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Research Article

A Novel Study Based on Fuzzy p-Ideals of BCI-Algebras

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In this paper, we propose the concept of $(\in, \in \vee(j^*, q_j))$-fuzzy p-ideals in "*BCI*-algebras." We show that "$(\in, \in \vee q)$)-fuzzy p-ideals" and "$(\in\vee(j^*, q_j), \in \vee(j^*, q_j))$-fuzzy *p*-ideals" are "$(\in, \in \vee(j^*, q_j))$-fuzzy *p*-ideals." However, the converse is not true, then presented examples. For a BCI-algebra $\widehat{Y}$, it has been shown that every $(\in, \in \vee(j^*, q_j))$-fuzzy p-ideal of $\widehat{Y}$ is an $(\in, \in \vee(j^*, q_j))$-fuzzy ideals of $\widehat{Y}$but not conversely, and then, an example is given. Furthermore in $\widehat{Y}$, a connection between $(\in, \in \vee(j^*, q_j))$-fuzzy p-ideals and p-ideals is established.

1. Introduction

The concepts of BCK and BCI-algebras were first introduced by Imai and Ise'ki in 1966 [1, 2]. The algebraic formulations of the BCK and BCI systems are BCK and BCI-algebras in combinatory logic. Eventually, the theory of these algebras has been developed rapidly and successfully with a specific focus on the ideal theory, for instance, Liu et al. [3] studied $q(a)$-ideals while fuzzy h-ideals are given in [4], and hybrid ideals are considered by Muhiuddin et al. [5, 6] in BCK/BCI-algebras. Recent research focused on several kinds of related ideals are studied in [7–10].

The theory of fuzzy set is given in [11] as a new discipline. Jun [12] initiated the study of fuzzy p-ideals in BCI-algebras and studied their various characteristics. Touqeer and Cagman [13] have given the notion of intuitionistic fuzzy p-ideals of BCI-algebras. Muhiuddin [14] investigated p-ideals of BCI-algebras related with neutrosophic N-structures.

In order to develop various kinds of fuzzy subgroups, the idea of "quasi-coincidence" of a fuzzy point with a fuzzy set is established in [15]. The same concepts were introduced and investigated by Jun [16, 17] in BCK/BCI-algebras. Zhan et al. [18] gave the idea of $(\in, \in \vee q)$-fuzzy ideal of BCI-algebra and explored their interesting results. Also, Zhang et al. [18] applied the idea of a quasicoincidence of a fuzzy point with a fuzzy set and introduced the concepts of $(\in, \in \vee q)$-fuzzy p(q and a)-ideals in BCI-algebras, while Ma et al. [19] present the ideas of various kinds of fuzzy ideals based on $(\in, \in \vee q)$-interval-valued fuzzy structures.

Al-Masarwah and Ahmad [20] developed the ideas of m-polar (α, β)-fuzzy ideals. Takallo et al. defined and presented m-polar $(\in, \in)$-fuzzy p-ideals in [21]. Numerous algebraic systems have been exposed to these structures, with a variety of outcomes [22–25].

The concept of generalized notion is natural to introduce. To do so, we introduced the concept of $(\in, \in \vee(j^*, q_j))$-fuzzy p-ideals. Furthermore, we presented the relationship between $(\in, \in \vee q)$)-fuzzy p-ideals and $(\in\vee(j^*, q_j), \in \vee(j^*, q_j))$-fuzzy *p*-ideals. Besides, we investigated the correspondence among these notions.

2. Preliminaries

An algebra "$\widehat{Y} = (\widehat{Y}; *, 0)$" is a BCI-algebra if $\forall \widehat{\mathfrak{v}}, \widehat{\omega}, \widehat{\mathfrak{z}} \in \widehat{Y}$,

(1) $((\widehat{\mathfrak{v}} * \widehat{\omega}) * (\widehat{\mathfrak{v}} * \widehat{\mathfrak{z}})) * (\widehat{\mathfrak{z}} * \widehat{\omega}) = 0$,

(2) $(\widehat{\mathfrak{v}} * (\widehat{\mathfrak{v}} * \widehat{\omega})) * \widehat{\omega} = 0$,

(3) $\widehat{\mathfrak{v}} * \widehat{\mathfrak{v}} = 0$,

(4) $\widehat{\mathfrak{v}} * \widehat{\omega} = 0$ and $\widehat{\omega} * \widehat{\mathfrak{v}} = 0 \Longrightarrow \widehat{\mathfrak{v}} = \widehat{\omega}$.

A partially ordered "$\leq$" on $\widehat{Y}$ is defined as $\widehat{\mathfrak{v}} \leq \widehat{\omega} \Longleftrightarrow \widehat{\mathfrak{v}} * \widehat{\omega} = 0$.

From now we mean $\widehat{Y}$ as a BCI-algebra. By a fuzzy subset (in brief, FS), we mean a function $\mathfrak{B} : \widehat{Y} \longrightarrow [0, 1]$.

Definition 1 (see [26]). Let $Z \in \widehat{Y}$ and $\tilde{\varsigma} \in [0, 1]$. The "ordered fuzzy point" (in brief, OFP) $Z_{\tilde{\varsigma}}$ of $\widehat{Y}$ is given as:

$$Z_{\tilde{\varsigma}}(\widehat{\omega}) = \begin{cases} \tilde{\varsigma}, & \text{if } \widehat{\omega} \in (z] \\ 0, & \text{if } \widehat{\omega} \in (z] \end{cases}, \quad \forall \widehat{\omega} \in \widehat{Y}. \tag{1}$$

It is obvious that $Z_{\tilde{\varsigma}}$ is an FS of $\widehat{Y}$. In the sequel, we indicate $Z_{\tilde{\varsigma}} \subseteq \mathfrak{B}$ as $Z_{\tilde{\varsigma}} \in \mathfrak{B}$ for any FS $\mathfrak{B}$. In other words, $Z_{\tilde{\varsigma}} \in \mathfrak{B} \Longleftrightarrow \mathfrak{B}(Z) \geq \tilde{\varsigma}$.

Definition 2 (see [26]). A FS $\mathfrak{B}$ of $\widehat{Y}$ is called an $(\in, \in \vee (j^*, q_j))$-fuzzy subalgebra (in brief, $(\in, \in \vee (j^*, q_j))$-FSA) of $\widehat{Y}$ if $\widehat{\mathfrak{z}}_{\tilde{\varsigma}} \in \mathfrak{B}$ and $\widehat{\mathfrak{v}}_{\tilde{\imath}} \in \mathfrak{B}$ implies $(\widehat{\mathfrak{z}} * \widehat{\mathfrak{v}})_{\tilde{\varsigma} \wedge \tilde{\imath}} \in \vee (j^*, q_j)\mathfrak{B} \forall \tilde{\varsigma}, \tilde{\imath} \in [0, 1]$ and $\widehat{\mathfrak{z}}, \widehat{\mathfrak{v}} \in \widehat{Y}$.

