Hindawi
Journal of Function Spaces
Volume 2023, Article ID 9053616, 11 pages
https://doi.org/10.1155/2023/9053616

Research Article

An Analytical View of Nonlinear Fractional Burger's Equations Using Conformable Double Elzaki Transform

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Received 27 February 2023; Revised 5 October 2023; Accepted 20 October 2023; Published 24 November 2023

Academic Editor: Abdul Rauf Khan

The conformable double Elzaki composition technique (CDET) and the Adomian decomposition technique are combined in this work to provide a novel approach for dealing with nonlinear partial issues under certain specified conditions. The conformable double Elzaki composition (CDEC) approach is the name we give to this novel technique. We also outline and discuss the main traits and major conclusions connected to the recommended technique. The new technique provides an estimated succession of answers that finally get close to the exact solution. This method has the advantage of generating findings rapidly since it generates analytical series solutions for the target equations without the requirement for discretization, transformation, or limited assumptions. We also present some numerical applications to back up our conclusions. The results demonstrate the strength and potency of the recommended strategy in dealing with a variety of problems in the fields of engineering and physics in symmetry with other strategies.

1. Introduction

Fractional calculus has been used to show that fractional differential equations are the best way to describe physical and technical processes. The failure of nonlinear mathematical models of the standard integer-order derivative should be obvious. Recently, fractional calculus has been more widely used, particularly in the domains of control theory, signal and image processing, mechanics, electricity, chemistry, biology, and economics. Continuous-time random walk, vibration and control, anomaly diffusion, nonlocal phenomena, historical processes—one of the topics covered is a fractional neutron point kinetic model. Others include porous media, fractional filters, biomedical engineering, power laws, the Riesz potential, fractional derivatives and fractals, computational fractional derivative equations, fractional Brownian motion, and Levy statistics. All sorts of fractions include fractional trajectories, fractional phase-locked loops, and fractional trajectories. He is an expert on this subject [1–3].

The conformable fractional derivative offers two benefits over traditional fractional derivatives: its definition is straightforward, and it meets most of the requirements for the classical integral derivative. It is also practical when used to model physical issues, making it easier to numerically solve differential equations than with the Riemann-Liouville or Caputo fractional derivatives. A large number of researchers have previously used conformable fractional derivatives in a variety of domains. The classic derivative and CFD are pretty similar. Because of its dependence on the basic limit notion and the resulting simplicity, it makes a few classic calculus theorems simpler to extend than the previous definitions of fractional derivatives. The CFDs also satisfy a number of classical characteristics, including the mean value theorem, product, quotient, and chain rules. Abdeljawad published the fractional derivatives and integrals of higher-order concepts with left and right conformability in a distinct study [4]. Due to the many important applications it has in several scientific fields, the conformable fractional derivative is gaining more and more attention. Recently, Arqub et al. provided multiple vitalizations of the development of COVID-19 [5] using controller principles based on the concept of conformable calculus. The most important reason for this is that fractional-order models have memory properties [6–8].

There have been applications of Burger's equation, a well-known partial differential equation, in several physical circumstances, such as shock waves, acoustic waves, the ISNA, CCA, Florida dynamic gas, and transmission lines. The time-fractional Burgers' problem with an Atangana-Baleanu derivative is numerically solved using cubic B-spline functions and a weighted technique [9]. Shafiq et al. [10] created an iterative technique employing a Caputo-Fabrizio fractional derivative to handle the damped Burger's problem (CFFD). Burgers-Korteweg-de Vries equations, modified Burger's equations, and nonlinear time-fractional Burger's equations are all modeled using the residual power series method (RPSM) [11]. The spectral collocation approach is studied using the Chebyshev polynomials, space reactional Korteweg-de Vries equations, and space fractional Korteweg-de Vries-Burgers equations based on the Caputo-Fabrizio fractional derivative [12]. Furthermore, the conformable double Laplace decomposition method has been looked at to find solutions to the fractionally coupled Burgers' one-dimensional conformable regular and singular equation [13]. The authors in [14, 15] solved the fractional differential equations using the conformable Laplace transform approach. The authors describe the conformable double Laplace transform method in [16] and use it to solve fractional partial differential equations. The precise solutions of the time-fractional Burgers' equations have been established using the first integral technique [17].

The main goal of this work is to extend the traditional double Elzaki formula to the conformable fractional order, or CDEDM, and to derive a number of fascinating conclusions from this new fractional version, including an alluring connection between CDET and the CFLT. The CDED and ADM are also used to find the analytical solution of a generalized conformable coupled Burger's fractional differential equation. A variety of significant nonlinear conformable fractional differential equations (CFDE) issues are also addressed using our ground-breaking method for space-time Caputo fractional derivatives, including Eq. (12). As a consequence, ξ and ζ satisfy $0<\xi,\ \zeta\le 1$, and $\phi\ge 0$ and are the parameters that characterize the order of the fractional space-time derivatives. When at least one of the factors changes, alternative reaction systems may be developed. When it is $\xi=\zeta=1$ applied, the fractional equations are reduced to the conventionally coupled Burger's equation. The results indicate that our innovative approach is a simple, effective, and impactful technique that can be successfully used to identify the general solutions to several distinct CFDEs. The remainder of the essay is structured as follows: Section 2 covers conformable fractional derivative (CFD) definitions, characteristics, theorems, along with certain fundamental CFDEs. (DET) definitions, characteristics, and theorem proofs. The conformable double Elzaki decomposition approach's specifics and convergence are also covered. Section 3 gives a description of the model and shows how CDET is utilized to provide precise analytical answers to the given conformable fractional Burgers' equation. We provide two illustrative examples to show the dependability, convergence, and effectiveness of the suggested technique in Section 4. Section 5 concludes with some remarks.

2. Conformable Double Elzaki Transforms and Some Properties

We will look at a few CDET definitions and characteristics to see if they can help us find additional modified data $h(\phi^{\xi}/\phi^{\xi},\varphi^{\zeta}/\zeta), H(\omega,\varpi)$ rather than having to consider the following.

Definition 1. If h : $(0,\infty)\longrightarrow\mathbb{R}$ is true, the CFD (conformable fractional derivative) of h with order ζ may be expressed as follows:

$$\frac{d^{\xi}}{d\phi^{\xi}}h\left(\frac{\phi^{\xi}}{\xi}\right)=\lim_{\varepsilon\longrightarrow 0}\frac{h\left(\phi^{\xi}/\xi+\varepsilon\phi^{1-\xi}\right)}{\varepsilon},\ \frac{\phi^{\xi}}{\xi}>0,0<\xi\le 1. \quad (1)$$

See [13, 18].

Definition 2 (see [19]). Let $h(\phi^{\xi}/\xi,\varphi^{\zeta}/\zeta)$: $\mathfrak{R}\times(0,\infty)\longrightarrow\mathfrak{R}$. Then, the conformable space fractional partial derivative of order ζ and ζ a function $h(\phi^{\xi}/\xi,\varphi^{\zeta}/\zeta)$ is defined as

$$\begin{aligned}
\frac{\partial^{\xi}}{\partial\phi^{\xi}}h\left(\frac{\phi^{\xi}}{\xi},\frac{\varphi^{\zeta}}{\zeta}\right)&=\lim_{\varepsilon\longrightarrow 0}\frac{h\left(\phi^{\xi}/\xi+\varepsilon\phi^{1-\xi},\varphi^{\zeta}/\zeta\right)-h\left(\phi^{\xi}/\xi,\varphi^{\zeta}/\zeta\right)}{\varepsilon},\ \frac{\phi^{\xi}}{\xi},\frac{\varphi^{\zeta}}{\zeta}>0,0<\xi,\zeta\le 1,\\
\frac{\partial^{\zeta}}{\partial\varphi^{\zeta}}h\left(\frac{\phi^{\xi}}{\xi},\frac{\varphi^{\zeta}}{\zeta}\right)&=\lim_{\upsilon\longrightarrow 0}\frac{h\left(\phi^{\xi}/\xi,\varphi^{\zeta}/\zeta+\upsilon\varphi^{1-\zeta}\right)-h\left(\phi^{\xi}/\xi,\varphi^{\zeta}/\zeta\right)}{\upsilon},\ \frac{\phi^{\xi}}{\xi},\frac{\varphi^{\zeta}}{\zeta}>0,0<\xi,\zeta\le 1.
\end{aligned} \quad (2)$$

Definition 3. Conformable Elzaki transform of a function $h:[0,\infty)\longrightarrow\mathbb{R}$ is defined by

$$S_{\phi}^{\xi}\left[h\left(\frac{\phi^{\xi}}{\xi}\right)\right]=H(\omega)=\omega\int_{0}^{\infty}e^{-\phi^{\xi}/\omega\xi}h\left(\frac{\phi^{\xi}}{\xi}\right)d\phi, \quad (3)$$

when there is a conformable meaning to the integral with regard to ω [20].

