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Research Article

Nonlinear n-Order m-Point Semipositive Boundary Value Problems and Applications

Hua Su

School of Statistics and Mathematics, Shandong University of Finance and Economics, Jinan, Shandong 250014, China

Correspondence should be addressed to Hua Su; jnsuhua@163.com

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In our paper, we consider the positive solutions of the nonlinear n-order m-point semipositive BVP. In this BVP equation, we allow that f can change the symbol for $0 < t < 1$; by using the fixed point index theory, the existence of positive solutions and many positive solutions are obtained under the condition that f is superlinear or sublinear.

1. Introduction

In our paper, we study the nonlinear n-order m-point semipositive problems:

$$(L\phi)(t) = f(t, \phi(t)), \quad 0 < t < 1, \tag{1}$$

with the following boundary value conditions:

$$\begin{cases} \phi(1) = \sum_{i=1}^{m-2} a_i \phi(\eta_i), \\ \phi^{(i)}(0) = \phi^{(j)}(1) = 0, \end{cases} \tag{2}$$

where $(L\phi)(t) = (-1)^{(n-k)} \phi^{(n)}(t)$, $0 \le i \le k-1, 0 \le j \le n-k-1$, $n \ge 2, 1 < k < n-1$; $a_i \in [0,\infty), i = 1, 2, \cdots, m-2, 0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$, are constants, $m \ge 3$.

For the differential equations, I offer wonderful tools for describing various natural phenomena arising from natural sciences, for example, [1–16]. Very few authors discussed cases (1) and (2). In this paper, under the condition that f is superlinear or sublinear, we focus on the existence of positive solutions for the nonlinear n-order m-point semipositive BVP (1) and (2) under the conditions that $f(t, \varphi)$ is continuous.

2. Preliminaries and Lemmas

Let $E = C[I, R]$ is a Banach space and $\|x\| = \max_{t\in I}|x(t)|$ where $I = [0, 1]$. And $L^1(0, 1)$ with norm $\|x\|_1 = \int_0^1 |x(t)| dt$.

Throughout this paper, we shall use the following notation:

$$G(t,s) = \frac{1}{\rho} \begin{cases} (1-t)^{n-k} \int_0^s \eta^{n-k-1} ((1-t)\eta + t - s)^{k-1} d\eta, \\ 0 \le s \le t \le 1, \\ t^k \int_s^1 (1-\eta)^{k-1} (s - t\eta)^{n-k-1} d\eta, \\ 0 \le t \le s \le 1, \end{cases} \tag{3}$$

where $\rho = (k-1)!(n-k-1)!$.

It is well known from papers [6, 7] that $G(t, s)$ is a nonnegative continuous function, and $G(t, s)$ is the Green's function of the BVP:

$$(L\phi)(t) = 0, t \in I,$$
$$\phi^{(i)}(0) = 0, 0 \le i \le k-1, \phi^{(j)}(1) = 0, 0 \le j \le n-k-1. \tag{4}$$

Let

$$\Phi_2(t) = \frac{(n-1)!}{\rho} \int_0^t s^{k-1}(1-s)^{n-k-1} ds. \tag{5}$$

It is obvious that $\Phi_2(t) \ge 0$ for $t \in I$ and by the properties of Euler integral, we have $\Phi_2(0) = 0, \Phi_2(1) = 1$ and $\|\Phi_2\| = 1$. Surely, for $t \in I$,

$$\Phi_2(t) \ge t^k (1-t)^{n-k} \|\Phi_2\|. \tag{6}$$

and for $t \in I$, we have

$$\Phi_2(t) \le \frac{(n-1)!}{\rho} t^{k-1} \left[1 - (1-t)^{n-k}\right] \le \frac{(n-1)!}{\rho} t^k. \tag{7}$$

Suppose the following conditions hold:
(H_1): $\sum_{i=1}^{m-2} a_i \Phi_2(\eta_i) < 1$.
For $0 \le t, s \le 1$, let

$$K(t,s) = G(t,s) + \left(1 - \sum_{i=1}^{m-2} a_i \Phi_2(\eta_i)\right)^{-1} \Phi_2(t) \sum_{i=1}^{m-2} a_i G(\eta_i, s). \tag{8}$$

$K(t,s)$ is the Green's function of the BVP:

$$(L\phi)(t) = 0, 0 < t < 1,$$
$$\begin{cases} \phi(1) = \sum_{i=1}^{m-2} a_i \phi(\eta_i), & 0 \le i \le k-1, \\ \phi^{(i)}(0) = \phi^{(j)}(1) = 0, & 0 \le j \le n-k-1. \end{cases} \tag{9}$$

By direct computation, we know that.

