Hindawi
Journal of Function Spaces
Volume 2023, Article ID 9748605, 10 pages
https://doi.org/10.1155/2023/9748605

Research Article

Method of Particular Solutions for Second-Order Differential Equation with Variable Coefficients via Orthogonal Polynomials

Huantian Xie, Zhaozhong Zhang, Ziwu Jiang, and Jianwei Zhou

School of Mathematics and Statistics, Linyi University, Linyi, 276000 Shandong, China

Correspondence should be addressed to Jianwei Zhou; jwzhou@yahoo.com

Received 12 July 2022; Revised 7 September 2022; Accepted 13 September 2022; Published 26 April 2023

Academic Editor: Yusuf Gurefe

In this paper, with classic Legendre polynomials, a method of particular solutions (MPS, for short) is proposed to solve a kind of second-order differential equations with a variable coefficient on a unit interval. The particular solutions, satisfying the natural Dirichlet boundary conditions, are constructed with orthogonal Legendre polynomials for the variable coefficient case. Meanwhile, we investigate the a-priori error estimates of the MPS approximations. Two a-priori error estimations in H^1- and L^∞-norms are shown to depict the convergence order of numerical approximations, respectively. Some numerical examples and convergence rates are provided to validate the merits of our proposed meshless method.

1. Introduction

In the past decades, various numerical methods are designed for solving kinds of differential equations, such as finite element method [1–3], spectral method [4–6], shifted Legendre approximation [7, 8], and differential transformation method [9, 10]. To avoid the constraints and workload of region divisions, a new family of computational methods has emerged. The so-called meshless or mesh-free methods have been investigated and used by many researchers. The advantage of meshless methods reads that the interpolation accuracy is not significantly affected by the nodal distribution. And hence meshless methods attract great attentions in various disciplines for treating a large variety of engineering problems. In fact, the MPS is originally proposed with the radial basis functions for solving various kinds of differential equations. Recently, the MPS has been continuously employed to solve various interesting models and proven to be an effective method in numerical simulations. For more details about this numerical scheme, please refer to [11–13] and the references cited therein.

To the best of our current knowledge, the meshless schemes, including Kansa method [14], method of fundamental solutions [15], method of particular solutions [16, 17], element-free Galerkin method [18], local point interpolation [19], and boundary knot method [20], are widely used to approximate a large class of partial differential equations in science and engineering fields. As reported in the literatures, the MPS has been applied to solve the Navier-Stokes problem [21], wave propagation problem [22], and time-fractional diffusion problem [23]. Despite the effectiveness of the MPS, there are some disadvantages such as the ill-conditioned collocation matrix, the uncertainty of the shape parameters, and difficulties in deriving the closed-form particular solutions for general differential operators, and for more details, please refer to [12, 17, 24–26] and the references cited therein.

In order to overcome these disadvantages, lots of works have been done on efficient numerical schemes for the MPS. And many basis functions have been designed to discretize partial differential equations. Chebyshev polynomials [11, 27], polynomials basis functions [16, 18, 28], and trigonometric functions [29] were employed with their closed-form particular solutions to approximate kinds of models. However, few results about error estimates of the MPS are illustrated in the current literatures.

In this paper, Legendre polynomials are used to design the particular solutions for the MPS. Specially, boundary conditions are naturally imposed, and the corresponding discretized scheme is constructed in a collocation scheme.

The closed-form particular solutions for given differential operators with variable coefficients are derived via recursive relationships of Legendre polynomials. Compared with the radial basis functions for the MPS, our proposed scheme provides a simple approach to effectively solve a kind of differential equations with variable coefficients.

Meanwhile, with an orthogonal projector and the Aubin-Nitsche duality argument, we provide rigorous studies on two a-priori error estimates for this numerical method. For sufficiently smooth solutions, the a-priori error estimations show that asymptotic super-exponential convergence orders of the MPS approximations are readily achieved in H^1- and L^∞-norms.

The remainder of this paper is organized as follows. Some preliminaries and a brief review of the MPS are presented in Section 2. The numerical procedures of the MPS for solving differential equations with variable coefficients are proposed in Section 3. In Section 4, two a-priori error estimates are given in different norms with rigorous proofs. And three numerical examples are provided with numerical errors and convergence orders to demonstrate the effectiveness of the proposed methods in Section 5. Furthermore, some conclusions and discussions are listed in Section 6. And in the last part, an appendix is given to sketch a rigorous proof for the recalled lemma.

2. Preliminaries

Let us introduce some basic notations which will be used in the sequel. Hereafter, we select a unit interval $I = (-1, 1)$ to show the sketch of the MPS approximations and a-priori error estimates and adopt the standard notation $W^{m,q}(I)$ for Sobolev space on I. Setting $W_0^{m,q}(I) = \{v \in W^{m,q}(I): (d^k v/dx^k)(\pm 1) = 0, 0 \le k \le m-1\}$, we denote $H_0^m(I) = W_0^{m,2}(I)$ and $\|\cdot\|_m = \|\cdot\|_{m,2}$. Specially, $\|\cdot\|_\infty$ and $\|\cdot\|$ denote the norms in $L^\infty(I)$ and $L^2(I)$, respectively. We use C and c to denote different constants in different formulae. For simplicity, we omit subscripts if $m = 0$. Particularly, if $m = 1$, we set

$$H_0^1(I) = \left\{v \in W^{1,2}(I): v(\pm 1) = 0\right\}. \tag{1}$$

Thereby, the scalar product in $L^2(I)$ and bilinear form in $H^1(I)$ are defined as

$$(v, w) = \int_I v(x)w(x)dx, \forall v, w \in L^2(I), \tag{2}$$

$$a(v, w) = \int_I \omega(x)v'(x)w'(x)dx, \forall v, w \in H^1(I). \tag{3}$$

We define the following polynomial sets:

$$\begin{aligned} \tilde{P}_N &= \{p_N(x): \text{the degree of } p_N(x) \le N\}, \\ P_N &= \left\{v \in \tilde{P}_N : v(\pm 1) = 0\right\}. \end{aligned} \tag{4}$$