Definition 4. Let $h(\phi^{\xi}/\xi,\varphi^{\zeta}/\zeta)$, $f(x,t)$ be a piecewise continuous function on the interval $[0,\infty)\times[0,\infty)$ of exponential order. Consider for some $a,b\in\mathfrak{R}$ sup $(\phi^{\xi}/\xi),(\varphi^{\zeta}/\zeta)>0$, $e^{|h(\phi^{\xi}/\xi),(\varphi^{\zeta}/\zeta)|/a(\phi^{\xi}/\xi)+b(\varphi^{\zeta}/\zeta)}$. The conformable double Elzaki transform is defined as follows under these presumptions:

$$E_{\phi}^{\xi}E_{\varphi}^{\zeta}\left[h\left(\frac{\phi^{\xi}}{\xi},\frac{\varphi^{\zeta}}{\zeta}\right)\right]=H(\omega,\varpi)=\omega\varpi\int_{0}^{\infty}\int_{0}^{\infty}e^{-\phi^{\xi}/\omega\xi-\varphi^{\zeta}/\varpi\zeta}h\left(\frac{\phi^{\xi}}{\xi},\frac{\varphi^{\zeta}}{\zeta}\right)d\varphi d\phi, \quad (4)$$

where $\omega,\varpi\in\cancel{c}$, $0<\xi,\zeta\le 1$. In terms of conformability, the fractional integrals are conformable fractional integrals ϕ^{ξ}/ξ and φ^{ξ}/ξ, respectively.

Lemma 5. *A function is transformed using the double fractional Elzaki transform.*

(1) $E_{\phi}^{\xi}E_{\varphi}^{\zeta}[(\phi^{\xi}/\xi)^{n}(\varphi^{\zeta}/\zeta)^{m}]=n!m!\omega^{n+2}\varpi^{m+2}$

(2) $E_{\phi}^{\xi}E_{\varphi}^{\zeta}[e^{a(\phi^{\xi}/\xi)+b(\varphi^{\zeta}/\zeta)}]=(\omega/1-a\omega)(\varpi/1-b\varpi)$

(3) $E_{\phi}^{\xi}E_{\varphi}^{\zeta}[\sin(a(\phi^{\xi}/\xi))\sin(b(\varphi^{\zeta}/\zeta))]=(a\omega^{2}/1-(a\omega)^{2})(b\varpi^{2}/1-(b\varpi)^{2})$

(4) *The double fractional Elzaki transform of the function $h(\phi^{\xi}/\xi,\varphi^{\zeta}/\zeta)$ is given by:*

$$E_{\phi}^{\xi}E_{\varphi}^{\zeta}\left[\left(\frac{\phi^{\xi}}{\xi}\right)^{n}\left(\frac{\varphi^{\zeta}}{\zeta}\right)^{m}\right]=\Gamma(n+1)\Gamma(m+1)\omega^{n+2}\varpi^{m+2},\quad a,b\in\mathbb{R}. \quad (5)$$

Theorem 6. *Let the conformable Elzaki transform of function $h(\phi^{\xi}/\xi,\varphi^{\zeta}/\zeta)$, where $0<\xi,\zeta\le 1$, such that $h(\phi^{\xi}/\xi,\varphi^{\zeta}/\zeta)\in C^{l}(\mathbb{R}^{+}\times\mathbb{R}^{+})$, $\partial^{m\xi}h/\partial\phi^{m\xi}$, and $\partial^{n\zeta}h/\partial\varphi^{n\zeta}$ denote m, n time function conformable fraction derivatives h, then*

$$\begin{aligned}E_{\phi}^{\xi}E_{\varphi}^{\zeta}\left[\frac{\partial^{m\xi}h}{\partial\phi^{m\xi}}\right]&=\frac{H(\omega,\varpi)}{\omega^{m}}-\frac{H(0,\varpi)}{\omega^{-2+m}}-\sum_{k=1}^{m-1}\omega^{2-m+k}E_{\varphi}^{\zeta}\left[\frac{\partial^{k\xi}h}{\partial\phi^{k\xi}}\left(0,\frac{\varphi^{\zeta}}{\zeta}\right)\right],\\ E_{\phi}^{\xi}E_{\varphi}^{\zeta}\left[\frac{\partial^{n\zeta}h}{\partial\varphi^{n\zeta}}\right]&=\frac{H(\omega,\varpi)}{\varpi^{n}}-\frac{H(\omega,0)}{\varpi^{-2+n}}-\sum_{k=1}^{n-1}\varpi^{2-n+k}E_{\phi}^{\xi}\left[\frac{\partial^{n\zeta}h}{\partial\varphi^{n\zeta}}\left(\frac{\phi^{\xi}}{\xi},0\right)\right].\end{aligned} \quad (6)$$

Theorem 7. *If $E_{\phi}^{\xi}E_{\varphi}^{\zeta}(h_{1}(\phi^{\xi}/\xi,\varphi^{\zeta}/\zeta))=H_{1}(\omega,\varpi)$ and $E_{\phi}^{\xi}E_{\varphi}^{\zeta}(h_{2}(\phi^{\xi}/\xi,\varphi^{\zeta}/\zeta))=H_{2}(\omega,\varpi)$ exist, in which case the following conformable fractional double Elzaki transform convolution theorem applies:*

$$\begin{aligned}&E_{\varphi}^{\varepsilon}E_{\phi}^{\varepsilon}\left[h_{1}\left(\frac{\phi^{\varepsilon}}{\varepsilon},\frac{\varphi^{\varepsilon}}{\varepsilon}\right)**h_{2}\left(\frac{\phi^{\varepsilon}}{\varepsilon},\frac{\varphi^{\varepsilon}}{\varepsilon}\right)\right]\\&=\omega\varpi\int_{0}^{\infty}\int_{0}^{\infty}e^{\phi^{\varepsilon}/\omega^{\varepsilon}-\varphi^{\varepsilon}/\omega^{\varepsilon}}h_{2}\left(\frac{\phi^{\varepsilon}}{\varepsilon},\frac{\varphi^{\varepsilon}}{\varepsilon}\right)d\phi d\varphi.\end{aligned} \quad (7)$$

Proof. Using the fractional partial derivatives' formulation of the double Elzaki transform

$$\begin{aligned}&E_{\varphi}^{\varepsilon}E_{\phi}^{\varepsilon}\left[h_{1}\left(\frac{\phi^{\varepsilon}}{\varepsilon},\frac{\varphi^{\varepsilon}}{\varepsilon}\right)**h_{2}\left(\frac{\phi^{\varepsilon}}{\varepsilon},\frac{\varphi^{\varepsilon}}{\varepsilon}\right)\right]\\&=\omega\varpi\int_{0}^{\infty}\int_{0}^{\infty}e^{\phi^{\varepsilon}/\varpi^{\varepsilon}-\varphi^{\varepsilon}/\varpi^{\varepsilon}}h_{2}\left(\frac{\phi^{\varepsilon}}{\varepsilon},\frac{\varphi^{\varepsilon}}{\varepsilon}\right)d\phi d\varphi.\end{aligned} \quad (8)$$