Lemma 1 (see [6]). *$G(t,s)$ defined as above have the following properties:*

$$Q(s) \ge G(t,s) \ge Q(s)q(t), 0 \le t, s \le 1, \tag{10}$$

where

$$Q(s) = \frac{1}{\rho} s^{n-k}(1-s)^k, q(t) = \frac{t^k(1-t)^k}{n-1}. \tag{11}$$

By Lemma 1 and (7) and (8), it is obvious that

$$AQ(s) \ge K(t,s) \ge Q(s)q(t), 0 \le t, s \le 1, \tag{12}$$

where $A = 1 + \sum_{i=1}^{m-2} a_i (1 - \sum_{i=1}^{m-2} a_i \Phi_2(\eta_i))^{-1}$, $0 \le t, s \le 1$.

Lemma 2. *Suppose $u(t) \in C^n[0,1]$ satisfies the following problem:*

$$(L\phi)(t) = h(t), t \in I,$$
$$\begin{cases} \phi(1) = \sum_{i=1}^{m-2} a_i \phi(\eta_i), & 0 \le i \le k-1, \\ \phi^{(i)}(0) = \phi^{(j)}(1) = 0, & 0 \le j \le n-k-1, \end{cases} \tag{13}$$

where $h \in L^1(0,1), h \ge 0$. Then,

$$\phi(t) \ge \|\phi\| \frac{q(t)}{A}, 0 \le t \le 1. \tag{14}$$

Proof. By $q(t)Q(s) \le K(t,s) \le AQ(s), 0 \le t, s \le 1$, we know that

$$\phi(t) = \int_0^1 K(t,s)h(s)ds \le A \int_0^1 Q(s)h(s)ds, \tag{15}$$

so,

$$\|\phi\| \le A \int_0^1 Q(s)h(s)ds. \tag{16}$$

Therefore,

$$\phi(t) = \int_0^1 K(t,s)h(s)ds \ge q(t) \int_0^1 Q(s)h(s)ds \ge \|\phi\| \frac{q(t)}{A}. \tag{17}$$

□

Lemma 3. *Suppose $\phi(t) \in C^n[0,1]$ satisfies the following problem:*

$$(L\phi)(t) = h(t), 0 < t < 1,$$
$$\begin{cases} \phi(1) = \sum_{i=1}^{m-2} a_i \phi(\eta_i), & 0 \le i \le k-1, \\ \phi^{(i)}(0) = \phi^{(j)}(1) = 0, & 0 \le j \le n-k-1, \end{cases} \tag{18}$$

where $h \in L^1(0,1), h \ge 0$. Then, for any $\theta \in (0, 1/2)$, there exists constant $\eta > 0$ such that

$$\phi(t) \ge \eta \|\phi\|, \theta \le t \le 1 - \theta. \tag{19}$$

Proof. Let $\eta = \max_{\theta \le t \le 1-\theta} q(t)/A$, and then by Lemma 2, we can obtain the results. □

Lemma 4. *Suppose $\bar{w}(t) \in C^n[0,1]$ satisfies the following problem:*

$$\begin{gathered}(L\phi)(t) = M(t), \quad 0 < t < 1, \\ \begin{cases}\phi(1) = \sum_{i=1}^{m-2} a_i \phi(\eta_i), & 0 \le i \le k-1, \\ \phi^{(i)}(0) = \phi^{(j)}(1) = 0, & 0 \le j \le n-k-1,\end{cases}\end{gathered} \tag{20}$$

where $W(t) \in L^1(0,1)$, $W(t) > 0$. Then, there exists constant $C \ge 1$ such that

$$\bar{w}(t) \le C\|W\|_1 q(t), 0 \le t \le 1. \tag{21}$$

Proof. For $t \in [0,1]$, we can have

$$\bar{w}(t) = \int_0^1 G(t,s)W(s)ds + \int_0^1 \left(1 - \sum_{i=1}^{m-2} a_i \Phi_2(\eta_i)\right)^{-1} \Phi_2(t) \sum_{i=1}^{m-2} a_i G(\eta_i, s) W(s) ds. \tag{22}$$

