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Research Article

Critical Fractional p-Laplacian System with Negative Exponents

Qinghao Zhu and Jianming Qi

School of Business, Shanghai Dianji University, Shanghai 201306, China

Correspondence should be addressed to Jianming Qi; qijianmingdaxia@163.com

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In this paper, we consider a class of fractional p-Laplacian problems with critical and negative exponents. By decomposition of the Nehari manifold, the existence and multiplicity of nontrivial solutions for the above problems are established with respect to a sufficiently small parameter.

1. Introduction and Main Result

Laplace transformation is an integral transformation commonly used in engineering mathematics. The transformation is a linear transformation that transforms a function with a real parameter number $t(t \geq 0)$ into a function with complex parameters. The Laplace transform has extensive applications in many fields of engineering technology and scientific research, especially when it plays an important role in mechanical systems, electrical systems, automatic control systems, reliability systems, and random service systems. In circuit analysis, it is often necessary to solve the differential equation or the integral equation, which can be solved by the Laplace transformation. The application of nonlinear equations promoted the development of nonlinear sensitive electronic devices on the load side and grid side of the power system. The stable operation of the power system at each level can be effectually protected by exploring the nonlinear phenomena in the case of ferromagnetic resonance overvoltage situation. So, studying the Laplacian system is an important topic.

In this paper, we study the following fractional p-Laplacian system:

$$\begin{cases} (-\Delta)_p^s u = \dfrac{\lambda\alpha}{\alpha+\beta} u^{\alpha-1} v^{\beta} + f(x) u^{-\gamma}, & \text{in } \Omega, \\ (-\Delta)_p^s v = \dfrac{\lambda\beta}{\alpha+\beta} u^{\alpha} v^{\beta-1} + g(x) v^{-\gamma}, & \text{in } \Omega, \\ u > 0, v > 0, & \text{in } \Omega, \\ u = v = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N > ps$ with $s \in (0,1)$, $0 < \gamma < 1 < p$, $\alpha > 1, \beta > 1$ with $\alpha + \beta = p_s^*$, where $p_s^* = Np/N - ps$ is the fractional critical Sobolev exponent, and λ is a parameter. $f(x)$ and $g(x)$ satisfy the conditions $f, g : \Omega \longrightarrow \mathbb{R}$ such that $0 < f, g \in L^{\gamma^*}(\Omega)$, where $\gamma^* = p_s^*/p_s^* - 1 + \gamma$. The fractional p-Laplacian operator $(-\Delta)_p^s$ is defined as

$$(-\Delta)_p^s u(x) = 2 \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(y) - u(x)|^{p-2}(u(y) - u(x))}{|x-y|^{N+ps}} dy, \quad x \in \mathbb{R}^N. \tag{2}$$

We define

$$X = \left\{ u \middle| u : \mathbb{R}^N \longrightarrow \mathbb{R} \text{ is measurable}, u|_\Omega \in L^p(\Omega) \right. \\ \left. \int_Q \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} dxdy < +\infty \right\}, \tag{3}$$

where $Q = \mathbb{R}^{2N} \setminus (\mathscr{C}\Omega \times \mathscr{C}\Omega)$ with $\mathscr{C}\Omega = \mathbb{R}^N \setminus \Omega$ and $X_0 = \{ u \in X \mid u = 0 \; on \, \mathscr{C}\Omega \}$. The space X is equipped with the norm $\|u\|_X = \|u\|_{L^p(\Omega)} + \|u\|_{X_0}$, where

$$\|u\|_{X_0}^p = \int_Q \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} dxdy. \tag{4}$$

Set $E = X_0 \times X_0$ with the norm $\|u, v\|^p = \|u\|_{X_0}^p + \|v\|_{X_0}^p$. $(u, v) \in E$ is called a weak solution of problem (1) if

$$\int_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi_1(x)-\varphi_1(y))}{|x-y|^{N+ps}}dxdy + \int_Q \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\varphi_2(x)-\varphi_2(y))}{|x-y|^{N+ps}}dxdy = \frac{\lambda\alpha}{\alpha+\beta}\int_\Omega u^{\alpha-1}v^\beta\varphi_1 dx + \frac{\lambda\beta}{\alpha+\beta}\int_\Omega u^\alpha v^{\beta-1}\varphi_2 dx + \int_\Omega \left(f(x)u^{-\gamma}\varphi_1 + g(x)v^{-\gamma}\varphi_2\right)dx \tag{5}$$

for all $(\varphi_1, \varphi_2) \in E$. The best fractional critical Sobolev constant S is defined as

$$S = \inf_{u\in X_0\setminus\{0\}} \frac{\int_{\mathbb{R}^{2N}} |u(x)-u(y)|^p/|x-y|^{N+ps}dxdy}{\left(\int_\Omega |u(x)|^{Np/N-ps}dx\right)^{N-ps/N}}. \tag{6}$$

In recent years, fractional Laplacian and p-Laplacian systems with subcritical and critical nonlinearities have been studied widely. Chen and Deng [1] and Li and Yang [2] studied the following critical fractional Laplacian system with a lower-order term:

$$\begin{cases} (-\Delta)^s u = \lambda|u|^{q-2}u + \dfrac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^\beta, & \text{in } \Omega, \\ (-\Delta)^s v = \mu|v|^{q-2}v + \dfrac{2\beta}{\alpha+\beta}|u|^\alpha|v|^{\beta-2}v, & \text{in } \Omega, \\ u = v = 0, & \text{in } \mathbb{R}^N\setminus\Omega, \end{cases} \tag{7}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain with a smooth boundary, $0 < s < 1$, $1 < q < 2$, $\alpha, \beta > 1$ satisfy $\alpha+\beta = 2_s^*$, where $2_s^* = 2N/N-2s$ and $N > 2s$, $\alpha, \beta > 0$ are parameters. The main difficulty lies in finding the interval of c where the local $(PS)_c$ condition is satisfied. The authors both adopted the explicit formula of extremal function related to the best Sobolev constant and some useful estimates established in Barrios et al. [3] (Lemma 3.8) to overcome this difficulty. Compared with the fractional Laplacian system with critical nonlinearities, for the critical fractional p-Laplacian system with $p \neq 2$, we must face the difficulty that the explicit formula for minimizers of critical Sobolev constant S does not exist. Chen and Squassina [4] overcame this difficulty by borrowing the asymptotic estimates for minimizers of S, which were obtained in Brasco et al. [5].

On the other hand, much attention has been focused on discussing the fractional p-Laplacian system with negative exponent and subcritical nonlinearity. In [6], Goyal first investigated the following fractional Laplacian system with sign-changing nonlinearity:

$$\begin{cases} (-\Delta)^s u = \lambda f(x)u^{-q} + \dfrac{\alpha}{\alpha+\beta}b(x)u^{\alpha-1}v^\beta, & \text{in } \Omega, \\ (-\Delta)^s w = \mu g(x)w^{-q} + \dfrac{\beta}{\alpha+\beta}b(x)u^\alpha w^{\beta-1}, & \text{in } \Omega, \\ u > 0, w > 0, & \text{in } \Omega, \\ u = w = 0, & \text{in } \mathbb{R}^N\setminus\Omega, \end{cases} \tag{8}$$

where $0 < q < 1$, $\alpha > 1$, $\beta > 1$, $2 < \alpha+\beta < 2_s^* - 1$ and $b(x)$ is a sign-changing function. Using the decomposition of the Nehari manifold, the multiplicity of positive solutions for (8) with respect to the pair of parameters (λ, μ) was established. Furthermore, the author extended the above same result to the following p-fractional Laplacian system:

$$\begin{cases} (-\Delta)_p^s u = \lambda f(x)u^{-q} + \dfrac{\alpha}{\alpha+\beta}b(x)u^{\alpha-1}v^\beta, & \text{in } \Omega, \\ (-\Delta)_p^s w = \mu g(x)w^{-q} + \dfrac{\beta}{\alpha+\beta}b(x)u^\alpha w^{\beta-1}, & \text{in } \Omega, \\ u > 0, w > 0, & \text{in } \Omega, \\ u = w = 0, & \text{in } \mathbb{R}^N\setminus\Omega, \end{cases} \tag{9}$$

where $\alpha > 1$, $\beta > 1$, $0 < q < 1 \le p-1 < \alpha+\beta < p_s^* - 1$, $b(x)$ is a sign-changing function. Very recently, Saoudi [7] investigated the following fractional p-Laplacian system:

$$\begin{cases} (-\Delta)_p^s u = \lambda a(x)|u|^{q-2}u + \dfrac{1-\alpha}{2-\alpha-\beta}c(x)|u|^{-\alpha}|v|^{1-\beta}, & \text{in } \Omega, \\ (-\Delta)_p^s v = \mu b(x)|v|^{q-2}v + \dfrac{1-\beta}{2-\alpha-\beta}c(x)|u|^{1-\alpha}|v|^{-\beta}, & \text{in } \Omega, \\ u = v = 0, & \text{in } \mathbb{R}^N\setminus\Omega, \end{cases} \tag{10}$$