Using Lemma 5

$$\omega\varpi\int_{0}^{\infty}\int_{0}^{\infty}e^{\varphi^{\varepsilon}/\varpi^{2}-\phi^{\varepsilon}/\varpi^{2}}\left[\int_{0}^{\varphi^{\varepsilon}/\varepsilon}\int_{0}^{\phi^{\varepsilon}/\varepsilon}h_{1}\left(\frac{\phi^{\varepsilon}}{\varepsilon}-\delta,\frac{\varphi^{\varepsilon}}{\varepsilon}-\xi\right)h_{2}(\delta,\xi)d\delta d\xi\right]d\phi d\varphi. \quad (9)$$

Using Heaviside's unit step function

$$\begin{aligned}&=\omega\varpi\int_{0}^{\infty}\int_{0}^{\infty}e^{\varphi^{\varepsilon}/\omega^{2}-\phi^{\varepsilon}/\omega^{2}}\left[\int_{0}^{\varphi^{\varepsilon}/\varepsilon}\int_{0}^{\phi^{\varepsilon}/\varepsilon}h_{1}\left(\frac{\phi^{\varepsilon}}{\varepsilon}-\delta,\frac{\varphi^{\varepsilon}}{\varepsilon}-\xi\right),H\left(\frac{\phi^{\varepsilon}}{\varepsilon}-\delta,\frac{\varphi^{\varepsilon}}{\varepsilon}-\xi\right).h_{2}(\delta,\xi)d\delta d\xi\right]d\phi d\varphi,\\&=\int_{0}^{\infty}\int_{0}^{\infty}h_{2}(\delta,\xi)d\delta d\xi\left[\omega\varpi\int_{0}^{\infty}\int_{0}^{\infty}e^{\varphi^{\varepsilon}/\omega^{2}-\phi^{\varepsilon}/\omega^{2}}h_{1}\left(\frac{\phi^{\varepsilon}}{\varepsilon}-\delta,\frac{\varphi^{\varepsilon}}{\varepsilon}-\xi\right).H\left(\frac{\phi^{\varepsilon}}{\varepsilon}-\delta,\frac{\varphi^{\varepsilon}}{\varepsilon}-\xi\right)d\phi d\varphi\right],\\ H&=\begin{cases}1, & \dfrac{\phi^{\varepsilon}}{\varepsilon}>\delta>\dfrac{\varphi^{\varepsilon}}{\varepsilon}>\xi,\\ 0, & \text{other}.\end{cases}\end{aligned} \quad (10)$$

Heaviside's unit step

$$= \int_0^\infty \int_0^\infty h_2(\delta,\xi) d\delta d\xi \left[\omega\varpi \int_0^\infty \int_0^\infty e^{\varphi^\varepsilon/\omega^2 - \phi^\varepsilon/\varpi^2} h_1\left(\frac{\phi^\varepsilon}{\varepsilon} - \delta, \frac{\varphi^\varepsilon}{\varepsilon} - \xi\right) d\phi d\varphi\right],$$
$$= \omega\varpi \int_0^\infty \int_0^\infty e^{\varphi^\varepsilon/\omega^2 - \phi^\varepsilon/\varpi^2} h_2(\delta,\xi).H_1(\omega,\varpi) d\phi d\varphi = \frac{1}{\omega\varpi}.H_2(\omega,\varpi) H_1(\omega,\varpi). \tag{11}$$

□

3. Burgers' Equation in Fractionally Coupled Burgers in One Dimension

This section describes methods for using conformable double Elzaki decomposition (CDEDM) to solve the regular and specific 1-D (CFBE). We note that the difficulties stated in [21] can be attained if ξ and ζ in the following issues:

Conformable in one dimension coupled Burgers' equation is as follows:

$$\begin{aligned} &\frac{\partial^\zeta h}{\partial \varphi^\zeta} - \frac{\partial^{2\xi} h}{\partial \phi^{2\xi}} + \theta h \frac{\partial^\xi h}{\partial \phi^\xi} + \varsigma \frac{\partial^{2\xi}}{\partial \phi^\xi}(hz) = k\left(\frac{\phi^\xi}{\xi}, \frac{\varphi^\zeta}{\zeta}\right), \\ &\frac{\partial^\zeta z}{\partial \varphi^\zeta} - \frac{\partial^{2\xi} z}{\partial \phi^{2\xi}} + \theta z \frac{\partial^\xi z}{\partial \phi^\xi} + \delta \frac{\partial^{2\xi}}{\partial \phi^\xi}(hz) = j\left(\frac{\phi^\xi}{\xi}, \frac{\varphi^\zeta}{\zeta}\right). \end{aligned} \tag{12}$$

Subject to

$$\begin{aligned} h\left(\frac{\phi^\xi}{\xi}, 0\right) &= k_1\left(\frac{\phi^\xi}{\xi}\right), \\ z\left(\frac{\phi^\xi}{\xi}, 0\right) &= j_1\left(\frac{\phi^\xi}{\xi}\right). \end{aligned} \tag{13}$$

For $\varphi > 0$, where $k(\phi^\xi/\xi, \varphi^\xi/\varphi)$, $k_1(\phi^\xi/\xi)$, $j(\phi^\xi/\xi, \varphi^\xi/\varphi)$ $j_1(\phi^\xi/\xi)$, and $j_1(\phi^\xi/\xi)$ are the supplied functions, while θ, ς and δ are the arbitrary constants based on system variables such as the Peclet number, the particle-gravitational Stokes velocity, and the Brownian diffusivity, see [22]. A conformable single Laplace transform is used to get Eq. (13), while a conformable double Laplace transform is used to acquire both sides of Equation (12).

$$H(\omega,\varpi) = \frac{K_1(\omega)}{\varpi^{-2}} + \varpi^\zeta K(\omega,\varpi) + \varpi^\zeta E_\phi^\xi E_\varphi^\zeta \left[\frac{\partial^{2\xi} h}{\partial \phi^{2\zeta}} - \theta h \frac{\partial^\xi h}{\partial \phi^\zeta} - \varsigma \frac{\partial^{2\xi}}{\partial \phi^\zeta}(hz)\right], \tag{14}$$

$$Z(\omega,\varpi) = \frac{j_1(\omega)}{\varpi^{-2}} + \varpi^\zeta j(\omega,\varpi) + \varpi^\zeta E_\phi^\xi E_\varphi^\zeta \left[\frac{\partial^{2\xi} z}{\partial \phi^{2\zeta}} - \theta z \frac{\partial^\xi z}{\partial \phi^\zeta} - \delta \frac{\partial^{2\xi}}{\partial \phi^\zeta}(hz)\right]. \tag{15}$$

The 1-D (CFBE) solution is described by the CDEDM as $h(\phi^\xi/\xi, \phi^\xi/\varphi)$ and $z(\phi^\xi/\xi, \phi^\xi/\varphi)$ via the infinite series.

$$h\left(\frac{\phi^\xi}{\xi}, \frac{\varphi^\xi}{\varphi}\right) = \sum_{\rho=0}^{\infty} h_\rho\left(\frac{\phi^\xi}{\xi}, \frac{\varphi^\xi}{\varphi}\right), \tag{16}$$

$z((\phi^\xi/\xi), (\varphi^\xi/\varphi)) = \sum_{\rho=0}^{\infty} z_\rho((\phi^\xi/\xi), (\varphi^\xi/\varphi))$.