Obviously, for $t \in I$,

$$\begin{aligned}\int_0^1 G(t,s)W(s)ds &= \frac{1}{\rho}\left[\int_0^t (1-t)^{n-k} W(s)ds \int_0^s \eta^{n-k-1}((1-t)\eta + t - s)^{k-1} d\eta + \int_t^1 t^k W(s) ds \int_s^1 (1-\eta)^{k-1}(s - t\eta)^{n-k-1} d\eta\right] \\ &\le \frac{1}{\rho}\left[\int_0^t W(s)(1-t)^{n-k} s^{n-k-1}[(1-s)t]^{k-1} ds + \int_t^1 W(s) t^k (1-s)^{k-1}[s(1-t)]^{n-k-1} ds\right] \\ &\le \frac{1}{\rho}\left[\int_0^t W(s)(1-t)^{n-k} t^k ds + \int_t^1 W(s) t^k (1-t)^{n-k} ds\right] \le \frac{n-1}{\rho} q(t) \int_0^1 W(s) ds.\end{aligned} \tag{23}$$

By the same method, we can get that

$$\begin{aligned}&\int_0^1 \left(1 - \sum_{i=1}^{m-2} a_i \Phi_2(\eta_i)\right)^{-1} \Phi_2(t) \sum_{i=1}^{m-2} a_i G(\eta_i, s) W(s) ds \\ &\quad \le \frac{n-1}{\rho}\left(1 - \sum_{i=1}^{m-2} a_i \Phi_2(\eta_i)\right)^{-1} \sum_{i=1}^{m-2} a_i q(t) \int_0^1 W(s) ds.\end{aligned} \tag{24}$$

So, we can choose the constant

$$C \ge \frac{n-1}{\rho} + \frac{n-1}{\rho}\left(1 - \sum_{i=1}^{m-2} a_i \Phi_2(\eta_i)\right)^{-1} \sum_{i=1}^{m-2} a_i, \tag{25}$$

And we have

$$\bar{w}(t) \le C\|W\|_1 q(t), 0 \le t \le 1. \tag{26}$$

This completes the proof of Lemma 4. □

In the rest of the paper, we also make the following assumptions:

(H_2) $f \in C([0,1] \times [0,+\infty), [-\infty,+\infty))$, and $W(t) > 0 \in L^1(0,1)$

$$f(t,\phi) \ge -W(t), \forall t \in (0,1), \phi \ge 0, \tag{27}$$

where $0 < \int_0^1 Q(s)W(s)ds < \infty$, $C\|W\|_1 < 1$, C is constant in Lemma 4, and $Q(s)$ is the function in Lemma 1.

We denote a cone K:

$$K = \{\phi \in E : \phi(t) \ge \|\phi\| q(t), \theta \le t \le 1 - \theta\}, \tag{28}$$

where $\theta \in (0, 1/2)$.

Set

$$\begin{aligned}R^* &= 2\left(A\eta^2 \int_\theta^{1-\theta} Q(s)ds\right)^{-1}, \\ R_* &= \left(A\int_0^1 (Q(s) + M(s))ds\right)^{-1}.\end{aligned} \tag{29}$$

By Lemma 4, we set $w(t) = \bar{w}(t)$, and for $t \in I$,

$$\begin{gathered}F(t,\phi) = B(t,\phi) + W(t), \\ B(t,\phi) = \begin{cases} f(t,\phi), & \phi \ge 0, \\ \phi < 0, & f(t,0).\end{cases}\end{gathered} \tag{30}$$