2.1. Legendre Polynomials. We denote by $L_i(x)$ the i-th degree Legendre polynomial with $x \in I$. Three-term recurrence relationship for Legendre polynomials reads

$$(i+1)L_{i+1}(x) = (2i+1)xL_i(x) - iL_{i-1}(x), i \ge 1, \tag{5}$$

and $L_0(x) = 1, L_1(x) = x$.

We recall that $\{L_i(x)\}_{i\ge 1}$ satisfy

$$L_i(\pm 1) = (\pm 1)^i, i \ge 1, \tag{6}$$

and hence there holds

$$L_i(x) - L_{i+2}(x) \in P_N(x), 0 \le i \le N-2. \tag{7}$$

Also, there is an orthogonality

$$\left(L_i(x), L_j(x)\right) = \begin{cases} 0, i \ne j, \\ \dfrac{2}{2i+1}, i = j. \end{cases} \tag{8}$$

And for $i \le N$, it is obvious that $L'_i(x) \in \tilde{P}_{N-1}$ and

$$(2i+1)L_i(x) = L'_{i+1}(x) - L'_{i-1}(x), i \ge 1. \tag{9}$$

2.2. The Method of Particular Solutions. In this subsection, we consider the second-order differential equation with homogeneous Dirichlet boundary condition:

$$\begin{cases} \left(\omega(x)(u(x))'\right)' = f(x), x \in I, \\ u(\pm 1) = 0, \end{cases} \tag{10}$$

and the constraint on $\omega(x)$ will be stated in the sequel.

By (3), we obtain the equivalent weak formulation of (10) reads: finding $u \in H_0^1(I)$ such that

$$a(u, v) = -(f, v), \forall v \in H_0^1(I). \tag{11}$$

In view of (9), we design the corresponding particular solutions for (10) as

$$\psi_i(x) = \frac{L_{i+1}(x) - L_{i-1}(x)}{2i+1}, i \ge 1, \tag{12}$$

which guarantee $\psi_i(\pm 1) = 0$.

And then we define $\mathscr{P}_N$ as

$$\mathscr{P}_N = \text{span}\{\psi_1(x), \psi_2(x), \cdots, \psi_{N-1}(x)\}, \tag{13}$$

where $\psi_i(x)$ satisfies the homogeneous Dirichlet boundary conditions in (10). For more details about the completeness of $\mathscr{P}_N$ in (13), please refer to [30].

According to (13), the MPS approximation of $u(x)$ can be stated as

$$u_N(x) = \sum_{j=1}^{N-1} c_j \psi_j(x), \forall x \in I, \tag{14}$$

where $\{c_j\}_{i=1}^{N-1}$ are the coefficients to be determined. For the sake of convenience, we use $\{x_k\}_{k=1}^{M}$ to denote the collocations in the interval. And then the corresponding equivalent collocation scheme for (10) reads: finding $u_N \in \mathscr{P}_N$ such that

$$\left(\left(\omega(x_k)(u_N(x_k))'\right)', v_k\right) = (f(x_k), v_k), k = 1, 2, \cdots, M, \tag{15}$$

where $v_k = \delta(x - x_k)$ denotes the Dirac delta distribution on x_k. For readers interested in the collocation approximations, please refer to [31].

3. The Model Problem and Its Approximation Scheme

3.1. The Model Problem with $\omega(x) = 1 - x^2$. In the following parts, we focus on $\omega(x) = 1 - x^2$. Since there does not exist any positive constant c satisfying $\omega(x) \geq c$ in I, we miss the sufficient conditions for the uniqueness of (10). And hence we have to restate the uniqueness of the solution for (11) with some novel techniques.

Theorem 1. *For $\omega(x) = 1 - x^2$, there exists a unique weak solution $u \in H_0^1(I)$ of (11).*

Proof. For any $v, w \in H^1(I)$, there holds

$$|a(v, w)| \leq \|v\|_1 \|w\|_1, \tag{16}$$

where we used $|\omega(x)| \leq 1$. One directly states the continuation of the bilinear form and also the existence of solutions.

Now we are at the point to investigate the uniqueness of the solution for (10). Obviously, the bilinear form is not elliptic. We have to prove the uniqueness with new techniques. Assuming there exist two solutions u_1 and u_2 satisfying (10), one readily gets that for all $x \in I$, there almost holds

$$\left(\omega(x)u_1'(x)\right)' = \left(\omega(x)u_2'(x)\right)', \tag{17}$$

which means

$$\begin{cases} \left(\left(1 - x^2\right)\mathscr{U}'(x)\right)' = 0, x \in I, \\ \mathscr{U}(\pm 1) = 0, \end{cases} \tag{18}$$

where $\mathscr{U}(x) = u_1(x) - u_2(x)$.