The Adomian polynomials Q_ρ, M_ρ, and D_ρ are expressed as follows:

$$\begin{aligned} Q_\rho &= \sum_\rho^\infty h_\rho h_{\phi\rho}, \\ M_\rho &= \sum_\rho^\infty z_\rho z_{\phi\rho}, \\ D_\rho &= \sum_\rho^\infty h_\rho z_{\phi\rho}. \end{aligned} \tag{17}$$

The formulas below can be used to determine the Adomian polynomials for the nonlinear terms hh_ϕ, zz_ϕ, and hz_ϕ.

$$\begin{aligned} Q_0 &= h_0 h_{0x}, \\ Q_1 &= h_{0x} h_1 + h_0 h_{1x}, \\ Q_2 &= h_{0x} h_2 + h_{1x} h_1 + h_{2x} h_0, \\ Q_3 &= h_{0x} h_3 + h_{1x} h_2 + h_{2x} h_1 + h_{3x} h_0, \\ &\vdots \quad \vdots \quad \vdots, \\ M_0 &= z_0 z_{0x}, \\ M_1 &= z_{0x} z_1 + z_0 z_{1x}, \\ M_2 &= z_{0x} z_2 + z_{1x} z_1 + z_{2x} z_0, \\ M_3 &= z_{0x} z_3 + z_{1x} z_2 + z_{2x} z_1 + z_{3x} z_0, \\ &\vdots \quad \vdots \quad \vdots, \\ D_0 &= h_0 z_0, \\ D_1 &= h_0 z_1 + h_1 z_0, \\ D_2 &= h_0 z_2 + h_1 z_1 + h_2 z_0, \\ D_3 &= h_0 z_3 + h_1 z_2 + h_2 z_1 + h_3 z_0, \\ &\vdots \quad \vdots \quad \vdots. \end{aligned} \tag{18}$$

Using Eq. (17), we can apply the inverse conformable double Elzaki transform, which may be used on both sides of Eqs. (14) and (15).

$$\sum_{\rho=0}^{\infty} h_\rho\left(\frac{\phi^\xi}{\xi}, \frac{\varphi^\xi}{\varphi}\right) = k_1\left(\frac{\phi^\xi}{\xi}\right) + E_\phi^{-1}E_\varphi^{-1}[\omega^m H(\omega, \varpi)] + E_\phi^{-1}E_\varphi^{-1}\left[\varpi^\zeta E_\phi^\xi E_\varphi^\zeta\left[\frac{\partial^{2\xi} h}{\partial\phi^{2\zeta}}\right]\right] - E_\phi^{-1}E_\varphi^{-1}\left[\varpi^\zeta E_\phi^\xi E_\varphi^\zeta\left[\theta Q_\rho\right]\right] - E_\phi^{-1}E_\varphi^{-1}\left[\varpi^\zeta E_\phi^\xi E_\varphi^\zeta\left[\varsigma D_\rho\right]\right], \tag{19}$$

$$\sum_{\rho=0}^{\infty} z_\rho\left(\frac{\phi^\xi}{\xi}, \frac{\varphi^\xi}{\varphi}\right) = j_1\left(\frac{\phi^\xi}{\xi}\right) + E_\phi^{-1}E_\varphi^{-1}[\omega^m Z(\omega, \varpi)] + E_\phi^{-1}E_\varphi^{-1}\left[\varpi^\zeta E_\phi^\xi E_\varphi^\zeta\left[\frac{\partial^{2\xi} z}{\partial\phi^{2\zeta}}\right]\right] - E_\phi^{-1}E_\varphi^{-1}\left[\varpi^\zeta E_\phi^\xi E_\varphi^\zeta\left[\theta M_\rho\right]\right] - E_\phi^{-1}E_\varphi^{-1}\left[\varpi^\zeta E_\phi^\xi E_\varphi^\zeta\left[\delta D_\rho\right]\right]. \tag{20}$$

When all sides of Eqs. (19) and (20) are compared, we get

$$\begin{aligned} h_0 &= k_1\left(\frac{\phi^\xi}{\xi}\right) + E_\phi^{-1}E_\varphi^{-1}[\omega^m H(\omega, \varpi)], \\ z_0 &= j_1\left(\frac{\phi^\xi}{\xi}\right) + E_\phi^{-1}E_\varphi^{-1}[\omega^m Z(\omega, \varpi)]. \end{aligned} \tag{21}$$

The recursive relation is represented by the equations below in general.

$$h_{\rho+1} = E_\phi^{-1}E_\varphi^{-1}\left[\varpi^\zeta E_\phi^\xi E_\varphi^\zeta\left[\frac{\partial^{2\xi} h}{\partial\phi^{2\zeta}}\right]\right] - E_\phi^{-1}E_\varphi^{-1}\left[\varpi^\zeta E_\phi^\xi E_\varphi^\zeta\left[\theta Q_\rho\right]\right] - E_\phi^{-1}E_\varphi^{-1}\left[\varpi^\zeta E_\phi^\xi E_\varphi^\zeta\left[\varsigma D_\rho\right]\right], \tag{22}$$

$$z_{\rho+1} = E_\phi^{-1}E_\varphi^{-1}\left[\varpi^\zeta E_\phi^\xi E_\varphi^\zeta\left[\frac{\partial^{2\xi} z}{\partial\phi^{2\zeta}}\right]\right] - E_\phi^{-1}E_\varphi^{-1}\left[\varpi^\zeta E_\phi^\xi E_\varphi^\zeta\left[\theta M_\rho\right]\right] - E_\phi^{-1}E_\varphi^{-1}\left[\varpi^\zeta E_\phi^\xi E_\varphi^\zeta\left[\delta D_\rho\right]\right]. \tag{23}$$

The previous equations must include the double inverse Elzaki transform with regard to ω and ϖ. The following example for the 1-D (CFBE) illustrates this method.

4. Illustrative Examples

Example 1. Consider the Burgers equation for a homogeneous, conformable fractionally coupled Burgers [23].

$$\begin{aligned} &\frac{\partial^\zeta h}{\partial\varphi^\zeta} - \frac{\partial^{2\xi} h}{\partial\phi^{2\zeta}} - 2h\frac{\partial^\xi h}{\partial\phi^\zeta} + \varsigma\frac{\partial^{2\xi}}{\partial\phi^\zeta}(hz) = 0, \\ &\frac{\partial^\zeta z}{\partial\varphi^\zeta} - \frac{\partial^{2\xi} z}{\partial\phi^{2\zeta}} - 2z\frac{\partial^\xi z}{\partial\phi^\zeta} + \delta\frac{\partial^{2\xi}}{\partial\phi^\zeta}(hz) = 0, \end{aligned} \tag{24}$$

with I.C

$$h\left(\frac{\varphi^\xi}{\xi}, 0\right) = \sin\left(\frac{\phi^\xi}{\xi}\right), z\left(\frac{\phi^\xi}{\xi}, 0\right) = \sin\left(\frac{\phi^\xi}{\xi}\right). \tag{25}$$

By using Eqs. (21), (22), and (23), we have

$$\begin{aligned} h_0 &= \sin\left(\frac{\phi^\xi}{\xi}\right), \\ z_0 &= \sin\left(\frac{\phi^\xi}{\xi}\right), \end{aligned} \tag{26}$$