Then, $\phi(t) > 0$ is solution of BVP (1) if and only if $\widetilde{\phi}(t) = \phi(t) + w(t)$ is the positive solution of the following BVP(*)

$$\begin{gathered}(L\phi)(t) = F(t, \phi(t) - w(t)), 0 < t < 1, \\ \begin{cases}\phi(1) = \sum_{i=1}^{m-2} a_i \phi(\eta_i), & 0 \le i \le k-1, \\ \phi^{(i)}(0) = \phi^{(j)}(1) = 0, & 0 \le j \le n-k-1.\end{cases}\end{gathered} \tag{31}$$

Clearly, BVP(*) is equivalent to the equation

$$\phi(t) = \int_0^1 K(t,s)F(s,\phi(s)-w(s))ds, \tag{32}$$

i.e.,the fixed point problem $\phi = T\phi$ with operator $T : E \longrightarrow E$ given by

$$(T\phi)(t) = \int_0^1 K(t,s)F(s,\phi(s)-w(s))ds. \tag{33}$$

3. The Existence of Single Positive Solution

In this section, we present our main results by fixed point index theory.

Theorem 5. *If conditions* (H_1) *and* (H_2) *hold. For* $C\|W\|_1 < r < 2C\|W\|_1 < R$, *and* f *also satisfies*

(A_1) *For* $\eta AR/2 \le \phi \le R$, *there has* $f(t,\phi) \ge NR$

(A_2) *For* $0 \le \phi \le r$, *there has* $f(t,\phi) \le mr$

where $N \in [R^*, \infty)$, $m \in (0, R_*]$, $mr \ge 1$

Then, the higher-order nonlinear m-point semipositive BVP (1) and (2) has at least one solution $\phi \in K$ *such that* $\|\phi\|$ *lies between* r *and* R.

Theorem 6. *If conditions* (H_1) *and* (H_2) *hold. And* f *also satisfies*

(A_3) $f_0 = \varphi \in [0, R_* - \alpha)$

(A_4) $f_\infty = \psi \in (4R^*/\eta, \infty)$ *where*

$$f_0 = \lim_{\phi \longrightarrow 0} \max_{0 \le t \le 1} \frac{f(t,\phi)}{\phi}, \qquad f_\infty = \lim_{\phi \longrightarrow \infty} \min_{0 \le t \le 1} \frac{f(t,\phi)}{\phi}. \tag{34}$$

Then, the higher-order nonlinear m-point semipositive boundary value problem (1) and (2) have a solution $\phi \in K$ *such that* $\|\phi\|$ *lies between* r *and* R.

Proof of Theorem 5. Firstly, let Ω_1 and Ω_2 of E:

$$\Omega_1 = \{\phi \in K : \|\phi\| < R\}, \Omega_2 = \{\phi \in K : \|\phi\| < r\}. \tag{35}$$

Then, for $\phi \in \partial\Omega_1$,

$$\phi(t) - w(t) \le \phi(t) \le \|\phi\| = R, t \in I, \tag{36}$$

$$\phi(t) - w(t) \ge \phi(t) - C\|W\|_1 q(t) \ge \phi(t) - \frac{C\|W\|_1}{R}\phi(t) \ge \frac{1}{2}\phi(t), \tag{37}$$

so, for $\theta \le t \le 1-\theta$,

$$\phi(t) - w(t) \ge \frac{\eta R}{2}. \tag{38}$$

And then by (A_1), for $\forall \phi \in \partial\Omega_1$,

$$\begin{aligned}\|T\phi\| &\ge (T\phi)(t) \ge \int_0^1 Q(s)q(t)(f(s,\phi(s)-w(s)) + W(s))ds \\ &\ge A\eta\int_\theta^{1-\theta} Q(s)f(s,\phi(s)-w(s))ds \ge \frac{A}{2}\eta^2 NR\int_\theta^{1-\theta} Q(s)ds \ge R = \|\phi\|.\end{aligned} \tag{39}$$

