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Research Article

Fredholm and Schatten Class Operators on Bergman Spaces with Exponential Weights

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In this paper, we give a characterization of Fredholmness of the Toeplitz operators on the Bergman spaces A^p_φ with exponential weights in $\mathbb{D}$ when $0 < p < \infty$. Also, we obtain the sufficient and necessary conditions which the Toeplitz and Hankel operators on A^2_φ belong to h-Schatten class, where h is a continuous increasing convex function.

1. Introduction

Let $\mathbb{C}$ denote the complex plane, $\mathbb{R}$ be the real line, and $\mathscr{H}(\mathbb{D})$ be the space of analytic functions in the unit disc $\mathbb{D} = \{z \in \mathbb{C} | |z| < 1\}$. For $0 < p \leq \infty$ and a subharmonic function φ on $\mathbb{D}$, the weighted Lebesgue space L^p_φ consists of measurable functions f such that

$$\begin{aligned} \|f\|_{p,\varphi} &= \left\{ \int_{\mathbb{D}} \left| f(z) e^{-\varphi(z)} \right|^p dA(z) \right\}^{1/p} < \infty, \\ \|f\|_{\infty,\varphi} &= ess \sup_{z\in\mathbb{D}} |f(z)| e^{-\varphi(z)} < \infty, \end{aligned} \tag{1}$$

where $dA = (1/\pi)dxdy$ denotes the normalized Lebesgue area measure on $\mathbb{D}$. Then, the weighted Bergman space A^p_φ is defined as

$$A^p_\varphi = \mathscr{H}(\mathbb{D}) \cap L^p_\varphi. \tag{2}$$

Let $\mathscr{C}_0$ be the space of all continuous functions ρ on $\mathbb{D}$ satisfying $\lim_{|z|\longrightarrow 1} \rho(z) = 0$. The class $\mathscr{L}$ is set to be

$$\mathscr{L} = \left\{ \rho : \mathbb{D} \longrightarrow \mathbb{R} | \|\rho\|_{\mathscr{L}} = \sup_{\substack{z,w\in\mathbb{D} \\ z\neq w}} \frac{|\rho(z) - \rho(w)|}{|z - w|} < \infty, \rho \in \mathscr{C}_0 \right\}. \tag{3}$$

$\mathscr{L}_0$ is defined to be the family of those $\rho \in \mathscr{L}$ with a property that for each $\varepsilon > 0$, there exists a compact set $E \subset \mathbb{D}$ such that

$$|\rho(z) - \rho(w)| \leq \varepsilon |z - w|, \tag{4}$$

where $z, w \in \mathbb{D} \setminus E$.

We say that φ belongs to the weight class $\mathscr{W}_0$ if φ satisfies the following statement:

(a) $\varphi \in \mathscr{C}^2$, $\Delta\varphi > 0$

(b) there exists $\rho \in \mathscr{L}_0$, such that $(\Delta\varphi)^{-(1/2)} \simeq \rho$

where Δ denotes the Laplacian operator and notation $a \simeq b$ indicates that there exists nonessential positive constant C such that $C^{-1} a \leq b \leq Ca$.

In what follows, we are focused on A^p_φ with $\varphi \in \mathscr{W}_0$, and the collection $\mathscr{W}_0$ contains nonradial weights, two classes of weights related closely to $\mathscr{W}_0$. One was introduced by Oleinik and Pavlov [1], denoted by $\mathscr{OP}$, and the other was introduced by Borichev et al. [2], denoted by $\mathcal{BDK}$. As stated by Hu et al. [3], the weight $\mathcal{W}_0$ covers $\mathcal{BDK}$, but there is no inclusion relationship between $\mathcal{W}_0$ and $\mathcal{OP}$.

It is easy to know that A^p_φ is a Banach space with norm $\|\cdot\|_{p,\varphi}$ if $1 \leq p \leq \infty$, whereas A^p_φ is a Fréchet space with metric $d(f, g) = \|f - g\|^p_{p,\varphi}$ if $0 < p < 1$. In particular, A^2_φ is a Hilbert space. Let $K_\varphi(w, z)$ be the Bergman kernel of A^2_φ and $K_z(w) = K(w, z) = K_\varphi(w, z)$ for short, and it is obvious that $\overline{K_z(w)} = K_w(z)$. For $0 < p < \infty$ and $z \in \mathbb{D}$, set $k_{p,z}(w) = K_z(w)/\|K_z\|_{p,\varphi}$ to be the normalized Bergman kernel for A^p_φ, and $k_z(w) = k_{2,z}(w)$ for short. The Bergman projection P can be represented as

$$Pf(z) = \int_{\mathbb{D}} K_w(z) f(w) e^{-2\varphi(w)} dA(w), z \in \mathbb{D}, \tag{5}$$

Moreover, P is bounded from $L^p_\varphi \longrightarrow A^p_\varphi$ and $Pf = f$ for any $f \in A^p_\varphi$ when $0 < p \leq \infty$. Let $\Gamma = \{\sum_{j=1}^m a_j K(\cdot, z_j) | m \in \mathbb{N}, z_j \in \mathbb{D}$ and $a_j \in \mathbb{C}$ for $j = 1, 2, \cdots m\}$. For $\varphi \in \mathcal{W}_0$ and $0 < p < \infty$, we know that Γ is dense in A^p_φ under the A^p_φ-norm $\|\cdot\|_{p,\varphi}$. Then, for $f \in L^p_\varphi$, we define densely the Toeplitz operator and Hankel operator, respectively, with symbol f as

$$\begin{aligned} T_f g &= P(fg), g \in \Gamma, \\ H_f g &= (I - P)(fg) = fg - P(fg), g \in \Gamma, \end{aligned} \tag{6}$$

where I is the identity operator.

In this paper, we will study the Fredholm properties of the Toeplitz operators on the Bergman spaces with exponential weights. Berger and Coburn [4] were the first to study the Fredholm Toeplitz operators on the Fock spaces F^2; recently, the Fredholm theory was extended to the doubling Fock spaces in Hu and Virtanen's study [5]. Hagger [6] studied the Fredholm properties of the Toeplitz operators acting on the weighted Bergman spaces $A^p_\nu(\mathbb{B}_n)$, Zorboska [7] determined the Fredholm Toeplitz operators with BMO[1] symbols when the Berezin transforms of the operators are bounded and of vanishing oscillation. Taskinen and Virtanen [8] combined some of the best known results on the compactness of the Toeplitz and Hankel operators in order to generalize the results on the Fredholm properties of the Toeplitz operators. Our goals of the present paper is to characterize the Fredholm properties of the Toeplitz operators T_f with vanishing mean oscillation symbols on exponential Bergman spaces with $\varphi \in \mathcal{W}_0$. In addition, as we know, the Toeplitz operators T_μ belong to the Schatten p-class $S_p(A^2_\varphi)$ when $0 < p < \infty$ was first considered by Luecking [9], and later, Arroussi et al. [10] considered the same problem and described the membership in Schatten p-class $S_p(A^2_w)$. Recently Zhang et al. [11] described Schatten p-class Toeplitz operators on A^2_φ. Also, Luecking [12] characterized the Schatten p-class of the Hankel operators H_f on the Bergman spaces, and Zhang et al. [11] characterized the Schatten p-class Hankel operators with general symbols on the Bergman spaces with exponential weights when $0 < p < \infty$. The definition of S_h was first introduced by EI-Falla et al. [13], and Arroussi et al. [14] characterized the h-Schatten class Toeplitz operators on large Fock spaces, where h is a continuous increasing convex function. At present, we will characterize h-Schatten class Toeplitz and Hankel operators on A^2_φ. It is worth mentioning that the result of h-Schatten class Hankel operators has not been studied before. And the details of our characterizations are shown in section 5.

This paper is organized as follows. In section 2, we will give some useful results which contain mainly Bergman kernel estimates, etc. Section 3 provides the proofs of boundedness and compactness of the Topelitz and Hankel operators. In section 4, we characterize the Fredholm properties of the Topelitz operators on A^p_φ. Section 5 contains the characterization of h-Schatten class Toeplitz and Hankel operators on A^2_φ.

2. Preliminaries

Let $\varphi \in \mathcal{W}_0$, we define the distance $d_\rho(z, w)$ as

$$d_\rho(z, w) = \inf_r \int_0^1 |\gamma'(t)| \frac{dt}{\rho(\gamma(t))}, \tag{7}$$

where the infimum is taken over all piecewise $\mathcal{C}^1$ curves $\gamma : [0, 1] \longrightarrow \mathbb{D}$ with $\gamma(0) = z$ and $\gamma(1) = w$. In fact, $d_\rho(\cdot, \cdot)$ is equivalent to the Bergman distance $\beta_\varphi(\cdot, \cdot)$ induced by the Bergman metric $1/2(\partial^2 \log K_z(z)/\partial z \partial \bar{z}) dz \otimes d\bar{z}$.

For $z \in \mathbb{D}$ and $r > 0$, define the disks $B_\rho(z, r) = \{w \in \mathbb{D} | d_\rho(w, z) < r\}$, $D(z, r) = \{w \in \mathbb{D} || w - z| < r\}$, and $D^r(z) = D(z, r\rho(z))$, and for more information, refer to [3].

Lemma 1. *Let $\rho \in \mathcal{L}$ be positive. Then, there exists $\alpha > 0$ with the following properties:*

(a) There exists constants C_1 and C_2 such that

$$C_1 \rho(w) \leq \rho(z) \leq C_2 \rho(w), \tag{8}$$

for $z \in \mathbb{D}$ and $w \in D^\alpha(z)$.

(b) There exists a constant $B > 0$ such that

$$D^r(z) \subseteq D^{Br}(w), D^r(w) \subseteq D^{Br}(z), \tag{9}$$

for $w \in D^r(z)$ and $0 < r \leq \alpha$.

(c) There exist positive constants C_1 and C_2 such that

$$B_\rho(z, C_1 r) \subseteq D^r(z) \subseteq B_\rho(z, C_2 r), \tag{10}$$

for $r \in \mathbb{D}$, and $0 < r \leq \alpha$.