Definition 3 (see [26]). A FS $\mathfrak{B}$ of $\widehat{Y}$ is said to be an $(\in, \in \vee (j^*, q_j))$-FI (briefly, fuzzy ideal) of $\widehat{Y}$ if

(1) $\widehat{\mathfrak{z}}_{\tilde{\varsigma}} \in \mathfrak{B}$ imply $0_{\tilde{\varsigma}} \in \vee (j^*, q_j)\mathfrak{B}$, and

(2) $(\widehat{\mathfrak{z}} * \widehat{\mathfrak{v}})_{\tilde{\varsigma}} \in \mathfrak{B}$ and $\widehat{\mathfrak{v}}_{\tilde{\imath}} \in \mathfrak{B}$ imply $\widehat{\mathfrak{z}}_{\tilde{\varsigma} \wedge \tilde{\imath}} \in \vee (j^*, q_j)\mathfrak{B}$,

$\forall \widehat{\mathfrak{z}}, \widehat{\mathfrak{v}} \in \widehat{Y}$ and $\tilde{\varsigma}, \tilde{\imath} \in (0, 1]$.

Lemma 4 (see [26]). *Let $\mathfrak{B}$ be a FS of $\widehat{Y}$. Then, $\widehat{\mathfrak{z}}_{\tilde{\varsigma}} \in \mathfrak{B}$ implies $0_{\tilde{\varsigma}} \in \vee (j^*, q_j)\mathfrak{B} \Longleftrightarrow \forall \widehat{\mathfrak{z}} \in \widehat{Y}, \mathfrak{B}(0) \geq \mathfrak{B}(\widehat{\mathfrak{z}}) \wedge j^* - j/2$.*

Lemma 5 (see [26]). *Let $\mathfrak{B}$ be an $(\in, \in \vee (j^*, q_j))$-FI in $\widehat{Y}$ such that $\widehat{\mathfrak{z}} \leq v$. Then, $\mathfrak{B}(\widehat{\mathfrak{z}}) \geq \mathfrak{B}(\widehat{\mathfrak{v}}) \wedge j^* - j/2$.*

Lemma 6 (see [26]. *Let $\mathfrak{B}$ be an $(\in, \in \vee (j^*, q_j))$-FI of $\widehat{Y}$. Then, $\forall \widehat{\mathfrak{z}}, \widehat{\mathfrak{v}}, \widehat{\omega} \in \widehat{Y}$, $\widehat{\mathfrak{z}} * \widehat{\mathfrak{v}} \leq \widehat{\omega} \Longrightarrow \mathfrak{B}(\widehat{\mathfrak{z}}) \geq \mathfrak{B}(\widehat{\mathfrak{v}}) \wedge \mathfrak{B}(\widehat{\omega}) \wedge j^* - j/2$.*

3. $(\in, \in \vee (j^*, q_j))$-Fuzzy p-Ideals

Definition 7. Let $\widehat{\mathfrak{z}}_{\tilde{\varsigma}}$ be in OFP of $\widehat{Y}$ and $j^* \in (0, 1]$. Then, $\widehat{\mathfrak{z}}_{\tilde{\varsigma}}$ is called (j^*, q)-quasicoincident with a FS $\mathfrak{B}$ of $\widehat{Y}$, denoted as $\widehat{\mathfrak{z}}_{\tilde{\varsigma}}(j^*, q)\mathfrak{B}$, if

$$\mathfrak{B}(\widehat{\mathfrak{z}}) + \tilde{\varsigma} > j^*. \tag{2}$$

Suppose that $0 \leq j < j^* \leq 1$. For OFP $\widehat{\mathfrak{z}}_{\tilde{\varsigma}}$, we define

(1) $\widehat{\mathfrak{z}}_{\tilde{\varsigma}}(j^*, q_j)\mathfrak{B}$, if $\mathfrak{B}(\widehat{\mathfrak{z}}) + \tilde{\varsigma} + j > j^*$

(2) $\widehat{\mathfrak{z}}_{\tilde{\varsigma}} \in \vee (j^*, q_j)\mathfrak{B}$, if $\widehat{\mathfrak{z}}_{\tilde{\varsigma}} \in \mathfrak{B}$ or $\widehat{\mathfrak{z}}_{\tilde{\varsigma}}(j^*, q_j)\mathfrak{B}$

(3) $\widehat{\mathfrak{z}}_{\tilde{\varsigma}} \bar{\gamma} \mathfrak{B}$, if $\widehat{\mathfrak{z}}_{\tilde{\varsigma}} \gamma \mathfrak{B}$ does not hold for $\gamma \in \{(j^*, q_j), \in \vee (j^*, q_j)\}$.

Definition 8. A FS $\mathfrak{B}$ of $\widehat{Y}$ is called an $(\in, \in \vee (j^*, q_j))$-fuzzy p-ideal (in brief, $(\in, \in \vee (j^*, q_j))$-FPI) of $\widehat{Y}$ if

(1) $\widehat{\mathfrak{z}}_{\tilde{\varsigma}} \in \mathfrak{B}$ imply $0_{\tilde{\varsigma}} \in \vee (j^*, q_j)\mathfrak{B}$

(2) $((\widehat{\mathfrak{z}} * \widehat{\mathfrak{v}}) * (\widehat{\omega} * \widehat{\mathfrak{v}}))_{\tilde{\varsigma}} \in \mathfrak{B}$ and $\widehat{\omega}_{\tilde{\imath}} \in \mathfrak{B}$ imply $\widehat{\mathfrak{z}}_{\tilde{\varsigma} \wedge \tilde{\imath}} \in \vee (j^*, q_j)\mathfrak{B}$

$\forall \widehat{\mathfrak{z}}, \widehat{\mathfrak{v}}, \widehat{\omega} \in \widehat{Y}$ and $\tilde{\varsigma}, \tilde{\imath} \in (0, 1]$.

Example 9. Consider a BCI-algebra $\widehat{Y} = \{0, \widehat{\imath}, \widehat{\ell}, \widehat{\omega}\}$, defined by Table 1.

Define a FS $\mathfrak{B}$ on $\widehat{Y}$ as

$$\mathfrak{B}(\widehat{\mathfrak{v}}) = \begin{cases} 0.6 & \text{if } \widehat{\mathfrak{v}} = 0 \\ 0.5 & \text{if } \widehat{\mathfrak{v}} = \widehat{\imath} \\ 0.3 & \text{if } \widehat{\mathfrak{v}} \in \left\{\widehat{\ell}, \widehat{\omega}\right\} \end{cases}. \tag{3}$$

It is easy to evaluate that $\mathfrak{B}$ is an $(\in, \in \vee (j^*, q_j))$-FPI for $j^* = 0.85$ and $j = 0.15$ of $\widehat{Y}$.

Definition 10. A FS $\mathfrak{B}$ of $\widehat{Y}$ is called an $(\in, \in \vee q)$-FPI (briefly, fuzzy p-ideal) of $\widehat{Y}$ if

(1) $\widehat{\mathfrak{z}}_{\tilde{\varsigma}} \in \mathfrak{B}$ imply $0_{\tilde{\varsigma}} \in \vee q\mathfrak{B}$

(2) $((\widehat{\mathfrak{z}} * \widehat{\mathfrak{v}}) * (\widehat{\omega} * \widehat{\mathfrak{v}}))_{\tilde{\varsigma}} \in \mathfrak{B}$ and $\widehat{\omega}_{\tilde{\imath}} \in \mathfrak{B}$ imply $\widehat{\mathfrak{z}}_{\tilde{\varsigma} \wedge \tilde{\imath}} \in \vee q\mathfrak{B}$

$\forall \widehat{\mathfrak{z}}, \widehat{\mathfrak{v}}, \widehat{\omega} \in \widehat{Y}$ and $\tilde{\varsigma}, \tilde{\imath} \in (0, 1]$.