$$\begin{aligned} h_1 &= E_\phi^{-1}E_\varphi^{-1}\left[\bar{\omega}E_\phi^\xi E_\varphi^\zeta\left[\frac{\partial^{2\xi} h_0}{\partial\phi^{2\zeta}} + 2h_0\frac{\partial^\xi h_0}{\partial\phi^\zeta} - \varsigma\frac{\partial^{2\xi} h_0 z_0}{\partial\phi^\zeta}\right]\right], \\ &= E_\phi^{-1}E_\varphi^{-1}\left[\bar{\omega}E_\phi^\xi E_\varphi^\zeta\left[-\sin\left(\frac{\phi^\xi}{\xi}\right)\right]\right], \\ &= E_\phi^{-1}E_\varphi^{-1}\left[\bar{\omega}^2\frac{\omega^2}{1-\omega^2}\right] \\ &= -\frac{\varphi^\zeta}{\zeta}\sin\left(\frac{\phi^\xi}{\xi}\right), \\ z_1 &= E_\phi^{-1}E_\varphi^{-1}\left[\bar{\omega}E_\phi^\xi E_\varphi^\zeta\left[\frac{\partial^{2\xi} z_0}{\partial\phi^{2\zeta}} + 2z_0\frac{\partial^\xi z_0}{\partial\phi^\zeta} - \varsigma\frac{\partial^{2\xi} h_0 z_0}{\partial\phi^\zeta}\right]\right], \\ &= E_\phi^{-1}E_\varphi^{-1}\left[\bar{\omega}E_\phi^\xi E_\varphi^\zeta\left[-\sin\left(\frac{\phi^\xi}{\xi}\right)\right]\right], \\ &= E_\phi^{-1}E_\varphi^{-1}\left[\bar{\omega}^2\frac{\omega^2}{1-\omega^2}\right] \\ &= -\frac{\varphi^\zeta}{\zeta}\sin\left(\frac{\phi^\xi}{\xi}\right), \\ h_2 &= E_\phi^{-1}E_\varphi^{-1}\left[\bar{\omega}E_\phi^\xi E_\varphi^\zeta\left[\frac{\partial^{2\xi} h_1}{\partial\phi^{2\zeta}} + 2\left(h_0\frac{\partial^\xi h_1}{\partial\phi^\zeta} + h_1\frac{\partial^\xi h_0}{\partial\phi^\zeta}\right)\right.\right. \\ &\quad \left.\left. - \varsigma\frac{\partial^\xi(h_0 z_1 + h_1 z_0)}{\partial\phi^\zeta}\right]\right], = E_\phi^{-1}E_\varphi^{-1}\left[\bar{\omega}E_\phi^\xi E_\varphi^\zeta\left[\frac{\varphi^\zeta}{\zeta}\sin\left(\frac{\phi^\xi}{\xi}\right)\right]\right], \\ &= E_\phi^{-1}E_\varphi^{-1}\left[\bar{\omega}^3\frac{\omega^2}{1-\omega^2}\right] = \frac{\left(\varphi^\zeta/\zeta\right)^2}{2}\sin\left(\frac{\phi^\xi}{\xi}\right), \end{aligned} \tag{27}$$

$$
\begin{aligned}
z_2 &= E_\phi^{-1} E_\varphi^{-1}\left[\bar{\omega} E_\phi^\xi E_\varphi^\zeta\left[\frac{\partial^{2\xi} z_1}{\partial\phi^{2\zeta}} + 2\left(z_0 \frac{\partial^\xi z_1}{\partial\phi^\zeta} + h_1 \frac{\partial^\xi z_0}{\partial\phi^\zeta}\right)\right.\right.\\
&\quad \left.\left. - \varsigma \frac{\partial^\xi (h_0 z_1 + h_1 z_0)}{\partial\phi^\zeta}\right]\right],\\
&= E_\phi^{-1} E_\varphi^{-1}\left[\bar{\omega} E_\phi^\xi E_\varphi^\zeta\left[\frac{\varphi^\zeta}{\zeta} \sin\left(\frac{\phi^\xi}{\xi}\right)\right]\right],\\
&= E_\phi^{-1} E_\varphi^{-1}\left[\bar{\omega}^3 \frac{\omega^2}{1-\omega^2}\right] = \frac{(\varphi^\zeta/\zeta)^2}{2} \sin\left(\frac{\phi^\xi}{\xi}\right),
\end{aligned}
\tag{28}
$$

$$
\begin{aligned}
h_3 &= E_\phi^{-1} E_\varphi^{-1}\left[\bar{\omega} E_\phi^\xi E_\varphi^\zeta\left[\frac{\partial^{2\xi} h_2}{\partial\phi^{2\zeta}} + 2\left(h_0 \frac{\partial^\xi h_2}{\partial\phi^\zeta} + h_1 \frac{\partial^\xi h_1}{\partial\phi^\zeta} + h_2 \frac{\partial^\xi}{\partial\phi^\zeta} h_0\right)\right.\right.\\
&\quad \left.\left. - \varsigma \frac{\partial^\xi}{\partial\phi^\zeta}(h_0 z_2 + h_1 z_1 + h_2 z_0)\right]\right]\\
&= -\frac{(\varphi^\zeta/\zeta)^3}{6} \sin\left(\frac{\phi^\xi}{\xi}\right),\\
z_3 &= E_\phi^{-1} E_\varphi^{-1}\left[\bar{\omega} E_\phi^\xi E_\varphi^\zeta\left[\frac{\partial^{2\xi} z_2}{\partial\phi^{2\zeta}} + 2\left(z_0 \frac{\partial^\xi z_2}{\partial\phi^\zeta} + z_1 \frac{\partial^\xi z_1}{\partial\phi^\zeta} + z_2 \frac{\partial^\xi}{\partial\phi^\zeta} z_0\right)\right.\right.\\
&\quad \left.\left. - \varsigma \frac{\partial^\xi}{\partial\phi^\zeta}(h_0 z_2 + h_1 z_1 + h_2 z_0)\right]\right]\\
&= -\frac{(\varphi^\zeta/\zeta)^3}{6} \sin\left(\frac{\phi^\xi}{\xi}\right).
\end{aligned}
\tag{29}
$$

And comparable to the other elements. Consequently, using Eqs. (22) and (23), the series solution is given by

$$
\begin{aligned}
h\left(\frac{\phi^\xi}{\xi}, \frac{\varphi^\zeta}{\zeta}\right) &= h_0 + h_1 + h_2 + h_3 + \cdots\\
&= \left(1 - \left(\frac{\phi^\xi}{\xi}\right) + \frac{(\varphi^\zeta/\zeta)^2}{2!} - \frac{(\varphi^\zeta/\zeta)^3}{3!} + \cdots\right) \sin\left(\frac{\phi^\xi}{\xi}\right),\\
z\left(\frac{\phi^\xi}{\xi}, \frac{\varphi^\zeta}{\zeta}\right) &= z_0 + z_1 + z_2 + z_3 + \cdots\\
&= \left(1 - \left(\frac{\phi^\xi}{\xi}\right) + \frac{(\varphi^\zeta/\zeta)^2}{2!} - \frac{(\varphi^\zeta/\zeta)^3}{3!} + \cdots\right) \sin\left(\frac{\phi^\xi}{\xi}\right).
\end{aligned}
\tag{30}
$$

And hence, exact solutions become

$$
\begin{aligned}
h\left(\frac{\phi^\xi}{\xi}, \frac{\varphi^\zeta}{\zeta}\right) &= e^{-\varphi^\zeta/\zeta} \sin\left(\frac{\phi^\xi}{\xi}\right),\\
z\left(\frac{\phi^\xi}{\xi}, \frac{\varphi^\zeta}{\zeta}\right) &= e^{-\varphi^\zeta/\zeta} \sin\left(\frac{\phi^\xi}{\xi}\right).
\end{aligned}
\tag{31}
$$

By taking $\xi = 1$ and $\zeta = 1$, the fractional solution become

$$
\begin{aligned}
h(\phi, \varphi) &= e^{-\zeta} \sin \xi,\\
z(\phi, \varphi) &= e^{-\zeta} \sin \xi.
\end{aligned}
\tag{32}
$$

Example 2. The following coupled space-fractional Burgers' equation is taken into consideration [23].