where $0 < \alpha < 1$, $0 < \beta < 1$, $2-\alpha-\beta < p < q < p_s^*$. Using the variational method, the author proved that (10) has at least two positive solutions when the pair of parameters (λ, μ) satisfies certain conditions. We have found that it is easier to deal with subcritical problems than critical ones because compact conditions for Sobolev embedding are satisfied in the subcritical case, while global $(PC)_c$ condition for the energy functional corresponding to (1) does not usually hold in the case of critical problems. Moreover, Arora and Fiscella [8] studied a class of double-phase problems with a negative exponent and a critical Sobolev nonlinearity. The proof of the main result is based on a suitable minimization argument on the Nehari manifold. Zuo et al. [9] combined the effects of a nonlocal operator with critical nonlinearity. Suitable embedding results were developed to establish the existence of infinitely many solutions and to provide an estimate of the boundedness of these solutions. For more recent results on fractional p-Laplacian problems with singular terms and critical terms simultaneously, see [10–13] and references therein. In fact, Ghanmi et al. [10] studied a class of nonlocal p-Kirchhoff problems with a negative exponent and critical nonlinearity. The authors established the multiplicity of positive solutions to the above problems by using a truncation argument. Sang [11] considered a fractional critical system with p-Laplacian operator and negative exponents. By applying fibering map analysis, the existence of two positive solutions for the above systems was obtained. Furthermore, Saoudi et al. [12] proved the existence of solutions to a nonlocal problem with a singular term and a discontinuous critical nonlinearity. Fiscella et al. [13] investigated the existence of nontrivial solutions for critical systems driven by the fractional p-Laplacian operator. The main features of this paper are the presence of critical nonlinearities and singular terms.

Motivated by the above results, we consider fractional critical p-Laplacian system (1). We combine critical problems with negative exponents. Note that the energy functional corresponding to (1) is not differentiable in the sense of Gâteaux; the method used in [1, 2, 4, 14–20] cannot be applied to our problem directly. Since $p \neq 2$, we cannot also extend the methods used in a single equation with critical and negative exponents [21, 22] when $p = 2$ to problem (1). We use the concentration compactness principle [23, 24] to avoid this barrier. Our idea comes from Wang et al. [25].

The energy functional associated with problem (1) is defined by

$$I_\lambda(u.v) = \frac{1}{p}\|u, v\|^p - \frac{\lambda}{\alpha+\beta}\int_\Omega u^\alpha v^\beta dx - \frac{1}{1-\gamma}\int_\Omega \left[f(x)u^{1-\gamma} + g(x)v^{1-\gamma}\right]dx. \quad (11)$$

We define a set

$$\mathcal{N}_\lambda = \left\{(u, v) \in E \,|\, \|u, v\|^p - \lambda\int_\Omega u^\alpha v^\beta dx - \int_\Omega \left[f(x)u^{1-\gamma} + g(x)v^{1-\gamma}\right]dx = 0\right\} \quad (12)$$

and decompose $\mathcal{N}_\lambda$ with the following subsets:

$$\mathcal{N}_\lambda^\pm = \left\{(u, v) \in \mathcal{N}_\lambda \,|\, (p-1+\gamma)\|u, v\|^p - \lambda(\alpha+\beta-1+\gamma)\int_\Omega u^\alpha v^\beta dx > (<) 0\right\},$$
$$\mathcal{N}_\lambda^0 = \left\{(u, v) \in \mathcal{N}_\lambda \,|\, (p-1+\gamma)\|u, v\|^p - \lambda(\alpha+\beta-1+\gamma)\int_\Omega u^\alpha v^\beta dx = 0\right\}. \quad (13)$$

Our main result is the following.

Theorem 1. *There exists $\lambda^* > 0$ such that for every $\lambda \in (0, \lambda^*)$, problem (1) has at least two nontrivial solutions (u_1, v_1) and (u_2, v_2) in E. More specifically, (u_1, v_1) is a local minimizer of I_λ in E with $I_\lambda(u_1, v_1) < 0$, and $(u_2, v_2) \in \mathcal{N}_\lambda^-$ is a minimizer of I_λ on $\mathcal{N}_\lambda^-$ with $I_\lambda(u_2, v_2) \geq 0$.*

2. Preliminaries and Some Lemmas

Let

$$S_{\alpha,\beta} \coloneqq \inf_{(u,v)\in E\setminus\{(0,0)\}} \frac{\|u, v\|^p}{\left(\int_\Omega |u|^\alpha |v|^\beta dx\right)^{p/\alpha+\beta}}, \quad (14)$$

and the relationship between $S_{\alpha,\beta}$ and S has been revealed in [4]. In order to prove our main result, the following lemmas are needed.

Lemma 2. *The functional I_λ has a local minimum m in E with m < 0.*

Proof. Since

$$\begin{aligned}\int_Q f(x)|u|^{1-\gamma} + \int_\Omega g(x)|v|^{1-\gamma}dx &\leq \|f\|_{\gamma^*}\|u\|_{p_s^*}^{1-\gamma} + \|g\|_{\gamma^*}\|v\|_{p_s^*}^{1-\gamma} \\ &\leq \|f\|_{\gamma^*}\left(\frac{\|u\|_{X_0}}{\sqrt[p]{S}}\right)^{1-\gamma} + \|g\|_{\gamma^*}\left(\frac{\|u\|_{X_0}}{\sqrt[p]{S}}\right)^{1-\gamma} \\ &\leq \left(\|f\|_{\gamma^*}^{p/1+\gamma} + \|g\|_{\gamma^*}^{p/1+\gamma}\right)^{1+\gamma/p}\left(\frac{\|u\|_{X_0}^p + \|v\|_{X_0}^p}{S}\right)^{1-\gamma/p} \\ &= \left(\|f\|_{\gamma^*}^{p/1+\gamma} + \|g\|_{\gamma^*}^{p/1+\gamma}\right)^{1+\gamma/p}\left(\frac{\|u, v\|}{\sqrt[p]{S}}\right)^{1-\gamma},\end{aligned} \quad (15)$$

$$\int_\Omega u^\alpha v^\beta dx \leq \left(\frac{\|u, v\|^p}{S_{\alpha,\beta}}\right)^{\alpha+\beta/p} \leq S_{\alpha,\beta}^{-\alpha+\beta/p}\|u, v\|^{\alpha+\beta}, \quad (16)$$

for every $(u, v) \in E$, where we have used the Hölder inequality, (6) and (14). We come to

$$I_\lambda(u, v) \geq \frac{1}{p}\|u, v\|^p - \frac{\lambda}{\alpha+\beta}S_{\alpha,\beta}^{-\alpha+\beta/p}\|u, v\|^{\alpha+\beta} - C_1\|u, v\|^{1-\gamma}, \quad \forall (u, v) \in E. \quad (17)$$

Hence, there exist $\lambda_1 > 0$ and $R > 0$ such that I_λ is bounded on $B_R = \{(u, v) \in E \,|\, \|u, v\| \leq R\}$ for every $\lambda \in (0, \lambda_1)$. It follows that $m = \inf_{(u.v)\in B_R} I_\lambda(u, v)$ is well defined for fixed $\lambda \in (0, \lambda_1)$. Furthermore, choosing $(u, v) \in B_R$ with all u, $v \neq 0$, we have

$$\lim_{t\longrightarrow 0^+} \frac{I_\lambda(tu, tv)}{t^{1-\gamma}} = -\frac{1}{1-\gamma}\int_\Omega \left[f(x)u^{1-\gamma} + g(x)v^{1-\gamma}\right]dx < 0; \quad (18)$$

thus, $I_\lambda(tu, tv) < 0$ for all u, $v \neq 0$ and t small enough. Consequently, $m < 0$. □

Lemma 3. *There exists $(u_1, v_1) \in B_R$ such that $I_\lambda(u_1, v_1) = m$.*

Proof. The definition of m tells us that there exists a minimizing sequence $\{(u_n, v_n)\}$ such that $\lim_{n\to\infty} I_\lambda(u_n, v_n) = m < 0$. We assume that $u_n \geq 0$ and $v_n \geq 0$. By $\|u_n, v_n\| \leq R$, there is a subsequence, which is still denoted by $\{(u_n, v_n)\}$, such that

$$(u_n, v_n) \rightharpoonup (u_1, v_1) \text{ in } E. \tag{19}$$

In terms of the fractional concentration compactness principle [10] (Theorem 2.5), there exist two Borel regular measures μ and ν, J denumerable, $x_j \in \bar{\Omega}$, $\mu_j \geq 0$, $\nu_j \geq 0$ with $\mu_j + \nu_j > 0$, $j \in J$, such that

$$\int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}} dy + \int_{\mathbb{R}^N} \frac{|v_n(x) - v_n(y)|^p}{|x-y|^{N+ps}} dy \rightharpoonup d\mu, u_n^\alpha v_n^\beta \rightharpoonup d\nu, \tag{20}$$

$$d\mu \geq \int_{\mathbb{R}^N} \frac{|u_1(x) - u_1(y)|^p}{|x-y|^{N+ps}} dy + \int_{\mathbb{R}^N} \frac{|v_1(x) - v_1(y)|^p}{|x-y|^{N+ps}} dy + \sum_{j\in J} \mu_j \delta_{x_j}, d\nu = u_1^\alpha v_1^\beta + \sum_{j\in J} \nu_j \delta_{x_j}, \mu_j \geq S\nu_j^{p/p_s^*}. \tag{21}$$

