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Research Article

Ground State Solutions of Schrödinger-Kirchhoff Equations with Potentials Vanishing at Infinity

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In this paper, we deal with the following Schrödinger-Kirchhoff equation with potentials vanishing at infinity: $-(\varepsilon^2 a + \varepsilon b\int_{\mathbb{R}^3} |\nabla u|^2)\Delta u + V(x)u = K(x)|u|^{p-1}u$ in $\mathbb{R}^3$and $u > 0, u \in H^1(\mathbb{R}^3)$, where $V(x) \sim |x|^{-\alpha}$ and $K(x) \sim |x|^{-\beta}$ with $0 < \alpha < 2$, and $\beta > 0$. We first prove the existence of positive ground state solutions $u_\varepsilon \in H^1(\mathbb{R}^3)$ under the assumption that $\sigma < p < 5$ for some $\sigma = \sigma_{\alpha,\beta}$, then we show that u_ε concentrates at a global minimum point of $\mathscr{A}(x) = V^{2/(p-1)-1/2}(x)/K^{2/(p-1)}(x)$.

1. Introduction

In this paper, the following Schrödinger-Kirchhoff equations with potentials vanishing at infinity are studied:

$$\begin{cases} -\left(\varepsilon^2 a + \varepsilon b\int_{\mathbb{R}^3} |\nabla u|^2\right)\Delta u + V(x)u = K(x)|u|^{p-1}u \text{ in } \mathbb{R}^3, \\ u > 0, u \in H^1\left(\mathbb{R}^3\right), \end{cases} \tag{1}$$

where $a, b > 0$ are constants, and $\varepsilon > 0$ is a small parameter. The potentials $V(x)$ and $K(x)$ satisfy

(V) $V : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is Hölder continuous and $\bar{a}/(1 + |x|^\alpha) \le V(x) \le A$ for some $\bar{a}, A > 0$, and $0 < \alpha < 2$

(K) $K : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is Hölder continuous and $0 < K(x) \le k/(1 + |x|^\beta)$ for some $\beta, k > 0$

Problem (1) is related to the stationary analogue of the equation

$$u_{tt} - \left(a + b\int_\Omega |\nabla u|^2\right)\Delta u = g(x, t), \tag{2}$$

proposed by Kirchhoff [1] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. Early studies on Kirchhoff equations (2) were Bernstein [2] and Pohozaev [3]. In recent years, lots of interesting results on the elliptic Kirchhoff equations have been obtained. Here, we only refer to [4–12] and references therein.

Let $b = 0$ in (1), then equation (1) becomes the well-known Schrödinger equation

$$-\varepsilon^2 \Delta u + V(x)u = f(u) \text{ in } \mathbb{R}^N. \tag{3}$$

There have been enormous results on (3). Since we cannot give a complete list of references here, we only refer to [13–22].

Recently, many authors considered the existence and concentration of positive solutions for Schrödinger-Kirchhoff equations. In [23], He and Zou studied the Schrödinger-Kirchhoff equation with subcritical nonlinearity. In [24], Wang et al. studied the Schrödinger-Kirchhoff equation with critical nonlinearity. In [25], Figueiredo et al. treated the Schrödinger-Kirchhoff equation with the almost optimal Berestycki-Lions type nonlinearity. In [26], Sun and Zhang investigated the existence and concentration of ground state solutions for Schrödinger-Kirchhoff equations with competing potentials. In [27], Sun and Zhang studied the ground state solutions for Schrödinger-Kirchhoff equations with a critical frequency. Further results can be seen in [28–36] etc.

In [37], Ambrosetti et al. studied a class of nonlinear Schrödinger equations with potentials vanishing at infinity. Inspired by [37], we consider the Schrödinger-Kirchhoff equation (1) which contains a nonlocal term $\int_{\mathbb{R}^3}|\nabla v|^2$ in it. Because of the nonlocal term, several special difficulties will occur in our arguments. For instance, the estimate (34) in our paper would not be obtained if we follow the proof of Lemma 17 in [37] directly. As a result, we can only prove Lemma 6 for small $b > 0$ in (1). To overcome this difficulty, new definition of $R_{n,\varepsilon}$ is given, and the desired inequality (34) is finally proved by delicate analysis.

Let

$$\sigma := \sigma_{\alpha,\beta} = \begin{cases} 5 - \dfrac{4\beta}{\alpha}, & \text{if } 0 < \beta < \dfrac{\alpha}{2}, \\ 3, & \text{otherwise}. \end{cases} \tag{4}$$

Our main results are as follows.

Theorem 1. *Assume that (V) and (K) are satisfied. Let*

$$\sigma < p < 5. \tag{5}$$

Then, for every $\varepsilon > 0$, there exists a positive ground state solution $u_\varepsilon \in H^1(\mathbb{R}^3)$ of (1).

Concerning the concentration behaviour of the ground state solutions u_ε obtained in Theorem 1, we have the following theorem.

Theorem 2. *Under the same assumptions as in Theorem 1, the positive ground state solutions $\{u_\varepsilon\}$ concentrate at a global minimum point x^* of $\mathscr{A}(x) = V^{2/(p-1)-1/2}(x)/K^{2/(p-1)}(x)$. That is, u_ε has a unique maximum point x_ε and $x_\varepsilon \longrightarrow x^*$ as $\varepsilon \longrightarrow 0^+$, and*

$$u_\varepsilon(x) = U^*\left(\frac{x - x_\varepsilon}{\varepsilon}\right) + \omega_\varepsilon(x) \text{ as } \varepsilon \longrightarrow 0^+, \tag{6}$$

where $\omega_\varepsilon \longrightarrow 0$ in $L^\infty(\mathbb{R}^3)$ and in $C^2_{loc}(\mathbb{R}^3)$ as $\varepsilon \longrightarrow 0^+$ and U^* is the unique positive ground state solution of

$$-\left(a + b\int_{\mathbb{R}^3}|\nabla U^*|^2\right)\Delta U^* + V(x^*)U^* = K(x^*)(U^*)^p. \tag{7}$$

Remark 3. The uniqueness of ground state solution of (7) can be seen in [26].

2. Existence of Positive Ground State Solutions

2.1. Preliminaries. To prove our results, we work in the following weighted Sobolev spaces:

$$\mathscr{H}_\varepsilon := \left\{v \in \mathscr{D}^{1,2}(\mathbb{R}^3) : \int_{\mathbb{R}^3}\left(\varepsilon^2 a|\nabla v|^2 + V(x)v^2\right) < +\infty\right\}. \tag{8}$$

$\mathscr{H}_\varepsilon$ is a Hilbert space, and the scalar product and norm are as follows:

$$\begin{aligned} \|v\|^2_\varepsilon &= \int_{\mathbb{R}^3}\left(\varepsilon^2 a|\nabla v|^2 + V(x)v^2\right), \\ (u, v)_\varepsilon &= \int_{\mathbb{R}^3}\left(\varepsilon^2 a\nabla u\nabla v + V(x)uv\right). \end{aligned} \tag{9}$$

Set $\mathscr{H} = \mathscr{H}_1$ with norm $\|\cdot\|_{\mathscr{H}}$. Let L^q_K be the weighted space of measurable functions $u : \mathbb{R}^3 \longrightarrow \mathbb{R}$ such that

$$|u|_{q,K} = \left(\int_{\mathbb{R}^3}K(x)|u|^q dx\right)^{1/q} < \infty. \tag{10}$$

The following result can be seen in [38].