Theorem 11. *In $\widehat{Y}$, every $(\in, \in \vee q)$-FPI is an $(\in, \in \vee (j^*, q_j))$-FPI, but converse may not be true in general.*

Proof. Assume that $\mathfrak{B}$ is an $(\in, \in \vee q)$-FPI of $\widehat{Y}$. Take $\widehat{\mathfrak{z}}_{\tilde{\varsigma}} \in \mathfrak{B}$ for $\widehat{\mathfrak{z}} \in \widehat{Y}$ and $\tilde{\varsigma} \in (0, 1]$. So by hypothesis, $0_{\tilde{\varsigma}} \in \vee q\mathfrak{B}$. It implies that $\mathfrak{B}(0) \geq \tilde{\varsigma}$ or $\mathfrak{B}(0) + u \geq 1$, and so, $\mathfrak{B}(0) \geq \tilde{\varsigma}$ or $\mathfrak{B}(0) +$

Table 1: Cayley table of the binary operation $*$.

$*$	0	$\hat{\imath}$	$\hat{l}$	$\hat{\omega}$
0	0	$\hat{\imath}$	$\hat{l}$	$\hat{\omega}$
$\hat{\imath}$	$\hat{\imath}$	0	$\hat{\omega}$	$\hat{\ell}$
$\hat{\ell}$	$\hat{\ell}$	$\hat{\omega}$	0	$\hat{\imath}$
$\hat{\omega}$	$\hat{\omega}$	$\hat{\ell}$	$\hat{\imath}$	0

$j+\tilde{\varsigma} \geq j^*$. Thus, $0_{\tilde{\varsigma}} \in \vee(j^*, q_j)\mathfrak{B}$. Further, take any $((\hat{\mathfrak{z}} * \hat{\mathfrak{b}}) * (\hat{\omega} * \hat{\mathfrak{b}}))_{\tilde{\varsigma}} \in \mathfrak{B}$ and $\hat{\omega}_{\tilde{\imath}} \in \mathfrak{B}$. So, $z_{\tilde{\varsigma}\wedge\tilde{\imath}} \in \vee q\mathfrak{B}$ implies $\mathfrak{B}(\hat{\mathfrak{z}}) \geq \tilde{\varsigma}\wedge\tilde{\imath}$ or $\mathfrak{B}(\hat{\mathfrak{z}}) + \tilde{\varsigma}\wedge\tilde{\imath} > 1$. Therefore, $\mathfrak{B}(\hat{\mathfrak{z}}) \geq \tilde{\varsigma}\wedge\tilde{\imath}$ or $\mathfrak{B}(\hat{\mathfrak{z}}) + j + \tilde{\varsigma}\wedge\tilde{\imath} > j^*$. Thus, $z_{\tilde{\varsigma}} \in \vee(j^*, q_j)\mathfrak{B}$, as required. □

Example 12. Consider a BCI-algebra $\hat{Y} = \{0, \hat{\imath}, \hat{\ell}, \hat{\mathfrak{z}}, \hat{\omega}\}$ which is defined by Table 2.

We define a FS

$$\mathfrak{B}(\hat{\mathfrak{b}}) = \begin{cases} 0.6, & \text{if } \hat{\mathfrak{b}} = 0 \\ 0.1, & \text{if } \hat{\mathfrak{b}} \in \{\hat{\imath}, \hat{\omega}\} \\ 0.3, & \text{if } \hat{\mathfrak{b}} = \hat{\ell} \\ 0.2, & \text{if } \hat{\mathfrak{b}} = z \end{cases}. \quad (4)$$

It is easy to evaluate that $\mathfrak{B}$ is an $(\epsilon, \in\vee(j^*, q_j))$-FPI of $\hat{Y}$ but not an $(\epsilon, \in\vee q)$-FPI as $((\hat{\ell} * \hat{\omega}) * (0 * \hat{\omega}))_{0.4} \in \mathfrak{B}$ and $0_{0.4} \in \mathfrak{B}$ but $\hat{\ell}_{0.4} \overline{\in\vee q} \mathfrak{B}$, where $j = 0.81$ and $j^* = 0.41$.

Definition 13. A FS $\mathfrak{B}$ of $\hat{Y}$ is said to be an $(\in\vee(j^*, q_j), \in\vee(j^*, q_j))$-FPI of $\hat{Y}$ if

(1) $\hat{\mathfrak{z}}_{\tilde{\varsigma}} \in \vee(j^*, q_j)\mathfrak{B}$ imply $0_{\tilde{\varsigma}} \in \vee(j^*, q_j)\mathfrak{B}$

(2) $((\hat{\mathfrak{z}} * \hat{\mathfrak{b}}) * (\hat{\omega} * \hat{\mathfrak{b}}))_{\tilde{\varsigma}} \in \vee(j^*, q_j)\mathfrak{B}$ and $\hat{\omega}_{\tilde{\imath}} \in \vee(j^*, q_j)\mathfrak{B}$ imply $\hat{\mathfrak{z}}_{\tilde{\varsigma}\wedge\tilde{\imath}} \in \vee(j^*, q_j)\mathfrak{B}$

$\forall \hat{\mathfrak{z}}, \hat{\mathfrak{b}}, \hat{\omega} \in \hat{Y}$ and $\tilde{\varsigma}, \tilde{\imath} \in (0, 1]$.

Lemma 14. *In $\hat{Y}$, every $(\in\vee(j^*, q_j), \in\vee(j^*, q_j))$-FPI is $(\epsilon, \in\vee(j^*, q_j))$-FPI.*

Proof. Let $\mathfrak{B}$ be any $(\in\vee(j^*, q_j), \in\vee(j^*, q_j))$-FPI of $\hat{Y}$. Take any $\hat{\mathfrak{z}}_{\tilde{\varsigma}} \in \mathfrak{B}$ for $z \in \hat{Y}$ and $\tilde{\varsigma} \in (0, 1]$. Then, $\hat{\mathfrak{z}}_{\tilde{\varsigma}} \in \vee(j^*, q_j)\mathfrak{B}$. Therefore, by hypothesis, $0_{\tilde{\varsigma}} \in \vee(j^*, q_j)\mathfrak{B}$. Assume that $((\hat{\mathfrak{z}} * \hat{\mathfrak{b}}) * (\hat{\omega} * \hat{\mathfrak{b}}))_{\tilde{\varsigma}} \in \mathfrak{B}$ and $\hat{\omega}_{\tilde{\imath}} \in \mathfrak{B}$ for any $\hat{\mathfrak{z}}, \hat{\mathfrak{b}}, \hat{\omega} \in \hat{Y}$. Then, $((\hat{\mathfrak{z}} * \hat{\mathfrak{b}}) * (\hat{\omega} * \hat{\mathfrak{b}}))_{\tilde{\varsigma}} \in \vee(j^*, q_j)\mathfrak{B}$ and $\hat{\omega}_{\tilde{\imath}} \in \vee(j^*, q_j)\mathfrak{B}$. So, $\hat{\mathfrak{z}}_{\tilde{\varsigma}\wedge\tilde{\imath}} \in \vee(j^*, q_j)\mathfrak{B}$, as required. □

Example 15. Consider a BCI-algebra $\hat{Y} = \{0, \hat{\imath}, \hat{\ell}, \hat{\mathfrak{z}}, \hat{\omega}\}$ which is defined by Table 3:

Table 2: Cayley table of the binary operation $*$ in $\hat{Y}$.