Proof. See Lemma 3.1 in Hu et al.'s study [3]. □

Lemma 2. *If $\rho \in \mathscr{L}$ is positive, then there exist positive constants α and s, depending only on $\|\rho\|_{\mathscr{L}}$ such that for $0 < r \le \alpha$, there exist a sequence $\{w_k\} \subset \mathbb{D}$ satisfying*

(a) $\mathbb{D} = \bigcup_k D^r(w_k)$

(b) $D^{sr}(w_k) \bigcap D^{sr}(w_j) = \varnothing$ *for* $k \ne j$

(c) $\{D^{2\alpha}(w_k)\}_k$ *is a covering of* $\mathbb{D}$ *of finite multiplicity* N

Proof. See Lemma 2.1 in Zhang et al.'s study [11]. □

A sequence $\{w_k\}$ satisfying $(A)-(C)$ of Lemma 2 will be called a (ρ, r)-lattice. The set of (ρ, r)-lattices will be denoted by $L(\rho, r)$. The statement (c) of Lemma 2 says that for $\{w_k\} \in \mathrm{L}(\rho, r)$, there exists an integer N such that

$$1 \le \sum_{k=1}^{\infty} \chi_{D^{2\alpha}(w_k)}(z) \le N \text{ for } z \in \mathbb{D}. \tag{11}$$

Lemma 3. *Let $\varphi \in \mathscr{W}_0$. There are positive constants C_1, C_2, and σ such that*

$$\begin{aligned} |K_z(w)| &\le C_1 \frac{e^{\varphi(z)+\varphi(w)}}{\rho(z)\rho(w)} e^{-\sigma d_\rho(z,w)} \text{ for } z, w \in \mathbb{D}, \\ |K_z(w)| &\ge C_2 \frac{e^{\varphi(z)+\varphi(w)}}{\rho(z)\rho(w)} \text{ for } d_\rho(z,w) \le \alpha. \end{aligned} \tag{12}$$

Proof. See Theorem 12 in Hu et al.'s study [3]. □

Lemma 4. *Let $\varphi \in \mathscr{W}_0$. Then*

$$\begin{aligned} \|K_z\|_{p,\varphi} &\simeq e^{\varphi(z)} \rho^{(2/p)-2}(z), \\ \|k_z\|_{p,\varphi} &\simeq \rho^{(2/p)-1}(z), \\ \|(\cdot, -z)k_z\|_{p,\varphi} &\simeq \rho^{2/p}(z) \text{ for } z \in \mathbb{D}. \end{aligned} \tag{13}$$

Proof. See Corollary 3.2 in Hu et al.'s study [3]. □

Lemma 5. *Suppose $\rho \in \mathscr{L}_0, k > -2, \sigma > 0, -\infty < l < \infty$. Then*

$$\int_{\mathbb{D}} |\xi - z|^k \rho^l(\xi) e^{-\sigma d_\rho(z,\xi)} dA(\xi) \le C\rho^{k+l+2}(z) \text{ for } z \in \mathbb{D}. \tag{14}$$

Proof. See Corollary 3.1 in Hu et al.'s study [3]. □

Lemma 6. *Suppose $\rho \in \mathscr{L}_0$, $k, l \in \mathbb{R}, \sigma > 0$. Then, there is a constant $C > 0$ such that*

$$\int_{\mathbb{D}} \left(d_\rho(z,\xi) + 1\right)^k \rho^l(\xi) e^{-\sigma d_\rho(z,\xi)} dA(\xi) \le C\rho^{l+2}(z). \tag{15}$$

Proof. Since $d_\rho(z,\xi) + 1 \ge 1$, it is obvious that for $z, \xi \in \mathbb{D}$, there is a constant $C_1 > 0$ such that

$$\left(d_\rho(z,\xi) + 1\right)^k \le C_1 e^{(\sigma/2) d_\rho(z,\xi)}, \tag{16}$$

for any $k \in \mathbb{R}$. Then, for $z \in \mathbb{D}$, we have

$$\begin{aligned} &\int_{\mathbb{D}} \left(d_\rho(z,\xi) + 1\right)^k \rho^l(\xi) e^{-\sigma d_\rho(z,\xi)} dA(\xi) \\ &\le C_1 \int_{\mathbb{D}} \rho^l(\xi) e^{-(\sigma/2) d_\rho(z,\xi)} dA(\xi) \\ &\le C_1 C \rho^{l+2}(z). \end{aligned} \tag{17}$$

□

Lemma 7. *For $p > 0$ and $l \in \mathbb{R}$, the following statements hold*

(a) $\int_{\mathbb{D}} (d_\rho(z,w) + 1)^l |K_z(w)|^p e^{-p\varphi(w)} dA(w) \lesssim C\rho^{2(1-p)}(z) e^{p\varphi(z)}$ *for* $z \in \mathbb{D}$

(b) $\lim_{R \longrightarrow \infty} \sup_{z \in \mathbb{D}} \rho^{2(p-1)}(z) e^{-p\varphi(z)} \int_{(D^R(z))^c} (d_\rho(z,w) + 1)^l |K_z(w)|^p e^{-p\varphi(w)} dA(w) = 0$

Proof.

(a) Lemma 3 implies that

$$\begin{aligned} &\int_{\mathbb{D}} \left(d_\rho(z,w) + 1\right)^l |K_z(w)|^p e^{-p\varphi(w)} dA(w) \\ &\le e^{p\varphi(z)} C_1 \int_{\mathbb{D}} \left(d_\rho(z,w) + 1\right)^l \rho^{-p}(z) \rho^{-p}(w) e^{-\sigma p d_\rho(z,w)} dA(w). \end{aligned} \tag{18}$$

It follows from Lemma 6 that

$$\begin{aligned} &e^{p\varphi(z)} \int_{\mathbb{D}} \left(d_\rho(z,w) + 1\right)^l \rho^{-p}(z) \rho^{-p}(w) e^{-\sigma p d_\rho(z,w)} dA(w) \\ &\le c e^{p\varphi(z)} \rho^{2-2p}(z). \end{aligned} \tag{19}$$

(b) Lemma 2.3 and Theorem 3.3 in Hu et al.'s study [3] show that

$$\begin{aligned} &\rho^{2(p-1)}(z) e^{-p\varphi(z)} \int_{(D^R(z))^c} \left(d_\rho(z,w) + 1\right)^l |K_z(w)|^p e^{-p\varphi(w)} dA(w) \\ &\le \rho^{p-2}(z) \int_{(D^R(z))^c} \left(d_\rho(z,w) + 1\right)^l e^{-(\sigma/2) p d_\rho(z,w)} \rho^{-p}(w) \frac{(\min\{\rho(z), \rho(w)\})^p}{|z-w|^p} dA(w). \end{aligned} \tag{20}$$

Note that $\lim_{|w| \longrightarrow 1} \rho(w) = 0$, and then, for sufficient lager number $R > 0$, we have

$$\rho^{p-2}(z)\int_{(D^R(z))^c}(d_\rho(z,w)+1)^l e^{-(\sigma/2)pd_\rho(z,w)}\rho^{-p}(w)\frac{(\min\{\rho(z),\rho(w)\})^p}{|z-w|^p}dA(w)$$
$$\lesssim \rho^{p-2}(z)\int_{(D^R(z))^c}(d_\rho(z,w)+1)^l e^{-(\sigma p/2)d_\rho(z,w)}|z-w|^{-p}dA(w). \tag{21}$$

Since $|z-w|\geq R\rho(z)$, we obtain that $|z-w|^{-p}\leq R^{-p}\rho(z)^{-p}$. Then

$$\rho^{p-2}(z)\int_{(D^R(z))^c}(d_\rho(z,w)+1)^l e^{-(\sigma p/2)d_\rho(z,w)}|z-w|^{-p}dA(w)$$
$$\lesssim R^{-p}\rho^{-2}(z)\int_{\mathbb{D}}(d_\rho(w,z)+1)^l e^{-(\sigma p/2)d_\rho(z,w)}dA(w). \tag{22}$$

Moreover, Lemma 6 implies that

$$R^{-p}\rho^{-2}(z)\int_{\mathbb{D}}(d_\rho(w,z)+1)^l e^{-(\sigma p/2)d_\rho(z,w)}dA(w)$$
$$\lesssim R^{-p}\longrightarrow 0(R\longrightarrow +\infty). \tag{23}$$

Thus

$$\lim_{R\longrightarrow+\infty}\sup_{z\in\mathbb{D}}\rho^{2(p-1)}(z)e^{-p\varphi(z)}\int_{(D^R(z))^c}(d_\rho(z,w)+1)^l|K_z(w)|^p e^{-p\varphi(w)}dA(w)=0. \tag{24}$$

□

For $1\leq p<\infty$, let $L^p_{loc}(\mathbb{D})$ be the set of all p-th locally Lebesgue integrable function on $\mathbb{D}$. For $f\in L^1_{\text{loc}}(\mathbb{D})$ and $r>0$, the averaging function $\widehat{f}_r$ is defined as

$$\widehat{f}_r(z)=\frac{1}{|D^r(z)|}\int_{D^r(z)}f(w)dA(w), \tag{25}$$

where $|D^r(z)|$ is the volume of $D^r(z)$. Suppose $f\in L^p_{\text{loc}}(\mathbb{D})$ and $r>0$, the p-th mean oscillation of f given by

$$MO_{p,r}(f)(z)=\left(\frac{1}{|D^r(z)|}\int_{D^r(z)}\left|f(w)-\widehat{f}_r(z)\right|^p dA(w)\right)^{1/p}, z\in\mathbb{D}. \tag{26}$$

The space BMO^p_r consists of the functions $f\in L^p_{\text{loc}}(\mathbb{D})$ such that $\|f\|_{\text{BMO}^p_r}=\sup_{z\in\mathbb{D}}\text{MO}^p_{p,r}(f)(z)<\infty$, where $1\leq p<\infty$ and $r>0$. When $\lim_{|z|\longrightarrow 1}\text{MO}_{p,r}(f)(z)=0$, we say $f\in\text{VMO}^p_r$.

For a continuous function f on $\mathbb{D}$, and $r>0$, let

$$\omega_r(f)(z)=\sup_{w\in D^r(z)}\{|f(z)-f(w)|\}, z\in\mathbb{D} \tag{27}$$

be the oscillation of f.

For $r>0$, BO_r denote the space of all continuous functions f on $\mathbb{D}$ such that $\|f\|_{\text{BO}_r}=\sup_{z\in\mathbb{D}}\omega_r(f)(z)<\infty$. The space VO_r consists of functions f in BO_r satisfying $\lim_{|z|\longrightarrow 1}\omega_r(f)(z)=0$.

Suppose $1\leq p<\infty$, and $r>0$, let BA^p_r denote the space of all locally p-th integrable functions f on $\mathbb{D}$ such that $\|f\|_{BA^p_r}=\sup_{z\in\mathbb{D}}[\widehat{(|f|^p)}_r(z)]^{1/p}<\infty$. The space VA^p_r consists of the functions $f\in BA^p_r$ for which $\lim_{|z|\longrightarrow 1}[\widehat{(|f|^p)}_r(z)]^{1/p}=0$.