Theorem 4. *Assume that (V) and (K) are satisfied. Then $\mathscr{H}_\varepsilon \subset L^{p+1}_K$ provided $\sigma \le p \le 5$, and*

$$|v|_{p+1,K} \le C_\varepsilon\|v\|_\varepsilon, \forall v \in \mathscr{H}_\varepsilon, \tag{11}$$

where $C_\varepsilon > 0$ is a constant. Furthermore, the embedding is compact if (5) holds.

2.2. Proof of Theorem 1. In this section, we will prove Theorem 1.

Define $I_\varepsilon : \mathscr{H}_\varepsilon \longrightarrow \mathbb{R}$ by

$$\begin{aligned} I_\varepsilon(u) := &\frac{1}{2}\int_{\mathbb{R}^3}\left(\varepsilon^2 a|\nabla u|^2 + V(x)u^2\right) \\ &+ \frac{b}{4}\varepsilon\left(\int_{\mathbb{R}^3}|\nabla u|^2\right)^2 - \frac{1}{p+1}\int_{\mathbb{R}^3}K(x)|u|^{p+1}. \end{aligned} \tag{12}$$

Then, $I_\varepsilon \in C^1(\mathscr{H}_\varepsilon, \mathbb{R})$ and the critical point of the energy functional I_ε are just a weak solution of problem (1).

We have the following theorem.

Theorem 5. *Assume that (V) and (K) are satisfied, and let (5) hold. Then*

$$c_\varepsilon = \inf_{v \in \mathscr{H}_\varepsilon\setminus\{0\}} \max_{t \ge 0} I_\varepsilon(tv) \tag{13}$$

is the ground energy level of I_ε, and equation (1) has a positive ground state solution $u_\varepsilon \in \mathscr{H}_\varepsilon$. Furthermore,

$$\|u_\varepsilon\|^2_\varepsilon \le Cc_\varepsilon, \tag{14}$$

where $C > 0$ is a constant.

Proof. As Lemma 3.2 in [26] (or Lemma 2.2 in [27]), we know that c_ε is the ground energy level of I_ε, i.e.,

$$c_\varepsilon = \inf_{v \in \mathscr{N}_\varepsilon} I_\varepsilon(v), \tag{15}$$

where $\mathscr{N}_\varepsilon$ is the Nehari manifold of $I_\varepsilon : \mathscr{H}_\varepsilon \longrightarrow \mathbb{R}$ defined by

$$\mathcal{N}_\varepsilon = \left\{ v \in \mathcal{H}_\varepsilon \setminus \{0\}: \int_{\mathbb{R}^3} \left(\varepsilon^2 a|\nabla v|^2 + V(x)v^2\right) + \varepsilon b\left(\int_{\mathbb{R}^3} |\nabla v|^2\right)^2 = \int_{\mathbb{R}^3} K(x)|v|^{p+1} \right\}. \tag{16}$$

□

It is easy to verify the Mountain-Pass geometry of $I_\varepsilon : \mathcal{H}_\varepsilon \longrightarrow \mathbb{R}$ (see [26] or [27] for example). Now, we prove that I_ε satisfies the Palais-Smale condition.

Let $\{v_n\} \subset \mathcal{H}_\varepsilon$ be such that $I_\varepsilon(v_n) \longrightarrow c$ for some $c > 0$ and $I_\varepsilon{}'(v_n) \longrightarrow 0$. First standard arguments show that $\{v_n\}$ is bounded in $\mathcal{H}_\varepsilon$. Thus, there exists $v \in \mathcal{H}_\varepsilon$ and if necessary a subsequence of $\{v_n\}$ such that $v_n \rightharpoonup v$ in $\mathcal{H}_\varepsilon$. Since the embedding of $\mathcal{H}_\varepsilon$ into L_K^{p+1} is compact (see Theorem 4), we can get a subsequence of $\{v_n\}$ (also denoted by $\{v_n\}$) such that $v_n \longrightarrow v$ in L_K^{p+1}. Note that

$$\begin{aligned}
&\left\langle I_\varepsilon{}'(v_n) - I_\varepsilon{}'(v), v_n - v \right\rangle \\
&= \left(\varepsilon^2 a + \varepsilon b\int_{\mathbb{R}^3}|\nabla v_n|^2\right)\int_{\mathbb{R}^3}\nabla v_n \cdot \nabla(v_n - v) + \int_{\mathbb{R}^3} V(x)(v_n - v)^2 \\
&\quad - \left(\varepsilon^2 a + \varepsilon b\int_{\mathbb{R}^3}|\nabla v|^2\right)\int_{\mathbb{R}^3}\nabla v \cdot \nabla(v_n - v) - \int_{\mathbb{R}^3} K(x)\left(|v_n|^{p-1}v_n - |v|^{p-1}v\right) \\
&\quad \cdot (v_n - v) = \left(\varepsilon^2 a + \varepsilon b\int_{\mathbb{R}^3}|\nabla v_n|^2\right)\cdot\int_{\mathbb{R}^3}|\nabla(v_n - v)|^2 + \int_{\mathbb{R}^3} V(x)(v_n - v)^2 \\
&\quad + \varepsilon b\left(\int_{\mathbb{R}^3}|\nabla v_n|^2 - \int_{\mathbb{R}^3}|\nabla v|^2\right)\cdot\int_{\mathbb{R}^3}\nabla v \cdot \nabla(v_n - v) \\
&\quad - \int_{\mathbb{R}^3} K(x)\left(|v_n|^{p-1}v_n - |v|^{p-1}v\right)(v_n - v) \ge \|v_n - v\|_\varepsilon^2 \\
&\quad - \varepsilon b\left(\int_{\mathbb{R}^3}|\nabla v|^2 - \int_{\mathbb{R}^3}|\nabla v_n|^2\right)\int_{\mathbb{R}^3}\nabla v \cdot \nabla(v_n - v) \\
&\quad - \int_{\mathbb{R}^3} K(x)\left(|v_n|^{p-1}v_n - |v|^{p-1}v\right)(v_n - v),
\end{aligned} \tag{17}$$

which implies that

$$\begin{aligned}
\|v_n - v\|_\varepsilon^2 \le{} & \left\langle I_\varepsilon{}'(v_n) - I_\varepsilon{}'(v), v_n - v \right\rangle \\
& + \varepsilon b\left(\int_{\mathbb{R}^3}|\nabla v|^2 - \int_{\mathbb{R}^3}|\nabla v_n|^2\right) \\
& \cdot \int_{\mathbb{R}^3}\nabla v \cdot \nabla(v_n - v) \\
& + \int_{\mathbb{R}^3} K(x)\left(|v_n|^{p-1}v_n - |v|^{p-1}v\right)(v_n - v).
\end{aligned} \tag{18}$$