Therefore, we know that

$$i(T, \Omega_1, K) = 0. \tag{40}$$

Another, for $\phi \in \partial\Omega_2$, we know that

$$\begin{aligned}&\phi(t) - w(t) \le \phi(t) \le \|\phi\| = r, \\ &\phi(t) - w(t) \ge \phi(t) - C\|W\|_1 q(t) \ge \phi(t) - \frac{C\|W\|_1}{r}\phi(t) \ge 0,\end{aligned} \tag{41}$$

and then by (A_2),

$$\begin{aligned}(T\phi)(t) &= \int_0^1 K(t,s)F(s,\phi(s)-w(s))ds \\ &\le A\int_0^1 Q(s)(f(s,\phi(s)-w(s)) + W(s))ds \\ &\le A\int_0^1 (Q(s)mr + W(s))ds \le Amr\int_0^1 (Q(s) + W(s))ds \\ &\le r = \|\phi\|.\end{aligned} \tag{42}$$

Therefore,

$$\|T\phi\| \le \|\phi\|, \forall \phi \in \partial\Omega_2. \tag{43}$$

Then, we know that

$$i(T, \Omega_2, K) = 1. \tag{44}$$

Therefore, by (40) and (44), $r < R$,

$$i\left(T, \Omega_1 \setminus \bar{\Omega}_2, K\right) = -1. \tag{45}$$

Then, operator T has a fixed point $\widetilde{\phi} \in (\Omega_1 \setminus \bar{\Omega}_2)$ and $r \le \|\widetilde{\phi}\| \le R$.

Finally, using Lemmas 3 and 4, we know

$$\widetilde{\phi}(t) \ge \left\|\widetilde{\phi}\right\| q(t) \ge rq(t) > C\|W\|_1 q(t) \ge \bar{w}(t) = w(t), t \in (\theta, 1-\theta), \tag{46}$$

i.e., $\phi(t) = \widetilde{\phi}(t) - w(t) > 0$ is the solution of BVP (1) and (2). This completes the proof of Theorem 5. □

Proof of Theorem 6. Copying Theorem 5. First, by $f_0 = \varphi \in [0, R_* - \alpha)$, for $\epsilon = R_* - \alpha - \varphi$, there exists the number $\rho_1 >$

$C\|W\|_1, \rho(R_* - \alpha) \geq 1$, as $0 \leq u \leq \rho, u \neq 0$, we know that

$$f(t,\phi) \leq (\varphi + \epsilon)\phi = (R_* - \alpha)\rho_1. \quad (47)$$

So, for $r = \rho_1, m = R_* - \alpha \in (0, R_*)$, thus by (47), we know that

$$f(t,\phi) \leq mr, 0 \leq \phi \leq r. \quad (48)$$

So, condition (A_2) holds.

Next, using (A_4), $f_\infty = \psi \in (4R^*/\eta, \infty)$, then for $\epsilon = \lambda - (2R^*/\eta)$, there exists a number $R \neq r$, as $\phi \geq \eta R/2$, we know that

$$f(t,\phi) \geq (\psi - \epsilon)\phi \geq \left(\frac{4R^*}{\eta}\right) \cdot \frac{\eta R}{2} = 2R^* R, \quad (49)$$

Let $N = 2R^* > R^*$, thus, by (49), (A_1) holds. Then, we have that the results of Theorem 6 holds. □

4. The Existence of Many Positive Solutions

Next, we will discuss the existence of many positive solutions.

Theorem 7. *If (H_1), (H_2), and (A_2) hold. And f also satisfies the following conditions:*

(A_5) $f_0 = +\infty$

(A_6) $f_\infty = +\infty$where

$$f_0 = \lim_{\phi \longrightarrow 0} \max_{0 \leq t \leq 1} \frac{f(t,\phi)}{\phi}, f_\infty = \lim_{\phi \longrightarrow \infty} \min_{0 \leq t \leq 1} \frac{f(t,\phi)}{\phi}. \quad (50)$$

Then, the semipositive boundary value problems (1) and (2) have at least two solutions $\phi_1, \phi_2 \in K$ such that $0 < \|\phi_1\| < r < \|\phi_2\|$.

Theorem 8. *If (H_1), (H_2), and (A_1) hold. And f also satisfies the following conditions:*

(A_7) $f_0 = 0$

(A_8) $f_\infty = 0$where

$$f_0 = \lim_{\phi \longrightarrow 0} \max_{0 \leq t \leq 1} \frac{f(t,\phi)}{\phi}, f_\infty = \lim_{\phi \longrightarrow \infty} \min_{0 \leq t \leq 1} \frac{f(t,\phi)}{\phi}. \quad (51)$$

Then, the semipositive boundary value problems (1) and (2) have at least two solutions $\phi_1, \phi_2 \in K$ such that $0 < \|\phi_1\| < R < \|\phi_2\|$.