For $f\geq 0$ and $f|k_z|^2\in L^1_\varphi$ for any $z\in\mathbb{D}$, we define the Berezin transform $\tilde{f}$ of f by

$$\tilde{f}(z)=\int_{\mathbb{D}}f(w)|k_z(w)|^2e^{-2\varphi(w)}dA(w), \tag{28}$$

for $f\in L^p_{\text{loc}}(\mathbb{D})$. When $d\mu=|f|^p dA$, it follows from Theorem 11 and Theorem 12 in Zhang et al.'s study [11] with $p=q$ that

$$\sup_{z\in\mathbb{D}}\widehat{(|f|^p)}_r(z)\simeq\sup_{z\in\mathbb{D}}\widetilde{(|f|^p)}(z),$$
$$\lim_{|z|\longrightarrow 1}\widehat{(|f|^p)}_r(z)=0\Leftrightarrow\lim_{|z|\longrightarrow 1}\widetilde{(|f|^p)}(z)=0. \tag{29}$$

Theorem 8. *Let $0<r<\alpha, 1\leq p<\infty$ and $0<t,l<\infty$. Then, the following statements are all equivalent:*

(a) $f\in BMO^p_r$;

(b) f admits a decomposition $f=f_1+f_2$, where $f_1\in BO_r$ and $f_2\in BA^p_r$, moreover

$$\|f_1\|_{BO_r}+\|f_2\|_{BA^p_r}\simeq\|f\|_{BMO^p_r}, \tag{30}$$

(c) The function $\int_{\mathbb{D}}|f(w)-\widetilde{f}_t(z)|^p|k_{l,z(w)}|^l e^{-l\varphi(w)}dA(w)$ is bounded, where $\widetilde{f}_t(z)=\int_{\mathbb{D}}f(w)|k_{t,z}(w)|^t e^{-t\varphi(w)}dA(w)$ and $k_{t,z}(w)=K_z(w)/\|K_z\|_{t,\varphi}$

Proof. $(A)\Rightarrow(B)$. For $w\in D^{r/B}(z)$ with B as in Lemma 1(B), we have $D^{r/B}(w)\subseteq D^r(z)$. Set $f_1=\widehat{f}_{r/B}$ and $f_2=f-f_1$. Suppose $f\in\text{BMO}^p_r$ with $1\leq p<\infty$, and $w\in D^{r/B}(z)$, by the triangle inequality and Hölder's inequality, one can get

$$\begin{aligned}|f_1(w)-f_1(z)| &\le \left|f_1(z)-\widehat{f}_r(z)\right| + \left|\widehat{f}_r(z)-f_1(w)\right| \\ &\le \frac{1}{|D^{r/B}(z)|}\int_{D^{r/B}(z)}\left|f(u)-\widehat{f}_r(z)\right|dA(u) \\ &\quad + \frac{1}{|D^{r/B}(w)|}\int_{D^{r/B}(w)}\left|f(u)-\widehat{f}_r(z)\right|dA(u) \\ &\lesssim \left(\frac{1}{|D_{r/B}(z)|}\int_{D^{r}(z)}\left|f(u)-\widehat{f}_r(z)\right|^p dA(u)\right)^{1/p}.\end{aligned} \quad (31)$$

Thus $f_1 \in \mathrm{BO}_{r/B}$, it is easy to know that $f_1 \in BO_r$ and $\|f_1\|_{\mathrm{BO}_r} \lesssim \|f\|_{\mathrm{BMO}_r^p}$.

By the triangle inequality, we have

$$\begin{aligned}\mathrm{MO}_{r/B}^p(f)(z) &\le \left(\frac{1}{|D^{r/B}(z)|}\int_{D^{r/B}(z)}\left|f(u)-\widehat{f}_r(z)\right|^p dA(u)\right)^{1/p} \\ &\quad + \left|\widehat{f}_r(z)-\widehat{f_{r/B}}(z)\right|.\end{aligned} \quad (32)$$

If $1 \le p < \infty$, the Hölder's inequality implies that

$$\begin{aligned}\left|\widehat{f}_r(z)-\widehat{f_{r/B}}(z)\right| &= \left|\frac{1}{|D^{r/B}(z)|}\int_{D^{r/B}(z)} f(u)-\widehat{f}_r(z)dA(u)\right| \\ &\le \left|\frac{1}{|D^{r/B}(z)|}\int_{D^{r/B}(z)}\left|f(u)-\widehat{f}_r(z)\right|^p dA(u)\right|^{1/p}.\end{aligned} \quad (33)$$

Then, for $z \in \mathbb{D}$, there holds

$$\begin{aligned}\left(\widehat{(|f_2|^p)}_{r/B}(z)\right)^{1/p} &= \left(\frac{1}{|D^{r/B}(z)|}\int_{D^{r/B}(z)}|f(u)-f_1(u)|^p dA(u)\right)^{1/p} \\ &\le \left(\frac{1}{|D^{r/B}(z)|}\int_{D^{r/B}(z)}\left|f(u)-\widehat{f_{r/B}}(z)\right|^p dA(u)\right)^{1/p} \\ &\quad + \left(\frac{1}{|D^{r/B}(z)|}\left|\int_{D^{r/B}(z)}|f_1(u)-f_1(z)|^p dA(u)\right)^{1/p} \\ &\le \mathrm{MO}_{r/B}^p(f)(z) + \omega_{r/B}(f_1)(z) \\ &\lesssim \mathrm{MO}_r^p(f)(z) + \omega_r(f_1)(z).\end{aligned} \quad (34)$$

Thus, $f_2 \in BA_{r/B}^p$ and $\|f_2\|_{BA_r^p} \lesssim \|f\|_{\mathrm{BMO}_r^p}$.

$(B) \Rightarrow (C)$. Suppose $f = f_1 + f_2$ where $f_1 \in \mathrm{BO}_r$ and $f_2 \in \mathrm{BA}_r^p$, by the triangle inequality, there holds

$$\begin{aligned}&\left(\int_{\mathbb{D}}\left|f_1(w)-\widetilde{(f_1)}_t(z)\right|^p\left|k_{l,z}(w)\right|^l e^{-l\varphi(w)}dA(w)\right)^{1/p} \\ &\le \left(\int_{\mathbb{D}}|f_1(w)-f_1(z)|^p\left|k_{l,z}(w)\right|^l e^{-l\varphi(w)}dA(w)\right)^{1/p} \\ &\quad + \left(\int_{\mathbb{D}}\left|f_1(z)-\widetilde{(f_1)}_t(z)\right|^p\left|k_{l,z}(w)\right|^l e^{-l\varphi(w)}dA(w)\right)^{1/p} \\ &= \left(\int_{\mathbb{D}}|f_1(w)-f_1(z)|^p\left|k_{l,z}(w)\right|^l e^{-l\varphi(w)}dA(w)\right)^{1/p} \\ &\quad + \left|f_1(z)-\widetilde{(f_1)}_t(z)\right|.\end{aligned} \quad (35)$$

Lemma 3 and Lemma 4 imply that

$$\left|k_{l,z}(w)\right|^l e^{-l\varphi(w)} \lesssim \rho^{l-2}(z)\rho^{-l}(w)e^{-\sigma l d_\rho(z,w)}. \quad (36)$$

Note that $\sup_{z,w\in\mathbb{D}}(d_\rho(z,w)+1)^p e^{-(l\sigma/2)d_\rho(z,w)} \le C$ for some constant $C > 0$, and

$$|f_1(w)-f_1(z)| \lesssim \left(d_\rho(z,w)+1\right)\|f_1\|_{\mathrm{BO}_r}. \quad (37)$$

By Lemma 6, we have

$$\begin{aligned}&\int_{\mathbb{D}}|f_1(w)-f_1(z)|^p\left|k_{l,z}(w)\right|^l e^{-l\varphi(w)}dA(w) \\ &\le \|f_1\|_{\mathrm{BO}_r}^p\int_{\mathbb{D}}\left(d_\rho(z,w)+1\right)^p\rho^{l-2}(z)\rho^{-l}(w)e^{-\sigma l d_\rho(z,w)}dA(w) \\ &\lesssim \|f_1\|_{\mathrm{BO}_r}^p.\end{aligned} \quad (38)$$

For $1 \le p < \infty$, by the same reason, we have

$$\begin{aligned}\left|f_1(z)-\widetilde{(f_1)}_t(z)\right| &= \left|\int_{\mathbb{D}}(f_1(z)-f_1(w))\left|k_{t,z}(w)\right|^t e^{-t\varphi(w)}dA(w)\right| \\ &\lesssim \left(\int_{\mathbb{D}}|f_1(w)-f_1(z)|^p\left|k_{t,z}(w)\right|^t e^{-t\varphi(w)}dA(w)\right)^{1/p} \\ &\lesssim \|f_1\|_{\mathrm{BO}_r}.\end{aligned} \quad (39)$$

It is obvious that $|\widetilde{(f_2)}_t(z)| \lesssim [\widetilde{(|f_2|^p)}_t(z)]^{1/p}$ for $1 \le p < \infty$. Next, we prove the part with respect to f_2. By triangle inequality, there holds

$$\begin{aligned}&\left(\int_{\mathbb{D}}\left|f_2(w)-\widetilde{(f_2)}_t(z)\right|^p\left|k_{l,z}(w)\right|^l e^{-l\varphi(w)}dA(w)\right)^{1/p}\\&\quad\le\left(\int_{\mathbb{D}}|f_2(w)|^p\left|k_{l,z}(w)\right|^l e^{-l\varphi(w)}dA(w)\right)^{1/p}\\&\qquad+\left(\int_{\mathbb{D}}\left|\widetilde{(f_2)}_t(z)\right|^p\left|k_{l,z}(w)\right|^l e^{-l\varphi(w)}dA(w)\right)^{1/p}\\&\quad=\left[\widetilde{\left(|f_2|^p\right)}_l(z)\right]^{1/p}+\left|\widetilde{(f_2)}_t(z)\right|\le\|f_2\|_{\mathrm{BA}_r^p}.\end{aligned}\tag{40}$$

Thus

$$\begin{aligned}&\left(\int_{\mathbb{D}}\left|f(w)-\widehat{f}_t(z)\right|^p\left|k_{l,z}(w)\right|^l e^{-l\varphi(w)}dA(w)\right)^{1/p}\\&\quad\lesssim\|f_1\|_{\mathrm{BO}_r}+\|f_2\|_{\mathrm{BA}_r^p}.\end{aligned}\tag{41}$$

$(C)\Rightarrow(A)$. By Lemma 3, for $w\in D^r(z)$

$$\left|k_{l,z}(w)\right|^l e^{-l\varphi(w)}\simeq\frac{1}{|D^r(z)|}\simeq\rho^{-2}(z).\tag{42}$$

Hence

$$\begin{aligned}&\frac{1}{|D^r(z)|}\int_{D^r(z)}\left|f(w)-\widetilde{f}_t(z)\right|^p dA(w)\\&\quad\simeq\int_{D^r(z)}\left|f(w)-\widetilde{f}_t(z)\right|^p\left|k_{l,z}(w)\right|^l e^{-l\varphi(w)}dA(w)\\&\quad\le\int_{\mathbb{D}}\left|f(w)-\widetilde{f}_t(z)\right|^p\left|k_{l,z}(w)\right|^l e^{-l\varphi(w)}dA(w).\end{aligned}\tag{43}$$

Note that

$$\begin{aligned}\mathrm{MO}_r^p(f)(z)&=\left(\frac{1}{|D^r(z)|}\int_{D^r(z)}\left|f(w)-\widehat{f}_r(z)\right|^p dA(w)\right)^{1/p}\\&\le\left(\frac{1}{|D^r(z)|}\int_{D^r(z)}\left|f(w)-\widetilde{f}_t(z)\right|^p dA(w)\right)^{1/p}\\&\quad+\left(\frac{1}{|D^r(z)|}\int_{D^r(z)}\left|\widehat{f}_r(z)-\widetilde{f}_t(z)\right|dA(w)\right)^{1/p}\\&=\left(\frac{1}{|D^r(z)|}\int_{D^r(z)}\left|f(w)-\widetilde{f}_t(z)\right|^p dA(w)\right)^{1/p}\\&\quad+\left|\widehat{f}_r(z)-\widetilde{f}_t(z)\right|\\&\le\left(\frac{1}{|D^r(z)|}\int_{D^r(z)}\left|f(w)-\widetilde{f}_t(z)\right|^p dA(w)\right)^{1/p}\\&\quad+\left(\frac{1}{|D^r(z)|}\int_{D^r(z)}\left|f(w)-\widetilde{f}_t(z)\right|^p dA(w)\right)^{1/p}\\&\le 2\left(\frac{1}{|D^r(z)|}\int_{D^r(z)}\left|f(w)-\widetilde{f}_t(z)\right|^p dA(w)\right)^{1/p}.\end{aligned}\tag{44}$$

Thus, $f\in\mathrm{BMO}_r^p$. □

3. Boundedness and Compactness of Operators

For $l\in\mathbb{R}$, we define an integral operator G_l by

$$G_l f(z)=\int_{\mathbb{D}}f(\xi)\left(d_\rho(z,\xi)+1\right)^l|K_z(\xi)|e^{-2\varphi(\xi)}dA(\xi).\tag{45}$$

Lemma 9. *Suppose $\varphi\in\mathscr{W}$ with $(\Delta\varphi)^{-1/2}\simeq\rho\in\mathscr{L}$. Let $0<p<\infty$, and there exist positive constants α and C such that for $0<r\le\alpha$ and $f\in\mathscr{H}(\mathbb{D})$, there holds*

(a) $|f(z)e^{-\varphi(z)}|^p\le C/|D^r(z)|\int_{D^r(z)}|f(w)e^{-\varphi(w)}|^p dA(w)$,

(b) $|f'(z)e^{-\varphi(z)}|^p\le C/|D^r(z)|^{1+(p/2)}\int_{D^r(z)}|f(w)e^{-\varphi(w)}|^p dA(w)$.