Since $\{v_n\}$ is bounded in $\mathcal{H}_\varepsilon$, $I_\varepsilon'(v_n) \longrightarrow 0$ and $v_n \rightharpoonup v$ in $\mathcal{H}_\varepsilon$, we have that $\langle I_\varepsilon{}'(v_n) - I_\varepsilon{}'(v), v_n - v\rangle \longrightarrow 0$ and $\varepsilon b(\int_{\mathbb{R}^3}|\nabla v|^2 - \int_{\mathbb{R}^3}|\nabla v_n|^2)\int_{\mathbb{R}^3}\nabla v \cdot \nabla(v_n - v) \longrightarrow 0$ as $n \longrightarrow \infty$. Furthermore,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3} K(x)\left(|v_n|^{p-1}v_n - |v|^{p-1}v\right)(v_n - v)\right| \\
&= \left|\int_{\mathbb{R}^3} K^{p/(p+1)}(x)\left(|v_n|^{p-1}v_n - |v|^{p-1}v\right)K^{1/(p+1)}(x)(v_n - v)\right| \\
&\le \left(\int_{\mathbb{R}^3} K(x)\left||v_n|^{p-1}v_n - |v|^{p-1}v\right|^{(p+1)/p}\right)^{p/(p+1)} \\
&\quad \cdot \left(\int_{\mathbb{R}^3} K(x)|v_n - v|^{p+1}\right)^{1/(p+1)}.
\end{aligned} \tag{19}$$

Since $\{v_n\}$ is bounded in $L_K^{p+1}(\mathbb{R}^3)$ and $v_n \longrightarrow v$ in $L_K^{p+1}(\mathbb{R}^3)$, we have

$$\int_{\mathbb{R}^3} K(x)\left(|v_n|^{p-1}v_n - |v|^{p-1}v\right)(v_n - v) \longrightarrow 0, \tag{20}$$

as $n \longrightarrow \infty$. Thus, we have $\|v_n - v\|_\varepsilon \longrightarrow 0$ as $n \longrightarrow \infty$, i.e., $v_n \longrightarrow v$ in $\mathcal{H}_\varepsilon$.

From the above arguments, we know that $I_\varepsilon : \mathcal{H}_\varepsilon \longrightarrow \mathbb{R}$ satisfies the Mountain-Pass geometry and the Palais-Smale condition; hence, by the Mountain-Pass theorem, we can get a critical point $u_\varepsilon \in \mathcal{H}_\varepsilon$ of I_ε with $c_\varepsilon = I_\varepsilon(u_\varepsilon)$. As in [26], we can also know that $u_\varepsilon > 0$. From

$$-\left(\varepsilon^2 a + \varepsilon b\int_{\mathbb{R}^3}|\nabla u_\varepsilon|^2\right)\Delta u_\varepsilon + V(x)u_\varepsilon = K(x)u_\varepsilon^p, \tag{21}$$

we have

$$\int_{\mathbb{R}^3}\left(\varepsilon^2 a|\nabla u_\varepsilon|^2 + V(x)u_\varepsilon^2\right) + \varepsilon b\left(\int_{\mathbb{R}^3}|\nabla u_\varepsilon|^2\right)^2 = \int_{\mathbb{R}^3} K(x)|u_\varepsilon|^{p+1}. \tag{22}$$

Thus,

$$\begin{aligned}
c_\varepsilon = I_\varepsilon(u_\varepsilon) ={}& \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}^3}\left(\varepsilon^2 a|\nabla u_\varepsilon|^2 + V(x)u_\varepsilon^2\right) \\
& + \left(\frac{1}{4} - \frac{1}{p+1}\right)\varepsilon b\left(\int_{\mathbb{R}^3}|\nabla u_\varepsilon|^2\right)^2 \ge \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u_\varepsilon\|_\varepsilon^2,
\end{aligned} \tag{23}$$

which implies (14).

Next, we will show that the positive ground state solution u_ε of (1) found in Theorem 5 belongs indeed to $H^1(\mathbb{R}^3)$. We first prove

Lemma 6. *Let u_ε be solutions of (1) found in Theorem 5 and suppose there exists $\Gamma > 0$ such that*

$$\|u_\varepsilon\|_\varepsilon^2 \le \Gamma\varepsilon^3. \tag{24}$$

Then, there exist $K_\Gamma > 0$ and $R_\Gamma > 0$ such that for $R \ge R_\Gamma$ and $\Omega_{n,\varepsilon} \subset \mathbb{R}^3 \setminus B_R$,

$$\int_{\Omega_{n+1,\varepsilon}} \left(\varepsilon^2 a|\nabla u_\varepsilon|^2 + V(x)u_\varepsilon^2\right) \le \frac{3}{4}\int_{\Omega_{n,\varepsilon}} \left(\varepsilon^2 a|\nabla u_\varepsilon|^2 + V(x)u_\varepsilon^2\right), \tag{25}$$

where $\Omega_{n,\varepsilon} = \mathbb{R}^3 \setminus B_{R_{n,\varepsilon}}$ and $R_{n,\varepsilon} = \varepsilon K_\Gamma n^{2/(2-\alpha)}$.where $\bar{C}_\Gamma > 0$ is a constant.