Now, we turn to prove that the solution of boundary value problem (18) is zero. And hence we employ integrations by parts to get the unique solution

$$\mathscr{U}(x) = \frac{c_1}{2} \ln \frac{1 + x}{1 - x} + c_2, a.e. x \in I. \tag{19}$$

Considering the boundary conditions and properties of function $\ln(x)$ at $x = \pm 1$, we easily declare that $c_1 = 0$ and $c_2 = 0$, which means $\mathscr{U}(x) = 0, a.e. x \in I$. Then, we readily depict that $u_1(x) = u_2(x), a.e. x \in I$, which directly verifies the uniqueness of solution of (11). $\square$

3.2. The MPS with Legendre Polynomials. Noticing that, one of the challenges of the MPS is how to derive closed-form particular solutions for given differential operators. Although the particular solutions are not unique, it is always a complicated task to find appropriate particular solutions for given differential operators. In general, finding or designing closed-form particular solutions are nontrivial (for more details on this topic, please refer to [32] and the references therein).

It is well-known that the size of globally dense matrices in the MPS grows with the increase of collocation points and will cause bigger condition numbers of resultant matrices. Hence, the crucial task of the MPS is to choose pertinent $\tilde{P}_N$ such that the basis functions are as simple as possible. According to the recursive relationships of Legendre polynomials, we derive efficient basis functions for corresponding particular solutions bit by bit.

With (9), it is direct to state that

$$\left(\frac{L_{i+1}(x) - L_{i-1}(x)}{2i + 1}\right)' = L_i(x), i \geq 1. \tag{20}$$

And then we have

$$\left(\left(1 - x^2\right)\left(\frac{L_{i+1}(x) - L_{i-1}(x)}{2i + 1}\right)'\right)' = \left(\left(1 - x^2\right)L_i(x)\right)'. \tag{21}$$

Hence, the basis functions for the approximations of the right hand term can be set as

$$\begin{aligned} \phi_i(x) &= \left(\left(1 - x^2\right)L_i(x)\right)' \\ &= \left(1 - x^2\right)L_i'(x) - 2xL_i(x), 1 \leq i \leq N - 1, \end{aligned} \tag{22}$$

which satisfy the following identity

$$\left(\left(1 - x^2\right)(\psi_i(x))'\right)' = \phi_i(x), 1 \leq i \leq N - 1. \tag{23}$$

One readily gets that the discretized formulation of (11) reads: finding $u_N \in \mathscr{P}_N$ such that

$$a(u_N, v_N) = -(f, v_N), \forall v_N \in \mathscr{P}_N. \tag{24}$$

The details about the equivalent weak formulation can be found in [31]. Meanwhile, the existence and uniqueness of the numerical solution in $\mathscr{P}_N$ of (24) can be readily proved by the same techniques given in Theorem 1.

4. *The* A-Priori Error Estimates

In this section, we study *a-priori* error estimates of the MPS approximations by an orthogonal projector. For any $w \in H_0^1(I)$, there holds

$$w(x) = \sum_{i=1}^{\infty} \widehat{w}_i \psi_i(x). \tag{25}$$

In view of orthogonal properties of Legendre polynomials and (20), we get the following identities:

$$\psi_i'(x) = L_i(x), i \geq 1. \tag{26}$$

We recall the first derivative orthogonal projector $1\Pi_0^N : H_0^1(I) \mapsto \mathscr{P}_N$ such that

$$\left(\left(w - 1\Pi_0^N w \right)', v'_N \right) = 0, v_N \in \mathscr{P}_N. \tag{27}$$

Here, an error estimate for this first derivative orthogonal projector is shown in the following lemma.

Lemma 2 (see [4, 33]). *For all $v \in H_0^1(I) \cap H^m(I) (m \geq 1)$, there holds*

$$\left\| v - 1\Pi_0^N v \right\|_l \leq cN^{l-m} \|v\|_m, l = 0, 1. \tag{28}$$

4.1. The A-Priori Error Estimate in $\boldsymbol{H}^1$-Norm. The Aubin-Nitsche duality argument is employed to investigate error estimates of the MPS approximations in H^1-norm.

Lemma 3 (See [34, 35]). *For bounded interval I and $F \in H^{-1}(I)$, we set y_F as the unique solution of the following homogeneous boundary value problem*

$$\left(y_F', v' \right) = \langle F, v \rangle, \tag{29}$$

where $\langle \cdot, \cdot \rangle$ stands for the dual product on $H^{-1}(I) \times H_0^1(I)$. Then, $y_F \in H^1(I)$, and there holds

$$\|y_F\|_1 \leq c\|F\|_{-1}. \tag{30}$$

By the above results, we derive the following a-priori error estimate.

Theorem 4. *Let u and u_N be the solutions of (11) and (15), respectively. Then for all $u \in H_0^1(I) \cap H^m(I)$, it holds that*

$$\|u - u_N\|_1 \leq CN^{1-m} \|u\|_m. \tag{31}$$

Proof. It follows that

$$\begin{aligned}
\|u - u_N\|_1 &== \sup_{F \in H^{-1}(I)} \frac{|\langle F, u - u_N \rangle|}{\|F\|_{-1}} \\
&\overset{(4.3)}{=} \sup_{F \in H^{-1}(I)} \frac{\left((u - u_N)', y'_F \right)}{\|F\|_{-1}} \\
&= \sup_{F \in H^{-1}(I)} \frac{\left((u - u_N)', \left(y_F - 1\Pi_0^N y_F \right)' \right)}{\|F\|_{-1}} \\
&= \sup_{F \in H^{-1}(I)} \frac{\left(\left(u - 1\Pi_0^{N-1} u \right)', \left(y_F - 1\Pi_0^N y_F \right)' \right)}{\|F\|_{-1}} \\
&\leq \left\| \left(u - 1\Pi_0^N u \right)' \right\| \cdot \sup_{F \in H^{-1}(I)} \frac{\left\| y_F - 1\Pi_0^N y_F \right\|_1}{\|F\|_{-1}} \\
&\overset{(4.2)(4.4)}{\leq} c\left\| \left(u - 1\Pi_0^N u \right)' \right\| \\
&\overset{(4.2)}{\leq} CN^{1-m} \|u\|_m.
\end{aligned} \tag{32}$$

Then, the a-priori error estimation in (31) is yielded. □

4.2. The A-Priori Error Estimate in $\boldsymbol{L}^\infty$-Norm. In this subsection, we give the corresponding error estimate in L^∞-norm with a rigorous relationship during $L^\infty(I)$ and $H^1(I)$.