It follows from Vitali's theorem that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \left[f(x)u_n^{1-\gamma} + g(x)v_n^{1-\gamma}\right] dx = \int_{\mathbb{R}^N} \left[f(x)u_1^{1-\gamma} + g(x)v_1^{1-\gamma}\right] dx, \tag{22}$$

which, coupled with (20) and (21), gives

$$\begin{aligned} m &= \lim_{n\to\infty} \left[\frac{1}{p}\|u_n, v_n\|^p - \frac{1}{1-\gamma}\int_{\mathbb{R}^N} \left[f(x)u_n^{1-\gamma} + g(x)v_n^{1-\gamma}\right] dx \right. \\ &\quad \left. - \frac{\lambda}{\alpha+\beta}\int_\Omega u_n^\alpha v_n^\beta dx\right] \geq \frac{1}{p}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_1(x) - u_1(y)|^p}{|x-y|^{N+ps}} dydx \right. \\ &\quad \left. + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_1(x) - v_1(y)|^p}{|x-y|^{N+ps}} dydx + \sum_{j\in J} S\nu_j^{p/p_s^*}\right) \\ &\quad - \frac{\lambda}{\alpha+\beta}\left(\int_{\mathbb{R}^N} u_1^\alpha v_1^\beta dx + \sum_{j\in J}\nu_j\right) - \frac{1}{1-\gamma}\int_{\mathbb{R}^N} \\ &\quad \left[f(x)u_1^{1-\gamma} + g(x)v_1^{1-\gamma}\right] dx. \end{aligned} \tag{23}$$

If $J = \varnothing$, then

$$\begin{aligned} m &\geq \frac{1}{p}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_1(x) - u_1(y)|^p}{|x-y|^{N+ps}} dydx + \int_{\mathbb{R}^N} \right. \\ &\quad \left. \int_{\mathbb{R}^N} \frac{|v_1(x) - v_1(y)|^p}{|x-y|^{N+ps}} dydx\right) - \frac{\lambda}{\alpha+\beta}\int_{\mathbb{R}^N} u_1^\alpha v_1^\beta dx - \frac{1}{1-\gamma}\int_{\mathbb{R}^N} \\ &\quad \left[f(x)u_1^{1-\gamma} + g(x)v_1^{1-\gamma}\right] dx. \end{aligned} \tag{24}$$

Combining with the definition of m, we deduce that

$$\begin{aligned} m &= \frac{1}{p}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_1(x) - u_1(y)|^p}{|x-y|^{N+ps}} dydx + \int_{\mathbb{R}^N} \right. \\ &\quad \left. \int_{\mathbb{R}^N} \frac{|v_1(x) - v_1(y)|^p}{|x-y|^{N+ps}} dydx\right) - \frac{\lambda}{\alpha+\beta}\int_{\mathbb{R}^N} u_1^\alpha v_1^\beta dx - \frac{1}{1-\gamma}\int_{\mathbb{R}^N} \\ &\quad \left[f(x)u_1^{1-\gamma} + g(x)v_1^{1-\gamma}\right] dx. \end{aligned} \tag{25}$$

In the following, we focus on showing $J = \varnothing$. Firstly, by (21), we have $\int_{\mathbb{R}^N} d\nu \geq \int_{\mathbb{R}^N} u_1^\alpha v_1^\beta dx$, so we only need to prove that $\int_{\mathbb{R}^N} d\nu = \int_{\mathbb{R}^N} u_1^\alpha v_1^\beta dx$. We assume by contradiction that $\int_{\mathbb{R}^N} d\nu > \int_{\mathbb{R}^N} u_1^\alpha v_1^\beta dx$. Since $\sum_{j\in J}\nu_j = \int_{\mathbb{R}^N} d\nu - \int_{\mathbb{R}^N} u_1^\alpha v_1^\beta dx > 0$, we derive that

$$m \leq m - \frac{1}{p}\sum_{j\in J} S\nu_j^{p/p_s^*} + \frac{\lambda}{\alpha+\beta}\sum_{j\in J}\nu_j. \tag{26}$$

If $0 < \nu_j < 1$, for all $j \in J$, it follows that

$$m \leq m - \frac{1}{p}\sum_{j\in J} S\nu_j + \frac{\lambda}{\alpha+\beta}\sum_{j\in J}\nu_j = m + \left(\frac{\lambda}{\alpha+\beta} - \frac{S}{p}\right)\sum_{j\in J}\nu_j, \tag{27}$$

which is wrong if we choose $\lambda < (\alpha+\beta)S/p$.

If there exists a subsequence $\{\nu_j\}(j \in \{1, 2, \cdots\})$ such that $\nu_j \geq 1$, then

$$m \leq m + \left(\frac{\lambda}{\alpha+\beta} - \frac{S}{p}\right)\sum_{j\in J\backslash\{j|\nu_j\geq 1\}}\nu_j + \frac{\lambda}{\alpha+\beta}\sum_{\nu_j\geq 1}\nu_j - \frac{1}{p}\sum_{\nu_j\geq 1} S\nu_j^{p/\alpha+\beta} < m, \tag{28}$$

where $\lambda < \widehat{\lambda}_2$ with $(\widehat{\lambda}_2/\alpha+\beta - S/p)\sum_{j\in J\backslash\{j|\nu_j\geq 1\}}\nu_j + \widehat{\lambda}_2/\alpha+\beta \sum_{\nu_j\geq 1}\nu_j < 0$, which is a contradiction. Set $\lambda < \lambda_2 = \min\{(\alpha+\beta)S/p, \widehat{\lambda}_2\}$, we have that $J = \varnothing$. This completes the proof of Lemma 3. □

Lemma 4. *I is coercive in Λ.*

Proof. For $(u, v) \in \Lambda$, we have

$$\begin{aligned} I_\lambda(u, v) &= \left(\frac{1}{p} - \frac{1}{\alpha+\beta}\right)\|u, v\|^p - \left(\frac{1}{1-\gamma} - \frac{1}{\alpha+\beta}\right)\int_\Omega \\ &\quad \left[f(x)u^{1-\gamma} + g(x)v^{1-\gamma}\right] dx \geq \left(\frac{1}{p} - \frac{1}{\alpha+\beta}\right)\|(u, v)\|^p - \\ &\quad \left(\frac{1}{1-\gamma} - \frac{1}{\alpha+\beta}\right)\left(\|f\|_{\gamma^*}^{p/1+\gamma} + \|g\|_{\gamma^*}^{p/1+\gamma}\right)^{1+\gamma/p} S^{\gamma-1/p}\|u, v\|^{1-\gamma}, \end{aligned} \tag{29}$$

it follows from $p > 1-\gamma$ that I_λ is coercive on Λ. □

Let

$$\lambda_3 = \frac{p-1+\gamma}{\alpha+\beta-1+\gamma} S_{\alpha,\beta}^{\alpha+\beta/p} \left[\frac{\alpha+\beta-p}{\alpha+\beta-1+\gamma}\left(\|f\|_{\gamma^*}^{p/1+\gamma}+\|g\|_{\gamma^*}^{p/1+\gamma}\right)^{-1+\gamma/p} S^{1-\gamma/p}\right]^{\alpha+\beta-p/\gamma+p-1}. \tag{30}$$

Lemma 5. *There exist two and only two numbers t_1 and t_2 with $0<t_1<t_2$ such that*

$$(t_1u, t_1v) \in \Lambda^+, (t_2u, t_2v) \in \Lambda^-, \tag{31}$$

for all $\lambda \in (0, \lambda_3)$ and $(u, v) \in E$.