Proof. By (24), we have

$$\int_{\mathbb{R}^3} a|\nabla u_\varepsilon|^2 \le \Gamma\varepsilon. \tag{26}$$

Then

$$\varepsilon^2 a + \varepsilon b\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 \le \varepsilon^2\left(a + b\frac{\Gamma}{a}\right). \tag{27}$$

Let $M_1 := a + b(\Gamma/a)$, and choose $K_\Gamma > 0$ large enough such that

$$\frac{M_1^2}{a}K_\Gamma^{\alpha-2} \le \frac{\bar{a}}{2}, \tag{28}$$

where $\bar{a}$ is defined in (V). Let $R_{n,\varepsilon} := \varepsilon K_\Gamma n^{2/(2-\alpha)}$, and let $\chi_{n,\varepsilon}(r)$ be piecewise affine functions such that

$$\begin{aligned} &\chi_{n,\varepsilon}(r) = 0 \text{ for } r \le R_{n,\varepsilon}, \\ &\chi_{n,\varepsilon}(r) = 1 \text{ for } r \ge R_{n+1,\varepsilon}. \end{aligned} \tag{29}$$

Then

$$\begin{aligned} \left|R_{n+1,\varepsilon} - R_{n,\varepsilon}\right| &= \varepsilon K_\Gamma\left|(n+1)^{2/(2-\alpha)} - n^{2/(2-\alpha)}\right| \\ &= \varepsilon K_\Gamma (n+1)^{\alpha/(2-\alpha)}\left|(n+1) - n\left(\frac{n}{n+1}\right)^{\alpha/(2-\alpha)}\right| \\ &= \varepsilon^{(2-\alpha)/2}K_\Gamma^{(2-\alpha)/2}\left(\varepsilon K_\Gamma (n+1)^{2/(2-\alpha)}\right)^{\alpha/2} \\ &\quad \cdot \left|(n+1) - n\left(\frac{n}{n+1}\right)^{\alpha/(2-\alpha)}\right| \\ &\ge \varepsilon^{(2-\alpha)/2}K_\Gamma^{(2-\alpha)/2}R_{n+1,\varepsilon}^{\alpha/2}, \end{aligned} \tag{30}$$

which yields

$$\left|R_{n+1,\varepsilon} - R_{n,\varepsilon}\right|^{-2} \le \varepsilon^{\alpha-2}K_\Gamma^{\alpha-2}R_{n+1,\varepsilon}^{-\alpha}. \tag{31}$$

By (28) and (31), we have

$$\begin{aligned} \frac{M_1^2}{a}\varepsilon^2\left|\nabla\chi_{n,\varepsilon}(x)\right|^2 &= \frac{M_1^2}{a}\varepsilon^2\left|R_{n+1,\varepsilon} - R_{n,\varepsilon}\right|^{-2} \le \frac{M_1^2}{a}\varepsilon^\alpha K_\Gamma^{\alpha-2}R_{n+1,\varepsilon}^{-\alpha} \\ &\le \frac{M_1^2}{a}K_\Gamma^{\alpha-2}R_{n+1,\varepsilon}^{-\alpha} \le \frac{\bar{a}}{2}R_{n+1,\varepsilon}^{-\alpha} \le V(x), x \in \mathbb{R}^3. \end{aligned} \tag{32}$$

Now, we test (1) on $\chi_{n,\varepsilon}u_\varepsilon$. We get

$$\begin{aligned} &\left(\varepsilon^2 a + \varepsilon b\int_{\mathbb{R}^3}|\nabla u_\varepsilon|^2\right)\int_{\Omega_{n,\varepsilon}}\chi_{n,\varepsilon}|\nabla u_\varepsilon|^2 + \int_{\Omega_{n,\varepsilon}}\chi_{n,\varepsilon}Vu_\varepsilon^2 \\ &\quad = \int_{\Omega_{n,\varepsilon}}\chi_{n,\varepsilon}Ku_\varepsilon^{p+1} - \left(\varepsilon^2 a + \varepsilon b\int_{\mathbb{R}^3}|\nabla u_\varepsilon|^2\right)\int_{\Omega_{n,\varepsilon}}\nabla u_\varepsilon\nabla\chi_{n,\varepsilon}u_\varepsilon. \end{aligned} \tag{33}$$

Now, by (32), we have

$$\begin{aligned} &-\left(\varepsilon^2 a + \varepsilon b\int_{\mathbb{R}^3}|\nabla u_\varepsilon|^2\right)\int_{\Omega_{n,\varepsilon}}\nabla u_\varepsilon\nabla\chi_{n,\varepsilon}u_\varepsilon \\ &\quad \le \varepsilon^2 M_1\int_{\Omega_{n,\varepsilon}}|\nabla u_\varepsilon|\left|\nabla\chi_{n,\varepsilon}\right||u_\varepsilon| \\ &\quad = \int_{\Omega_{n,\varepsilon}}\left(\varepsilon\sqrt{a}|\nabla u_\varepsilon|\right)\left(\varepsilon\frac{1}{\sqrt{a}}M_1\left|\nabla\chi_{n,\varepsilon}\right||u_\varepsilon|\right) \\ &\quad \le \frac{1}{2}\int_{\Omega_{n,\varepsilon}}\varepsilon^2 a|\nabla u_\varepsilon|^2 + \frac{M_1^2}{a}\varepsilon^2\left|\nabla\chi_{n,\varepsilon}(x)\right|^2u_\varepsilon^2 \\ &\quad \le \frac{1}{2}\int_{\Omega_{n,\varepsilon}}\varepsilon^2 a|\nabla u_\varepsilon|^2 + V(x)u_\varepsilon^2. \end{aligned} \tag{34}$$

Then (33) and (34) imply that

$$\begin{aligned} \int_{\Omega_{n+1,\varepsilon}}\left(\varepsilon^2 a|\nabla u_\varepsilon|^2 + V(x)u_\varepsilon^2\right) &\le \int_{\Omega_{n,\varepsilon}}\chi_{n,\varepsilon}\left(\varepsilon^2 a|\nabla u_\varepsilon|^2 + V(x)u_\varepsilon^2\right) \\ &\le \left(\varepsilon^2 a + \varepsilon b\int_{\mathbb{R}^3}|\nabla u_\varepsilon|^2\right)\int_{\Omega_{n,\varepsilon}}\chi_{n,\varepsilon}|\nabla u_\varepsilon|^2 \\ &\quad + \int_{\Omega_{n,\varepsilon}}\chi_{n,\varepsilon}Vu_\varepsilon^2 \le \int_{\Omega_{n,\varepsilon}}Ku_\varepsilon^{p+1} \\ &\quad + \frac{1}{2}\int_{\Omega_{n,\varepsilon}}\left(\varepsilon^2 a|\nabla u_\varepsilon|^2 + V(x)u_\varepsilon^2\right). \end{aligned} \tag{35}$$

Then, by Proposition 11 in [37], let $\delta > 0$ be fixed, and for sufficiently large $R > 0$, we have

$$\begin{aligned} &\int_{\Omega_{n+1,\varepsilon}}\left(\varepsilon^2 a|\nabla u_\varepsilon|^2 + V(x)u_\varepsilon^2\right) \\ &\le \frac{1}{2}\int_{\Omega_{n,\varepsilon}}\left(\varepsilon^2 a|\nabla u_\varepsilon|^2 + V(x)u_\varepsilon^2\right) \\ &\quad + \delta\varepsilon^{-3(p-1)/2}\left(\int_{\Omega_{n,\varepsilon}}\varepsilon^2 a|\nabla u_\varepsilon|^2 + V(x)u_\varepsilon^2\right)^{(p+1)/2}. \end{aligned} \tag{36}$$