Lemma 5. *For all $v \in H^1(I)$, there holds the following estimate*

$$\|v\|_\infty^2 \leq \|v\|^2 + 4\left\| v' \right\|^2. \tag{33}$$

Proof. Since the interval is bounded, one gets that $W^{m,2}(I) \subset W^{m,1}(I)$. By the embedding theorems (refer to Chapter 12 in [36]), we know that $W^{1,1}(I)$ is embedded in $L^\infty(I)$. Furthermore, $H^1(I)$ is a subset of $W^{1,1}(I)$ due to the bounded interval I. Hence, $H^1(I)$ is embedded in $L^\infty(I)$. About the constants within the above estimate, please refer to the Theorem 1.9 in [37] for further details. And a theoretical proof is listed in the appendix, which improves the proof given in [38]. □

Theorem 6. *Let u and u_N be the solutions of (11) and (15), respectively. Then, for all $u \in H_0^1(I) \cap H^m(I)$, there holds*

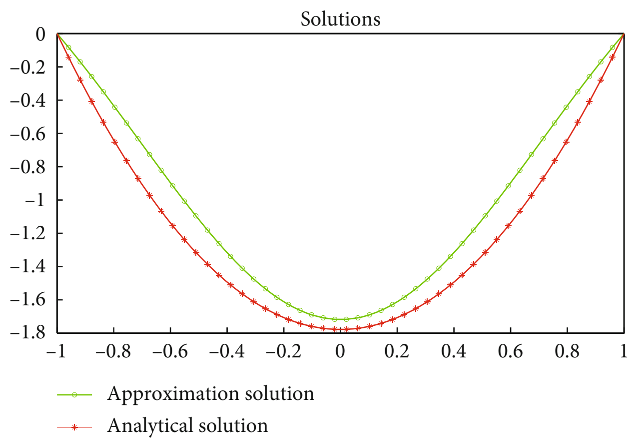
$$\|u - u_N\|_\infty \leq CN^{1-m} \|u\|_m. \tag{34}$$

Proof. It is clear that $u - u_N \in H^1(I)$. Then,

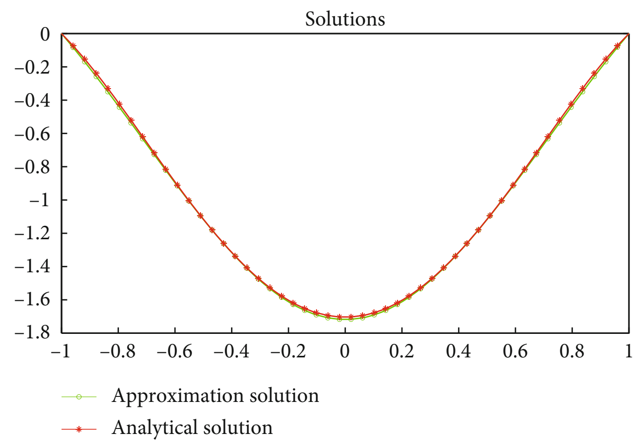
$$\begin{aligned}
\|u - u_N\|_\infty^2 &\overset{(4.6)}{\leq} \|u - u_N\|^2 + 4\left\| (u - u_N)' \right\|^2 \\
&\leq \left(C_I^2 + 4 \right) \left\| (u - u_N)' \right\|^2 \\
&\overset{(4.5)}{\leq} cN^{2(1-m)} \|u\|_m^2,
\end{aligned} \tag{35}$$

Table 1: Errors of $u - u_N$ and orders of convergence for Example 7.

N	$\|u-u_N\|_\infty$	$\|(u-u_N)'\|_\infty$	$\|u-u_N\|$	$\|(u-u_N)'\|$	Order
2	2.6781e-01	1.6331e-00	1.3512e-00	4.3813e-00	/
4	2.6248e-02	3.6964e-01	1.3282e-01	7.1212e-01	2.6620
8	9.9027e-05	3.9703e-03	4.8942e-04	5.9872e-03	6.9139
16	1.1428e-10	2.0526e-08	5.5757e-10	2.9599e-08	17.6304
32	9.9920e-16	7.9936e-15	3.5289e-15	1.4668e-14	20.9041
64	6.6613e-16	5.5511e-15	2.2505e-15	6.9826e-15	1.0401



(a) $N = 2$



(b) $N = 4$

Figure 1: Pointwise curve of u and u_N at two different N.

where C_I denotes the constant within the Poincaré inequality. One readily gets that the desired result listed in (34) holds. $\square$

The above two a-priori error estimations, which are given in H^1- and L^∞- norms, show that an asymptotic super-exponential convergence order for the MPS approximations can be achieved for any sufficiently smooth solution.