Proof. See Lemma 13 in Hu et al.'s study [3]. □

It follows from Lemma 9 that $\rho^{2/p}(z)|f(z)e^{-\varphi(z)}|\lesssim\|f\|_{p,\varphi}$ where $0<p<\infty$, $z\in\mathbb{D}$.

Lemma 10. *The operator G_l is bounded on L_φ^p with $1\le p\le\infty$; meanwhile, G_l is bounded from F_φ^p to L_φ^p with $0<p<1$.*

Proof. First, we consider the case $1\le p\le\infty$. Lemma 7 implies that

$$\begin{aligned}\|G_l f\|_{1,\varphi}&\le\int_{\mathbb{D}}e^{-\varphi(z)}dA(z)\int_{\mathbb{D}}|f(\xi)|(d(\xi,z)+1)^l|K_z(\xi)|e^{-2\varphi(\xi)}dA(\xi)\\&=\int_{\mathbb{D}}|f(\xi)|e^{-2\varphi(\xi)}dA(\xi)\int_{\mathbb{D}}(d(z,\xi)+1)^l|K_z(\xi)|e^{-\varphi(z)}dA(z)\\&\le C\int_{\mathbb{D}}|f(\xi)|e^{-\varphi(\xi)}dA(\xi)\le C\|f\|_{1,\varphi}.\end{aligned}\tag{46}$$

Similarly,

$$\begin{aligned}\|G_l f\|_{\infty,\varphi}&=\sup_{z\in\mathbb{D}}e^{-\varphi(z)}\int_{\mathbb{D}}|f(\xi)|(d(z,\xi)+1)^l|K_z(\xi)|e^{-2\varphi(\xi)}dA(\xi)\\&\le\|f\|_{\infty,\varphi}\sup_{z\in\mathbb{D}}e^{-\varphi(z)}\int_{\mathbb{D}}(d(\xi,z)+1)^l|K_z(\xi)|e^{-\varphi(\xi)}dA(\xi)\\&\le C\|f\|_{\infty,\varphi}.\end{aligned}\tag{47}$$

By interpolation theorem, G_l is bounded on L_φ^p with $1\le p\le\infty$.

Next, we consider the case$0<p<1$. We know that$d_\rho(\cdot,\cdot)$is equivalent to the Bergman distance$\beta_\varphi(\cdot,\cdot)$, and then, by Lemma 1 in Hu and Virtanen's study [5], we have $d_\rho(w,\xi)\le C$ for $\xi\in D(w)$. For $0<r\le\alpha$, we choose an (ρ,r)-lattice $\{z_j\}_{j=1}^\infty\subset\mathbb{D}$ as in Lemma 2, and then by Lemma 2 and Lemma 9 for $f\in\mathscr{H}(\mathbb{D})$, we obtain

$$|G_l f(z)|^p \le \left(\sum_{j=1}^{\infty}\int_{D^r(z_j)} |f(\xi)|\left(d_\rho(z,\xi)+1\right)^l |K_z(\xi)| e^{-2\varphi(\xi)} dA(\xi)\right)^p$$
$$\le \sum_{j=1}^{\infty}\left(\int_{D^r(z_j)} |f(\xi)|\left(d_\rho(z,\xi)+1\right)^l |K_z(\xi)| e^{-2\varphi(\xi)} dA(\xi)\right)^p$$
$$\le C\sum_{j=1}^{\infty}\sup_{\xi\in D^r(z_j)}\left(|f(\xi)|\left(d_\rho(z,\xi)+1\right)^l |K_z(\xi)| e^{-2\varphi(\xi)}\right)^p \rho^{2p}(z_j)$$
$$\le C\sum_{j=1}^{\infty}\sup_{\xi\in D^r(z_j)}\int_{D^r(\xi)} \cdot\left(|f(w)|\left(d_\rho(z,w)+1\right)^l |K_z(w)| e^{-2\varphi(w)}\right)^p \rho^{2p-2}(w)dA(w)$$
$$\le C\sum_{j=1}^{\infty}\int_{D^{Br}(z_j)}\left(|f(w)|\left(d_\rho(z,w)+1\right)^l |K_z(w)| e^{-2\varphi(w)}\right)^p \rho^{2p-2}(w)dA(w)$$
$$\le CN\int_{\mathbb{D}}\left(|f(w)|\left(d_\rho(z,w)+1\right)^l |K_z(w)| e^{-2\varphi(w)}\right)^p \rho^{2p-2}(w)dA(w). \tag{48}$$

Then, Lemma 7 implies that

$$\|G_l f\|_{p,\varphi}^p \le C\int_{\mathbb{D}} e^{-p\varphi(z)}dA(z)\int_{\mathbb{D}} \cdot\left(|f(w)|\left(d_\rho(z,w)+1\right)^l |K_z(w)| e^{-2\varphi(w)}\right)^p \rho^{2p-2}(w)dA(w)$$
$$\le C\int_{\mathbb{D}}\left(|f(w)|^p e^{-2p\varphi(w)}\right)\rho^{2(p-1)}(w)dA(w)\int_{\mathbb{D}} \cdot\left(d_\rho(z,w)+1\right)^l |K_z(w)|^p e^{-p\varphi(z)}dA(z)$$
$$\le C\int_{\mathbb{D}} |f(w)|^p e^{-p\varphi(w)}dA(w) = C\|f\|_{p,\varphi}. \tag{49}$$

The proof is completed. □

Theorem 11. *Let $0 < p < \infty$. If $f \in L^{\infty}(\mathbb{D})$ has compact support, the Hankel operator H_f is compact from $A_\varphi^p \longrightarrow L_\varphi^p$.*

Proof. For $1 \le p < \infty$, refer to Theorem 4.3 in Hu and Pau [15]. Next, we prove the case $0 < p < 1$. Without loss of generality, we assume that the support of f is contained in some $D(0,R)$, $0 < R < 1$. Write $d\mu = |f|dA$; there is an $0 < R_1 < 1$ so that $\widehat{\mu}(w) = 1/|D^\alpha(w)|\int_{D^\alpha(w)} d\mu = 0$ when $|w| \ge R_1$.

Then, for any bounded sequence $\{g_j\}_{j=1}^{\infty}$ in A_φ^p converging to 0 uniformly on any compact subset of $\mathbb{D}$, we get

$$|P(fg_i)| \le \int_{\mathbb{D}}\left|f(w)g_j(w)K_z(w)\right| e^{-2\varphi(w)}dA(w)$$
$$= \|f\|_{L^\infty(\mathbb{D})}\int_{\mathbb{D}}\left|g_j(w)K_z(w)\right| e^{-2\varphi(w)}d\mu(w)$$
$$\lesssim \|f\|_{L^\infty(\mathbb{D})}\int_{\mathbb{D}}\left|g_j(w)K_z(w)\right| e^{-2\varphi(w)}\widehat{\mu}(w)dA(w)$$
$$\lesssim \|f\|_{L^\infty(\mathbb{D})}\int_{D(0,R_1)}\left|g_j(w)K_z(w)\right| e^{-2\varphi(w)}dA(w). \tag{50}$$

Since $\rho(z) \longrightarrow 0$, when $|z| \longrightarrow 1$, for any $\varepsilon > 0$, there exists $0 < R < 1$ such that $\rho(z) < \varepsilon$ when $|z| \ge R$. Consider the (ρ, r) -lattice $Z : \{z_j\} \in \mathbb{D}$ with $0 < r \le \alpha$. Since $\rho(w) \simeq \rho(z_j)$ whenever $w \in D^r(z_j)$, it follows from (B) in Lemma 1 that

$$D^r(z_j) \subset D^{Br}(w) \subset D^{B^2 r}(z_j), \tag{51}$$

And then, we have

$$\left|\int_{D(0,R)}\left|g_j(w)K_z(w)\right| e^{-2\varphi(w)}dA(w)\right|^p$$
$$\le \left(\sum_{z_j\in Z}\int_{D(0,R)\cap D^r(z_j)}\left|g_j(w)K_z(w)\right| e^{-2\varphi(w)}dA(w)\right)^p$$
$$\lesssim \sum_{z_j\in Z, d_\rho(z_j, D(0,R))<r}\ \sup_{w\in D^r(z_j)}\left|g_j(w)K_z(w)\right| e^{-2\varphi(w)}\rho^{2p}(w)$$
$$\lesssim \sum_{z_j\in Z, d_\rho(z_j, D(0,R))<r}\int_{D^{B^2 r}(z_j)}\left|g_j(w)K_z(w)\right| e^{-2\varphi(w)}\rho^{2p-2}(w)dA(w)$$
$$\lesssim \sum_{z_j\in Z\bigcap D(0,R+\varepsilon)}\int_{D^{B^2}(z_j)}\left|g_j(w)K_z(w)\right| e^{-2\varphi(w)}\rho^{2p-2}(w)dA(w)$$
$$\le N\int_{D(0,R+\varepsilon)}\left|g_j(w)K_z(w)\right| e^{-2\varphi(w)}\rho^{2p-2}(w)dA(w). \tag{52}$$

Thus,

$$\left\|P\left(fg_j\right)\right\|_{p,\varphi}^p \lesssim \|f\|_{L^\infty(\mathbb{D})}^p\int_{\mathbb{D}} e^{-p\varphi(z)}dA(z)\int_{D(0,R+\varepsilon\alpha)} \cdot\left|g_j(w)K_z(w)\right|^p e^{-2p\varphi(w)}\rho^{2p-2}(w)dA(w)$$
$$= \|f\|_{L^\infty}^p\int_{D(0,R+\varepsilon\alpha)}\left|g_j(w)\right|^p e^{-2p\varphi(w)}\rho^{2p-2}(w)dA(w)\int_{\mathbb{D}} \cdot |K_z(w)|^p e^{-p\varphi(z)}dA(z)$$
$$\lesssim \|f\|_{L^\infty}^p\int_{D(0,R+\varepsilon\alpha)}\left|g_j(w)\right|^p e^{-p\varphi(w)}dA(w) \longrightarrow 0, \tag{53}$$

when $j \longrightarrow \infty$. Therefore, H_f is compact. □

Theorem 12. *Suppose $0 < p < \infty$, and $0 < r < \alpha$, the following statements hold.*

(a) *If $f \in BO_r$. then $H_f : F_\varphi^p \longrightarrow L_\varphi^p$ is bounded, moreover $\|H_f\|_{F_\varphi^p \longrightarrow L_\varphi^p} \lesssim \|f\|_{BO_r}$*

and

(b) *If $f \in VO_r$ then $H_f : F_\varphi^p \longrightarrow L_\varphi^p$ is compact*

Proof.