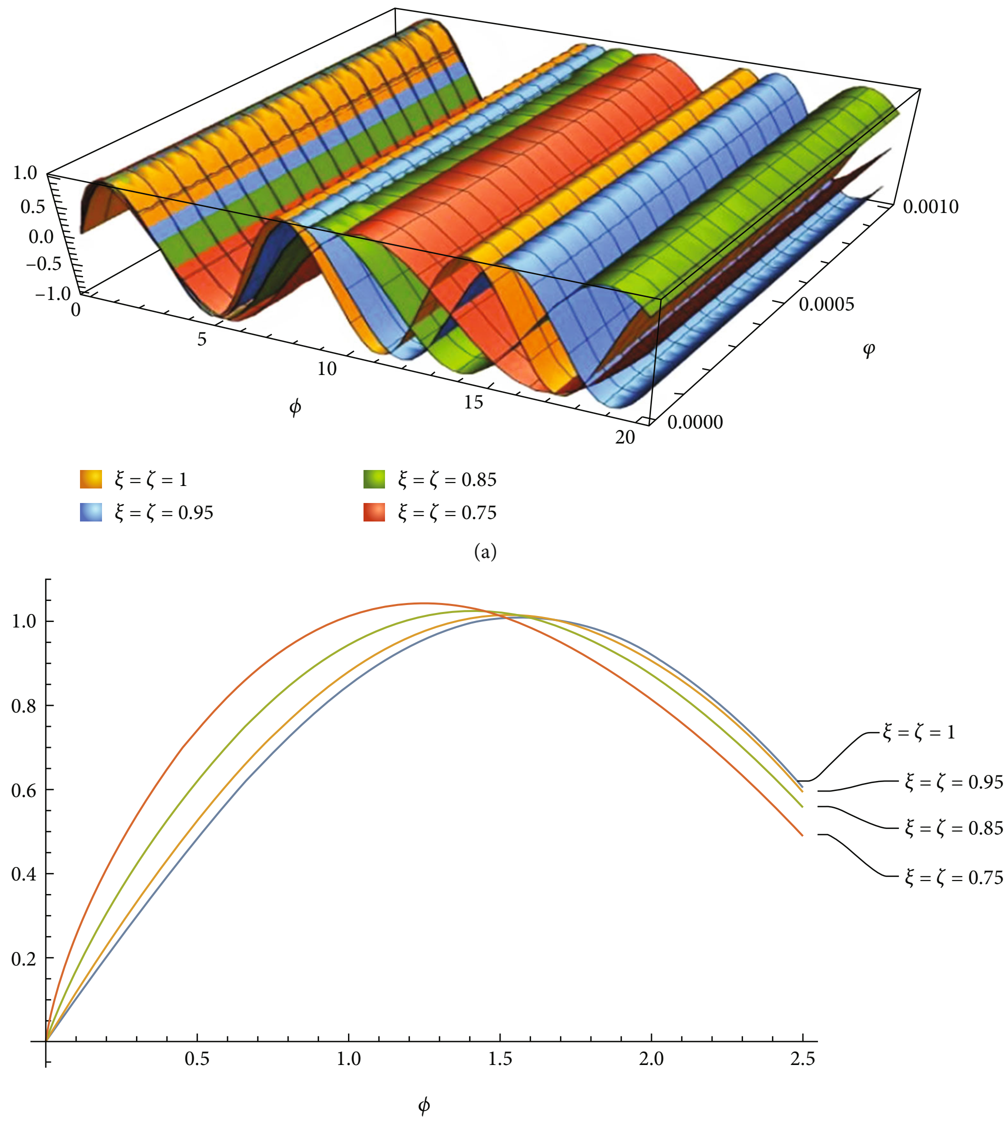
$$
\begin{aligned}
\frac{\partial^\zeta h}{\partial\varphi^\zeta} - \frac{\partial^{2\xi} h}{\partial\phi^{2\zeta}} - 2h \frac{\partial^\xi h}{\partial\phi^\zeta} + \varsigma \frac{\partial^{2\xi}}{\partial\phi^\zeta}(hz) &= 0,\\
\frac{\partial^\zeta z}{\partial\varphi^\zeta} - \frac{\partial^{2\xi} z}{\partial\phi^{2\zeta}} - 2z \frac{\partial^\xi z}{\partial\phi^\zeta} + \delta \frac{\partial^{2\xi}}{\partial\phi^\zeta}(hz) &= 0.
\end{aligned}
\tag{33}
$$

With I.C

$$
\begin{aligned}
h\left(\frac{\phi^\xi}{\xi}, 0\right) &= \left(\frac{\phi^\xi}{\xi}\right)^2,\\
z\left(\frac{\phi^\xi}{\xi}, 0\right) &= \left(\frac{\phi^\xi}{\xi}\right)^3.
\end{aligned}
\tag{34}
$$

By using Eqs. (22), (23), and (24), we have

$$
\begin{aligned}
h_0 &= \left(\frac{\phi^\xi}{\xi}\right)^2,\\
z_0 &= \left(\frac{\phi^\xi}{\xi}\right)^3,\\
h_1 &= E_\phi^{-1} E_\varphi^{-1}\left[\bar{\omega} E_\phi^\xi E_\varphi^\zeta\left[\frac{\partial^{2\xi} h_0}{\partial\phi^{2\zeta}} + 2h_0 \frac{\partial^\xi h_0}{\partial\phi^\zeta} - \varsigma \frac{\partial^{2\xi} h_0 z_0}{\partial\phi^\zeta}\right]\right],\\
&= E_\phi^{-1} E_\varphi^{-1}\left[\bar{\omega} E_\phi^\xi E_\varphi^\zeta\left[2 + 4\frac{\phi^{3\xi}}{\xi} - 5\frac{\phi^{4\xi}}{\xi}\right]\right],\\
&= E_\phi^{-1} E_\varphi^{-1}\left[\bar{\omega}^3\left(2\omega^2 + 4.3!\omega^3 - 5 \cdot 4!\omega^4\right)\right]\\
&= \frac{\varphi^\zeta}{\zeta}\left(2 + 4\frac{\phi^{3\xi}}{\xi} - 5\frac{\phi^{4\xi}}{\xi}\right),\\
z_1 &= E_\phi^{-1} E_\varphi^{-1}\left[\bar{\omega} E_\phi^\xi E_\varphi^\zeta\left[\frac{\partial^{2\xi} z_0}{\partial\phi^{2\zeta}} + 2z_0 \frac{\partial^\xi z_0}{\partial\phi^\zeta} - \varsigma \frac{\partial^{2\xi} h_0 z_0}{\partial\phi^\zeta}\right]\right]\\
&= E_\phi^{-1} E_\varphi^{-1}\left[\bar{\omega} E_\phi^\xi E_\varphi^\zeta\left[6\frac{\phi^\xi}{\xi} + 6\frac{\phi^{5\xi}}{\xi} - 5\frac{\phi^{4\xi}}{\xi}\right]\right]\\
&= E_\phi^{-1} E_\varphi^{-1}\left[\bar{\omega}^3\left(6\omega^3 + 6.5!\omega^3 - 5 \cdot 4!\omega^4\right)\right]\\
&= \frac{\varphi^\zeta}{\zeta}\left(6\frac{\phi^\xi}{\xi} + 6\frac{\phi^{5\xi}}{\xi} - 5\frac{\phi^{4\xi}}{\xi}\right),\\
h_2 &= E_\phi^{-1} E_\varphi^{-1}\left[\bar{\omega} E_\phi^\xi E_\varphi^\zeta\left[\frac{\partial^{2\xi} h_1}{\partial\phi^{2\zeta}} + 2\left(h_0 \frac{\partial^\xi h_1}{\partial\phi^\zeta} + h_1 \frac{\partial^\xi h_0}{\partial\phi^\zeta}\right)\right.\right.\\
&\quad \left.\left. - \varsigma \frac{\partial^\xi (h_0 z_1 + h_1 z_0)}{\partial\phi^\zeta}\right]\right],\\
&= E_\phi^{-1} E_\varphi^{-1}\left[\bar{\omega} E_\phi^\xi E_\varphi^\zeta\left[\frac{\varphi^\zeta}{\zeta}\left(26\frac{\phi^\xi}{\xi} - 84\frac{\phi^{2\xi}}{\xi} + 32\frac{\phi^{4\xi}}{\xi} - 44\frac{\phi^{5\xi}}{\xi} - 7\frac{\phi^{6\xi}}{\xi}\right)\right]\right],\\
&= E_\phi^{-1} E_\varphi^{-1}\left[\bar{\omega}^4\left(26\omega^3 - 84\Gamma(3)\omega^4 + 32\Gamma(5)\omega^6 - 44\Gamma(6)\omega^7 - 76\Gamma(7)\omega^8\right)\right],\\
&= \left(\frac{\varphi^\zeta}{\zeta}\right)^2\left(26\frac{\phi^\xi}{\xi} - 84\frac{\phi^{2\xi}}{\xi} + 32\frac{\phi^{4\xi}}{\xi} - 44\frac{\phi^{5\xi}}{\xi} - 7\frac{\phi^{6\xi}}{\xi}\right),
\end{aligned}
$$



(b)

FIGURE 1: Approximate solution of $h(\phi, \varphi)$ and $z(\phi, \varphi)$ in Example 1.