5. Numerical Results

In the following different kinds of numerical examples, we show the approximation data in tables and figures, which illustrate the efficiency of the MPS for (10). For simplicity, we evenly select distributed nodes as the collocation points.

Example 7. Setting the boundary value problem (10) with

$$f(x) = (4x^4 - 10x^2 + 2)e^{1-x^2}, \tag{36}$$

we get the analytic solution

$$u(x) = 1 - e^{1-x^2}. \tag{37}$$

Obviously, this analytic solution is sufficient smooth on I. The numerical data listed in Table 1 show error estimates of numerical approximations and the first derivatives of numerical solutions versus N, respectively. And two a-priori error estimations with L^∞- and L^2-norms verify our theoretical analyses. Hence, by the numerical data in the first five columns, we obtain the high accuracy property of the MPS approximations.

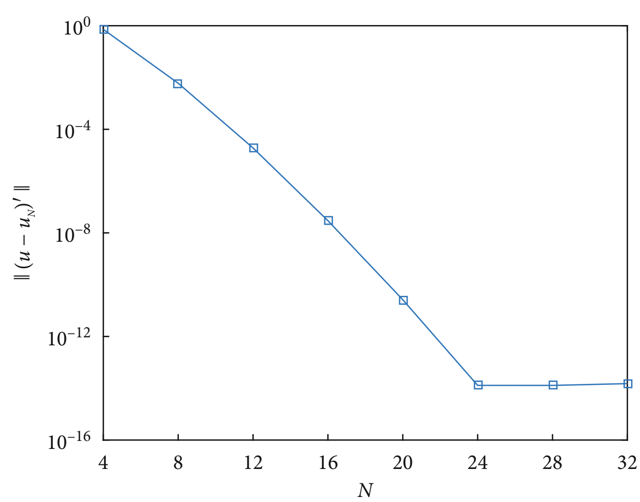
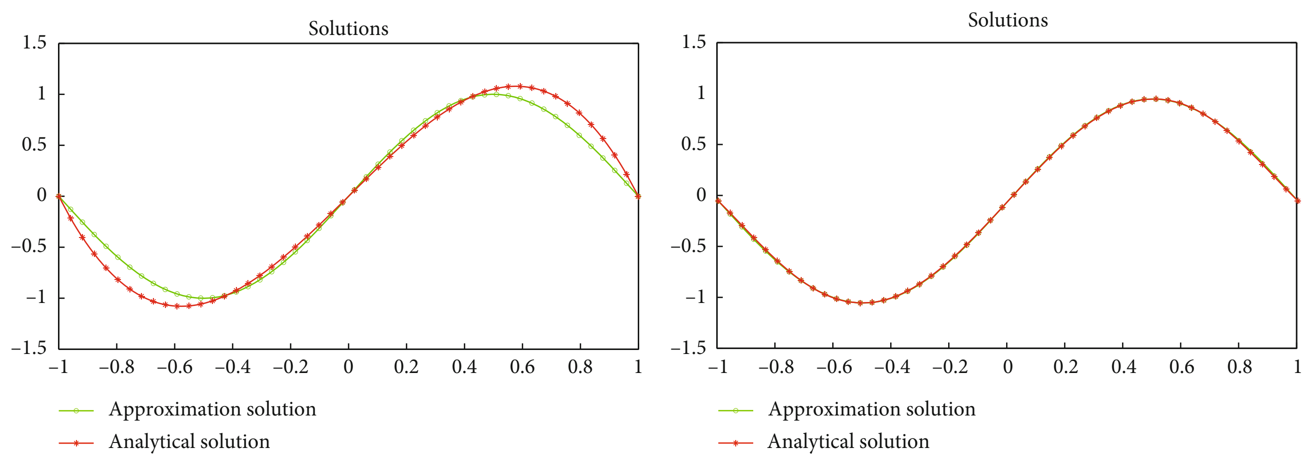


Figure 2: Errors of $\|(u-u_N)'\|$ versus N in the semi-logarithmic scale.

Table 2: Errors of $u - u_N$ and orders of convergence for Example 8.

N	$\|u - u_N\|_\infty$	$\|(u - u_N)'\|_\infty$	$\|u - u_N\|$	$\|(u - u_N)'\|$	Order
2	2.2879e-01	2.5276e+00	8.6146e-01	5.7626e+00	/
4	1.7595e-02	3.8416e-01	5.6339e-02	6.8030e-01	3.0934
8	1.5157e-05	8.6969e-04	3.3639e-05	1.3211e-03	9.0126
16	4.1144e-13	2.4593e-11	1.7072e-12	3.8911e-11	25.0161
32	4.4408e-16	1.7763e-15	1.3686e-15	4.3076e-15	13.0729
64	1.1102e-15	4.4408e-15	4.8534e-15	6.9881e-15	0. 9125



(a) $N = 2$ (b) $N = 4$

Figure 3: u and u_N at two different N.

The last column depicts the convergent orders, which will validate the high efficiency of the MPS. Since the errors arrive at the machine accuracy, the convergence order, 1.0401 in the last column, has no essential significance. Here, the convergence order is calculated by

$$\log_{N_{i+1}/N_i} \frac{error_i}{error_{i+1}}, \tag{38}$$

where the subscripts denote corresponding i-th and $(i+1)$-th information. It is obvious that for any sufficiently smooth analytic solution, the convergence orders of the MPS can be sharply enhanced by the increased N.

For the given right-hand side function f in (36), the analytic solutions and the MPS approximations of $N = 2$ and $N = 4$ are pointwise delineated in Figure 1.