Proof. For $t>0$, we define $\phi_{u,v}(t)$: $(0,+\infty)\times E \longrightarrow \mathbb{R}$ by

$$\phi_{u,v}(t) = t^{p-(\alpha+\beta)}\|u,v\|^p - \lambda\int_\Omega u^\alpha v^\beta dx - t^{1-\gamma-(\alpha+\beta)}\int_\Omega \left[f(x)u^{1-\gamma}+g(x)v^{1-\gamma}\right]dx. \tag{32}$$

$$t \coloneqq t_{\max} = \left(\frac{(\alpha+\beta-p)\|u,v\|^p}{(\alpha+\beta-1+\gamma)\int_\Omega[f(x)u^{1-\gamma}+g(x)v^{1-\gamma}]dx}\right)^{-1/p-1+\gamma},$$
$$\phi''_{u,v}(t_{\max}) = \|u,v\|^p(p-\alpha-\beta)(p-1+\gamma)\left(\frac{(\alpha+\beta-p)\|(u,v)\|^p}{(\alpha+\beta-1+\gamma)\int_\Omega[f(x)u^{1-\gamma}+g(x)v^{1-\gamma}]dx}\right)^{p-\alpha-\beta-2/1-\gamma-p} < 0. \tag{33}$$

Therefore, $\phi_{u,v}(t)$ achieves its maximum at $t_{\max}$, and

$$\begin{aligned}\phi_{u,v}(t_{\max}) &= \left(\frac{p-1+\gamma}{\alpha+\beta-1+\gamma}\right)\left(\frac{\alpha+\beta-p}{\alpha+\beta-1+\gamma}\right)^{\alpha+\beta-p/p-1+\gamma} \frac{\|(u,v)\|^{p\alpha+\beta-1+\gamma/p-1+\gamma}}{\left(\int_\Omega[f(x)u^{1-\gamma}+g(x)v^{1-\gamma}]dx\right)^{\alpha+\beta-p/p-1+\gamma}} - \lambda\int_\Omega u^\alpha v^\beta dx > \\ &\left(\frac{p-1+\gamma}{\alpha+\beta-1+\gamma}\right)\left(\frac{\alpha+\beta-p}{\alpha+\beta-1+\gamma}\right)^{\alpha+\beta-p/p-1+\gamma} \frac{\|u,v\|^{p\alpha+\beta-1+\gamma/p-1+\gamma}}{\left[\left(\|f\|_{\gamma^*}^{p/1+\gamma}+\|g\|_{\gamma^*}^{p/1+\gamma}\right)^{1+\gamma/p}\left(\|(u,v)\|/\sqrt[p]{S}\right)^{1-\gamma}\right]^{\alpha+\beta-p/p-1+\gamma}} \\ &- \lambda S_{\alpha,\beta}^{-\alpha+\beta/p}\|u,v\|^{\alpha+\beta} = \left[\left(\frac{p-1+\gamma}{\alpha+\beta-1+\gamma}\right)\left(\frac{\alpha+\beta-p}{\alpha+\beta-1+\gamma}\right)^{\alpha+\beta-p/p-1+\gamma} \left(\sqrt[p]{S}\right)^{(1-\gamma)(\alpha+\beta-p)/p-1+\gamma}\left(\|f\|_{\gamma^*}^{p/1+\gamma}+\|g\|_{\gamma^*}^{p/1+\gamma}\right)^{-(1+\gamma)(\alpha+\beta-p)/p(p-1+\gamma)} \right.\\ &\left. - \lambda S_{\alpha,\beta}^{-\alpha+\beta/p}\right]\|u,v\|^{\alpha+\beta} \coloneqq E(\lambda)\|u,v\|^{\alpha+\beta}.\end{aligned} \tag{34}$$

Note that $E(\lambda)=0$ if and only if $\lambda=\lambda_3$. By the condition that $\lambda<\lambda_3$, we deduce that $E(\lambda)>0$ and $\phi_{u,v}(t_{\max})>0$. A simple computation leads to

$$\lim_{t\longrightarrow 0^+}\phi = \infty, \lim_{t\longrightarrow +\infty}\phi = -\lambda\int_\Omega u^\alpha v^\beta dx < 0. \tag{35}$$

Consequently, $\phi_{u,v}(t)$ has exactly two zero points t_1 and t_2 with $0<t_1<t_2$ such that $\phi'(t_2)<0<\phi'(t_1)$.

Let $\phi'_{u,v}(t)=0$, which yields
If $\phi_{u,v}(t)=0$ and $\phi_{u,v}(t)>0$, then

$$\|tu,tv\|^p - \lambda\int_\Omega (tu)(tv)^\beta dx - \int_\Omega\left[f(x)(tu)^{1-\gamma}+g(x)(tv)^{1-\gamma}\right]dx = 0, \tag{36}$$

so $(tu,tv)\in\Lambda$.

Furthermore, $\phi'_{u,v}(t)>0$ implies that

$$(p-(\alpha+\beta))t^{p-\alpha-\beta-1}\|u,v\|^p + (\gamma+\alpha+\beta-1)t^{-\gamma-\alpha-\beta}\int_\Omega \left[f(x)u^{1-\gamma}+g(x)v^{1-\gamma}\right]dx > 0; \tag{37}$$

by the definition of Λ, we have

$$(p-1+\gamma)\|(tu,tv)\|^p - \lambda(\alpha+\beta-1+\gamma)\int_\Omega (tu)^\alpha(tv)^\beta dx > 0, \tag{38}$$

namely, $(tu,tv)\in\Lambda^+$. Therefore, $(t_1u, t_1v)\in\Lambda^+$. Similarly, we get $(t_2, u, t_2v)\in\Lambda^-$. □

Lemma 6. *For $\lambda\in(0,\lambda_3)$, we have $\Lambda^0=\{(0,0)\}$.*

Proof. Assume that $\Lambda^0 \neq \{(0,0)\}$. If $(u,v)\in\Lambda^0$, then

$$\|u,v\|^p - \lambda\int_\Omega u^\alpha v^\beta dx - \int_\Omega\left[f(x)\|u^{1-\gamma}+g(x)v^{1-\gamma}\right]dx = 0, \tag{39}$$

$$(p-1+\gamma)\|u,v\|^p-\lambda(\alpha+\beta-1+\gamma)\int_\Omega u^\alpha v^\beta dx=0. \quad (40)$$

By (39) and (40), we get

$$\begin{aligned}\|u,v\|^p&=\frac{\alpha+\beta-1+\gamma}{\alpha+\beta-p}\int_\Omega\left[f(x)u^{1-\gamma}+g(x)v^{1-\gamma}\right]dx\\&\le\frac{\alpha+\beta-1+\gamma}{\alpha+\beta-p}\left(\|f\|_{\gamma^*}^{p/1+\gamma}+\|g\|_{\gamma^*}^{p/1+\gamma}\right)^{1+\gamma/p}S^{-1-\gamma/p}\|u,v\|^{1-\gamma},\end{aligned} \quad (41)$$

$$\begin{aligned}\|u,v\|^p&=\lambda\frac{\alpha+\beta-1+\gamma}{p-1+\gamma}\int_\Omega u^\alpha v^\beta dx\\&\le\lambda\frac{\alpha+\beta-1+\gamma}{p-1+\gamma}S_{\alpha,\beta}^{-\alpha+\beta/p}\|(u,v)\|^{\alpha+\beta}.\end{aligned} \quad (42)$$

Combining with (41) and (42), we derive that

$$\begin{aligned}\lambda&\ge\frac{p-1+\gamma}{\alpha+\beta-1+\gamma}S_{\alpha,\beta}^{\alpha+\beta/p}\|u,v\|^{p-\alpha-\beta}\\&\ge\frac{p-1+\gamma}{\alpha+\beta-1+\gamma}S_{\alpha,\beta}^{\alpha+\beta/p}\\&\left[\frac{\alpha+\beta-p}{\alpha+\beta-1+\gamma}\left(\|f\|_{\gamma^*}^{p/1+\gamma}+\|g\|_{\gamma^*}^{p/1+\gamma}\right)^{-1+\gamma/p}S^{1-\gamma/p}\right]^{\alpha+\beta-p/\gamma+p-1}\\&=\lambda_3,\end{aligned} \quad (43)$$

which contradicts with $\lambda\in(0,\lambda_3)$. □

Lemma 7. *Assume that $\lambda\in(0,\lambda_3)$, then Λ^- is a closed set in E-topology.*

Proof. Assume that $\{(u_n,v_n)\}\subset\Lambda^-$ and $(u_n,v_n)\longrightarrow(u,v)$ strongly in E. In the following, we prove that $(u,v)\in\Lambda^-$. For $\{(u_n,v_n)\}\subset\Lambda^-$, we have

$$(p-1+\gamma)\|u_n,v_n\|^p-\lambda(\alpha+\beta-1+\gamma)\int_\Omega u_n^\alpha v_n^\beta<0, \quad (44)$$

and so,

$$(p-1+\gamma)\|u,v\|^p-\lambda(\alpha+\beta-1+\gamma)\int_\Omega u^\alpha v^\beta\le0. \quad (45)$$

Therefore, $(u,v)\in\Lambda^-\cup\Lambda^0$. We can show that $(u,v)\in\Lambda^0$ by contradiction. Otherwise, if $(u,v)\in\Lambda^0$, we get that from the definition of Λ^0 that

$$\|u,v\|^p-\lambda\int_\Omega u^\alpha v^\beta dx-\int_\Omega\left[f(x)u^{1-\gamma}+g(x)v^{1-\gamma}\right]dx=0, \quad (46)$$

$$(p-1+\gamma)\|u,v\|^p-\lambda(\alpha+\beta-1+\gamma)\int_\Omega u^\alpha v^\beta dx=0. \quad (47)$$