$*$	0	$\hat{\imath}$	$\hat{l}$	$\hat{\mathfrak{z}}$	$\hat{\omega}$
0	0	0	0	$\hat{\mathfrak{z}}$	$\hat{\mathfrak{z}}$
$\hat{\imath}$	$\hat{\imath}$	0	$\hat{\imath}$	$\hat{\omega}$	$\hat{\mathfrak{z}}$
$\hat{\ell}$	$\hat{\ell}$	$\hat{\ell}$	0	$\hat{\mathfrak{z}}$	$\hat{\mathfrak{z}}$
$\hat{\mathfrak{z}}$	$\hat{\mathfrak{z}}$	$\hat{\mathfrak{z}}$	$\hat{\mathfrak{z}}$	0	0
$\hat{\omega}$	$\hat{\omega}$	$\hat{z}$	$\hat{\omega}$	$\hat{\imath}$	0

Table 3: Cayley table of the binary operation $*$.

$*$	0	$\hat{\imath}$	$\hat{l}$	$\hat{\mathfrak{z}}$	$\hat{\omega}$
0	0	0	0	0	0
$\hat{\imath}$	$\hat{\imath}$	0	$\hat{\imath}$	0	$\hat{\imath}$
$\hat{\ell}$	$\hat{\ell}$	$\hat{\ell}$	0	$\hat{\ell}$	0
$\hat{\mathfrak{z}}$	$\hat{\mathfrak{z}}$	$\hat{\imath}$	$\hat{\mathfrak{z}}$	0	$\hat{\mathfrak{z}}$
$\hat{\omega}$	$\hat{\omega}$	$\hat{\omega}$	$\hat{\ell}$	$\hat{\omega}$	0

Define $B : \hat{Y} \longrightarrow [0, 1]$ by

$$\mathfrak{B}(\hat{\mathfrak{b}}) = \begin{cases} 0.4, & \text{if } \hat{\mathfrak{b}} = 0 \\ 0.6, & \text{if } \hat{\mathfrak{b}} \in \{\hat{\imath}, \hat{\mathfrak{z}}\} \\ 0.1, & \text{if } \hat{\mathfrak{b}} = \{\hat{\ell}, \hat{\omega}\} \end{cases}. \quad (5)$$

$\mathfrak{B}$ is an $(\epsilon, \in\vee(j^*, q_j))$-FPI of $\hat{Y}$ with $j = 0.5$ and $j^* = 0.7$, although it is not an "$(\in\vee(j^*, q_j), \in\vee(j^*, q_j))$-FPI" of $\hat{Y}$ as $\hat{\ell}_{\tilde{\varsigma}=0.95} = (\hat{\ell} * \hat{\imath}) * (0 * \hat{\imath}))_{\tilde{\varsigma}=0.95} \in \vee(j^*, q_j)\mathfrak{B}$ and $0_{\tilde{\imath}=0.5} \in \vee(j^*, q_j)\mathfrak{B}$ but $\hat{\ell}_{\tilde{\varsigma}\wedge\tilde{\imath}=0.5} \overline{\in\vee(j^*, q_j)}\mathfrak{B}$.

Lemma 16. *Let $\mathfrak{B}$ be a FS of $\hat{Y}$. Then, $\forall \hat{\mathfrak{z}}, \hat{\mathfrak{b}}, \hat{\omega} \in \hat{Y}$, $((\hat{\mathfrak{z}} * \hat{\mathfrak{b}}) * (\hat{\omega} * \hat{\mathfrak{b}}))_{\tilde{\varsigma}} \in \mathfrak{B}$, and $\hat{\omega}_{\tilde{\imath}} \in \mathfrak{B}$ imply $\hat{\mathfrak{z}}_{\tilde{\varsigma}\wedge\tilde{\imath}} \in \vee(j^*, q_j)\mathfrak{B} \Longleftrightarrow \mathfrak{B}(\hat{\mathfrak{z}}) \geq \mathfrak{B}((\hat{\mathfrak{z}} * \hat{\mathfrak{b}}) * (\hat{\omega} * \hat{\mathfrak{b}}))\wedge\mathfrak{B}(\hat{\omega})\wedge j^* - j/2$.*

Proof. ($\Longrightarrow$) Contrary assume that for some $\hat{\mathfrak{z}}, \hat{\mathfrak{b}} \in \hat{Y}$, $\mathfrak{B}(\hat{\mathfrak{z}}) < \mathfrak{B}((\hat{\mathfrak{z}} * \hat{\mathfrak{b}}) * (\hat{\omega} * \hat{\mathfrak{b}}))\wedge\mathfrak{B}(\hat{\omega})\wedge j^* - j/2$. Take $\tilde{\varsigma} \in (0, j^* - j/2]$s.t. $\mathfrak{B}(\hat{\mathfrak{z}}) < \tilde{\varsigma} \leq \mathfrak{B}((\hat{\mathfrak{z}} * \hat{\mathfrak{b}}) * (\hat{\omega} * \hat{\mathfrak{b}}))\wedge\mathfrak{B}(\hat{\omega})\wedge j^* - j/2$. Then, $((\hat{\mathfrak{z}} * \hat{\mathfrak{b}}) * (\hat{\omega} * \hat{\mathfrak{b}}))_{\tilde{\varsigma}} \in \mathfrak{B}$ and $\hat{\omega}_{\tilde{\varsigma}} \in \mathfrak{B}$, but $\hat{\mathfrak{z}}_{\tilde{\varsigma}} \overline{\in\vee(j^*, q_j)}\mathfrak{B}$, which is impossible. Hence, $\mathfrak{B}(\hat{\mathfrak{z}}) \geq \mathfrak{B}((\hat{\mathfrak{z}} * \hat{\mathfrak{b}}) * (\hat{\omega} * \hat{\mathfrak{b}}))\wedge\mathfrak{B}(\hat{\omega})\wedge j^* - j/2$.

($\Longleftarrow$) Let $((\hat{\mathfrak{z}} * \hat{\mathfrak{b}}) * (\hat{\omega} * \hat{\mathfrak{b}}))_{\tilde{\varsigma}} \in \mathfrak{B}$ and $\hat{\omega}_{\tilde{\imath}} \in \mathfrak{B}$, $\forall \tilde{\varsigma}, \tilde{\imath} \in (0, 1]$. Then, $\mathfrak{B}((\hat{\mathfrak{z}} * \hat{\mathfrak{b}}) * (\hat{\omega} * \hat{\mathfrak{b}})) \geq \tilde{\varsigma}$ and $\mathfrak{B}(\hat{\omega}) \geq \tilde{\imath}$. Thus,

$$\mathfrak{B}(\hat{\mathfrak{z}}) \geq \mathfrak{B}((\hat{\mathfrak{z}} * \hat{\mathfrak{b}}) * (\hat{\omega} * \hat{\mathfrak{b}}))\wedge\mathfrak{B}(\hat{\omega})\wedge\frac{j^* - j}{2} \geq \tilde{\varsigma}\wedge\tilde{\imath}\wedge\frac{j^* - j}{2}. \quad (6)$$

Now, if $\tilde{\varsigma}\wedge\tilde{\imath} \leq j^* - j/2$, then $\mathfrak{B}(\hat{\mathfrak{z}}) \geq \tilde{\varsigma}\wedge\tilde{\imath}$ implies $\hat{\mathfrak{z}}_{\tilde{\varsigma}\wedge\tilde{\imath}} \in \mathfrak{B}$. If $\tilde{\varsigma}\wedge\tilde{\imath} > j^* - j/2$, then $\mathfrak{B}(\hat{\mathfrak{z}}) \geq j^* - j/2$. So, we have

$$\mathfrak{B}(\widehat{\mathfrak{z}}) + \tilde{\varsigma}\wedge\tilde{\imath} > \frac{j^* - j}{2} + \frac{j^* - j}{2} = j^* - j. \tag{7}$$

It follows that $\widehat{\mathfrak{z}}_{\tilde{\varsigma}\wedge\tilde{\imath}}(j^*, q_j)\mathfrak{B}$. Therefore, $\widehat{\mathfrak{z}}_{\tilde{\varsigma}\wedge\tilde{\imath}} \in \vee(j^*, q_j)\mathfrak{B}$, as needed. □

On combining Lemmas 4 and 16, we get the following result.