And numerical results of $\|(u - u_N)'\|$ are shown by the semi-logarithmic scale in Figure 2. By the Poincaré inequality, we know that the approximation errors in H^1-norm are naturally consistent with our proposed a-priori error estimates. These figures show the efficiency of the MPS approximations for this example.

Following the above numerical data shown in Table 1 and Figures 1 and 2, it is clear that the numerical errors decrease exponentially with increased N. And hence, the convergence and high accuracy of our proposed numerical scheme are demonstrated.

Example 8. We consider the boundary value problem (10) with

$$f(x) = -2\pi x \cos(\pi x) - \pi^2 (1 - x^2) \sin(\pi x), \tag{39}$$

and the corresponding analytic solution reads

$$u(x) = \sin(\pi x). \tag{40}$$

By our proposed MPS schemes, corresponding numerical errors are listed in Table 2. Also convergence orders are given, which depict the finite algebraic convergence properties. Since numerical data of $N = 32$ and $N = 64$ approach the machine accuracy, which lead to that the last convergence order 0.9125 is unworthy of consideration. And the curves of numerical solution and analytic solution are shown in Figure 3.

Considering the above results and figures, we readily know the sharply approximation properties of the MPS.

Example 9. In this example, we consider the boundary value problem (10) with

$$f(x) = 5(6x^2 - 1)(1 - x^2)^{3/2}, \tag{41}$$

Table 3: Errors of $u - u_N$ and orders of convergence for Example 9.

N	$\|u-u_N\|_\infty$	$\|(u-u_N)'\|_\infty$	$\|u-u_N\|$	$\|(u-u_N)'\|$	Order
2	4.0971e-01	2.2952e-00	2.0526e-00	6.5743e-00	/
4	4.7082e-02	7.9970e-01	2.4208e-01	1.4037e-00	2.2734
8	9.9885e-04	5.8015e-02	4.9670e-03	8.3329e-02	4.0929
16	1.8621e-05	7.0230e-03	7.9775e-05	9.9683e-03	3.0659
32	2.2089e-06	7.4329e-04	1.3539e-05	1.0517e-03	3.2444
64	3.7025e-07	3.2285e-05	2.3867e-06	4.5686e-05	4.5230

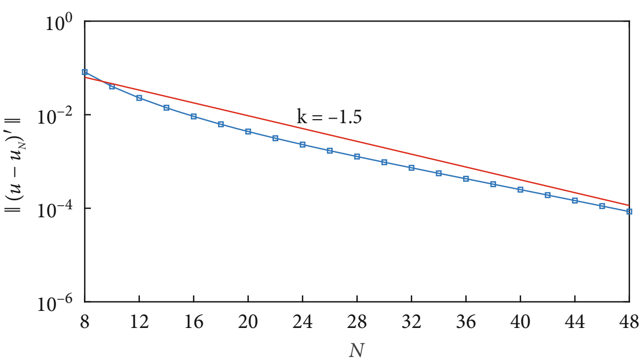


Figure 4: Errors of $\|(u-u_N)'\|$ versus N in semi-logarithmic scale and reference convergence line.

and the corresponding analytic solution

$$u(x) = \left(1 - x^2\right)^{5/2}. \tag{42}$$

Since the third derivative of this solution is singular at the boundary points $x = \pm 1$, the convergence order is not exponential. By the MPS schemes, numerical errors of our proposed approximations are listed in Table 3. And the convergence orders are shown in the last column, which depict the finite algebraic convergence properties.

Furthermore, considering accumulations of round-off errors and convergence orders, we show that the MPS approximations perform well for this kind of second-order differential equations.

Here, we present Figure 4 to show the errors of numerical solutions against various N by semi-logarithmic scale. The error curve of $\|(u-u_N)'\|$ is around the reference line, whose slope reads $\mathrm{k} = -1.5$. This indicates that our error estimates uniformly predict the numerical errors of the MPS, which is consistent with the regularity of the given solution.

In the light of kinds of classical solutions with different smoothness, we demonstrate that our a-priori error estimates uniformly predict the errors of the MPS approximations. Furthermore, considering accumulations of round-off errors, the current section verifies our theoretical results for the model problems with the proposed MPS approximations.

6. Conclusions

The highlight of this work is that we skillfully employed Legendre polynomials to solve second-order differential equations by the MPS. To investigate the efficiency and accuracy of proposed numerical schemes, we study the errors of corresponding numerical approximations. By orthogonal projector and Aubin-Nitsche duality argument, we obtain the a-priori error estimate in H^1-norm with rigorous analyses. Meanwhile, with the help of relationships between L^∞- and H^1-norms on any bounded interval, we readily get corresponding *a-priori* error estimate in L^∞-norm. In the numerical examples, three analytic solutions with different regularity are selected: One is with finite smoothness and others are with infinite regularity. Furthermore, convergence orders and numerical errors are listed to confirm our theoretical results, which also validate the efficiency and high accuracy of the MPS.

The success of dealing with this typical model problem by the MPS will pave the way for solving other more challenging models in science and engineering applications. In our ongoing researches, corresponding further discussions have been listed for the MPS in high dimensional domains, such as how to design the basis functions and corresponding particular solutions based on orthogonal polynomials and how to select collocation points for singular domains. Fortunately, the tensor product of orthogonal polynomials will help us to reformulate the particular solutions and corresponding discretizations. We believe that this method will be applicable for a large amount of partial differential equations and is an efficient numerical scheme in applications.

Appendix

A. The Proof of Lemma 4.3

This appendix follows the proof of both Theorem 7.10 in [39] and (3.9) in [38] and gives a rigorous proof for Lemma 5 on any bounded interval (a, b).