By (34), (46), and (47), we deduce that

$$\begin{aligned}E(\lambda)\|u,v\|^{\alpha+\beta}&<\left(\frac{p-1+\gamma}{\alpha+\beta-1+\gamma}\right)\left(\frac{\alpha+\beta-p}{\alpha+\beta-1+\gamma}\right)^{\alpha+\beta-p/p-1+\gamma}\\&\frac{\|u,v\|^{p\alpha+\beta-1+\gamma/p-1+\gamma}}{\left(\int_\Omega[f(x)u^{1-\gamma}+g(x)v^{1-\gamma}]dx\right)^{\alpha+\beta-p/p-1+\gamma}}\\&-\lambda\int_\Omega u^\alpha v^\beta dx=\left(\frac{p-1+\gamma}{\alpha+\beta-1+\gamma}\right)\\&\left(\frac{\alpha+\beta-p}{\alpha+\beta-1+\gamma}\right)^{\alpha+\beta-p/p-1+\gamma}\\&\frac{\|u,v\|^{p\alpha+\beta-1+\gamma/p-1+\gamma}}{\left((\alpha+\beta-p)\|u,v\|^p/\alpha+\beta-1+\gamma\right)^{\alpha+\beta-p/p-1+\gamma}}\\&-\frac{p-1+\gamma}{\alpha+\beta-1+\gamma}\|u,v\|^p=0,\end{aligned} \quad (48)$$

which contradicts with $E(\lambda)>0$ for all $\lambda\in(0,\lambda_3)$. □

Lemma 8. *Let $(u,v)\in\mathcal{N}_\lambda^-$, then for every $(\phi_1,\phi_2)\in E$, there exist a number $\varepsilon>0$ and a continuous function $h:[0,+\infty)\longrightarrow(0,+\infty)$ such that*

$$h(0)=1,h(t)(u+t\phi_1,v+t\phi_2)\in\mathcal{N}_\lambda^-,|t|<\varepsilon. \quad (49)$$

Proof. We define $F:E\times\mathbb{R}^+\longrightarrow\mathbb{R}$ as follows:

$$\begin{aligned}F(t,r)&=r^{p-1+\gamma}\|u+t\phi_1,v+t\phi_2\|^p-\lambda r^{\alpha+\beta-1+\gamma}\int_\Omega\\&(u+t\phi_1)^\alpha(v+t\phi_2)^\beta dx-\int_\Omega\left[f(x)(u+t\phi_1)^{1-\gamma}+g(x)\right.\\&\left.(v+t\phi_2)^{1-\gamma}\right]dx.\end{aligned} \quad (50)$$

For $(u,v)\in\mathcal{N}_\lambda^-\subset\mathcal{N}$, we have

$$\begin{aligned}F(0,1)&=\|u,v\|^p-\lambda\int_\Omega u^\alpha v^\beta dx-\int_\Omega\left[f(x)u^{1-\gamma}+g(x)v^{1-\gamma}\right]dx=0,\\F_r(0,1)&=(p-1+\gamma)\|u,v\|^p-\lambda(\alpha+\beta-1+\gamma)\int_\Omega u^\alpha v^\beta dx<0.\end{aligned} \quad (51)$$

We deduce that there exists $\bar{\varepsilon}>0$ such that $F(t,r)=0$ has a unique continuous solution $r=h(t)>0$ for $|t|<\bar{\varepsilon}$, $t\in\mathbb{R}$ by using the implicit function theorem at the point $(0,1)$. Since $F(0,1)=0$, then $h(0)=1$. By $F(t,h(t))=0$ for $|t|<\bar{\varepsilon}$, $t\in\mathbb{R}$, we obtain

$$0 = (h(t))^{p-1+\gamma}\|u + t\phi_1, v + t\phi_2\|^p - \lambda(h(t))^{\alpha+\beta-1+\gamma}\int_\Omega (u + t\phi_1)^\alpha (v + t\phi_2)^\beta dx - \int_\Omega \left[f(x)(u + t\phi_1)^{1-\gamma} + g(x)(v + t\phi_2)^{1-\gamma}\right] dx = \frac{1}{(h(t))^{1-\gamma}}\left[\|h(t)(u + t\phi_1, v + t\phi_2)\|^p - \lambda\int_\Omega (h(t)(u + t\phi_1))^\alpha (h(t)(v + t\phi_2))^\beta dx - \int_\Omega \left[f(x)(h(t)(u + t\phi_1))^{1-\gamma} + g(x)(h(t)(v + t\phi_2))^{1-\gamma} dx\right]\right], \tag{52}$$

that is,

$$h(t)(u + t\phi_1, v + t\phi_2) \in \mathcal{N}_\lambda, \quad \forall t \in \mathbb{R}, |t| < \bar{\varepsilon}. \tag{53}$$

Since $F_r(0, 1) > 0$ and

$$F_r(t, h(t)) = (p - 1 + \gamma)(h(t))^{p-2+\gamma}\|u + t\phi_1, v + t\phi_2\|^p - \lambda(\alpha + \beta - 1 + \gamma)(h(t))^{\alpha+\beta-2+\gamma}\int_\Omega (u + t\phi_1)^\alpha (v + t\phi_2)^\beta dx = \frac{1}{(h(t))^{2-\gamma}}\left[(p - 1 + \gamma)\|h(t)(u + t\phi_1, v + t\phi_2)\|^p - \lambda(\alpha + \beta - 1 + \gamma)\int_\Omega (h(t)(u + t\phi_1))^\alpha (h(t)(v + t\phi_2))^\beta dx\right], \tag{54}$$

we can choose $\varepsilon > 0$ sufficiently small $(\varepsilon < \bar{\varepsilon})$ such that for every $t \in \mathbb{R}$ with $|t| < \varepsilon$,

$$(p - 1 + \gamma)\|h(t)(u + t\phi_1, v + t\phi_2)\|^p - \lambda(\alpha + \beta - 1 + \gamma)\int_\Omega (h(t)(u + t\phi_1))^\alpha (h(t)(v + t\phi_2))^\beta dx < 0, \tag{55}$$

that is,

$$h(t)(u + t\phi_1, v + t\phi_2) \in \mathcal{N}_\lambda^-, \quad \forall t \in \mathbb{R}. \tag{56}$$

This completes the proof of Lemma 8. □

Lemma 9. *There exists a constant $\lambda_4 > 0$ such that $I_\lambda(u, v) \geq 0$ for each $(u, v) \in \mathcal{N}_\lambda^-$ and all $\lambda \in (0, \lambda_4)$.*

Proof. Assume that there exists $(\bar{u}, \bar{v}) \in \mathcal{N}_\lambda^-$ such that $I_\lambda(\bar{u}, \bar{v}) < 0$; thus,

$$\lambda\left(\frac{1}{p} - \frac{1}{\alpha + \beta}\right)\int_\Omega \bar{u}^\alpha \bar{v}^\beta dx - \left(\frac{1}{1 - \gamma} - \frac{1}{p}\right)\int_\Omega \left[f(x)\bar{u}^{1-\gamma} + g(x)\bar{v}^{1-\gamma}\right] dx < 0. \tag{57}$$

Equations (14) and (15) lead

$$\int_\Omega \bar{u}^\alpha \bar{v}^\beta dx < \left[\frac{(p - 1 + \gamma)(\alpha + \beta)}{\lambda(1 - \gamma)(\alpha + \beta - p)}\left(\|f\|_{\gamma^*}^{p/1+\gamma} + \|g\|_{\gamma^*}^{p/1+\gamma}\right)^{1+\gamma/p}\left(\sqrt[p]{\frac{S_{\alpha,\beta}}{S}}\right)^{1-\gamma}\right]^{\alpha+\beta/\alpha+\beta-1+\gamma}. \tag{58}$$

On the other hand, from (6) and (14), we have

$$\int_\Omega u^\alpha v^\beta dx \geq \left[\frac{(p - 1 + \gamma)S_{\alpha,\beta}}{\lambda(\alpha + \beta - 1 + \gamma)}\right]^{\alpha+\beta/\alpha+\beta-p}, \text{ for every } (u, v) \in \mathcal{N}_\lambda^-. \tag{59}$$

Choosing

$$\lambda < \lambda_4 = \left[\frac{(p - 1 + \gamma)(\alpha + \beta)}{(1 - \gamma)(\alpha + \beta - p)}\left(\|f\|_{\gamma^*}^{p/1+\gamma} + \|g\|_{\gamma^*}^{p/1+\gamma}\right)^{1+\gamma/p}\left(\sqrt[p]{\frac{S_{\alpha,\beta}}{S}}\right)^{1-\gamma}\right]^{(\alpha+\beta)(\alpha+\beta-p)/1-\gamma-p} \cdot \left[\frac{\alpha + \beta - 1 + \gamma}{(p - 1 + \gamma)S_{\alpha,\beta}}\right]^{(\alpha+\beta)(\alpha+\beta-1+\gamma)/1-\gamma-p}, \tag{60}$$

we have

$$\int_\Omega \bar{u}^\alpha \bar{v}^\beta dx < \left[\frac{(p - 1 + \gamma)S_{\alpha,\beta}}{\lambda(\alpha + \beta - 1 + \gamma)}\right]^{\alpha+\beta/\alpha+\beta-p}, \tag{61}$$

which is a contradiction. This completes the proof of Lemma 6. □

3. The Proof of Theorem 1

Our proof is divided into the following three steps.

Step 1. Problem (1) has a weak solution (u_2, v_2) in E.