Proof of Theorem 7. Copying Theorem 5. First, for $N > R^*$, From (A_7), there exists a constant $\rho_* \in (2C\|W\|_1, r)$ which satisfy

$$f(t,\phi) \geq M\phi, 0 < \phi \leq \rho_*, \phi \neq 0, 0 < t < 1. \quad (52)$$

Let $\Omega_{\rho_*} = \{\phi \in K : \|\phi\| < \rho_*\}$. Then, for $t \in I$ and $\phi \in \partial\Omega_{\rho_*}$, we know that

$$\phi(t) - w(t) \leq \phi(t) \leq \|\phi\| = \rho_*, \quad (53)$$

$$\phi(t) - w(t) \geq \phi(t) - C\|W\|_1 q(t) \geq \phi(t) - \frac{C\|W\|_1}{\rho_*}\phi(t) \geq \frac{1}{2}\phi(t), \quad (54)$$

so, for $\theta \leq t \leq 1-\theta, \forall \phi \in \partial\Omega_{\rho_*}$, we know that

$$\phi(t) - w(t) \geq \frac{1}{2}\phi(t) \geq \frac{\|\phi\|}{2}q(t) \geq \frac{\delta\rho_*}{2}. \quad (55)$$

Therefore, from (52), we could obtain the following:

$$\begin{aligned} \|T\phi\| \geq (T\phi)(t) &= \int_0^1 K(t,s)F(s,\phi(s) - w(s))ds \\ &\geq \int_0^1 Q(s)q(t)(f(s,\phi(s) - w(s)) + W(s))ds \\ &\geq \frac{A}{2}\eta^2 N\rho_* \int_\theta^{1-\theta} Q(s)ds \geq \rho_* = \|u\|. \end{aligned}$$

Then,

$$i\left(T, \Omega_{\rho_*}, K\right) = 0. \quad (57)$$

Next, using condition (A_8), for any $\bar{M} > \theta^*$, there exists a constant $\rho_0 > 0$ which satisfies

$$f(t,u) \geq \bar{M}u, u \geq \rho_0, 0 < t < 1. \quad (58)$$

We can choose a constant $\rho^* > \max\{R, 2\rho_0/\eta\}$ which satisfies $\rho_* < R < \rho^*$.

Set $\Omega_{\rho^*} = \{\phi \in K : \|\phi\| < \rho^*\}$. Then,

$$\phi(t) - w(t) \leq \phi(t) \leq \|\phi\| = \rho^*, \quad (59)$$

$$\phi(t) - w(t) \geq \phi(t) - C\|W\|_1 q(t) \geq \phi(t) - \frac{C\|W\|_1}{\rho^*}\phi(t) \geq \frac{1}{2}u(t), \quad (60)$$

where $t \in I$ and $\phi \in \partial\Omega_{\rho^*}$. So, for $\theta \leq t \leq 1-\theta$, we have

$$\phi(t) - w(t) \geq \frac{1}{2}\phi(t) \geq \frac{\|\phi\|}{2}q(t) \geq \frac{\eta\rho^*}{2} \geq \rho_0. \quad (61)$$

And then for $\phi \in \partial\Omega_{\rho^*}$, we can easily obtain that

$$\begin{aligned} \|T\phi\| \geq (T\phi)(t) &\geq \int_0^1 Q(s)q(t)(f(s,\phi(s) - w(s)) + W(s))ds \\ &\geq \eta A \int_\theta^{1-\theta} Q(s)f(s,\phi(s) - w(s))ds \geq \frac{A}{2}\eta^2 \bar{M}R \int_\theta^{1-\theta} Q(s)ds \geq R = \|\phi\|. \end{aligned} \quad (62)$$