Firstly, we proceed from $\forall v \in C^\infty[a, b]$. By the first mean value theorem of integrals, we know that there exists a $\sigma \in (a, b)$ satisfying

$$\|v\|^2 = \int_a^b |v(x)|^2 dx = |v(\sigma)|^2 (b - a). \tag{A.1}$$

Meanwhile, by the Newton-Leibniz integration formula, we have

$$v(x) = v(\sigma) + \int_{\sigma}^{x} v'(t)dt. \tag{A.2}$$

One readily gets

$$\begin{aligned} |v(x)|^2 &= \left| v(\sigma) + \int_{\sigma}^{x} v'(t)dt \right|^2 \le 2\left[|v(\sigma)|^2 + \left| \int_{\sigma}^{x} v'(t)dt \right|^2 \right] \\ &\le 2\left[|v(\sigma)|^2 + \left| \left(\int_{\sigma}^{x} |v'(t)|^2 dt \right)^{1/2} \left(\int_{\sigma}^{x} dt \right)^{1/2} \right|^2 \right] \\ &= 2\left[|v(\sigma)|^2 + |x-\sigma| \int_{\sigma}^{x} |v'(t)|^2 dt \right] \\ &\overset{(A.1)}{\le} \frac{2}{b-a}\|v\|^2 + 2|b-a| \|v'\|^2. \end{aligned} \tag{A.3}$$

Hence, in view of $C^{\infty}[a,b]$ is dense in $H^1(a,b)$, then for any $v \in H^1(a,b)$, there exists $\{v_k(x)\} \in C^{\infty}[a,b]$ satisfying

$$\|v - v_k\|_1 \longrightarrow 0, k \longrightarrow \infty. \tag{A.4}$$

Now it is obvious that

$$v_k - v_l \in C^{\infty}[a,b], \forall k, l. \tag{A.5}$$

By (A.3), one arrives at

$$|v_k(x) - v_l(x)|^2 \le \frac{2}{b-a}\|v_k - v_l\|^2 + 2|b-a| \cdot \|(v_k - v_l)'\|^2. \tag{A.6}$$

Then for $\forall x \in [a,b]$ and $k, l \longrightarrow \infty$, there holds

$$\begin{aligned} &\max_{x\in[a,b]} |v_k(x) - v_l(x)|^2 \\ &\quad\overset{(A.4)}{\le} \frac{2}{b-a}\|v_k - v_l\|^2 + 2|b-a| \cdot \|(v_k - v_l)'\|^2 \overset{(A.3)}{\longrightarrow} 0, \end{aligned} \tag{A.7}$$

which means that $\{v_k(x)\}$ is a Cauchy sequence in $C[a,b]$.

Meanwhile, in the light of the completeness of $C[a,b]$, we know that there exists $\tilde{v} \in C[a,b]$ such that

$$v_k \overset{C[a,b]}{\longrightarrow} \tilde{v}, k \longrightarrow \infty, \tag{A.8}$$

i.e.,

$$\max_{x\in[a,b]} |v_k(x) - \tilde{v}(x)| \longrightarrow 0, k \longrightarrow \infty. \tag{A.9}$$

Secondly, we identify the relationship between v and $\tilde{v}$. By Minkowski's inequality and Lebesgue integration, we have

$$\begin{aligned} \left(\int_{(a,b)} |v - \tilde{v}|^2 \right)^{1/2} &\le \left(\int_{(a,b)} |v - v_k|^2 \right)^{1/2} \\ &\quad + \left(\int_{(a,b)} |v_k - \tilde{v}|^2 \right)^{1/2}, \end{aligned} \tag{A.10}$$

then for $k \longrightarrow \infty$, there holds

$$\begin{aligned} \left(\int_{(a,b)} |v - \tilde{v}|^2 \right)^{1/2} &\le \lim_{k\longrightarrow\infty} \max_{x\in[a,b]} |\tilde{v}(x) - v_k(x)| (b-a)^{1/2} \\ &\quad + \lim_{k\longrightarrow\infty} \|v_k - \tilde{v}\|_1 \\ &= 0, \end{aligned} \tag{A.11}$$

i.e.,

$$\int_{(a,b)} |v - \tilde{v}|^2 = 0. \tag{A.12}$$

Therefore,

$$v(x) = \tilde{v}(x), a.e. x \in [a,b]. \tag{A.13}$$

Finally, by (A.3), we know that for all v_k, there holds

$$|v_k(x)|^2 \le \frac{2}{b-a}\|v_k\|^2 + 2|b-a| \cdot \|v'_k\|^2. \tag{A.14}$$

Then,

$$\begin{aligned} |\tilde{v}(x)|^2 &\overset{(A.5)}{=} \lim_{k\longrightarrow\infty} |v_k(x)|^2 \\ &\le \frac{2}{b-a} \lim_{k\longrightarrow\infty} \|v_k\|^2 + 2|b-a| \cdot \lim_{k\longrightarrow\infty} \|v'_k\|^2 \\ &\overset{(A.3)}{=} \frac{2}{b-a}\|v\|^2 + 2|b-a| \cdot \|v'\|^2. \end{aligned} \tag{A.15}$$

With the help of (A.13), we directly get

$$|v(x)|^2 \le \frac{2}{b-a}\|v\|^2 + 2|b-a| \cdot \|v'\|^2, a.e. x \in (a,b), \tag{A.16}$$

which means

$$\|v\|_{\infty}^2 \le \frac{2}{b-a}\|v\|^2 + 2|b-a| \cdot \|v'\|^2. \tag{A.17}$$

This is the desired result in Lemma 5.