Let $m_- = \inf_{(u,v)\in\mathcal{N}_\lambda^-} I_\lambda(u, v) > -\infty$. In terms of Lemma 4, we know that I_λ is coercive in $\mathcal{N}_\lambda$; hence, m_- is well defined. Ekeland's variational principle guarantees to extract a minimizing sequence $\{(\tilde{u}_n, \tilde{v}_n)\} \subset \mathcal{N}_\lambda^-$ with

$$I_\lambda(\tilde{u}_n, \tilde{v}_n) < m_- + \frac{1}{n}, I_\lambda(\tilde{u}_n, \tilde{v}_n) \leq I_\lambda(u, v) + \frac{1}{n}\|u - \tilde{u}_n, v - \tilde{v}_n\|, \forall\ (u, v) \in \mathcal{N}_\lambda^-. \tag{62}$$

We assume that $\tilde{u}_n, \tilde{v}_n \geq 0$ in Ω and $(\tilde{u}_n, \tilde{v}_n)$ (up to subsequence if necessary) converges to a nonnegative function, denoted by (u_2, v_2) satisfying

$$\begin{aligned}(\tilde{u}_n, \tilde{v}_n) &\rightharpoonup (u_2, v_2) \text{ weakly in } E,\\ (\tilde{u}_n, \tilde{v}_n) &\longrightarrow (u_2, v_2) \text{ a.e.in } \Omega.\end{aligned} \tag{63}$$

Fix $(\phi_1, \phi_2) \in E$ with $\phi_1, \phi_2 \geq 0$, by using Lemma 5, there exists a sequence of functions $f_k : [0,+\infty) \longrightarrow (0,+\infty)$ such that $f_k(0) = 1$ and $f_k(t)(\tilde{u}_k + t\phi_1, \tilde{v}_k + t\phi_2) \in \mathcal{N}_\lambda^-$ for all $(\tilde{u}_k, \tilde{v}_k) \in \mathcal{N}_\lambda^-$ and t small enough. It follows from the definition of $\mathcal{N}_\lambda$ that

$$\begin{aligned} &f_k^p(t)\|\tilde{u}_k + t\phi_1, \tilde{v}_k + t\phi_2\|^p - \lambda f_k^{\alpha+\beta}(t)\int_\Omega (\tilde{u}_k + t\phi_1)^\alpha(\tilde{v}_k + t\phi_2)^\beta dx \\ &\quad - f_k^{1-\gamma}(t)\int_\Omega \left[f(x)(\tilde{u}_k + t\phi_1)^{1-\gamma} + g(x)(\tilde{v}_k + t\phi_2)^{1-\gamma}\right] dx = 0, \end{aligned} \tag{64}$$

$$\|\tilde{u}_k, \tilde{v}_k\|^p - \lambda\int_\Omega \tilde{u}_k^\alpha \tilde{v}_k^\beta dx - \int_\Omega \left[f(x)\tilde{u}_k^{1-\gamma} + g(x)\tilde{v}_k^{1-\gamma}\right] dx = 0, \tag{65}$$

so

$$\begin{aligned} &\frac{1}{k}\left[|f_k(t) - 1|\|\tilde{u}_k, \tilde{v}_k\| + t f_k(t)\|\phi_1, \phi_2\|\right] \geq \frac{1}{k}\|f_k(t)(\tilde{u}_k + t\phi_1, \tilde{v}_k + t\phi_2) \\ &\quad - (\tilde{u}_k, \tilde{v}_k)\| \geq I(\tilde{u}_k, \tilde{v}_k) - I[f_k(t)(\tilde{u}_k + t\phi_1, \tilde{v}_k + t\phi_2)] \\ &= \frac{1 - f_k^p(t)}{p}\|\tilde{u}_k, \tilde{v}_k\|^p + \lambda\frac{f_k^{\alpha+\beta}(t) - 1}{\alpha + \beta}\int_\Omega (\tilde{u}_k + t\phi_1)^\alpha(\tilde{v}_k + t\phi_2)^\beta dx \\ &\quad + \frac{f_k^{1-\gamma}(t) - 1}{1 - \gamma}\int_\Omega \left[f(x)(\tilde{u}_k + t\phi_1)^{1-\gamma} + g(x)(\tilde{v}_k + t\phi_2)^{1-\gamma}\right] dx \\ &\quad + \frac{f_k^p(t)}{p}\left(\|\tilde{u}_k, \tilde{v}_k\|^p - \|\tilde{u}_k + t\phi_1, \tilde{v}_k + t\phi_2\|^p\right) \\ &\quad + \frac{\lambda}{\alpha + \beta}\left[\int_\Omega (\tilde{u}_k + t\phi_1)^\alpha(\tilde{v}_k + t\phi_2)^\beta dx - \int_\Omega \tilde{u}_k^\alpha \tilde{v}_k^\beta dx\right] \\ &\quad + \frac{1}{1 - \gamma}\int_\Omega f(x)\left[(\tilde{u}_k + t\phi_1)^{1-\gamma} - \tilde{u}_k^{1-\gamma}\right] dx \\ &\quad + \frac{1}{1 - \gamma}\int_\Omega g(x)\left[(\tilde{v}_k + t\phi_2)^{1-\gamma} - \tilde{v}_k^{1-\gamma}\right] dx\,; \end{aligned} \tag{66}$$

thus,

$$\begin{aligned} &\frac{1}{k}\left[f_k'(0)\|\tilde{u}_k, \tilde{v}_k\| + \|\phi_1, \phi_2\|\right] \geq -f_k'(0)\|\tilde{u}_k, \tilde{v}_k\|^p \\ &\quad + \lambda f_k'(0)\int_\Omega \tilde{u}_k^\alpha \tilde{v}_k^\beta dx + f_k'(0)\int_\Omega \left[f(x)\tilde{u}_k^{1-\gamma} + g(x)\tilde{v}_k^{1-\gamma}\right] dx \\ &\quad - \int_Q \frac{1}{|x - y|^{N+ps}}\left[|\tilde{u}_k(x) - \tilde{u}_k(y)|^{p-2}(\tilde{u}_k(x) - \tilde{u}_k(y))(\phi_1(x)\right. \\ &\quad - \phi_1(y)) + |\tilde{v}_k(x) - \tilde{v}_k(y)|^{p-2}(\tilde{v}_k(x) - \tilde{v}_k(y))(\phi_2(x) \\ &\quad \left.- \phi_2(y))\right] dxdy + \frac{\lambda\alpha}{\alpha + \beta}\int_\Omega \tilde{u}_k^{\alpha-1}\tilde{v}_k^\beta \phi_1 dx + \frac{\lambda\beta}{\alpha + \beta}\int_\Omega \tilde{u}_k^\alpha \tilde{v}_k^{\beta-1}\phi_2 dx \\ &\quad + \int_\Omega \left[f(x)\tilde{u}_k^{-\gamma}\phi_1 + g(x)\tilde{v}_k^{-\gamma}\phi_2\right] dx = -f_k'(0)\left\{\|\tilde{u}_k, \tilde{v}_k\|^p - \lambda\int_\Omega \tilde{u}_k^\alpha \tilde{v}_k^\beta dx\right. \\ &\quad \left.- \int_\Omega \left[f(x)\tilde{u}_k^{1-\gamma} + g(x)\tilde{v}_k^{1-\gamma}\right] dx\right\} - \int_Q \frac{1}{|x - y|^{N+ps}}\left[|\tilde{u}_k(x) - \tilde{u}_k(y)|^{p-2}(\tilde{u}_k(x)\right. \\ &\quad \left.- \tilde{u}_k(y))(\phi_1(x) - \phi_1(y)) + |\tilde{v}_k(x) - \tilde{v}_k(y)|^{p-2}(\tilde{v}_k(x) - \tilde{v}_k(y))(\phi_2(x) - \phi_2(y))\right] dxdy \\ &\quad + \frac{\lambda\alpha}{\alpha + \beta}\int_\Omega \tilde{u}_k^{\alpha-1}\tilde{v}_k^\beta \phi_1 dx + \frac{\lambda\beta}{\alpha + \beta}\int_\Omega \tilde{u}_k^\alpha \tilde{v}_k^{\beta-1}\phi_2 dx + \int_\Omega \left[f(x)\tilde{u}_k^{-\gamma}\phi_1 + g(x)\tilde{v}_k^{-\gamma}\phi_2\right] dx. \end{aligned} \tag{67}$$

We derive that

$$\begin{aligned} &\int_\Omega \left[f(x)\tilde{u}_k^{-\gamma}\phi_1 + g(x)\tilde{v}_k^{-\gamma}\phi_2\right] dx \leq \frac{1}{k}\left[f_k'(0)\|\tilde{u}_k, \tilde{v}_k\| + \|\phi_1, \phi_2\|\right] \\ &\quad + \int_Q \frac{1}{|x - y|^{N+ps}}\left[|\tilde{u}_k(x) - \tilde{u}_k(y)|^{p-2}(\tilde{u}_k(x) - \tilde{u}_k(y))(\phi_1(x)\right. \\ &\quad \left.- \phi_1(y)) + |\tilde{v}_k(x) - \tilde{v}_k(y)|^{p-2}(\tilde{v}_k(x) - \tilde{v}_k(y))(\phi_2(x) - \phi_2(y))\right] dxdy \\ &\quad - \frac{\lambda\alpha}{\alpha + \beta}\int_\Omega \tilde{u}_k^{\alpha-1}\tilde{v}_k^\beta \phi_1 dx - \frac{\lambda\beta}{\alpha + \beta}\int_\Omega \tilde{u}_k^\alpha \tilde{v}_k^{\beta-1}\phi_2 dx. \end{aligned} \tag{68}$$