Theorem 17. *A FS $\mathfrak{B}$ of $\widehat{Y}$ is an $(\in, \in\vee(j^*, q_j))$-FPI of $\widehat{Y} \Longleftrightarrow$*

(1) $\mathfrak{B}(0) \geq \mathfrak{B}(\widehat{\mathfrak{z}})\wedge j^* - j/2$

(2) $\mathfrak{B}(\widehat{\mathfrak{z}}) \geq \mathfrak{B}((\widehat{\mathfrak{z}} * \widehat{\mathfrak{v}}) * (\widehat{\omega} * \widehat{\mathfrak{v}}))\wedge\mathfrak{B}(\widehat{\omega})\wedge j^* - j/2$,

$\forall \widehat{\mathfrak{z}}, \widehat{\mathfrak{v}}, \widehat{\omega} \in \widehat{Y}$.

Theorem 18. *Every $(\in, \in\vee(j^*, q_j))$-FPI of $\widehat{Y}$ is an $(\in, \in\vee(j^*, q_j))$-FI of $\widehat{Y}$.*

Proof. Assume that $\mathfrak{B}$ is an $(\in, \in\vee(j^*, q_j))$-FPI of $\widehat{Y}$. Then, $\forall \widehat{\mathfrak{z}}, \widehat{\mathfrak{v}}, \widehat{\omega} \in \widehat{Y}$; we have

$$\mathfrak{B}(\widehat{\mathfrak{z}}) \geq \mathfrak{B}\left(\left(\widehat{\mathfrak{z}} * \widehat{\mathfrak{v}}\right) * \left(\widehat{\omega} * \widehat{\mathfrak{v}}\right)\right)\wedge\mathfrak{B}\left(\widehat{\omega}\right)\wedge\frac{j^* - j}{2}. \tag{8}$$

Substitute $\widehat{\mathfrak{v}}$ by 0 in above inequality, so

$$\begin{aligned}\mathfrak{B}(\widehat{\mathfrak{z}}) &\geq \mathfrak{B}\left(\left(\widehat{\mathfrak{z}} * 0\right) * \left(\widehat{\omega} * 0\right)\right)\wedge\mathfrak{B}\left(\widehat{\omega}\right)\wedge\frac{j^* - j}{2} \\ &= \mathfrak{B}\left(\widehat{\mathfrak{z}} * \widehat{\omega}\right)\wedge\mathfrak{B}\left(\widehat{\omega}\right)\wedge\frac{j^* - j}{2}.\end{aligned} \tag{9}$$

Hence, $\mathfrak{B}$ is an $(\in, \in\vee(j^*, q_j))$-FI. □

Example 19. Take a BCI-algebra of Example 12 defined by Table 2. We define a FS

$$\mathfrak{B}\left(\widehat{\mathfrak{v}}\right) = \begin{cases} 0.4, & \text{if } \widehat{\mathfrak{v}} = 0 \\ 0.1, & \text{if } \widehat{\mathfrak{v}} \in \left\{\widehat{\imath}, \widehat{\mathfrak{z}}, \widehat{\omega}\right\} \\ 0.2, & \text{if } \widehat{\mathfrak{v}} = \widehat{\ell} \end{cases}. \tag{10}$$

It is easy to calculate that $\mathfrak{B}$ is an $(\in, \in\vee(j^*, q_j))$-FI of $\widehat{Y}$ for $j^* = 0.9$ and $j = 0$; however, it is not an $(\in, \in\vee(j^*, q_j))$-FPI as $0.3 = \mathfrak{B}(\widehat{\ell}) \ngeq \mathfrak{B}((\widehat{\ell} * \widehat{\omega}) * (0 * \widehat{\omega}))\wedge\mathfrak{B}(0)\wedge j^* - j/2 = \mathfrak{B}(0) = 0.6$.

Theorem 20. *If B is an $(\in, \in\vee(j^*, q_j))$-FPI, then*

(1) $\mathfrak{B}(\widehat{\mathfrak{z}}) \geq \mathfrak{B}(0 * (0 * \widehat{\mathfrak{z}}))\wedge j^* - j/2$, $\forall \widehat{\mathfrak{z}} \in \widehat{Y}$

(2) $\mathfrak{B}(\widehat{\mathfrak{z}}) = \mathfrak{B}(0 * (0 * \widehat{\mathfrak{z}}))\wedge j^* - j/2$, $\forall \widehat{\mathfrak{z}} \in \widehat{Y}$

Proof.

(1) Assume that $\mathfrak{B}$ is an $(\in, \in\vee(j^*, q_j))$-FPI of $\widehat{Y}$. So,

$$\mathfrak{B}(\widehat{\mathfrak{z}}) \geq \mathfrak{B}\left(\left(\widehat{\mathfrak{z}} * \widehat{\mathfrak{v}}\right) * \left(\widehat{\omega} * \widehat{\mathfrak{v}}\right)\right)\wedge\mathfrak{B}\left(\widehat{\omega}\right)\wedge\frac{j^* - j}{2}, \forall \widehat{\mathfrak{z}}, \widehat{\mathfrak{v}}, \widehat{\omega} \in \widehat{Y}. \tag{11}$$

Substitute $\widehat{\mathfrak{z}}$ for $\widehat{\mathfrak{v}}$ and 0 for $\widehat{\omega}$, so

$$\begin{aligned}\mathfrak{B}(\widehat{\mathfrak{z}}) &\geq \mathfrak{B}((\widehat{\mathfrak{z}} * \widehat{\mathfrak{z}}) * (0 * \widehat{\mathfrak{z}}))\wedge\mathfrak{B}(0)\wedge\frac{j^* - j}{2} \\ &= \mathfrak{B}(0 * (0 * \widehat{\mathfrak{z}}))\wedge\mathfrak{B}(0)\wedge\frac{j^* - j}{2} \\ &= \mathfrak{B}(0 * (0 * \widehat{\mathfrak{z}}))\wedge\frac{j^* - j}{2}.\end{aligned} \tag{12}$$

(2) Since $0 * (0 * \widehat{\mathfrak{z}}) \leq z$ for any $z \in \widehat{Y}$. So, by Theorem 18 and Lemma 5, we have

$$\mathfrak{B}(0 * (0 * \widehat{\mathfrak{z}})) \geq \mathfrak{B}(\widehat{\mathfrak{z}})\wedge\frac{j^* - j}{2}. \tag{13}$$

From (a), $\mathfrak{B}(\widehat{\mathfrak{z}}) \geq \mathfrak{B}(0 * (0 * \widehat{\mathfrak{z}}))\wedge j^* - j/2$. Thus,

$$\mathfrak{B}(\widehat{\mathfrak{z}}) = \mathfrak{B}(0 * (0 * \widehat{\mathfrak{z}}))\wedge\frac{j^* - j}{2}. \tag{14}$$

□

A condition for $(\in, \in\vee(j^*, q_j))$-FI to be $(\in, \in\vee(j^*, q_j))$-FPI is given in the following two results.