By (24), we have

$$\begin{aligned} &\left(\int_{\Omega_{n,\varepsilon}}\varepsilon^2 a|\nabla u_\varepsilon|^2 + V(x)u_\varepsilon^2\right)^{(p+1)/2} \\ &\quad \le \Gamma^{(p-1)/2}\varepsilon^{3(p-1)/2}\int_{\Omega_{n,\varepsilon}}\varepsilon^2 a|\nabla u_\varepsilon|^2 + V(x)u_\varepsilon^2. \end{aligned} \tag{37}$$

Thus, by the above two estimates, we know

$$\int_{\Omega_{n+1,\varepsilon}} \left(\varepsilon^2 a|\nabla u_\varepsilon|^2 + V(x)u_\varepsilon^2\right) \le \left(\frac{1}{2} + \delta\Gamma^{(p-1)/2}\right)\int_{\Omega_{n,\varepsilon}} \left(\varepsilon^2 a|\nabla u_\varepsilon|^2 + V(x)u_\varepsilon^2\right). \tag{38}$$

Choosing $\delta > 0$ sufficiently small, we can get that

$$\int_{\Omega_{n+1,\varepsilon}} \left(\varepsilon^2 a|\nabla u_\varepsilon|^2 + V(x)u_\varepsilon^2\right) \le \frac{3}{4}\int_{\Omega_{n,\varepsilon}} \left(\varepsilon^2 a|\nabla u_\varepsilon|^2 + V(x)u_\varepsilon^2\right). \tag{39}$$

□

Lemma 7. *Let u_ε be solutions of (1), and let Γ, R_Γ, and K_Γ be as in Lemma 6. Then, for all $\rho \ge 2R_\Gamma$,*

$$\int_{|x|>\rho} \left(\varepsilon^2 a|\nabla u_\varepsilon|^2 + V(x)u_\varepsilon^2\right) \le \bar{C}_\Gamma \varepsilon^3 \exp\left\{-\frac{1}{2}\left|\log\frac{3}{4}\right|(K_\Gamma\varepsilon)^{-(2-\alpha)/2}\left(\rho^{(2-\alpha)/2} - R_\Gamma^{(2-\alpha)/2}\right)\right\}, \tag{40}$$

Proof. The proof is just as the proof of Lemma 18 in [37], we give it here for the sake of completeness.

Let $\rho > 2R_\Gamma$ and choosing two positive integers $\tilde{n} > \bar{n}$ with

$$R_{\tilde{n},\varepsilon} \le \rho \le R_{\tilde{n}+1,\varepsilon}, \quad R_{\bar{n},\varepsilon} \le R_\Gamma \le R_{\bar{n}+1,\varepsilon}, \tag{41}$$

where $R_{n,\varepsilon}$ is defined in Lemma 6. From Lemma 6, we can know that

$$\int_{|x|>\rho} \left(\varepsilon^2 a|\nabla u_\varepsilon|^2 + V(x)u_\varepsilon^2\right) \le \int_{|x|>R_{\tilde{n},\varepsilon}} \left(\varepsilon^2 a|\nabla u_\varepsilon|^2 + V(x)u_\varepsilon^2\right) \le \left(\frac{3}{4}\right)^{\tilde{n}-\bar{n}}\int_{|x|>R_\Gamma} \left(\varepsilon^2 a|\nabla u_\varepsilon|^2 + V(x)u_\varepsilon^2\right). \tag{42}$$

Then, from (24), we have

$$\int_{|x|>\rho} \left(\varepsilon^2 a|\nabla u_\varepsilon|^2 + V(x)u_\varepsilon^2\right) \le \left(\frac{3}{4}\right)^{\tilde{n}-\bar{n}}\Gamma\varepsilon^3. \tag{43}$$

By the choices of $\tilde{n}, \bar{n}$,

$$\rho \sim \varepsilon K_\Gamma \tilde{n}^{2/(2-\alpha)}, \quad R_\Gamma \sim \varepsilon K_\Gamma \bar{n}^{2/(2-\alpha)}, \tag{44}$$

which implies

$$\tilde{n} - \bar{n} \ge \frac{1}{2}(K_\Gamma\varepsilon)^{-(2-\alpha)/2}\left(\rho^{(2-\alpha)/2} - R_\Gamma^{(2-\alpha)/2}\right). \tag{45}$$

Then, from (43) and the above formula, we can get the estimate (40).

Now, we are in position to prove Theorem 1. To prove $u_\varepsilon \in H^1(\mathbb{R}^3)$, we actually need to show that $u_\varepsilon \in L^2(\mathbb{R}^3)$. In the following, we follow [37] to give a proof here. □

First, for the simplicity of the notation, we can take ε to 1. Let $u \in \mathscr{H}$ be a solution of (1)(with $\varepsilon = 1$), and we choose $y \in \mathbb{R}^3$ such that $|y| > 2$. Then

$$\int_{B_1(y)} u^2 = \int_{B_1(y)} V(x)u^2 \cdot \frac{1}{V(x)} \le c_1|y|^\alpha \int_{B_1(y)} V(x)u^2. \tag{46}$$

For $R = (1/2)|y|$ we know that

$$\int_{B_1(y)} V(x)u^2 \le \int_{\mathbb{R}^3\setminus B_R} V(x)u^2. \tag{47}$$

From the above two estimates and Lemma 2.8, we have

$$\int_{B_1(y)} u^2 \le C_1|y|^\alpha \exp\left\{-C_2|y|^{1-\alpha/2}\right\}, \forall |y| \gg 1. \tag{48}$$

Let $B_5 \setminus B_2 \subset \cup_{i=1}^m B_1(y_i)$ where $m \in \mathbb{N}$ and $y_i \in \mathbb{R}^3$. Define $y_{i,k} := 2^k y_i$. Then

$$\int_{\mathbb{R}^3\setminus B_2} u^2 \le \sum_{k=0}^{\infty}\int_{2^k(B_5\setminus B_2)} u^2 \le \sum_{i,k}\int_{B_{2^k}(y_{i,k})} u^2. \tag{49}$$

By (48) and $0 < \alpha < 2$, we have

$$\int_{\mathbb{R}^3\setminus B_2} u^2 \le C_1 \sum_{i,k} |y_{i,k}|^\alpha \exp\left\{-C_2|y_{i,k}|^{1-\alpha/2}\right\} < \infty. \tag{50}$$

Thus, we have that $u \in L^2(\mathbb{R}^3)$, whence $u \in H^1(\mathbb{R}^3)$. Finally, standard elliptic estimate (see Theorem 4.1 in [39] for example) shows that $\lim_{|x|\longrightarrow\infty} u_\varepsilon(x) = 0$.