Furthermore, there exists a constant $C_2 > 0$ such that $|f_k'(0)| \leq C_2$. Taking $k \longrightarrow \infty$, we deduce from Fatou's lemma that

$$\begin{aligned} &\int_\Omega \left[f(x)u_2^{-\gamma}\phi_1 + g(x)v_2^{-\gamma}\phi_2\right] dx \leq \liminf_{k\longrightarrow\infty}\int_\Omega \left[f(x)\tilde{u}_k^{-\gamma}\phi_1\right. \\ &\quad \left.+ g(x)\tilde{v}_k^{-\gamma}\phi_2\right] dx \leq \int_Q \frac{1}{|x - y|^{N+ps}}\left[|u_2(x) - u_2(y)|^{p-2}(u_2(x)\right. \\ &\quad - u_2(y))(\phi_1(x) - \phi_1(y)) + |v_2(x) - v_2(y)|^{p-2}(v_2(x) - v_2(y))(\phi_2(x) \\ &\quad \left.- \phi_2(y))\right] dxdy - \frac{\lambda\alpha}{\alpha + \beta}\int_\Omega u_2^{\alpha-1}v_2^\beta \phi_1 dx - \frac{\lambda\beta}{\alpha + \beta}\int_\Omega u_2^\alpha v_2^{\beta-1}\phi_2 dx. \end{aligned} \tag{69}$$

Consequently,

$$\begin{aligned} &\int_Q \frac{1}{|x - y|^{N+ps}}\left[|u_2(x) - u_2(y)|^{p-2}(u_2(x) - u_2(y))(\phi_1(x) - \phi_1(y))\right. \\ &\quad \left.+ |v_2(x) - v_2(y)|^{p-2}(v_2(x) - v_2(y))(\phi_2(x) - \phi_2(y))\right] dxdy \\ &\quad - \frac{\lambda\alpha}{\alpha + \beta}\int_\Omega u_2^{\alpha-1}v_2^\beta \phi_1 dx - \frac{\lambda\beta}{\alpha + \beta}\int_\Omega u_2^\alpha v_2^{\beta-1}\phi_2 dx - \int_\Omega \left[f(x)u_2^{-\gamma}\phi_1\right. \\ &\quad \left.+ g(x)v_2^{-\gamma}\phi_2\right] dx \geq 0, \forall (\phi_1, \phi_2) \in C_E. \end{aligned} \tag{70}$$

Since (ϕ_1, ϕ_2) is arbitrary, this inequality also holds for $-(\phi_1, \phi_2)$; thus,

$$\begin{aligned} &\int_Q \frac{1}{|x - y|^{N+ps}}\left[|u_2(x) - u_2(y)|^{p-2}(u_2(x) - u_2(y))(\phi_1(x)\right. \\ &\quad - \phi_1(y)) + |v_2(x) - v_2(y)|^{p-2}(v_2(x) - v_2(y))(\phi_2(x) \\ &\quad \left.- \phi_2(y))\right] dxdy - \frac{\lambda\alpha}{\alpha + \beta}\int_\Omega u_2^{\alpha-1}v_2^\beta \phi_1 dx - \frac{\lambda\beta}{\alpha + \beta}\int_\Omega u_2^\alpha v_2^{\beta-1}\phi_2 dx \\ &\quad - \int_\Omega \left[f(x)u_2^{-\gamma}\phi_1 + g(x)v_2^{-\gamma}\phi_2\right] dx = 0, \forall \ (\phi_1, \phi_2) \in C_E, \end{aligned} \tag{71}$$

which implies that (u_2, v_2) is a weak solution of the problem (1).

Step 2. There exists a constant $\lambda_5 > 0$ such that $(u_2, v_2) \in \mathcal{N}_\lambda^-$ when $0 < \lambda < \lambda_5$.

For each $(u, v) \in \mathcal{N}_\lambda^-$, we have

$$\begin{aligned} I_\lambda(u,v) &= \frac{1}{p}\|(u,v)\|^p - \frac{\lambda}{\alpha+\beta}\int_\Omega u^\alpha v^\beta dx \\ &\quad - \frac{1}{1-\gamma}\left[\|(u,v)\|^p - \lambda\int_\Omega u^\alpha v^\beta dx\right] \\ &= \left(\frac{1}{p} - \frac{1}{1-\gamma}\right)\|(u,v)\|^p \\ &\quad - \lambda\left(\frac{1}{\alpha+\beta} - \frac{1}{1-\gamma}\right)\int_\Omega u^\alpha v^\beta dx \\ &< \frac{\lambda(\alpha+\beta-1+\gamma)}{p(p-1+\gamma)}\int_\Omega u^\alpha v^\beta dx \\ &\quad - \lambda\left(\frac{1}{\alpha+\beta} - \frac{1}{1-\gamma}\right)\int_\Omega u^\alpha v^\beta dx \\ &= \lambda\left[\frac{\alpha+\beta-1+\gamma}{p(p-1+\gamma)} - \frac{1}{\alpha+\beta} + \frac{1}{1-\gamma}\right]\int_\Omega u^\alpha v^\beta dx. \end{aligned} \tag{72}$$

We only need to prove that $m_- = \inf_{(u,v)\in\mathcal{N}_\lambda^-} I_\lambda(u,v) < 1/N\, S^{N/sp}$, when $\lambda < \tilde{\lambda}_5$ with

$$\tilde{\lambda}_5\left[\frac{\alpha+\beta-1+\gamma}{p(p-1+\gamma)} - \frac{1}{\alpha+\beta} + \frac{1}{1-\gamma}\right]\int_\Omega u^\alpha v^\beta dx < \frac{1}{N}S^{N/sp}. \tag{73}$$

Repeating the arguments as in Step 1, we have

$$\begin{aligned} &\int_Q \frac{|\tilde{u}_k(x)-\tilde{u}_k(y)|^p}{|x-y|^{N+ps}}dxdy + \int_Q \frac{|\tilde{v}_k(x)-\tilde{v}_k(y)|^p}{|x-y|^{N+ps}}dxdy \\ &\quad - \lambda\int_\Omega \tilde{u}_k^\alpha \tilde{v}_k^\beta dx - \int_\Omega\left[f(x)\tilde{u}_k^{1-\gamma} + g(x)\tilde{v}_k^{1-\gamma}\right]dx = o(1). \end{aligned} \tag{74}$$

In the following, we show that $(u_2, v_2) \in \mathcal{N}_\lambda^-$. Note that $\mathcal{N}_\lambda^-$ is closed and $(\tilde{u}_k, \tilde{v}_k) \rightharpoonup (u_2, v_2)$ in E, it suffices to prove that $\|\tilde{u}_k, \tilde{v}_k\| \longrightarrow \|u_2, v_2\|$. We suppose that $(\tilde{u}_k, \tilde{v}_k)$ and (u_2, v_2) satisfy the same properties as in (20) and (21). Let $x_j \in \Omega$ in the support of ν and μ. Define $\varphi_\varepsilon \in C_c^\infty(B_{2\varepsilon}(x_j))$ with

$$\varphi_\varepsilon \geq 0, \varphi_\varepsilon|_{B_\varepsilon} = 1, |\varphi_\varepsilon|_\infty \leq 1, |\nabla\varphi_\varepsilon|_\infty \leq \frac{C_3}{\varepsilon}. \tag{75}$$