Therefore,

$$i\left(T, \Omega_{\rho^*}, K\right) = 0. \quad (63)$$

Finally, letting $\Omega_r = \{\phi \in K : \|\phi\| < r\}$, for any $\phi \in \partial\Omega_r$, by (A_2), Lemma 2, we can also easy to obtain that

$$\|T\phi\| \le \|\phi\|, \forall \phi \in \partial\Omega_r. \tag{64}$$

Then,

$$i(T, \Omega_r, K) = 1. \tag{65}$$

Therefore, by (57)–(65) and $\rho_* < r < \rho^*$, we could obtain the following results:

$$\begin{aligned} &i\left(T, \Omega_r \setminus \bar{\Omega}_{\rho_*}, K\right) = 1, \\ &i\left(T, \Omega_{\rho^*} \setminus \bar{\Omega}_r, K\right) = -1. \end{aligned} \tag{66}$$

Then, T have fixed point $\widetilde{\phi}_1 \in \Omega_r \setminus \bar{\Omega}_{\rho_*}$, and fixed point $\widetilde{\phi}_2 \in \Omega_{\rho^*} \setminus \bar{\Omega}_r$ and $\rho_* < \|\widetilde{\phi}_1\| < r < \|\widetilde{\phi}_2\| \le \rho^*$.

Finally, using Lemmas 3 and 4, for $t \in (\theta, 1-\theta)$, we have

$$\begin{aligned} &\widetilde{\phi}_1(t) \ge \left\|\widetilde{\phi}\right\| q(t) \ge \rho_* q(t) > C\|W\|_1 q(t) \ge \bar{w}(t) = w(t), \\ &\widetilde{\phi}_2(t) \ge \left\|\widetilde{\phi}\right\| q(t) \ge \rho^* q(t) > C\|W\|_1 q(t) \ge \bar{w}(t) = w(t), \end{aligned} \tag{67}$$

i.e., $\phi_1(t) = \widetilde{\phi}_1(t) - w(t), s\phi_2(t) = \widetilde{\phi}_2(t) - w(t)$ are the positive solutions of (1) and (2). □

Proof of Theorem 8. Using the proof of Theorem 5. We only need to discuss the operator T which is given as (33).

First, by $f_0 = 0$, for $\epsilon_1 \in (0, R_*)$, there exists a constant $\rho_* \in (C\|W\|_1, R), \epsilon_1 \rho_* \ge 1$ which satisfy

$$f(t, \phi) \le \epsilon_1 \phi, 0 < \phi \le \rho_*. \tag{68}$$

Set $\Omega_{\rho_*} = \{\phi \in K : \|\phi\| < \rho_*\}$, then for $\phi \in \partial\Omega_{\rho_*}$, $\phi(t) - w(t) \le \phi(t) \le \|\phi\| = \rho_*$, and

$$\phi(t) - w(t) \ge \phi(t) - C\|W\|_1 q(t) \ge \phi(t) - \frac{C\|W\|_1}{\rho_*}\phi(t) \ge 0, \tag{69}$$

and then by (A_2), we have

$$\begin{aligned} (T\phi)(t) &= \int_0^1 K(t,s) F(s, \phi(s) - w(s)) ds \\ &\le A\int_0^1 Q(s)(f(s, \phi(s) - w(s)) + W(s)) ds \\ &\le A\epsilon_1 \rho_* \int_0^1 (Q(s) + W(s)) ds \le \rho_* = \|\phi\|. \end{aligned} \tag{70}$$

So, we have

$$\|T\phi\| \le \|\phi\|, \forall \phi \in \partial\Omega_{\rho_*}. \tag{71}$$

Then, by ((1)), we have

$$i\left(T, \Omega_{\rho_*}, K\right) = 1. \tag{72}$$

Next, letting

$$f^*(x) = \max_{0\le t\le 1, 0\le\phi\le x} f(t, \phi). \tag{73}$$

It is easy to know that $f^*(x)$ is monotone increasing for $x \ge 0$.