3. Concentration Behaviour of Ground State Solutions

In this section, we will study the concentration behaviour of the positive ground state solutions $\{u_\varepsilon\}$ obtained in Theorem 1 as $\varepsilon \longrightarrow 0^+$. Assume that (V) and (K) are satisfied, and let (5) hold. By (5), we can know that $\mathscr{A}(x) \longrightarrow \infty$ as $|x| \longrightarrow \infty$, where $\mathscr{A}(x) = V^{2/(p-1)-1/2}(x)/K^{2/(p-1)}(x)$ is defined in Theorem 2, and therefore, $\mathscr{A}(x)$ has a global minimum point x_0 in $\mathbb{R}^3$. Now, let $u_0 > 0$ be the unique positive ground state solution (see [26]) of

$$-\left(a + b\int_{\mathbb{R}^3}|\nabla u|^2\right)\Delta u + V(x_0)u = K(x_0)u^p, x \in \mathbb{R}^3, \tag{51}$$

and let c_0 be the ground energy level associated to (51), i.e., $c_0 = I_0(u_o)$ where

$$I_0(u) := \frac{1}{2}\int_{\mathbb{R}^3}\left(a|\nabla u|^2 + V(x_0)u^2\right) + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2\right)^2 - \frac{1}{p+1}\int_{\mathbb{R}^3}K(x_0)|u|^{p+1}. \tag{52}$$

Let c_ε be as in Theorem 5, then we have the following lemma.

Lemma 8.

$$\limsup_{\varepsilon\longrightarrow 0^+} \varepsilon^{-3}c_\varepsilon \le c_0. \tag{53}$$

Proof. Since the proof is just the same as the proof of Lemma 11 in [26], we omit it here. □

Remark 9. In particular, there exists $C^* > 0$ such that $c_\varepsilon \le C^*\varepsilon^3$ for sufficiently small $\varepsilon > 0$.

Corollary 10. *Let u_ε be the ground state solutions obtained in Theorem 1. Then by Lemma 8 and (14), we have that there exists $\Gamma > 0$ such that for sufficiently small $\varepsilon > 0$,*

$$\|u_\varepsilon\|_\varepsilon^2 \le \Gamma\varepsilon^3. \tag{54}$$

Now, we give a uniform pointwise decay estimate for the solutions u_ε:

Lemma 11. *Let Γ, R_Γ, $,K_\Gamma$ and u_ε be as in Lemma 6. Then, there exist constants $C > 0$ and $d > 0$, such that for $|x| \ge 2R_\Gamma + C$,*

$$|u_\varepsilon(x)| \le C|x|^d\varepsilon^{-d}\exp\left\{-\frac{1}{4}\left|\log\frac{3}{4}\right|(K_\Gamma\varepsilon)^{-(2-\alpha)/2}\left(|x|^{(2-\alpha)/2} - R_\Gamma^{(2-\alpha)/2}\right)\right\}, \tag{55}$$

where C depends only on Γ and p and d depends on p, α, and β.

Proof. Let $x_0 \in \mathbb{R}^3$ be such that $|x_0| \ge 2R_\Gamma + 2$, and let η be a smooth cut-off funtion satisfying $\eta = 1$ for $x \in B_1(x_0)$, $\eta = 0$ for $x \in \mathbb{R}^3 \setminus B_2(x_0)$, and $|\nabla\eta| \le 2$. For simplicity of notation, we let $v = u_\varepsilon$. For $L > 0$ and $s \ge 0$, define $\phi = \phi_{s,L} := v\min\{|v|^{2s}, L^2\}\eta^2$. The function v satisfies

$$-\left(\varepsilon^2 a + \varepsilon b\int_{\mathbb{R}^3}|\nabla v|^2\right)\Delta v + V(x)v = K(x)v^p, \tag{56}$$

and by (24), we have $\int_{\mathbb{R}^3}\varepsilon^2 a|\nabla v|^2 \le \Gamma\varepsilon^3$, which implies that $\int_{\mathbb{R}^3}|\nabla v|^2 \le (\Gamma/a)\varepsilon$. Now, test (56) on ϕ we can obtain that

$$\begin{aligned}
&\varepsilon^2\int a|\nabla v|^2\min\left\{|v|^{2s}, L^2\right\}\eta^2 \\
&\quad + \frac{s}{2}\varepsilon^2\int_{\{|v|^s\le L\}} a\left|\nabla\left(|v|^2\right)\right|^2 v^{2s-2}\eta^2 + \int V(x)v^2\eta^2\min\left\{|v|^{2s}, L^2\right\} \\
&\le -2\varepsilon^2\int av\eta\min\left\{|v|^{2s}, L^2\right\}\nabla v\cdot\nabla\eta - 2\varepsilon b\int_{\mathbb{R}^3}|\nabla v|^2 \\
&\quad\cdot\int v\eta\min\left\{|v|^{2s}, L^2\right\}\nabla v\cdot\nabla\eta + \int Kv^{p+1}\eta^2\min\left\{|v|^{2s}, L^2\right\} \\
&\le 2\varepsilon^2\int av\eta\min\left\{|v|^{2s}, L^2\right\}|\nabla v\cdot\nabla\eta| \\
&\quad + \frac{2\varepsilon^2 b\Gamma}{a}\int v\eta\min\left\{|v|^{2s}, L^2\right\}|\nabla v\cdot\nabla\eta| + \int Kv^{p+1}\eta^2 \\
&\quad\cdot\min\left\{|v|^{2s}, L^2\right\} \le \frac{1}{2}\varepsilon^2\int a|\nabla v|^2\min\left\{|v|^{2s}, L^2\right\}\eta^2 \\
&\quad + C\varepsilon^2\int av^2\min\left\{|v|^{2s}, L^2\right\}|\nabla\eta|^2 + \int Kv^{p+1}\eta^2\min\left\{|v|^{2s}, L^2\right\}.
\end{aligned} \tag{57}$$

□

Next, we can follow the proof of Lemma 22 in [37] directly, so we omit the details here.

By Lemma 11, we know that $\lim_{|x|\longrightarrow\infty}u_\varepsilon = 0$ for any small and fixed $\varepsilon > 0$. Thus, there exists a maximum point x_ε in $\mathbb{R}^3$ for u_ε. We have the following lemma.

