-1}(x_\varepsilon) \leq u_\varepsilon^{-1}(x_\varepsilon).
\]

(58)

Now, by (5) and (K), we know that

\[
| \varepsilon |^{d-a} \leq V(x_\varepsilon) K^{-1}(x_\varepsilon),
\]

(59)

where \( c > 0 \) is a constant. Combining (58), (59), and (55), we have that for \( |x_\varepsilon| \geq 2R_\Gamma \),

\[
| \varepsilon |^{d-a} \leq |x_\varepsilon|^{(d-1) - \varepsilon^{-d-1}} \exp \left\{ -\frac{1}{2} \log \frac{3}{4} (K_\Gamma e)^{-(2-a)/2} \left( |x_\varepsilon|^{(2-a)/2} - K_\Gamma^{-a+2}/2 \right) \right\}.
\]

(60)

\(\square\)

This shows that \( |x_\varepsilon| \) must be bounded as \( \varepsilon \to 0^+ \), and Lemma 12 has been proved.
Lemma 13. Let \( u_\epsilon \) be as in Lemma 12. Then, for sufficiently small \( \epsilon > 0 \), \( \| u_\epsilon \|_{L^\infty} \geq C \) where \( C > 0 \) is a constant.

Proof. From (1) we know that

\[
\| u_\epsilon \|_{L^p}^p = \int_{\mathbb{R}^3} (\epsilon^2 a |\nabla u_\epsilon|^2 + V(x)u_\epsilon^2) \leq \int_{\mathbb{R}^3} K(x)u_\epsilon^{p+1}.
\]

From Lemma 12, we know that going to a subsequence if necessary, there exists \( \{ u_\epsilon \} \) such that \( u_\epsilon \rightharpoonup u \) in \( \mathbb{R}^3 \). By the maximum principle, we have that

\[
\| u \|_{L^\infty} \leq 0.
\]

Combining (61), (62), and (63), we get

\[
\| u_\epsilon \|_{L^p}^p \leq \int_{|x| \leq R} K(x)u_\epsilon^{p+1} \leq \delta \epsilon^{-(p-1)/2} \| u_\epsilon \|_{L^\infty}^{p+1} + \frac{k}{a} (1 + R^a) \| u_\epsilon \|_{L^\infty}^{p+1}
\]

which implies that

\[
1 < \delta \epsilon^{-(p-1)/2} \| u_\epsilon \|_{L^\infty}^{p+1} + \frac{k}{a} (1 + R^a) \| u_\epsilon \|_{L^\infty}^{p+1}.
\]

By Corollary 10, we know that \( \| u_\epsilon \|_{L^p}^p \leq \| u \|_{L^\infty} \leq C \), where \( \delta > 0 \) is a constant. Therefore, we have

\[
| a + b \int_{\mathbb{R}^3} (\nabla \phi_\epsilon)^2 \Delta \phi_\epsilon + V(\epsilon x + x_\epsilon) \phi_\epsilon | = K(\epsilon x + x_\epsilon) \phi_\epsilon^2.
\]

From Corollary 10 and (V), we know that

\[
\Gamma \geq \epsilon^{-3} \| u_\epsilon \|_{L^\infty}^2 \geq \epsilon^{-3} \int_{\mathbb{R}^3} \epsilon^2 a |\nabla \phi_\epsilon|^2 + \frac{\bar{a}}{1 + |x|^2} u_\epsilon^2
\]

Thus, we have proved the lemma.

Proof of Theorem 14. Let \( x_\epsilon \) be a global maximum point of \( u_\epsilon \). From Lemma 12, we know that going to a subsequence if necessary, there exists \( x^* \in \mathbb{R}^3 \) such that \( x_\epsilon \rightharpoonup x^* \). Since \( x_0 \) is a global minimum point of \( A(x) \) in \( \mathbb{R}^3 \), then by Lemma 13 of [26], we have

\[
\| u_\epsilon \|_{L^p}^p \leq \int_{|x| > R} K(x)u_\epsilon^{p+1} \leq \delta \epsilon^{-(p-1)/2} \| u_\epsilon \|_{L^\infty}^{p+1} + \frac{k}{a} (1 + R^a) \| u_\epsilon \|_{L^\infty}^{p+1}.
\]

Thus, we have proved the lemma.

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From the above arguments, we know that
\[ \mathcal{A}(x) = V^{2(p-1)-1/2}(x)/K^{2(p-1)}(x). \] Since the proof of the uniqueness of maximum point is just similar as in [37], we omit it here. Now, we have proved Theorem 2. \qed

**Data Availability**

The author confirm that the data supporting the findings of this study are available within the article.

**Conflicts of Interest**

The author declares that they have no conflicts of interest.

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**References**