(a) Because of Theorem 4.2 in Hu and Pau [15], H_f bounded and $\|H_f\|_{F_\varphi^p \longrightarrow L_\varphi^p} \lesssim \|f\|_{\mathrm{BO}_r}$ when $1 \le p < \infty$. Next, we prove the case $0 < p < 1$, and it is easy to know that

$$|f(z)-f(w)|\leq \|f\|_{BO_r}\left(d_\rho(z,w)+1\right), z, w\in \mathbb{D},\quad P(g)=g, g\in A^p_\varphi. \tag{54}$$

Then

$$\begin{aligned}\left|H_f(g)(z)\right| &\leq \int_{\mathbb{D}} |f(w)-f(z)||g(w)||K_z(w)|e^{-2\varphi(w)}dA(w)\\ &\leq \|f\|_{BO_r}\int_{\mathbb{D}}\left(d_\rho(z,w)+1\right)|g(w)||K_z(w)|e^{-2\varphi(w)}dA(w).\end{aligned} \tag{55}$$

Lemma 10 implies that result hold.

(b) Because of Theorem 4.3 in Hu and Pau [15], H_f is compact when $1\leq p<\infty$. Suppose $0<p<1$, $f\in VO_r$, for $\varepsilon>0, \exists R_0>0$ such that $w_r(f)<\varepsilon$ when $d_\rho(w,0)\geq R_0$. Moreover, for $w\in\mathbb{D}$, and

$$d_\rho(w,0)>R_0, \tag{56}$$

there exists $\xi(w)\in\mathbb{D}$ such that $d_\rho(\xi,0)=R_0$, and $d_\rho(w,0)=d_\rho(w,\xi)+d_\rho(\xi,0)$. Hence

$$|f(w)-f(0)|\leq \|f\|_{BO_r}\left(d_\rho(\xi,0)+1\right)+\varepsilon\left(d_\rho(w,\xi)+1\right), \tag{57}$$

therefore, there is an $R/2>R_0$ such that

$$\frac{|f(w)|}{d_\rho(w,0)}\leq \frac{\|f\|_{BO_r}(R_0+1)+|f(0)|}{d_\rho(w,0)}+\varepsilon\frac{d_\rho(\xi,w)+1}{d_\rho(w,0)}<2\varepsilon, \tag{58}$$

with $d_\rho(w,0)>R/2$. Define a function

$$h_k=\begin{cases}1, & d_\rho(z,0)<R,\\ 2-\dfrac{d_\rho(z,0)}{R}, & R<d_\rho(z,0)<2R,\\ 0, & 2R\leq d_\rho(z,0).\end{cases} \tag{59}$$

Set $f_k=fh_k$; it is obvious that

$$w_r(f-f_k)(z)=\begin{cases}0, & \mathrm{d}_\rho(z,0)<\dfrac{\mathrm{R}}{2},\\ w_r(f)(z), & \mathrm{d}_\rho(z,0)>2\mathrm{R}+\dfrac{\mathrm{R}}{2},\end{cases} \tag{60}$$

when $R/2\leq d_\rho(z,0)\leq 5R/2$ and $w\in D^r(z)$. Then

$$\begin{aligned}&|(f(z)-f_R(w))-(f(w)-f_R(w))|\\ &\leq |f(w)||h_R(z)-h_R(w)|+(1-h_R(z))|f(w)-f(z)|\\ &\lesssim \frac{r}{R}|f(w)|+w_r(f)(z)\\ &\lesssim \frac{|f(w)|}{d_\rho(w,0)}\frac{d_\rho(w,0)}{R}+w_r(f)(z)\\ &\lesssim \frac{|f(w)|}{d_\rho(w,0)}\frac{\left(d_\rho(w,z)+d_\rho(z,0)\right)}{R}+w_r(f)(z)\lesssim\varepsilon.\end{aligned} \tag{61}$$

By (55), $\|H_f-H_{f_k}\|_{F^p_\varphi\longrightarrow L^p_\varphi}\lesssim\|f-f_k\|_{\mathrm{BO}_r}$. The compactness of H_f follows from the compactness of H_{f_k}. □

Lemma 13. *If $f\in VO_r$ with $r\in(0,\alpha)$. Then, $\lim_{|z|\longrightarrow 1}(f-\tilde{f})(z)=0$.*

Proof. Lemma 6 and Lemma 7 imply that

$$\sup_{z\in\mathbb{D}}\int_{\left(D^R(z)\right)^c}\left(d_\rho(z,\xi)+1\right)|K_z(\xi)|^2e^{-2\varphi(\xi)}dA(\xi)\longrightarrow 0, \tag{62}$$

as $R\longrightarrow\infty$. Then, for any $\varepsilon>0$, there exists a positive number R such that for all $z\in\mathbb{D}$,

$$\int_{\left(D^R(z)\right)^c}\left(d_\rho(z,\xi)+1\right)|K_z(\xi)|^2e^{-2\varphi(\xi)}dA(\xi)<\varepsilon. \tag{63}$$

Note that $f\in VO_r$, there is $0<k<1$ such that

$$\sup_{\xi\in D^r(z)}|f(\xi)-f(z)|<\varepsilon, \tag{64}$$

when $k<|z|<1$. Thus

$$\begin{aligned}\left|f-\tilde{f}\right|(z)&\leq\int_{\mathbb{D}}|f(z)-f(\xi)||K_z(\xi)|^2e^{-2\varphi(\xi)}dA(\xi)\\ &\leq\left(\int_{D^r(z)}+\int_{(D^r(z))^c}\right)|f(z)-f(\xi)||K_z(\xi)|^2e^{-2\varphi(\xi)}dA(\xi)\\ &\lesssim\varepsilon+\|f\|_{\mathrm{BO}_r}\int_{(D^r(z))^c}\left(d_\rho(z,\xi)+1\right)|K_z(\xi)|^2e^{-2\varphi(\xi)}dA(\xi)\\ &\lesssim\varepsilon.\end{aligned} \tag{65}$$

□

Notation 14. We use P^+ to represent the integral operator which is defined by

$$P^+f(z)=\int_{\mathbb{D}}f(\xi)|K_z(\xi)|e^{-2\varphi(\xi)}dA(\xi), z\in\mathbb{D}. \tag{66}$$

Lemma 10 implies that P^+ is a bounded operator from

$L^p_\varphi \longrightarrow A^p_\varphi$ with $1 \le p < \infty$, and from $A^p_\varphi \longrightarrow L^p_\varphi$ with $0 < p < 1$.

Lemma 15. *If $0 < p < \infty$, $f \in VO_r$, and $r \in (0, \alpha]$, for $z_j \in \mathbb{D}$ satisfying $\lim_{z_j \longrightarrow \partial\mathbb{D}} f(z_j) = 0$, then*

$$\lim_{j\longrightarrow\infty} \left\| T_f\left(k_{p,z_j}\right) \right\|_{p,\varphi} = 0. \tag{67}$$

Proof. When $0 < p < 1$, and $f \in VO_r$, by the same proof in Lemma 7, for $\varepsilon > 0$, there exists $R > 0$ such that

$$\begin{aligned} &\int_{D^R(z)} \left(d_\rho(z,\xi)+1\right)^p \rho^{2(p-1)}(\xi)\left|k_{p,z}(\xi)\right|^p e^{-p\varphi(z)} dA(\xi) \\ &\quad \le \left(\frac{\varepsilon}{2\|f\|_{BO_r}+1}\right)^p, \end{aligned} \tag{68}$$

for all $z \in \mathbb{D}$. Moreover, for the $\varepsilon > 0$ and $R > 0$, there exists if $0 < k < 1$ such that

$$\sup_{\xi\in D^{BR}(z)} |f(\xi) - f(z)| < \varepsilon, \tag{69}$$

when $k < |z| < 1$. Thus, when $|z_j| > k$, by Lemma 1 and proof in Theorem 11, there hold

$$\begin{aligned} &\left\{\int_{\mathbb{D}} |f(\xi)-f(z_j)|\left|k_{p,z_j}(\xi)\right||K_z(\xi)|e^{-2\varphi(\xi)}dA(\xi)\right\}^p \\ &\quad = \left\{\left(\int_{D^{\mathrm{BR}}(z_j)} + \int_{(D^{\mathrm{BR}}(z_j))^c}\right)|f(\xi)-f(z_j)|\left|k_{p,z_j}(\xi)\right||K_z(\xi)|e^{-2\varphi(\xi)}dA(\xi)\right\}^p \\ &\quad \le \left\{\varepsilon\int_{D^{\mathrm{BR}}(z_j)}\left|k_{p,z_j}(\xi)\right||K_z(\xi)|\left|e^{-2\varphi(\xi)}dA(\xi)\right\}^p \\ &\qquad + \left\{\|f\|_{\mathrm{BO}^r}\int_{(D^{\mathrm{BR}}(z_j))^c}\left(d_\rho(z_j,\xi)+1\right)\left|k_{p,z_j}(\xi)\right||K_z(\xi)|e^{-2\varphi(\xi)}dA(\xi)\right\}^p \\ &\quad \lesssim \varepsilon^p\int_{\mathbb{D}}\left|k_{p,z_j}(\xi)\right|^p|K_z(\xi)|^p e^{-2p\varphi(\xi)}\rho^{2(p-1)}(\xi)dA(\xi) \\ &\qquad + \|f\|^p_{\mathrm{BO}_r}\int_{(D^R(z_j))^c}\left(d_\rho(z_j,\xi)+1\right)^p \\ &\qquad \times\left|k_{p,z_j}(\xi)\right|^p|K_z(\xi)|^p e^{-2p\varphi(\xi)}\rho^{2(p-1)}(\xi)dA(\xi). \end{aligned} \tag{70}$$

Therefore, for $|z_j| > k$, by Lemma 7 and Fubini's theorem, we obtain

$$\begin{aligned} &\int_{\mathbb{D}}\left\{\int_{\mathbb{D}}|f(\xi)-f(z_j)|\left|k_{p,z_j}(\xi)\right||K_z(\xi)|e^{-2\varphi(\xi)}dA(\xi)\right\}^p e^{-p\varphi(z)}dA(z) \\ &\quad \lesssim \varepsilon^p\left\|k_{p,z_j}\right\|^p_{p,\varphi} + \|f\|^p_{\mathrm{BO}_r}\int_{(D^R(z_j))^c}\left(d_\rho(z_j,\xi)+1\right)^p \\ &\qquad \times\left|k_{p,z_j}(\xi)\right|^p e^{-p\varphi(\xi)}dA(\xi) \lesssim \varepsilon^p. \end{aligned} \tag{71}$$

It is obvious that

$$\begin{aligned} \left|T_f\left(k_{p,z_j}\right)(z)\right| \le &\int_{\mathbb{D}}\left(|f(\xi)-f(z_j)|+|f(z_j)|\right) \\ &\cdot\left|k_{p,z_j}(\xi)\right||K_z(\xi)|e^{-2\varphi(\xi)}dA(\xi). \end{aligned} \tag{72}$$