Lemma 12. *Let u_ε be the solution of (1) satisfying (24), and let x_ε be any maximum point of u_ε. Then, for sufficiently small $\varepsilon > 0$, $|x_\varepsilon| \le C$, where $C = C(\Gamma)$.*

Proof. As x_ε is a maximum point of u_ε, we know that $\Delta u_\varepsilon(x_\varepsilon) \le 0$, and furthermore, $-(\varepsilon^2 a + \varepsilon b\int_{\mathbb{R}^3}|\nabla u_\varepsilon|^2)\Delta u_\varepsilon(x_\varepsilon) \ge 0$. Therefore, from (1), we have that

$$V(x_\varepsilon)K^{-1}(x_\varepsilon) \le u_\varepsilon^{p-1}(x_\varepsilon). \tag{58}$$

Now, by (V) and (K), we know that

$$c|x_\varepsilon|^{\beta-\alpha} \le V(x_\varepsilon)K^{-1}(x_\varepsilon), \tag{59}$$

where $c > 0$ is a constant. Combining (58), (59), and (55), we have that for $|x_\varepsilon| \ge 2R_\Gamma$,

$$\begin{aligned}
c|x_\varepsilon|^{\beta-\alpha} &\le |x_\varepsilon|^{d(p-1)}\varepsilon^{-d(p-1)} \\
&\quad\cdot\exp\left\{-\frac{1}{4}(p-1)\left|\log\frac{3}{4}\right|(K_\Gamma\varepsilon)^{-(2-\alpha)/2}\left(|x_\varepsilon|^{(2-\alpha)/2} - R_\Gamma^{(2-\alpha)/2}\right)\right\}.
\end{aligned} \tag{60}$$

□

This shows that $|x_\varepsilon|$ must be bounded as $\varepsilon \longrightarrow 0^+$, and Lemma 12 has been proved.

Lemma 13. *Let u_ε be as in Lemma 12. Then, for sufficiently small $\varepsilon > 0$, $\|u_\varepsilon\|_{L^\infty} \geq C$ where $C > 0$ is a constant.*

Proof. From (1) we know that

$$\|u_\varepsilon\|_\varepsilon^2 = \int_{\mathbb{R}^3} \left(\varepsilon^2 a|\nabla u_\varepsilon|^2 + V(x)u_\varepsilon^2\right) \leq \int_{\mathbb{R}^3} K(x)u_\varepsilon^{p+1}. \quad (61)$$

□

Now, let $\delta < \Gamma^{-(p-1)/2}$ be fixed. By Proposition 11 in [37], we know that there exists $R > 0$ such that

$$\int_{|x|>R} K(x)u_\varepsilon^{p+1} \leq \delta\varepsilon^{-3(p-1)/2}\|u_\varepsilon\|_\varepsilon^{p+1}. \quad (62)$$

Then, from the proof of Lemma 25 in [37], we have

$$\int_{|x|\leq R} K(x)u_\varepsilon^{p+1} \leq \frac{k}{\bar{a}}(1+R^\alpha)\left\|u_\varepsilon^{p-1}\right\|_{L^\infty}\|u_\varepsilon\|_\varepsilon^2. \quad (63)$$

Combining (61), (62), and (63), we get

$$\begin{aligned}\|u_\varepsilon\|_\varepsilon^2 \leq \int_{\mathbb{R}^3} K(x)u_\varepsilon^{p+1} &\leq \delta\varepsilon^{-3(p-1)/2}\|u_\varepsilon\|_\varepsilon^{p+1} \\ &+ \frac{k}{\bar{a}}(1+R^\alpha)\left\|u_\varepsilon^{p-1}\right\|_{L^\infty}\|u_\varepsilon\|_\varepsilon^2,\end{aligned} \quad (64)$$

which implies that

$$1 \leq \delta\varepsilon^{-3(p-1)/2}\|u_\varepsilon\|_\varepsilon^{p-1} + \frac{k}{\bar{a}}(1+R^\alpha)\left\|u_\varepsilon^{p-1}\right\|_{L^\infty}. \quad (65)$$

By Corollary 10, we know that $\|u_\varepsilon\|_\varepsilon^{p-1} \leq \Gamma^{(p-1)/2}\varepsilon^{3(p-1)/2}$. Then, from (65), we get

$$1 \leq \delta\Gamma^{(p-1)/2} + \frac{k}{\bar{a}}(1+R^\alpha)\left\|u_\varepsilon^{p-1}\right\|_{L^\infty}. \quad (66)$$

Now, by $\delta < \Gamma^{-(p-1)/2}$, we have that

$$\|u_\varepsilon\|_\varepsilon^{p-1} \geq \left(1 - \delta\Gamma^{(p-1)/2}\right)\frac{\bar{a}}{k(1+R^\alpha)} > 0. \quad (67)$$

Thus, we have proved the lemma.

Proof of Theorem 14. Let x_ε be a global maximum point of u_ε. From Lemma 12, we know that going to a subsequence if necessary, there exists $x^* \in \mathbb{R}^3$ such that $x_\varepsilon \longrightarrow x^*$. Since x_0 is a global minimum point of $\mathscr{A}(x)$ in $\mathbb{R}^3$, then by Lemma 13 of [26], we have

$$c(x^*) \geq c_0, \quad (68)$$

where $c(x^*)$ is the ground energy level corresponding to equation (51) with x_0 replaced by x^* in it. Set $\phi_\varepsilon(x) := u_\varepsilon(\varepsilon x + x_\varepsilon)$, then ϕ_ε satisfies

$$-\left(a + b\int_{\mathbb{R}^3}|\nabla\phi_\varepsilon|^2\right)\Delta\phi_\varepsilon + V(\varepsilon x + x_\varepsilon)\phi_\varepsilon = K(\varepsilon x + x_\varepsilon)\phi_\varepsilon^p. \quad (69)$$

From Corollary 10 and (V), we know that

$$\begin{aligned}\Gamma \geq \varepsilon^{-3}\|u_\varepsilon\|_\varepsilon^2 &\geq \varepsilon^{-3}\int_{\mathbb{R}^3} \varepsilon^2 a|\nabla\phi_\varepsilon|^2 + \frac{\bar{a}}{1+|x|^\alpha}u_\varepsilon^2 \\ &= \int_{\mathbb{R}^3} a|\nabla\phi_\varepsilon|^2 + \frac{\bar{a}}{1+|\varepsilon x + x_\varepsilon|^\alpha}\phi_\varepsilon^2.\end{aligned} \quad (70)$$