Applying (74), we derive that

$$\begin{aligned} &\int_Q \frac{|\tilde{u}_k(x)-\tilde{u}_k(y)|^{p-2}(\tilde{u}_k(x)-\tilde{u}_k(y))(\phi_1(x)-\phi_1(y))}{|x-y|^{N+ps}}dxdy \\ &\quad + \int_Q \frac{|\tilde{v}_k(x)-\tilde{v}_k(y)|^{p-2}(\tilde{v}_k(x)-\tilde{v}_k(y))(\phi_2(x)-\phi_2(y))}{|x-y|^{N+ps}}dxdy \\ &\quad - \frac{\lambda\alpha}{\alpha+\beta}\int_\Omega \tilde{u}_k^{\alpha-1}\tilde{v}_k^\beta\phi_1 dx - \frac{\lambda\beta}{\alpha+\beta}\int_\Omega \tilde{u}_k^\alpha \tilde{v}_k^{\beta-1}\phi_2 dx \\ &\quad - \int_\Omega\left[f(x)\tilde{u}_k^{-\gamma}\phi_1 + g(x)\tilde{v}_k^{-\gamma}\phi_2\right]dx = o(1)(k\longrightarrow\infty), \end{aligned} \tag{76}$$

for every $(\phi_1, \phi_2) \in E$. Since $(\varphi_\varepsilon\tilde{u}_k, \varphi_\varepsilon\tilde{v}_k) \in E$, repeating the arguments as in Lemma 3, we derive that

$$\begin{aligned} &\int_Q \frac{|\tilde{u}_k(x)-\tilde{u}_k(y)|^{p-2}(\tilde{u}_k(x)-\tilde{u}_k(y))(\varphi_\varepsilon\tilde{u}_k(x)-\varphi_\varepsilon\tilde{u}_k(y))}{|x-y|^{N+ps}}dxdy \\ &\quad + \int_Q \frac{|\tilde{v}_k(x)-\tilde{v}_k(y)|^{p-2}(\tilde{v}_k(x)-\tilde{v}_k(y))(\varphi_\varepsilon\tilde{v}_k(x)-\varphi_\varepsilon\tilde{v}_k(y))}{|x-y|^{N+ps}}dxdy \\ &\quad - \frac{\lambda\alpha}{\alpha+\beta}\int_\Omega \tilde{u}_k^{\alpha-1}\tilde{v}_k^\beta\varphi_\varepsilon\tilde{u}_k dx - \frac{\lambda\beta}{\alpha+\beta}\int_\Omega \tilde{u}_k^\alpha\tilde{v}_k^{\beta-1}\varphi_\varepsilon\tilde{v}_k dx \\ &\quad - \int_\Omega\left[f(x)\tilde{u}_k^{-\gamma}\varphi_\varepsilon\tilde{u}_k + g(x)\tilde{v}_k^{-\gamma}\varphi_\varepsilon\tilde{v}_k\right]dx = o(1), \end{aligned} \tag{77}$$

$k \longrightarrow \infty$. Moreover,

$$\begin{aligned} &\int_Q \varphi_\varepsilon d\mu - \lambda\int_\Omega \varphi_\varepsilon d\nu - \int_\Omega\left[f(x)u_2^{1-\gamma}\varphi_\varepsilon + g(x)v_2^{1-\gamma}\varphi_\varepsilon\right]dx \\ &= -\int_Q \frac{|\tilde{u}_k(x)-\tilde{u}_k(y)|^{p-2}(\tilde{u}_k(x)-\tilde{u}_k(y))(\varphi_\varepsilon(x)-\varphi_\varepsilon(y))\tilde{u}_k(y)}{|x-y|^{N+ps}}dxdy \\ &\quad - \int_Q \frac{|\tilde{v}_k(x)-\tilde{v}_k(y)|^{p-2}(\tilde{v}_k(x)-\tilde{v}_k(y))(\varphi_\varepsilon(x)-\varphi_\varepsilon(y))\tilde{v}_k(y)}{|x-y|^{N+ps}}dxdy \\ &\quad + o(1) \quad (k\longrightarrow\infty). \end{aligned} \tag{78}$$

Thus,

$$\begin{aligned} 0 &= \lim_{\varepsilon\longrightarrow 0}\left\{\int_Q \varphi_\varepsilon d\mu - \lambda\int_\Omega \varphi_\varepsilon d\nu - \int_\Omega\left[f(x)u_2^{1-\gamma}\varphi_\varepsilon + g(x)v_2^{1-\gamma}\varphi_\varepsilon\right]dx\right\} \\ &= \mu_i - \lambda\nu_j, \end{aligned} \tag{79}$$

by $\mu_j \geq S\nu_j^{p/p_s^*}$, namely, $\lambda\nu_j \geq S\nu_j^{p/p_s^*}$, we get $\nu_j = 0$ or $\nu_j \geq (S/\lambda)^{N/sp}$.

Next, we prove that $\nu_j \geq (S/\lambda)c^{N/sp}$ does not hold. Otherwise, there exists j_0 such that $\nu_{j_0} \geq (S/\lambda)^{N/sp}$, then

$$\begin{aligned} m_- &= \lim_{k\longrightarrow\infty}\left[\frac{1}{p}\|\tilde{u}_k, \tilde{v}_k\|^p - \frac{\lambda}{\alpha+\beta}\int_\Omega \tilde{u}_k^\alpha\tilde{v}_k^\beta dx - \frac{1}{1-\gamma}\int_\Omega\right. \\ &\quad \left.\left(f(x)\tilde{u}_k^{1-\gamma} + g(x)\tilde{v}_k^{1-\gamma}\right)dx\right] \geq \lambda\left[\frac{1}{1-\gamma} - \frac{1}{\alpha+\beta}\right. \\ &\quad \left. + \left(\frac{1}{p} - \frac{1}{1-\gamma}\right)\frac{\alpha+\beta-1+\gamma}{p-1+\gamma}\right]\int_\Omega \tilde{u}_k^\alpha\tilde{v}_k^\beta dx \\ &\geq \lambda\left[\frac{1}{1-\gamma} - \frac{1}{\alpha+\beta} - \left(\frac{1}{1-\gamma} - \frac{1}{p}\right)\frac{\alpha+\beta-1+\gamma}{p-1+\gamma}\right] \\ &\quad \left(\int_\Omega u_2^\alpha v_2^\beta dx + \sum_{j\in J}\nu_j\right) \geq \left[\frac{1}{1-\gamma} - \frac{1}{\alpha+\beta} - \left(\frac{1}{1-\gamma} - \frac{1}{p}\right)\right. \\ &\quad \left.\frac{\alpha+\beta-1+\gamma}{p-1+\gamma}\right]\lambda^{1-N/sp}S^{N/sp} \geq \frac{1}{N}S^{N/sp}, \end{aligned} \tag{80}$$

where $\lambda < \bar{\lambda}_5$ with $\bar{\lambda}_5 < \{N[1/1-\gamma-1/\alpha+\beta-(1/1-\gamma-1/p)\alpha+\beta-1+\gamma/p-1+\gamma]\}^{sp/N-sp}$; this contradicts the fact $m_- < 1/NS^{N/ps}$. Choosing $\lambda_5 = \min\{\tilde{\lambda}_5, \bar{\lambda}_5\}$, we have $\tilde{u}_k^\alpha \tilde{v}_k^\beta \longrightarrow u_2^\alpha v_2^\beta$ and $(u_2, v_2) \in \mathcal{N}_\lambda^-$ when $0 < \lambda < \lambda_5$.

Step 3. (u_1, v_1) is a nontrivial solution to problem (1).

Let $\lambda^* = \min\{\lambda_i\}, i = 1, 2, 3, 4, 5$, then Lemmas 2–9 and Steps 1 and 2 hold for all $0 < \lambda < \lambda^*$. Using Lemma 3, we have

$$\min_{t\in\mathbb{R}} I_\lambda(u_1 + t\phi_1, v_1 + t\phi_2) = I_\lambda(u_1 + t\phi_1, v_1 + t\phi_2)|_t = 0 = I_\lambda(u_1, v_1), \quad \forall(\phi_1, \phi_2) \in E. \tag{81}$$

Hence, (u_1, v_1) is a weak solution of (1). Since

$$\begin{aligned} 0 \le I_\lambda(u_1 + t\phi_1, v_1 + t\phi_2) - I_\lambda(u_1, v_1) &= \frac{1}{p}\|u_1 + t\phi_1, v_1 + t\phi_2\|^p \\ &- \frac{\lambda}{\alpha+\beta}\int_\Omega (u_1 + t\phi_1)^\alpha (v_1 + t\phi_2)^\beta dx - \frac{1}{1-\gamma}\int_\Omega \left[f(x)(u_1 + t\phi_1)^{1-\gamma}\right. \\ &\left. + g(x)(v_1 + t\phi_2)^{1-\gamma}\right] dx - \frac{1}{p}\|u_1, v_1\|^p + \frac{\lambda}{\alpha+\beta}\int_\Omega u_1^\alpha v_1^\beta dx \\ &+ \frac{1}{1-\gamma}\int_\Omega \left[f(x)u_1^{1-\gamma} + g(x)v_1^{1-\gamma}\right] dx \le \frac{1}{p}\|u_1 + t\phi_1, v_1 + t\phi_2\|^p \\ &- \frac{1}{p}\|u_1, v_1\|^p, \phi_1 \ge 0, \phi_2 \ge 0. \end{aligned} \tag{82}$$

Dividing by $t > 0$ and passing to the limit as $t \longrightarrow 0$, we obtain

$$\begin{aligned} &\int_Q \frac{|u_1(x) - u_1(y)|^{p-2}(u_1(x) - u_1(y))(\phi_1(x) - \phi_1(y))}{|x-y|^{N+ps}} dxdy \\ &+ \int_Q \frac{|v_1(x) - v_1(y)|^{p-2}(v_1(x) - v_1(y))(\phi_2(x) - \phi_2(y))}{|x-y|^{N+ps}} dxdy \ge 0. \end{aligned} \tag{83}$$

Hence, (u_1, v_1) is a nontrivial solution of (1). This completes the proof of Theorem 1.

Data Availability

Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no competing interests.

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