Theorem 21. *Let $\mathfrak{B}$ be an $(\in, \in\vee(j^*, q_j))$-FI of $\widehat{Y}$ satisfying*

$$\left(\forall \widehat{\mathfrak{z}}, \widehat{\mathfrak{v}}, \widehat{\omega} \in \widehat{Y}\right)\mathfrak{B}\left(\widehat{\mathfrak{z}} * \widehat{\omega}\right) \geq \mathfrak{B}\left(\left(\widehat{\mathfrak{z}} * \widehat{\mathfrak{v}}\right) * \left(\widehat{\omega} * \widehat{\mathfrak{v}}\right)\right). \tag{15}$$

Then, $\mathfrak{B}$ is an "$(\in, \in\vee(j^*, q_j))$-FPI" of $\widehat{Y}$.

Proof. Suppose that $\mathfrak{B}$ is an $(\in, \in\vee(j^*, q_j))$-FI satisfying (15). Then, $\forall \widehat{\mathfrak{z}}, \widehat{\mathfrak{v}}, \widehat{\omega} \in \widehat{Y}$, following hold

$$\begin{aligned}\mathfrak{B}(\widehat{\mathfrak{z}}) &\geq \mathfrak{B}\left(\widehat{\mathfrak{z}} * \widehat{\omega}\right)\wedge\mathfrak{B}\left(\widehat{\omega}\right)\wedge\frac{j^* - j}{2} \\ &\geq \mathfrak{B}\left(\left(\widehat{\mathfrak{z}} * \widehat{\mathfrak{v}}\right) * \left(\widehat{\omega} * \widehat{\mathfrak{v}}\right)\right)\wedge\mathfrak{B}\left(\widehat{\omega}\right)\wedge\frac{j^* - j}{2}\end{aligned} \tag{16}$$

Hence, $\mathfrak{B}$ is an $(\in, \in\vee(j^*, q_j))$-FPI of $\widehat{Y}$. □

Theorem 22. *Let $\mathfrak{B}$ be an $(\in, \in\vee(j^*, q_j))$-FI of $\widehat{Y}$ satisfying*

$$\left(\forall \widehat{\mathfrak{z}} \in \widehat{Y}\right)\mathfrak{B}(\widehat{\mathfrak{z}}) \geq \mathfrak{B}(0 * (0 * \widehat{\mathfrak{z}}))\wedge\frac{j^* - j}{2}. \tag{17}$$

Then, B is an $(\in, \in\vee(j^*, q_j))$-FPI of $\widehat{Y}$.

Proof. Let $\mathfrak{z}, \widehat{\omega}, \widehat{\mathfrak{v}} \in \widehat{Y}$. Then, by ((17)) and Theorem 20

$$\begin{aligned}\mathfrak{B}(\widehat{\mathfrak{z}} * \widehat{\omega}) &\geq \mathfrak{B}(0 * (0 * (\widehat{\mathfrak{z}} * \widehat{\omega}))) \wedge \frac{j^* - j}{2} \\ &= \mathfrak{B}((0 * \widehat{\omega}) * (0 * \widehat{\mathfrak{z}})) \wedge \frac{j^* - j}{2} \\ &= \mathfrak{B}(0 * (0 * ((\widehat{\mathfrak{z}} * v) * (\widehat{\omega} * v)))) \wedge \frac{j^* - j}{2} \\ &= \mathfrak{B}((\widehat{\mathfrak{z}} * \widehat{\mathfrak{v}}) * (\widehat{\omega} * \widehat{\mathfrak{v}})) \wedge \frac{j^* - j}{2}.\end{aligned} \tag{18}$$

Hence from Theorem 21, $\mathfrak{B}$ is an $(\in, \in \vee(j^*, q_j))$-FPI of $\widehat{Y}$. □

Theorem 23. *A FS $\mathfrak{B}$ is an " $(\in, \in \vee(j^*, q_j))$-FPI" in $\widehat{Y} \Longleftrightarrow$; the set $\mathfrak{B}_{\tilde{\varsigma}}(\neq \varnothing)$ is a p-ideal of $\widehat{Y}$, $\forall \tilde{\varsigma} \in (0, (j^* - j/2)]$.*

Proof. ($\Longrightarrow$) Take $\tilde{\varsigma} \in (0, j^* - j/2]$ such that $\mathfrak{B}_{\tilde{\varsigma}} \neq \varnothing$. From Theorem 17, we have

$$\mathfrak{B}(0) \geq \mathfrak{B}(\widehat{\mathfrak{z}}) \wedge \frac{j^* - j}{2}, \tag{19}$$

with $\widehat{\mathfrak{z}} \in \mathfrak{B}_{\tilde{\varsigma}}$. Thus, $\mathfrak{B}(0) \geq \tilde{\varsigma} \wedge j^* - j/2 = \tilde{\varsigma}$. Therefore, $0 \in \mathfrak{B}_{\tilde{\varsigma}}$.

Next, assume that $(\widehat{\mathfrak{z}} * \widehat{\mathfrak{v}}) * (\widehat{\omega} * \widehat{\mathfrak{v}}) \in \mathfrak{B}_{\tilde{\varsigma}}$ and $\widehat{\omega} \in B_{\tilde{\varsigma}}$. Then, $\mathfrak{B}((\widehat{\mathfrak{z}} * \widehat{\mathfrak{v}}) * (\widehat{\omega} * \widehat{\mathfrak{v}})) \geq \tilde{\varsigma}$ and $\mathfrak{B}(\widehat{\omega}) \geq \tilde{\varsigma}$. Again, by Theorem 17, we have

$$\mathfrak{B}(\widehat{\mathfrak{z}}) \geq \mathfrak{B}((\widehat{\mathfrak{z}} * \widehat{\mathfrak{v}}) * (\widehat{\omega} * \widehat{\mathfrak{v}})) \wedge \mathfrak{B}(\widehat{\omega}) \wedge \frac{j^* - j}{2} \geq \tilde{\varsigma} \wedge \tilde{\varsigma} \wedge \frac{j^* - j}{2} = \tilde{\varsigma}. \tag{20}$$

So, $z \in \mathfrak{B}_{\tilde{\varsigma}}$. Consequently, $\mathfrak{B}_{\tilde{\varsigma}}$ is p-ideal of $\widehat{Y}$.