From Lemma 12, we know that $|\varepsilon x + x_\varepsilon| \leq C(1+|x|)$, and therefore,

$$\int_{\mathbb{R}^3} a|\nabla\phi_\varepsilon|^2 + \frac{\bar{a}}{1+|x|^\alpha}\phi_\varepsilon^2 \leq C', \quad (71)$$

where $C' > 0$ is a constant. Then $\{\phi_\varepsilon\}$ is bounded in $\bar{H} := \{v \in \mathscr{D}^{1,2}(\mathbb{R}^3): \int_{\mathbb{R}^3}\bar{a}/1+|x|^\alpha v^2 < +\infty\}$, and therefore, going to a subsequence if necessary there exists $U^* \in \bar{H}$ such that $\phi_\varepsilon \rightharpoonup U^*$ in $\bar{H}$, $\phi_\varepsilon \longrightarrow U^*$ in L_K^{p+1}, and $\phi_\varepsilon \longrightarrow U^*$ a.e. in $\mathbb{R}^3$. By standard arguments (see [37]), we have that ϕ_ε converges in $C_{\mathrm{loc}}^2(\mathbb{R}^3)$ to U^* and $\phi_\varepsilon \longrightarrow U^*$ in $L^\infty(\mathbb{R}^3)$. Now, we assume that up to a subsequence, $\int_{\mathbb{R}^3}|\nabla\phi_\varepsilon|^2 \longrightarrow A$, then $A \geq \int_{\mathbb{R}^3}|\nabla U^*|^2$ and

$$-(a+bA)\Delta U^* + V(x^*)U^* = K(x^*)(U^*)^p, x \in \mathbb{R}^3, \quad (72)$$

since 0 is the maximum point of ϕ_ε, so does U^*. Moreover, Lemma 13 shows that $\phi_\varepsilon(0) = u_\varepsilon(x_\varepsilon) = \|u_\varepsilon\|_{L^\infty} \geq C$ for some $C > 0$, thus $U^*(0) = \max U^* \geq C > 0$. Therefore, $U^* \not\equiv 0$ and by the maximum principle we have that $U^* > 0$. Now, for $R_n \longrightarrow \infty$, by Corollary 10, we have

$$\int_{\bar{B}_{R_n}} a|\nabla\phi_\varepsilon|^2 + V(\varepsilon x + x_\varepsilon)\phi_\varepsilon^2 \leq \varepsilon^{-3}\|v_\varepsilon\|_\varepsilon^2 \leq \Gamma. \quad (73)$$

Thus, by the Dominated Convergence Theorem, $\phi_\varepsilon \longrightarrow U^*$ in $C^1(\bar{B}_{R_n})$, and (73) we have

$$\int_{\bar{B}_{R_n}} a|\nabla U^*|^2 + V(x^*)(U^*)^2 \leq \Gamma. \quad (74)$$

Now, let $R_n \longrightarrow \infty$, we get that $U^* \in H^1(\mathbb{R}^3)$.

Next, we prove actually $A = \int_{\mathbb{R}^3}|\nabla U^*|^2$, where A is in (72). Otherwise, we assume $A > \int_{\mathbb{R}^3}|\nabla U^*|^2$. Then, from (72), we have

$$\int_{\mathbb{R}^3}\left(a|\nabla U^*|^2 + V(x^*)(U^*)^2\right) + Ab\int_{\mathbb{R}^3}|\nabla U^*|^2 - \int_{\mathbb{R}^3}K(x^*)|U^*|^{p+1} = 0, \quad (75)$$

which implies that $\langle I_{x^*}{}'(U^*), U^*\rangle < 0$, where $\langle I_{x^*}{}'(U^*), U^*\rangle$ is defined by

$$\left\langle I_{x^*}{}'(U^*), U^* \right\rangle = \int_{\mathbb{R}^3} \left(a|\nabla U^*|^2 + V(x^*)(U^*)^2\right) + b\left(\int_{\mathbb{R}^3} |\nabla U^*|^2\right)^2 - \int_{\mathbb{R}^3} K(x^*)|U^*|^{p+1}. \quad (76)$$

Then, there is a unique $0 < t < 1$ such that

$$\left\langle I_{x^*}{}'(tU^*), tU^* \right\rangle = 0. \quad (77)$$

Now,

$$\begin{aligned} c(x^*) &\le I_{x^*}(tU^*) - \frac{1}{4}\left\langle I_{x^*}{}'(tU^*), tU^* \right\rangle \\ &= \frac{t^2}{4}\int_{\mathbb{R}^3} \left(a|\nabla U^*|^2 + V(x^*)U^{*2}\right) + \left(\frac{t^{p+1}}{4} - \frac{t^{p+1}}{p+1}\right)\int_{\mathbb{R}^3} K(x^*)|U^*|^{p+1} \\ &< \frac{1}{4}\int_{\mathbb{R}^3} \left(a|\nabla U^*|^2 + V(x^*)U^{*2}\right) + \left(\frac{1}{4} - \frac{1}{p+1}\right)\int_{\mathbb{R}^3} K(x^*)|U^*|^{p+1} \\ &\le \varliminf_{\varepsilon \longrightarrow 0} \left(\frac{1}{4}\int_{\mathbb{R}^3} \left(a|\nabla \phi_\varepsilon|^2 + V(\varepsilon x + x_\varepsilon)\phi_\varepsilon{}^2\right) + \left(\frac{1}{4} - \frac{1}{p+1}\right)\int_{\mathbb{R}^3} K(\varepsilon x + x_\varepsilon)|\phi_\varepsilon|^{p+1}\right) \\ &= \varliminf_{\varepsilon \longrightarrow 0} \left(\bar{I}_\varepsilon(\phi_\varepsilon) - \frac{1}{4}\left\langle \bar{I}_\varepsilon'(\phi_\varepsilon), \phi_\varepsilon \right\rangle\right), \end{aligned} \quad (78)$$

where $\bar{I}_\varepsilon(v)$: $\mathscr{H}_\varepsilon \longrightarrow \mathbb{R}$ is defined by

$$\bar{I}_\varepsilon(v) \coloneqq \frac{1}{2}\int_{\mathbb{R}^3} \left(a|\nabla v|^2 + V(\varepsilon x + x_\varepsilon)v^2\right) + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla v|^2\right)^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} K(\varepsilon x + x_\varepsilon)|v|^{p+1}. \quad (79)$$

Then, by Lemma 8,

$$\varliminf_{\varepsilon \longrightarrow 0} \bar{I}_\varepsilon(\phi_\varepsilon) = \varliminf_{\varepsilon \longrightarrow 0} \varepsilon^{-3} I_\varepsilon(v_\varepsilon) = \varliminf_{\varepsilon \longrightarrow 0} \varepsilon^{-3} c_\varepsilon \le \limsup_{\varepsilon \longrightarrow 0} \varepsilon^{-3} c_\varepsilon \le c_0. \quad (80)$$

Thus, by (78), we have $c(x^*) < c_0$ which is a contradiction to (68), and hence, we have proved $A = \int_{\mathbb{R}^3} |\nabla U^*|^2$. From the above arguments, we know that U^* is the unique positive ground state solution of

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla U^*|^2\right)\Delta U^* + V(x^*)U^* = K(x^*)(U^*)^p. \quad (81)$$

Furthermore, from (78), we know that $c(x^*) = c_0$, which implies that x^* is a minimum point of the function $\mathscr{A}(x) = V^{2/(p-1)-1/2}(x)/K^{2/(p-1)}(x)$. Since the proof of the uniqueness of maximum point is just similar as in [37], we omit it here. Now, we have proved Theorem 2. □

Data Availability

The author confirm that the data supporting the findings of this study are available within the article.

Conflicts of Interest

The author declares that they have no conflicts of interest.

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