By Notation 14, we have

$$\begin{aligned} &\lim_{z_j\longrightarrow\mathbb{D}}\sup\left\|T_f\left(k_{p,z_j}\right)\right\|^p_{p,\varphi} \lesssim \varepsilon^p + \lim_{z_j\longrightarrow\partial\mathbb{D}}|f(z_j)|^p \\ &\quad + \left\|P^+\left(\left|k_{p,z_j}\right|\right)\right\|^p_{p,\varphi} \lesssim \varepsilon^p. \end{aligned} \tag{73}$$

When $1 \le p < \infty$,

$$\begin{aligned} &\int_{\mathbb{D}}|f(\xi)-f(z_j)|\left|k_{p,z_j}(\xi)\right||K_z(\xi)|e^{-2\varphi(\xi)}dA(\xi) \\ &\quad = \left\{\int_{D^{\mathrm{BR}}(z_j)} + \int_{(D^{\mathrm{BR}}(z_j))^c}\right\}|f(\xi)-f(z_j)| \\ &\qquad \times\left|k_{p,z_j}(\xi)\right||K_z(\xi)|e^{-2\varphi(\xi)}dA(\xi) \\ &\quad \lesssim \int_{\mathbb{D}}\left[\varepsilon + \|f\|_{\mathrm{BO}_r}\left(d_\rho(\xi,z_j)+1\right)\chi_{(D^{\mathrm{BR}}(z_j))^c}\right] \\ &\qquad \times\left|k_{p,z_j}(\xi)\right||K_z(\xi)|e^{-2\varphi(\xi)}dA(\xi) = \varepsilon P^+\left(\left|k_{p,z_j}(\xi)\right|\right) \\ &\qquad + \|f\|_{\mathrm{BO}_r}P_+\left(d_\rho(\cdot,z_j)+1\right)\chi_{(D^{\mathrm{BR}}(z_j))^c}\left|k_{p,z_j}(\xi)\right|(z). \end{aligned} \tag{74}$$

By Lemma 10, we have

$$\begin{aligned} &\int_{\mathbb{D}}\left\{\int_{\mathbb{D}}|f(\xi)-f(z_j)|\left|k_{p,z_j}(\xi)\right||K_z(\xi)|e^{-2\varphi(\xi)}dA(\xi)\right\}^p e^{-p\varphi(z)}dA(z) \\ &\quad \lesssim \varepsilon + \|f\|_{\mathrm{BO}_r}\|P^+\|\left\|\left(d_\rho(\cdot,z_j)+1\right)\chi_{(D^{\mathrm{BR}}(z_j))^c}\left|k_{p,z_j}\right|\right\|_{p,\varphi}. \end{aligned} \tag{75}$$

Now, the results follow as in the case $0 < p < 1$. □

Lemma 16. *If $0 < p < \infty$ and T_f is compact on A^p_φ, then $\lim_{|z|\longrightarrow 1}\widetilde{T_f}(z) = 0$, where $\widetilde{T_f} = \langle T_f k_z, k_z\rangle_\varphi$ is called Berezin transform of Toeplitz operator T_f for $z \in \mathbb{D}$.*

Proof. When $1 \le p < \infty$, the results follow from the fact $k_z \longrightarrow^w 0$ when $|z| \longrightarrow 1$. Recall that $E \subset A^p_\varphi$ is relatively compact if and only if for every $\varepsilon > 0$, there exists $0 < R < 1$ such that

$$\sup_{g\in E}\int_{R<|z|<1}\left|g(z)e^{-\varphi(z)}\right|^p dA(z) < \varepsilon. \tag{76}$$

If T_f is compact on A^p_φ, then

$$\lim_{R\longrightarrow 1}\sup_{g\in A^p_\varphi,\|g\|_{p,\varphi}\le 1}\int_{R<|z|\le 1}\left|T_f g(z)e^{-\varphi(z)}\right|^p dA(z)=0. \quad (77)$$

Note that

$$\widetilde{T_f}(z)=\frac{\|K_z\|_{p,\varphi}}{K_z(z)}\left\langle T_f\left(k_{p,z}\right),K_z\right\rangle=\frac{\|K_z\|_{p,\varphi}}{K_z(z)}T_f\left(k_{p,z}\right)(z). \quad (78)$$

Thus

$$\begin{aligned}\left|\widetilde{T_f}(z)\right|^p &\lesssim \left|\rho^{2/p}(z)T_f\left(k_{p,z}\right)(z)e^{-\varphi(z)}\right|^p\\ &\lesssim \int_{D^1(z)}\left|T_f\left(k_{p,z}\right)(z)e^{-\varphi(z)}\right|^p dA(\xi)\longrightarrow 0,\end{aligned} \quad (79)$$

when $|z|\longrightarrow 1$. □

Theorem 17. *If $0<p<\infty$, $f\in VO_r$ and $0<r<\alpha$. Then*

(a) $T_{f-\tilde f}$ is compact on A^p_φ

and

(b) T_f is compact on A^p_φ if and only if $\lim_{|z|\longrightarrow 1}\tilde f(z)=0$

Proof.

(a) Lemma 3 implies that for $w\in D^1(z)$, we have

$$\begin{aligned}\left|f(\xi)k_w^2(\xi)\right|e^{-2\varphi(\xi)} &\lesssim (|f(\xi)-f(z)|+|f(z)|)\rho^{-2}(w)e^{-2\sigma d_\rho(\xi,w)}\\ &\lesssim \left(\|f\|_{BO_r}+|f(z)|\right)\rho^{-2}(z)e^{-\sigma d_\rho(\xi,z)}.\end{aligned} \quad (80)$$

Then, Lebesgue's dominated convergence theorem shows that

$$\begin{aligned}\lim_{w\longrightarrow z}\tilde f(w) &= \lim_{w\longrightarrow z}\int_{\mathbb{D}} f(\xi)|k_w(\xi)|^2 e^{-2\varphi(\xi)}dA(\xi)\\ &=\int_{\mathbb{D}} f(\xi)|k_z(\xi)|^2e^{-2\varphi(\xi)}dA(\xi)\\ &=\tilde f(z).\end{aligned} \quad (81)$$

It shows that $\tilde f$ is continuous on $\mathbb{D}$. Hence, $f-\tilde f\in C(\mathbb{D})$, by Notation 14, we know that $f-\tilde f\in L^\infty\bigcap VO_r$. Theorem 4.2 Zorboska's study [7] implies that $T_{g\chi_{D(0,R)}}$ is compact on A^p_φ for $g\in L^\infty$. Note

$$\left\|T_g-T_{g\chi_{D(0,R)}}\right\|\lesssim\left\|g-g_{\chi_{B(0,R)}}\right\|_{L^\infty}\longrightarrow 0, \quad (82)$$

when $R\longrightarrow 1$, then T_g is compact, and thus $T_{f-\tilde f}$ is compact.

(b) If T_f is compact on A^p_φ, Lemma 13 shows that $\lim_{|z|\longrightarrow 1}\widetilde{T_f}(z)=0$. If $f\in VO_r\subset BO_r$, it is easy to know that

$$\int_{\mathbb{D}}|k_z(\xi)|e^{-2\varphi(\xi)}dA(\xi)\int_{\mathbb{D}}\left|f(w)k_z(w)K_\xi(w)\right|e^{-2\varphi(w)}dA(w)<\infty, \quad (83)$$

since $\widetilde{T_f}(z)=\tilde f(z)$ and $T_f=T_{\tilde f}+T_{f-\tilde f}$, the conclusion holds. □

4. Fredholm Theory

A linear mapping T on a topological vector space X is called to be Fredholm if

$$\dim\ker T<\infty,\dim\ker X/T(x)<\infty. \quad (84)$$

When X is a Banach space, it is well known that T is Fredholm if and only if $T+K(X)$ is invertible in the Calkin algebra $B(X)/K(X)$, where $B(X)$ and $K(X)$ represent, respectively, the spaces of bounded and compact operators. It shows that an operator T on a Banach space is Fredholm if and only if there are bounded operators A and B on X, such that

$$\begin{aligned}AT&=I+K_1,\\ TB&=I+K_2,\end{aligned} \quad (85)$$

for some compact operators K_1 and K_2 on X.

A pair $(X,\|\cdot\|)$ is said to be a quasi-Banach space if $\|\cdot\|$ satisfies all the properties of a norm except for the triangle inequality and if there is a constant $C>0$ such that

$$\|x+y\|\le C(\|x\|+\|y\|). \quad (86)$$

Note that Bergman space $A^p_\varphi (0<p<1)$ are quasi-Banach spaces.

Theorem 18. *Let $\varphi\in\mathscr{W}_0$, for $0<p\le 1$,*

$$\begin{aligned}\left(A^p_\varphi\right)^* &= A^\infty_{2-2/p,\varphi}\\ &=\left\{f\in\mathscr{H}(\mathbb{D})\,|\,\|f\|_{\infty,2-2/p,\varphi}=\sup_{z\in\mathbb{D}}|f(z)|\rho^{2-2/p}(z)e^{-\varphi(z)}<\infty\right\},\end{aligned} \quad (87)$$

under the pairing

$$\langle f, g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}e^{-2\varphi(z)}dA(z), \text{ where } f \in A^p_\varphi, g \in A^\infty_{2-2/p,\varphi}. \quad (88)$$

Proof. Refer to Hu et al. [3]. □

Theorem 19. *A bounded linear operator T on a dual rich quasi-Banach space X is Fredholm if and only if it has a regular; that is, there is a bounded linear operator S on X such that ST − I and TS − I are both compact on X.*

Proof. See Section 3.5.1 in Runst and Sickel's study [16]. □

Theorem 20. *Let $f \in VMO^1_r$, $0 < p < \infty$, and $0 < r < \alpha$. Then, the Toeplitz operator T_f is Fredholm on A^p_φ if and only if*

$$0 < \lim_{|z|\longrightarrow 1} \inf \left|\tilde{f}(z)\right| \le \lim_{|z|\longrightarrow 1} \sup \left|\tilde{f}(z)\right| < \infty. \quad (89)$$

Proof. Because of the decomposition $\mathrm{VMO}^1_r = \mathrm{VO}_r + \mathrm{VA}^1_r$, there are functions $f_1 \in \mathrm{VO}_r$ and $f_2 \in \mathrm{VA}^1_r$ such that $f = f_1 + f_2$. Set $d\mu = |f_2|dA$. Then, we have

$$\lim_{|z|\longrightarrow 1} \frac{1}{|D^r(z)|}\int_{D^r(z)} d\mu = 0, \quad (90)$$

which shows that μ is the vanishing Carleson measure. By Theorem 12 and Theorem 4.2 in Zhang et al.'s study [11], we have

$$\lim_{|z|\longrightarrow 1} \left|\widetilde{f_2}(z)\right| = 0, \quad (91)$$

and T_{f_2} is compact on A^p_φ for $0 < p < \infty$. Therefore, T_f is Fredholm if and only if T_{f_1} is Fredholm. (91) shows that

$$\begin{aligned} \lim_{|z|\longrightarrow 1} \inf \left|\tilde{f}(z)\right| &= \lim_{|z|\longrightarrow 1} \inf \left|\widetilde{f_1}(z)\right|, \\ \lim_{|z|\longrightarrow 1} \sup \left|\tilde{f}(z)\right| &= \lim_{|z|\longrightarrow 1} \sup \left|\tilde{f}_1(z)\right|. \end{aligned} \quad (92)$$