($\Longleftarrow$) Suppose that $\mathfrak{B}_{\tilde{\varsigma}}$ is a p-ideal of $\widehat{Y}$, $\forall \tilde{\varsigma} \in (0, j^* - k/2]$. If for some $\widehat{\mathfrak{z}} \in \widehat{Y}$, $\mathfrak{B}(0) < \mathfrak{B}(\widehat{\mathfrak{z}}) \wedge j^* - j/2$. Then $\exists \tilde{\varsigma} \in (0, j^* - k/2]$ "such that" $\mathfrak{B}(0) < \tilde{\varsigma} \leq \mathfrak{B}(\widehat{\mathfrak{z}}) \wedge j^* - j/2$. It implies that $\widehat{\mathfrak{z}} \in \mathfrak{B}_{\tilde{\varsigma}}$but $0 \in \mathfrak{B}_{\tilde{\varsigma}}$, a contradiction. So, $\mathfrak{B}(0) \geq \mathfrak{B}(\widehat{\mathfrak{z}}) \wedge j^* - j/2$. Also, if for some $\widehat{\mathfrak{z}}, \widehat{\mathfrak{v}}, \widehat{\omega} \in \widehat{Y}$, $\mathfrak{B}(\widehat{\mathfrak{z}}) < \mathfrak{B}((\widehat{\mathfrak{z}} * \widehat{\mathfrak{v}}) * (\widehat{\omega} * \widehat{\mathfrak{v}})) \wedge \mathfrak{B}(\widehat{\omega}) \wedge j^* - j/2$. Then $\exists \tilde{\varsigma} \in (0, (j^* - k/2)]$ s.t.

$$\mathfrak{B}(\widehat{\mathfrak{z}}) < \tilde{\varsigma} \leq \mathfrak{B}((\widehat{\mathfrak{z}} * \widehat{\mathfrak{v}}) * (\widehat{\omega} * \widehat{\mathfrak{v}})) \wedge \mathfrak{B}(\widehat{\omega}) \wedge \frac{j^* - j}{2}. \tag{21}$$

It implies that $(\widehat{\mathfrak{z}} * \widehat{\mathfrak{v}}) * (\widehat{\omega} * \widehat{\mathfrak{v}}) \in \mathfrak{B}_{\tilde{\varsigma}}$ and $\widehat{\omega} \in \mathfrak{B}_{\tilde{\varsigma}}$ but $\widehat{\mathfrak{z}} \notin B_{\tilde{\varsigma}}$, another contradiction. Hence, $\mathfrak{B}(\widehat{\mathfrak{z}}) \geq \mathfrak{B}((\widehat{\mathfrak{z}} * \widehat{\mathfrak{v}}) * (\widehat{\omega} * \widehat{\mathfrak{v}})) \wedge \mathfrak{B}(\widehat{\omega}) \wedge j^* - j/2$, as required. □

Definition 24. Let $\mathfrak{B}$ be a FS of $\widehat{Y}$. The set

$$\widetilde{[\mathfrak{B}]}_{\tilde{\varsigma}} = \left\{\widehat{\mathfrak{z}} \in \widehat{Y} \mid \widehat{\mathfrak{z}}_{\tilde{\varsigma}} \in \vee\left(j^*, q_j\right)\mathfrak{B}\right\}, \text{where } \tilde{\varsigma} \in (0, 1], \tag{22}$$

is said to be an $(\in \vee(j^*, q_j))$-level subset of $\mathfrak{B}$.

Theorem 25. *A FS $\mathfrak{B}$ of $\widehat{Y}$ is an $(\in, \in \vee(j^*, q_j))$-FPI of $\widehat{Y} \Longleftrightarrow$ the $(\in \vee(j^*, q_j))$-level subset $\widetilde{[\mathfrak{B}]}_{\tilde{\varsigma}}$ of $\mathfrak{B}$ is a p-ideal of $\widehat{Y}$, $\forall \tilde{\varsigma} \in (0, 1]$.*

Proof. ($\Longrightarrow$) Assume that $\mathfrak{B}$ is an $(\in, \in \vee(j^*, q_j))$-FPI of $\widehat{Y}$. Take any $\widehat{\mathfrak{z}} \in \widetilde{[\mathfrak{B}]}_{\tilde{\varsigma}}$. Then, $\widehat{\mathfrak{z}}_{\tilde{\varsigma}} \in \vee(j^*, q_j)\mathfrak{B}$. So, $\mathfrak{B}(\widehat{\mathfrak{z}}) \geq \tilde{\varsigma}$ or $\mathfrak{B}(\widehat{\mathfrak{z}}) + \tilde{\varsigma} > j^* - j$. By Theorem 17, $\mathfrak{B}(0) \geq \mathfrak{B}(\widehat{\mathfrak{z}}) \wedge j^* - j/2$. Thus $\mathfrak{B}(0) \geq \tilde{\varsigma} \wedge j^* - j/2$ when $\mathfrak{B}(\widehat{\mathfrak{z}}) \geq \tilde{\varsigma}$. If $u > j^* - j/2$, then $\mathfrak{B}(0) \geq j^* - j/2$ yields $0 \in \widetilde{[\mathfrak{B}]}_{\tilde{\varsigma}}$. Also, if $\tilde{\varsigma} \leq j^* - j/2$, then $\mathfrak{B}(0) \geq \tilde{\varsigma}$ implies $0 \in \widetilde{[\mathfrak{B}]}_{\tilde{\varsigma}}$. Similary $0 \in \widetilde{[\mathfrak{B}]}_{\tilde{\varsigma}}$ when $\mathfrak{B}(\widehat{\mathfrak{z}}) + \tilde{\varsigma} > j^* - j$.

Next, take any $((\widehat{\mathfrak{z}} * \widehat{\mathfrak{v}}) * (\widehat{\omega} * \widehat{\mathfrak{v}})) \in \widetilde{[\mathfrak{B}]}_{\tilde{\varsigma}}$ and $\widehat{\omega} \in \widetilde{[\mathfrak{B}]}_{\tilde{\varsigma}}$. Then, $((\widehat{\mathfrak{z}} * \widehat{\mathfrak{v}}) * (\widehat{\omega} * \widehat{\mathfrak{v}})) \in \vee(j^*, q_{\tilde{j}})\mathfrak{B}$ and $\widehat{\omega} \in \vee(j^*, q_{\tilde{j}})\mathfrak{B}$, i.e., either $\mathfrak{B}((\widehat{\mathfrak{z}} * \widehat{\mathfrak{v}}) * (\widehat{\omega} * \widehat{\mathfrak{v}})) \geq \tilde{\varsigma}$ or $\mathfrak{B}((\widehat{\mathfrak{z}} * \widehat{\mathfrak{v}}) * (\widehat{\omega} * \widehat{\mathfrak{v}})) + \tilde{\varsigma} > j^* - j$ and either $\mathfrak{B}(\widehat{\omega}) \geq \tilde{\varsigma}$ or $\mathfrak{B}(\widehat{\omega}) + \tilde{\varsigma} > j^* - j$. By assumption, $\mathfrak{B}(\widehat{\mathfrak{z}}) \geq \mathfrak{B}((\widehat{\mathfrak{z}} * \widehat{\mathfrak{v}}) * (\widehat{\omega} * \widehat{\mathfrak{v}})) \wedge \mathfrak{B}(\widehat{\omega}) \wedge j^* - j/2$. We have cases:

Case (i). Let $\mathfrak{B}((\widehat{\mathfrak{z}} * \widehat{\mathfrak{v}}) * (\widehat{\omega} * \widehat{\mathfrak{v}})) \geq \tilde{\varsigma}$ and $\mathfrak{B}(\widehat{\omega}) \geq \tilde{\varsigma}$. If $u > j^* - j/2$, then

$$\mathfrak{B}(\widehat{\mathfrak{z}}) \geq \mathfrak{B}((\widehat{\mathfrak{z}} * \widehat{\mathfrak{v}}) * (\widehat{\omega} * \widehat{\mathfrak{v}})) \wedge \mathfrak{B}(\widehat{\omega}) \wedge \frac{j^* - j}{2} \geq \tilde{\varsigma} \wedge \frac{j^* - j}{2} = \frac{j^* - j}{2}, \tag{23}$$

and so, $\widehat{\mathfrak{z}}_{\tilde{\varsigma}} \in (j^*, q_j)\mathfrak{B}$. If $\tilde{\varsigma} \leq j^* - j/2$, then

$$\mathfrak{B}(\widehat{\mathfrak{z}}) \geq \mathfrak{B}((\widehat{\mathfrak{z}} * \widehat{\mathfrak{v}}) * (\widehat{\omega} * \widehat{\mathfrak{v}})) \wedge \mathfrak{B}(\widehat{\omega}) \wedge \frac{j^* - j}{2} \geq \tilde{\varsigma} \wedge \frac{j^* - j}{2} = \tilde{\varsigma}. \tag{24}$$

So $\widehat{\mathfrak{z}}_{\tilde{\varsigma}} \in \mathfrak{B}$. Hence, $z_{\tilde{\varsigma}} \in \vee(j^*, q_j)\mathfrak{B}$.