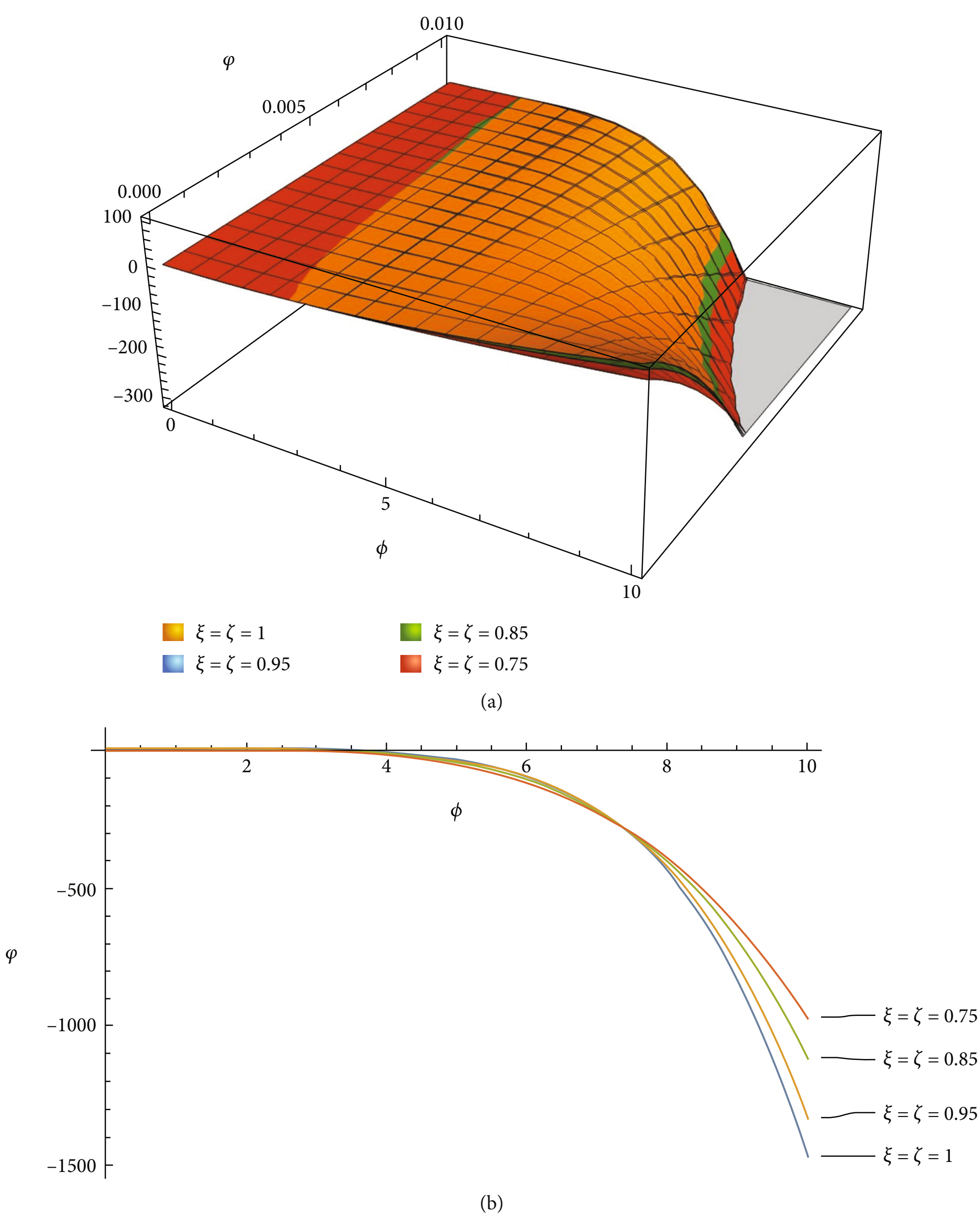
$$
\begin{aligned}
z_2 &= E_\phi^{-1} E_\varphi^{-1}\left[\bar{\omega} E_\phi^\xi E_\varphi^\zeta\left[\frac{\partial^{2\xi} z_1}{\partial \phi^{2\zeta}} + 2\left(z_0 \frac{\partial^\xi z_1}{\partial \phi^\zeta} + h_1 \frac{\partial^\xi z_0}{\partial \phi^\zeta}\right) - \varsigma \frac{\partial^\xi (h_0 z_1 + h_1 z_0)}{\partial \phi^\zeta}\right]\right], \\
&= E_\phi^{-1} E_\varphi^{-1}\left[\bar{\omega} E_\phi^\xi E_\varphi^\zeta\left[\frac{\varphi^\zeta}{\zeta}\left(-84\frac{\phi^{2\xi}}{\xi} + 132\frac{\phi^{3\xi}}{\xi} + 12\frac{\phi^{4\xi}}{\xi} + 4\frac{\phi^{5\xi}}{\xi} - 47\frac{\phi^{6\xi}}{\xi} - 10\frac{\phi^{7\xi}}{\xi} + 12\frac{\phi^{8\xi}}{\xi}\right)\right]\right], \\
&= E_\phi^{-1} E_\varphi^{-1}\left[\bar{\omega}^4\begin{pmatrix} -84\Gamma(3)\omega^4 + 132\Gamma(4)\omega^5 + 12\Gamma(5)\omega^6 + 4\Gamma(6)\omega^7 - 47\Gamma(7)\omega^8 \\ -10\Gamma(8)\omega^9 + 12\Gamma(9)\omega^{10} \end{pmatrix}\right], \\
&= \left(\frac{\varphi^\zeta}{\zeta}\right)^2\left(-84\frac{\phi^{2\xi}}{\xi} + 132\frac{\phi^{3\xi}}{\xi} + 12\frac{\phi^{4\xi}}{\xi} + 4\frac{\phi^{5\xi}}{\xi} - 47\frac{\phi^{6\xi}}{\xi} - 10\frac{\phi^{7\xi}}{\xi} + 12\frac{\phi^{8\xi}}{\xi}\right).
\end{aligned}
\tag{35}
$$

And comparable to the other elements. Consequently, using Eqs. (22) and (23), the series solution is given by

$$
\begin{aligned}
h\left(\frac{\phi^\xi}{\xi}, \frac{\varphi^\zeta}{\zeta}\right) &= h_0 + h_1 + h_2 + h_3 + \cdots \\
&= \left(\left(\frac{\phi^\xi}{\xi}\right)^2 + \frac{\varphi^\zeta}{\zeta}\left(2 + 4\frac{\phi^{3\xi}}{\xi} - 5\frac{\phi^{4\xi}}{\xi}\right) + \left(\frac{\varphi^\zeta}{\zeta}\right)^2 \right. \\
&\quad \left. \cdot \left(26\frac{\phi^\xi}{\xi} - 84\frac{\phi^{2\xi}}{\xi} + 32\frac{\phi^{4\xi}}{\xi} - 44\frac{\phi^{5\xi}}{\xi} - 7\frac{\phi^{6\xi}}{\xi}\right) + \cdots\right), \\
z\left(\frac{\phi^\xi}{\xi}, \frac{\varphi^\zeta}{\zeta}\right) &= z_0 + z_1 + z_2 + z_3 + \cdots \\
&= \left(\left(\frac{\phi^\xi}{\xi}\right)^3 + \frac{\varphi^\zeta}{\zeta}\left(6\frac{\phi^\xi}{\xi} + 6\frac{\phi^{5\xi}}{\xi} - 5\frac{\phi^{4\xi}}{\xi}\right) + \left(\frac{\varphi^\zeta}{\zeta}\right)^2 \right. \\
&\quad \left. \cdot \left(-84\frac{\phi^{2\xi}}{\xi} + 132\frac{\phi^{3\xi}}{\xi} + 12\frac{\phi^{4\xi}}{\xi} + 4\frac{\phi^{5\xi}}{\xi} - 47\frac{\phi^{6\xi}}{\xi} - 10\frac{\phi^{7\xi}}{\xi} + 12\frac{\phi^{8\xi}}{\xi}\right) + \cdots\right).
\end{aligned}
\tag{36}
$$

Table 1: Solution for the first three approximations with an exact solution with mesh points $\varphi = 0.01, 0.02$ for Eq. (24).