Thus, by $f_\infty = 0$, and $\lim_{x\longrightarrow\infty} f^*(x)/x = 0$. Therefore, for any $\epsilon_2 \in (0, \theta_*)$, there exists $\rho^* > r$ such

$$f^*(x) \le \epsilon_2 x, x \le \rho^*. \tag{74}$$

Set $\Omega_{\rho^*} = \{\phi \in K : \|\phi\| < \rho^*\}$, then for any $u \in \partial\Omega_{\rho^*}$,

$$\begin{aligned} T\phi(t) &\le A\int_0^1 Q(s)(f(s, \phi(s) - w(s)) + W(s)) ds \\ &\le A\int_0^1 Q(s)(f(\rho^*) + W(s)) ds \le \epsilon_2 A\rho^* \int_0^1 (Q(s) + W(s)) \le \rho^* = \|\phi\|. \end{aligned} \tag{75}$$

i.e., $\|T\phi\| \le \|\phi\|, \forall \phi \in \partial\Omega_{\rho^*}$. Then, by ((1)), we have

$$i\left(T, \Omega_{\rho^*}, K\right) = 1. \tag{76}$$

Next, similar to Theorem 5, we set $\Omega_R = \{\phi \in K : \|\phi\| < R\}$, and for any $\phi \in \partial\Omega_R$, by Lemma 2, condition (A_1), we can also know that

$$\|T\phi\| \ge \|\phi\|, \forall \phi \in \partial\Omega_R. \tag{77}$$

Then,

$$i(T, \Omega_R, K) = 0. \tag{78}$$

Therefore,

$$si\left(T, \Omega_R \setminus \bar{\Omega}_{\rho_*}, K\right) = -1, i\left(T, \Omega_{\rho^*} \setminus \bar{\Omega}_R, K\right) = 1. \tag{79}$$

Then, T have fixed point $\widetilde{\phi}_1 \in \Omega_R \setminus \bar{\Omega}_{\rho_*}$, and fixed point $\widetilde{\phi}_2 \in \Omega_{\rho^*} \setminus \bar{\Omega}_R$ and $\rho_* < \|\widetilde{\phi}_1\| < R < \|\widetilde{\phi}_2\| \le \rho^*$.

Finally, using Lemmas 3 and 4,

$$\begin{aligned} &\widetilde{\phi}_1(t) \ge \left\|\widetilde{\phi}\right\| q(t) \ge \rho_* q(t) > C\|W\|_1 q(t) \ge \bar{w}(t) = w(t), \\ &\widetilde{\phi}_2(t) \ge \left\|\widetilde{\phi}\right\| q(t) \ge \rho^* q(t) > C\|W\|_1 q(t) \ge \bar{w}(t) = w(t), \end{aligned} \tag{80}$$

here $t \in (\theta, 1-\theta)$. Then, $\phi_1(t) = \widetilde{\phi}_1(t) - w(t) > 0, \phi_2(t) = \widetilde{\phi}_2(t) - w(t) > 0$ are the solutions of BVP (1) and (2). The proof of Theorem 8 is complete. □

5. Application

Example 1. Let $I = [0, 1]$, we consider the following semipositive BVP for $t \in I$:

$$(-1)^3 \phi^{(5)} - \left[\sqrt{e^{\phi(t)} \ln (1 + \phi(t))} - t^5 \right] = 0, \quad (81)$$

with the following boundary value conditions:

$$\phi^{(i)}(0) = 0, i = 1, 2 ; \phi^{(j)}(1) = 0, j = 0, 1, 2. \quad (82)$$

Clearly,

$$f(t, \phi) = \sqrt{e^{\phi(t)} \ln (1 + u(t))} - t^5 \geq -t^5 = -W(t). \quad (83)$$

By direct calculating, we have

$$\int_0^1 t^5 ds = \int_0^1 \frac{1}{6} t^2 dt = \frac{1}{6} < 1. \quad (84)$$

Therefore, using Lemma 4, let $C = 3$ such that $C\|W\|_1 = 1/2 < 1$. Then, (H_2) holds.

By directly calculating, we can be easy to know that $f_0 = 0, f_\infty = \infty$. So, conditions (A_3) and (A_4) hold. Then, let r, R such that $C\|W\|_1 = 1/2 < r \leq 2C\|W\|_1 = 1 < R$. Then by Theorem 6, we have Example 1 has at least one positive solution $\phi(t)$ and $r \leq \|\phi\| \leq R$.

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Disclosure

The preprint of this manuscript can be found in the following: https://assets.researchsquare.com/files/rs-2387096/v1_covered.pdf?c=1671766530.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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