Now, we just need to prove the conclusion for $f \in \mathrm{VO}_r$. If $f \in \mathrm{VO}_r$, and T_f is Fredholm on A^p_φ, then T_f is bounded on A^p_φ. Suppose that $0 < p \le 1$, note that

$$\begin{aligned} &\left\|\rho^{2-2/p}(\cdot)\left(\rho^{2/p-1}(z)k_z(\cdot)\right)\right\|_{\infty,\varphi} < \infty, z \in \mathbb{D}, \\ &\left|\left\langle T_f k_z, k_z\right\rangle\right| \simeq \left|\left\langle T_f k_{p,z}, \rho^{2/p-1}(z)k_z\right\rangle\right|, \end{aligned} \quad (93)$$

then

$$\begin{aligned} \left|\tilde{f}(z)\right| &\le C\left\|T_f k_{p,z}\right\|_{p,\varphi}\left\|\rho^{2-(2/p)}(\cdot)\left(\rho^{2/p-1}(z)k_z(\cdot)\right)\right\|_{\infty,\varphi} \\ &\le \left\|T_f\right\|_{A^p_\varphi\longrightarrow A^p_\varphi}. \end{aligned} \quad (94)$$

Hence, $\lim_{|z|\longrightarrow 1} \sup |\tilde{f}(z)| < \infty$. When $1 < p < \infty$, $\widetilde{T_f} = \tilde{f}$ and Hölder's inequality show that $|\tilde{f}(z)| \le \|T_f\|_{A^p_\varphi\longrightarrow A^p_\varphi}$. If $\lim_{|z|\longrightarrow 1} \inf |\widetilde{f_i}(z)| > 0$ are false, there are some sequence $\{z_k\} \subset \mathbb{D}$ such that $\lim_{z_k\longrightarrow \partial\mathbb{D}} \tilde{f}(z_k) = 0$. By Lemma 15

$$\lim_{k\longrightarrow\infty} \left\|T_{\tilde{f}}\left(k_{p,z_k}\right)\right\|_{p,\varphi} = 0. \quad (95)$$

Then, for any bounded operator S on A^p_φ, we have

$$\lim_{k\longrightarrow\infty} \left\|\left(ST_{\tilde{f}}\right)\left(k_{p,z_k}\right)\right\|_{p,\varphi} = 0. \quad (96)$$

Because that

$$\begin{aligned} &\left|\widetilde{ST_{\tilde{f}}}(z_k)\right| \simeq \left|\left\langle\left(ST_{\tilde{f}}\right)k_{p,z_k}, \rho^{\left(\frac{2}{p}\right)-1}(z_k)k_{z_k}\right\rangle\right|, \text{ if } 0 < p \le 1, \\ &\left|\widetilde{ST_{\tilde{f}}}(z_k)\right| \simeq \left|\left\langle\left(ST_{\tilde{f}}\right)k_{p,z_k}, k_{p',z_k}\right\rangle\right|, \text{ if } 1 \le p < \infty, \end{aligned} \quad (97)$$

where p' is the conjugate exponent of p. We have

$$\lim_{k\longrightarrow\infty} \left|\widetilde{ST_{\tilde{f}}}(z_k)\right| = 0. \quad (98)$$

It is obvious that $T_f = T_{\tilde{f}} + T_{f-\tilde{f}}$ is Fredholm on A^p_φ; thus, there is a bounded operator S such that

$$ST_{\tilde{f}} = I + K, \quad (99)$$

where I is identity and K is the compact operator on A^p_φ. Hence, $\lim_{k\longrightarrow\infty} |\widetilde{ST_{\tilde{f}}}(z_k)| \ge 1 - \lim_{k\longrightarrow\infty} |\tilde{K}(z_k)| = 1$ which contradicts (98).

Conversely, if $f \in \mathrm{VO}_r$ and $\tilde{f}$ satisfies (89), then there are positive constants C, c, and R such that

$$c \le \left|\tilde{f}(z)\right| \le C, \text{ for } 0 < R \le |z| < 1. \quad (100)$$

By Lemma 13 and $\tilde{f} \in VO_r \bigcap L^\infty(\mathbb{D})$. Set function g satisfying

$$g(z) = \begin{cases} 0, & |z| < R \\ \dfrac{1}{\tilde{f}(z)}, & R \le |z| < 1. \end{cases} \quad (101)$$

Then, $g \in L^\infty$, and $w_r(g)(z) \lesssim w_r(\tilde{f})(z) \longrightarrow 0$ when $|z| \longrightarrow 1$. Theorem 12 shows that H_g is compact operator from $A^p_\varphi \longrightarrow L^p_\varphi$. Note that

$$\tilde{f}(z)g(z) = \begin{cases} 0, & |z| < R \\ 1, & R \le |z| < 1. \end{cases} \quad (102)$$

Therefore, $T_{\tilde{f}g} = I - T_{\chi_{D(0,R)}}$ on A^p_φ. Thus

$$\begin{aligned} T_{\tilde{f}} T_g &= PM_{\tilde{f}} PM_g = PM_{\tilde{f}}(I - (I-P))M_g \\ &= T_{\tilde{f}g} - PM\tilde{f}H_g = I - T_{\chi_{D(0,R)}} - PM_{\tilde{f}}H_g. \end{aligned} \quad (103)$$

It is obvious that $T_{\chi_{D(0,R)}}$ and $PM_{\tilde{f}}H_g$ are compact. Similarly, $T_{\tilde{f}}T_g = I + K_2$, where K_2 is a compact operator on A^p_φ. Then, $T_{\tilde{f}}$ is Fredholm. □

Corollary 21. *Let $0 < p < \infty, f \in VMO^1_r$ and $0 < r < \alpha$. If Theorem 20 holds, then*

$$\sigma_{ess}(T_f) = \bigcap_{0<R<1} \tilde{f}(\mathbb{D} \setminus \bar{D}(0,R)), \quad (104)$$

and essential spectrum $\sigma_{ess}(T_f)$ is connected.

5. Schatten Class Toeplitz and Hankel Operators

If T is a bounded linear operator $T : H_1 \longrightarrow H_2$, where H_1 and H_2 are two Hilbert spaces, the singular values $s_j(T)$ of T are defined by

$$s_j(T) = \inf \{\|T-K\| | K : H_1 \longrightarrow H_2, \text{rank } K < j\}, \quad (105)$$

where rank K denotes the rank of operator K. T is compact if and only if $s_j(T) \longrightarrow 0$ as $j \longrightarrow \infty$. For $0 < p < \infty$, T is in the Schatten class S_p, if

$$s_j(T) \in l^p. \quad (106)$$

$\|T\|^p_{S_p} = \sum_{j=1}^{\infty} s_j(T)^p$ is a norm when $1 \le p < \infty$, and a quasi-norm when $0 < p < 1$. In fact, we have

$$\begin{aligned} \|S+T\|_{S_p} &\le \|S\|_{S_p} + \|T\|_{S_p}, 1 \le p < \infty, \\ \|S+T\|^p_{S_p} &\le \|S\|^p_{S_p} + \|T\|^p_{S_p}, 0 < p < 1. \end{aligned} \quad (107)$$

In addition, $T \in S_p$ if and only if $T^*T \in S_{p/2}$.

Definition 22. Let T be a compact operator from H_1 to H_2 and $h : R^+ \longrightarrow R^+$ is a continuous increasing convex function, we say that $T \in S_h$ if there is a positive constant $c > 0$ such that

$$\sum_{j=1}^{\infty} h(c \cdot s_j(T)) < \infty. \quad (108)$$

Theorem 23. *Let $h : R^+ \longrightarrow R^+$ are continuously increasing convex function. Let μ be a positive Borel measure on $\mathbb{D}$ such that the Toeplitz operator $T_\mu : A^2_\varphi \longrightarrow A^2_\varphi$ is compact. Then, $T_\mu \in S_h$ if and only if there is a constant $c > 0$ such that*

$$\int_{\mathbb{D}} h(c\tilde{\mu}(z))\rho^{-2}(z)dA(z) < \infty, \quad (109)$$

where $T_\mu f(z) = \int_{\mathbb{D}} f(w)K_z(w)d\mu(w)$.

Proof. Suppose that $T_\mu \in S_h$ that is

$$\sum_{j=1}^{\infty} h(cs_j(T)) < \infty, \quad (110)$$

for some constant $c > 0$. Let $\{e_k\}_{k=1}^{\infty}$ be an orthogonality basis of A^2_φ. Then

$$T_\mu = \sum_{k=1}^{\infty} s_k \langle \cdot, e_k \rangle e_k, \quad (111)$$

where s_k are also the eigenvalues of T_μ. It is obvious that

$$\sum_{k=1}^{\infty} \left| \langle k_z, e_k \rangle_\varphi \right|^2 = 1. \quad (112)$$

Then, it follows from the convexity of h, Jensen's inequality, and Lemma 4 that

$$\begin{aligned} &\int_{\mathbb{D}} h(c\tilde{\mu}(z))\rho^{-2}(z)dA(z) \\ &= \int_{\mathbb{D}} h\left(c\langle T_\mu k_z, k_z \rangle_\varphi\right)\rho^{-2}(z)dA(z) \\ &= \int_{\mathbb{D}} h\left(\sum_{k=1}^{\infty} cs_k \left|\langle k_z, e_k\rangle_\varphi\right|^2\right)\rho^{-2}(z)dA(z) \\ &\le \int_{\mathbb{D}} \sum_{k=1}^{\infty} h(cs_k)\left|\langle k_z, e_k\rangle_\varphi\right|^2 \rho^{-2}(z)dA(z) \\ &= \int_{\mathbb{D}} \sum_{k=1}^{\infty} h(cs_k)\|K_z\|^2_{2,\varphi}|e_k(z)|^2\rho^{-2}(z)dA(z) \\ &\le \sum_{k=1}^{\infty} h(cs_k)\int_{\mathbb{D}} |e_k(z)|^2 e^{-2\varphi(z)}dA(z) \\ &= \sum_{k=1}^{\infty} h(cs_k) < \infty. \end{aligned} \quad (113)$$

Conversely, assume that $\int_{\mathbb{D}} h(c\tilde{\mu}(z))\rho^{-2}(z)dA(z) < \infty$ for some $c > 0$. Then, by Lemma 4, we have

$$\widehat{\mu_r}(z) = \int_{D^r(z)} \rho^{-2}(z)d\mu(w) \approx \int_{D^r(z)} |k_z(w)|^2 e^{-2\varphi(w)}d\mu(w) \le \tilde{\mu}(z). \quad (114)$$

Note that

$$\begin{aligned}\langle T_\mu e_k, e_k\rangle_\varphi &= \int_{\mathbb{D}} |e_k(z)|^2 e^{-2\varphi(z)} d\mu(z) \\ &\lesssim \int_{\mathbb{D}} \widehat{\mu_r}(z)|e_k(z)|^2 e^{-2\varphi(z)} dA(z) \\ &\lesssim \int_{\mathbb{D}} \tilde{\mu}(z)|e_k(z)|^2 e^{-2\varphi(z)} dA(z).\end{aligned} \quad (115)$$

Jensen's formula shows that

$$\begin{aligned}\sum_{k=1}^{\infty} h\left(c\langle T_\mu e_k, e_k\rangle_\varphi\right) &\le \int_{\mathbb{D}} h(c\tilde{\mu}(z))\left(\sum_{k=1}^{\infty}|e_k(z)|^2\right)e^{-2\varphi(z)}dA(z) \\ &= \int_{\mathbb{D}} h(c\tilde{\mu}(z))\|k_z\|_{2,\varphi}^2 e^{-2\varphi(z)} dA(z) \\ &= \int_{\mathbb{D}} h(c\tilde{\mu}(z))\rho^{-2}(z) dA(z) < \infty,\end{aligned} \quad (116)$$

Therefore, $T_\mu \in S_h$. □

Let

$$\Gamma = \left\{\sum_{j=1}^{N} a_j K(\cdot, z_j), N \in N^+, a_j \in \mathbb{C}, z_j \in \mathbb{D}, i \le j \le N\right\}, \quad (117)$$

and

$$S = \left\{f \text{ is measurable on } \mathbb{D} | fg \in L_\varphi^1 \text{ for } g \in \Gamma\right\}. \quad (118)$$

It follows from Hu et al. [3] that Γ is dense in A_φ^2. Define

$$G_r(f) = \inf\left\{\left(\frac{1}{|D^r(z)|}\int_{D^r(z)} |f-h|^2 dA(z)\right)^{1/2}, h \in \mathscr{H}(D^r(z))\right\}, \quad (119)$$

where $\mathscr{H}(D^r(z))$ is a set of all analytic functions on $D^r(z)$.