φ	ϕ	$\xi = \xi = 0.75$	$\xi = \xi = 0.85$	$\xi = \xi = 0.95$	$\xi = \xi = 1$
	0	0	0	0	0
	0.25	0.473696	0.362654	0.282034	0.249890
0.01	0.5	0.743001	0.621749	0.525227	0.484244
	0.75	0.917256	0.815280	0.727569	0.688489
	1	1.01379	0.945184	0.880319	0.849928
	0	0	0	0	0
	0.25	0.487511	0.369550	0.285538	0.252402
0.02	0.5	0.764670	0.633572	0.531753	0.489111
	0.75	0.944007	0.830784	0.736609	0.695409
	1	1.043360	0.963158	0.891257	0.858470



(a)

(b)

Figure 2: Approximate solution of Example 2 $h(\phi, \varphi)$.

Table 2: Solution for the first three approximations with an exact solution with mesh point $\varphi = 0.01, 0.02$ for Eq. (33) $h(\phi, \varphi)$.

φ	ϕ	$\xi = \xi = 0.75$	$\xi = \xi = 0.85$	$\xi = \xi = 0.95$	$\xi = \xi = 1$
	0	0.0843274	0.0469473	0.0265037	0.02
	0.25	0.310203	0.180159	0.106949	0.0830627
0.01	0.5	0.700438	0.472716	0.324629	0.271127
	0.75	1.17241	0.878232	0.663774	0.580623
	1	1.669	1.36099	1.10707	1.0023
	0	0.141821	0.0846226	0.0512018	0.04
	0.25	0.368019	0.219678	0.132811	0.103892
0.02	0.5	0.721289	0.501505	0.347784	0.290756
	0.75	1.10909	0.875062	0.674863	0.592884
	1	1.43787	1.28108	1.08182	0.9892

Table 3: Solution for the first three approximations with an exact solution with mesh points $\varphi = 0.01, 0.02$ for Eq. (33) $z(\phi, \varphi)$.

φ	ϕ	$\xi = \xi = 0.75$	$\xi = \xi = 0.85$	$\xi = \xi = 0.95$	$\xi = \xi = 1$
	0	0	0	0	0
	0.25	0.213391	0.0954739	0.044014	0.0301734
0.01	0.5	0.684713	0.363731	0.20247	0.153311
	0.75	1.52299	0.912165	0.576333	0.465761
	1	2.79768	1.83211	1.26733	1.0719
	0	0	0	0	0
	0.25	0.278178	0.130777	0.0630332	0.0440921
0.02	0.5	0.807105	0.429789	0.239092	0.180744
	0.75	1.73171	1.02198	0.636329	0.510585
	1	3.12774	2.01079	1.36744	1.1476

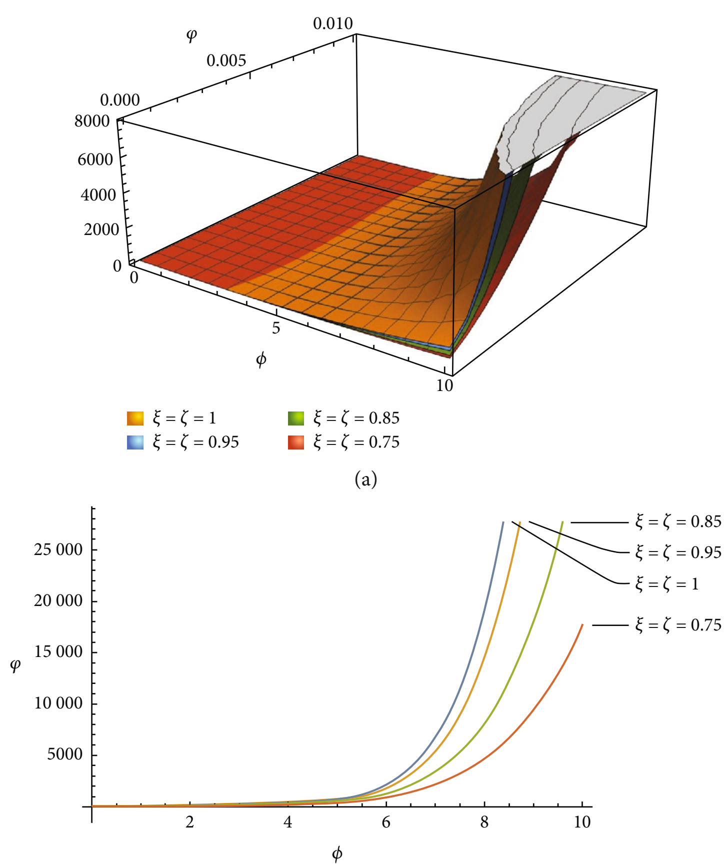


Figure 3: Approximate solution of Example 2 $z(\phi, \varphi)$.

5. Model Formulation and Numerical Calculation

In this section, we will use the numerical results of $H(\varphi, \phi)$ and $Z(\varphi, \phi)$ in Eqs. (24) and (33) for the exact solution when ($\xi = \zeta = 1$), and approximate solutions when ξ and ζ have different fractional values, to demonstrate the precision and effectiveness of the conformable double Elzaki transform method (CDET).

Figure 1 shows the plots of the truncated series solution using different numbers of terms together with the corresponding exact solution of Example 1 at $\varphi = 0.01, 0.02$. It shows that the truncated series solution of order $\xi = \zeta = 1$ is very close to the exact solution, which shows the rapid convergence of the proposed method.

Table 1 displays the differences in the precise and approximative solutions for the various orders in Example 1 for various ξ and ζ values. It demonstrates the method's derived approximation's quick convergence.

Figure 2 shows the plots of the truncated series solution using different numbers of terms together with the corresponding exact solution of Example 1 at $\varphi = 0.01, 0.02$. It shows that the truncated series solution of order $\xi = \zeta = 1$ is very close to the exact solution, which shows the rapid convergence of the proposed method.

Table 2 displays the differences in the precise and approximative solutions for the various orders in Example 1 for various ξ and ζ values. It demonstrates the method's derived approximation's quick convergence.

6. Conclusion

The CDEA concept is provided in this article, beginning with an application to a few particular tasks. Theorems and properties linked to CDEA were then presented and illustrated. We demonstrated the applicability and effectiveness of the suggested method by using the CDEA in conjunction with the iterative strategy to get precise solutions to a broad class of nonlinear conformable partial differential equations in the sense of conformable derivatives. To assess the reliability of the findings, we derived the conclusions and compared them to the exact solutions in the integer case. Our research led us to the conclusion that the suggested approach provides correct solutions to nonlinear conformable partial differential equations and is effective, appropriate, trustworthy, and adequate (Tables 1–3 and Figures 1–3). Additionally, compared to other similar approaches, the CDEA approach's computations use less computing resources.

Data Availability

On request for supporting data, the authors will provide the study's findings.

Conflicts of Interest

There is no conflict of interest between any authors and institutions.

Authors' Contributions

We all looked at the document and came to an agreement on the final version.

Acknowledgments

We would like to thank the Department of Mathematics for their assistance.

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