Case (ii). Let $\mathfrak{B}((\widehat{\mathfrak{z}} * \widehat{\mathfrak{v}}) * (\widehat{\omega} * \widehat{\mathfrak{v}})) \geq \tilde{\varsigma}$ and $\mathfrak{B}(\widehat{\omega}) + \tilde{\varsigma} \geq j^* - j$. If $u > j^* - j/2$, then

$$\begin{aligned}\mathfrak{B}(\widehat{\mathfrak{z}}) &\geq \mathfrak{B}((\widehat{\mathfrak{z}} * \widehat{\mathfrak{v}}) * (\widehat{\omega} * \widehat{\mathfrak{v}})) \wedge \mathfrak{B}(\widehat{\omega}) \wedge \frac{j^* - j}{2} \\ &\geq \tilde{\varsigma} \wedge j^* - j - \tilde{\varsigma} \wedge \frac{j^* - j}{2} = j^* - j - \tilde{\varsigma},\end{aligned} \tag{25}$$

i.e., $\mathfrak{B}(\widehat{\mathfrak{z}}) + \tilde{\varsigma} > j^* - j$ and, thus, $\widehat{\mathfrak{z}}_{\tilde{\varsigma}}(j^*, q_j)\mathfrak{B}$. If $\tilde{\varsigma} \leq j^* - j/2$. then

$$\begin{aligned}\mathfrak{B}(\widehat{\mathfrak{z}}) &\geq \mathfrak{B}((\widehat{\mathfrak{z}} * \widehat{\mathfrak{v}}) * (\widehat{\omega} * \widehat{\mathfrak{v}})) \wedge \mathfrak{B}(\widehat{\omega}) \wedge \frac{j^* - j}{2} \\ &\geq \tilde{\varsigma} \wedge \kappa^* - j - \tilde{\varsigma} \wedge \frac{j^* - j}{2} = \tilde{\varsigma},\end{aligned} \tag{26}$$

and so $\widehat{\mathfrak{z}}_{\tilde{\varsigma}} \in \mathfrak{B}$. Hence, $\widehat{\mathfrak{z}}_{\tilde{\varsigma}} \in \vee(j^*, q_j)\mathfrak{B}$.

Likewise, in the other two cases, i.e., when $\mathfrak{B}((\widehat{\mathfrak{z}} * \widehat{\mathfrak{v}}) * (\widehat{\omega} * \widehat{\mathfrak{v}})) + \tilde{\varsigma} > j^* - j$, $\mathfrak{B}(\widehat{\omega}) \geq \tilde{\varsigma}$ and $\mathfrak{B}((\widehat{\mathfrak{z}} * \widehat{\mathfrak{v}}) * (\widehat{\omega} * \widehat{\mathfrak{v}})) + \tilde{\varsigma} > j^* - j$, $\mathfrak{B}(\widehat{\omega}) + \tilde{\varsigma} > j^* - j$ implying that $\widehat{\mathfrak{z}}_{\tilde{\varsigma}} \in \vee(j^*, q_j)\mathfrak{B}$. Hence, in each case, $\widehat{\mathfrak{z}}_{\tilde{\varsigma}} \in \vee(j^*, q_j)\mathfrak{B}$, and thus $z \in \widetilde{[\mathfrak{B}]}_{\tilde{\varsigma}}$.

($\Longleftarrow$) Let $\widetilde{[\mathfrak{B}]}_{\tilde{\varsigma}}$ be "p-ideal" for all $\tilde{\varsigma} \in (0, 1]$. Contrary suppose that

$$\mathfrak{B}(0) < \mathfrak{B}(\hat{\mathfrak{z}}) \wedge \frac{j^* - j}{2}, \tag{27}$$

with $\hat{\mathfrak{z}} \in \mathfrak{B}_{\tilde{\varsigma}}$. Then, $\exists \tilde{\varsigma} \in (0, 1]$ "such that" $\mathfrak{B}(0) < \tilde{\varsigma} \leq \mathfrak{B}(\hat{\mathfrak{z}}) \wedge j^* - j/2$. It yields that $\hat{\mathfrak{z}} \in \widetilde{[\mathfrak{B}]}_{\tilde{\varsigma}}$, but $0 \notin \widetilde{[\mathfrak{B}]}_{\tilde{\varsigma}}$, which contradicts itself. Therefore

$$\mathfrak{B}(0) \geq \mathfrak{B}(\hat{\mathfrak{z}}) \wedge \frac{j^* - j}{2}. \tag{28}$$

Also, if $\mathfrak{B}(\hat{\mathfrak{z}}) < \mathfrak{B}((\hat{\mathfrak{z}} * \hat{\mathfrak{v}}) * (\widehat{\omega} * \hat{\mathfrak{v}})) \wedge \mathfrak{B}(\widehat{\omega}) \wedge j^* - j/2$ for some $\hat{\mathfrak{z}}, \hat{\mathfrak{v}}, \widehat{\omega} \in \widehat{Y}$. Then, $\exists \tilde{\varsigma} \in (0, 1]$ such that

$$\mathfrak{B}(\hat{\mathfrak{z}}) < \tilde{\varsigma} \leq \mathfrak{B}\left(\left(\hat{\mathfrak{z}} * \hat{\mathfrak{v}}\right) * \left(\widehat{\omega} * \hat{\mathfrak{v}}\right)\right) \wedge \mathfrak{B}\left(\widehat{\omega}\right) \wedge \frac{j^* - j}{2}. \tag{29}$$

Henceforth, $((\hat{\mathfrak{z}} * \hat{\mathfrak{v}}) * (\widehat{\omega} * \hat{\mathfrak{v}})) \in \widetilde{[\mathfrak{B}]}_{\tilde{\varsigma}}$ and $\widehat{\omega} \in \widetilde{[\mathfrak{B}]}_{\tilde{\varsigma}}$ but $\hat{\mathfrak{z}} \in \widetilde{[\mathfrak{B}]}_{\tilde{\varsigma}}$, another contradiction. Therefore, $\mathfrak{B}(\hat{\mathfrak{z}}) \geq \mathfrak{B}((\hat{\mathfrak{z}} * \hat{\mathfrak{v}}) * (\widehat{\omega} * \hat{\mathfrak{v}})) \wedge \mathfrak{B}(\widehat{\omega}) \wedge j^* - j/2$, as required. □

Data Availability

No underlying data was collected or produced in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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