Theorem 24. *Suppose that $\varphi \in \mathscr{W}_0$,$h(\sqrt{(\cdot)})$: $R^+ \longrightarrow R^+$ is a continuous increasing convex function, $0 < r < \alpha$,$f \in S$, and $G_r(f) \in L^\infty$. Then, the following statements are equivalent:*

(a) Hankel operator H_f belongs to S_h

(b) For some (any) r, there is a constant $c > 0$ such that

$$\int_{\mathbb{D}} h(cG_r(f)(z))\rho^{-2}(z)dA(z) < \infty. \quad (120)$$

Proof. (A)⇒(B). Let $\{e_j\}_{j=1}^\infty$ be orthogonality basis of A_φ^2. Define

$$T_{e_j} = \frac{\chi_{D^r(a_j)} H_f\left(k_{a_j}\right)}{\left(\int_{D^r(a_j)}\left|H_f\left(k_{a_j}\right)\right|^2 e^{-2\varphi} dA\right)^{1/2}} = t_j \chi_{D^r(a_j)} H_f\left(k_{a_j}\right), \quad (121)$$

where $\{a_j\}$ is the $(\rho, r/B)$-lattice of A_φ^2. It is obvious that $\|T_g\|_{2,\varphi}^2 \lesssim \|g\|_{2,\varphi}^2$, and then, T is bounded. The convexity of $h(\sqrt{(\cdot)})$ shows that $h(\cdot)$ is also a convex function. Set

$$A(g) = \sum_{j=1}^{\infty}\langle g, e_j\rangle k_{a_j}, \quad (122)$$

and then, we have

$$\begin{aligned}&\int_{\mathbb{D}} h(cG_{r/B}(f)(z))\rho^{-2}(z)dA(z) \\ &\le \sum_{j=1}^{\infty}\int_{D^r(a_j)} h(cG_{r/B}(f)(z))\rho^{-2}(z)dA(z) \\ &\le \sum_{j=1}^{\infty}\sup_{z\in D^{r/B}(a_j)} h(cG_r(f)(z)) \\ &\lesssim \sum_{j=1}^{\infty} h(c_1 G_r(f)(a_j)) \\ &\lesssim \sum_{j=1}^{\infty} h\left(\left(\frac{c_1}{|D^r(a_j)|}\int_{D^r(a_j)}\left|f - \frac{1}{k_{a_j}}P\left(fk_{a_j}\right)\right|^2 dA(z)\right)^{1/2}\right) \\ &\simeq \sum_{j=1}^{\infty} h\left(\left(c_2\int_{D^r(a_j)}\left|f - \frac{1}{k_{a_j}}P\left(fk_{a_j}\right)\right|^2\left|k_{a_j}\right|^2 e^{-2\varphi(z)}dA(z)\right)^{1/2}\right) \\ &= \sum_{j=1}^{\infty} h\left(\left(c_2\int_{D^r(a_j)}\left|H_f\left(k_{a_j}\right)\right|^2 e^{-2\varphi(z)}dA(z)\right)^{1/2}\right) \\ &= \sum_{j=1}^{\infty} h\left(c_2\left|t_j\left\langle H_f k_{a_j}, \chi_{D^r(a_j)}H_f k_{a_j}\right\rangle\right|\right) \\ &\le \sum_{j=1}^{\infty} h(c_2|\langle T^* H_f A e_j, e_j\rangle|) \\ &\le \sum_{j=1}^{\infty} h(c_3 s_j(T^* H_f A)) \\ &\lesssim \sum_{j=1}^{\infty} h(c_3 s_j(H_f)) < \infty,\end{aligned} \quad (123)$$

where c, c_1, c_2, and c_3 are positive constants.

(B)⇒(A). Suppose that $\int_{\mathbb{D}} h(cG_r(f)(z))\rho^{-2}(z)dA(z) < \infty$, we define the square mean of $|f|$ over $D^r(z)$ by setting

$$M_r(f)(z) = \left(\frac{1}{|D^r(z)|}\int_{D^r(z)}|f|^2 dA(z)\right)^{1/2}, \quad (124)$$

for $f \in L^p_{\text{loc}}(\mathbb{D})$ and $r > 0$, decomposing $f = f_1 + f_2$ as

$$f_1 = \sum_{j=1}^{\infty} h_j \psi_j \text{and} f_2 = f - f_1, \tag{125}$$

where $\{\psi_j\}_{j=1}^{\infty}$ is a partition of unity subordinate to $\{D^{r/2}(a_j)\}_{j=1}^{\infty}$, $h_j \in \mathscr{H}(D^r(a_j))$ and given $f \in L^p_{\text{loc}}(\mathbb{D})$ for $j = 1, 2, \cdots$, such that

$$M^r(f - h_j) = G_r(f)(a_j). \tag{126}$$

Then, $f_1 \in C^1(\mathbb{D})$ and

$$|\rho(z)\bar{\partial} f_1(z)| + M_{r/12}(\rho\bar{\partial} f_1)(z) + M_{r/12}(f_2)(z) \le cG_r(f)(z). \tag{127}$$

Therefore,

$$\begin{aligned}
&\int_{\mathbb{D}} h(M_r(\rho\bar{\partial} f_1))\rho^{-2}(z)dA(z) \\
&\quad < \int_{\mathbb{D}} h(cM_{r/12}(\rho\bar{\partial} f_1))\rho^{-2}(z)dA(z) \\
&\quad \le \int_{\mathbb{D}} h(cG_r(f)(z))\rho^{-2}(z)dA(z) < \infty, \\
&\int_{\mathbb{D}} h(M_r(f_2)\rho^{-2}(z)dA(z) \\
&\quad < \int_{\mathbb{D}} h(cM_{r/12}(f_2)\rho^{-2}(z)dA(z) \\
&\quad \le \int_{\mathbb{D}} h(cG_r(f)(z))\rho^{-2}(z)dA(z) < \infty.
\end{aligned} \tag{128}$$

Let$\theta$to be$f_2$or$\rho\bar{\partial} f_1$, considering the multiplication operatorsM_θ, and by the assumption$G_r(f)(z) \in L^\infty$and Lemma 3.4 in Zeng et al.'s study [17], we have$M_r(\theta)(z) \in L^\infty$, and$M_\theta$is bounded from$A^2_\varphi$to$L^2_\varphi$. Since for any$g, h \in A^2_\varphi$, there holds

$$\langle M^*_\theta M_\theta g, h\rangle = \langle M_\theta g, M_\theta h\rangle = \left\langle T_{|\theta|^2} g, h\right\rangle. \tag{129}$$

We know that $M^*_\theta M_\theta = T_{|\theta|^2}$ on A^2_φ. Thus, $M_\theta \in S_h$ if and only if $M^*_\theta M_\theta = T_{|\theta|^2} \in S_{h(\sqrt{(\cdot)})}$. According to Theorem 23 and the condition which $h(\sqrt{(\cdot)})$ is a convex function, $T_{|\theta|^2} \in S_{h(\sqrt{(\cdot)})}$ if and only if

$$\int_{\mathbb{D}} h\left(c\left(\widetilde{|\theta|^2}(z)\right)^{1/2}\right)\rho^{-2}(z)dA(z) < \infty. \tag{130}$$

In proof of Theorem 23, we get $\widehat{\mu_r}(z) \le \tilde{\mu}(z)$. On the other side, we assert that

$$\int_{\mathbb{D}} h(\tilde{\mu}(z))dA(z) \le c\int_{\mathbb{D}} h(c\widehat{\mu_r}(w))dA(w). \tag{131}$$

In fact, by proof in proposition 2.5 of Zeng et al. [17], we have $\tilde{\mu}(z) \le \widetilde{\widehat{\mu_r}}(z)$, and together with Jensen's inequality, there holds

$$\begin{aligned}
\int_{\mathbb{D}} h(\tilde{\mu}(z))dA(z) &\le \int_{\mathbb{D}} h\left(c\widetilde{\widehat{\mu_r}}(z)\right)dA(z) \\
&\le \int_{\mathbb{D}} h\left(c\int_{\mathbb{D}} \left|k_z e^{-\varphi(w)}\right|^2 \widehat{\mu_r}(w)dA(w)\right) \\
&\le \int_{\mathbb{D}} \left(\int_{\mathbb{D}} h(c\widehat{\mu_r}(w))\left|k_z e^{-\varphi(w)}\right|^2 dA(w)\right)dA(z) \\
&\le \int_{\mathbb{D}} h(c\widehat{\mu_r}(w))dA(w)\int_{\mathbb{D}} \left|k_z e^{-\varphi(w)}\right|^2 dA(z) \\
&\le \int_{\mathbb{D}} h(c\widehat{\mu_r}(w))dA(w).
\end{aligned} \tag{132}$$

Therefore, we conclude that

$$\int_{\mathbb{D}} h\left(c\left(\widetilde{|\theta|^2}(z)\right)^{1/2}\right)\rho^{-2}(z)dA(z) < \infty \tag{133}$$

is equivalent to

$$\int_{\mathbb{D}} h\left(c\left(\widehat{|\theta|^2}(z)\right)^{1/2}\right)\rho^{-2}(z)dA(z) < \infty, \tag{134}$$

or equivalently

$$\int_{\mathbb{D}} h(cM_r(\theta)(z))\rho^{-2}(z)dA(z) < \infty, \tag{135}$$

hence $M_\theta \in S_h$. Since $\|H_{f_1}(g)\|_{L^2_\varphi} \lesssim \|g\rho\bar{\partial} f_1\|_{L^2_\varphi}$ and $\|H_{f_2}(g)\|_{L^2_\varphi} \lesssim \|f_2 g\|_{L^2_\varphi}$, both H_{f_1} and H_{f_2} belong to S_h which leads to $H_f \in S_h$. The proof is complete. □

Data Availability

No data were used in this paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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