Data Availability

Data available on request from the authors.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The authors contributed equally to this work.

Acknowledgments

This work is partly supported by Natural Science Foundation of China (Nos. 12271233 and 11926355), and Natural Science Foundation of Shandong Province (Nos. ZR2019YQ05, 2019KJI003, and ZR2018LA008).

References

[1] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, Society for Industrial and Applied Mathematics, 2002.

[2] H. F. Niu, D. P. Yang, and J. W. Zhou, "Numerical analysis of an optimal control problem governed by the stationary Navier-Stokes equations with global velocity-constrained," *Communications in Computational Physics*, vol. 24, no. 5, pp. 1477–1502, 2018.

[3] Z. J. Zhou and Z. Y. Tan, "Finite element approximation of optimal control problem governed by space fractional equation," *Journal of Scientific Computing*, vol. 78, no. 3, pp. 1840–1861, 2019.

[4] C. Canuto, M. Y. Hussaini, A. Quarteroni, and T. A. Zang, *Spectral Methods in Fluid Dynamics*, Springer, Berlin, 1988.

[5] L. Zhang and Z. J. Zhou, "Spectral Galerkin approximation of optimal control problem governed by Riesz fractional differential equation," *Applied Numerical Mathematics*, vol. 143, pp. 247–262, 2019.

[6] J. W. Zhou, H. Y. Li, and Z. Z. Zhang, "A posteriori error estimates of spectral approximations for second order partial differential equations in spherical geometries," *Journal of Scientific Computing*, vol. 90, no. 1, 2022.

[7] S. Yüzbasi, "A shifted Legendre method for solving a population model and delay linear Volterra integro-differential equations," *International Journal of Biomathematics*, vol. 10, no. 7, article 1750091, 2017.

[8] S. Yüzbasi and M. Sezer, "Shifted Legendre approximation with the residual correction to solve pantograph-delay type differential equations," *Applied Mathematical Modelling*, vol. 39, no. 21, pp. 6529–6542, 2015.

[9] E. Hetmaniok, M. Pleszczyński, and Y. Khan, "Solving the integral-differential equations with delayed argument by using the DTM method," *Sensors*, vol. 22, no. 11, p. 4124, 2022.

[10] Z. M. Odibat, "Differential transform method for solving Volterra integral equation with separable kernels," *Mathematical and Computer Modelling*, vol. 48, no. 7-8, pp. 1144–1149, 2008.

[11] C. S. Chen, M. A. Golberg, and R. Schaback, "Recent developments in the dual reciprocity method using compactly supported radial basis functions," in *Transformation of Domain Effects to the Boundary*, Y. F. Rashed, Ed., WIT Press, 2003.

[12] H. D. Cheng, "Particular solutions of Laplacian, Helmholtz-type, and polyharmonic operators involving higher order radial basis functions," *Engineering Analysis with Boundary Elements*, vol. 24, no. 7-8, pp. 531–538, 2000.

[13] C. Franke and R. Schaback, "Solving partial differential equations by collocation using radial basis functions," *Applied Mathematics & Computation*, vol. 93, no. 1, pp. 73–82, 1998.

[14] E. J. Kansa, "Multiquadrics-a scattered data approximation scheme with applications to computational fluid-dynamics-II solutions to parabolic, hyperbolic and elliptic partial differential equations," *Computers & Mathematics with Applications*, vol. 19, no. 8-9, pp. 147–161, 1990.

[15] G. Fairweather and A. Karageorghis, "The method of fundamental solutions for elliptic boundary value problems," *Advances in Computational Mathematics*, vol. 9, no. 1/2, pp. 69–95, 1998.

[16] C. S. Chen, C. M. Fan, and P. H. Wen, "The method of approximate particular solutions for solving elliptic problems with variable coefficients," *International Journal of Computational Methods*, vol. 8, no. 3, pp. 545–559, 2011.

[17] M. A. Golberg, A. S. Muleshkov, and C. S. Chen, "Particular solutions of Helmholtz-type operators using higher order polyharmonic splines," *Computational Mechanics*, vol. 23, no. 5, pp. 411–419, 1999.

[18] T. Belytschko, Y. Lu, and L. Gu, "Element free Galerkin method," *International Journal for Numerical Methods in Engineering*, vol. 37, no. 2, pp. 229–256, 1994.

[19] G. R. Liu and Y. T. Gu, "Boundary meshfree methods based on the boundary point interpolation methods," *Engineering Analysis with Boundary Elements*, vol. 28, no. 5, pp. 475–487, 2004.

[20] W. Chen, "Symmetric boundary knot method," *Engineering Analysis with Boundary Elements*, vol. 26, no. 6, pp. 489–494, 2002.

[21] X. Zhang, M. Chen, C. S. Chen, and Z. Li, "Localized method of approximate particular solutions for solving unsteady Navier-Stokes problem," *Applied Mathematical Modelling*, vol. 40, no. 3, pp. 2265–2273, 2016.

[22] P. H. Wen and C. S. Chen, "The method of particular solutions for solving scalar wave equations, international journal for numerical methods in biomedical," *Engineering*, vol. 26, no. 12, pp. 1878–1889, 2010.

[23] L. Yan and F. L. Yang, "The method of approximate particular solutions for the time fractional diffusion equation with a nonlocal boundary condition," *Computers & Mathematics with Applications*, vol. 70, no. 3, pp. 254–264, 2015.

[24] M. A. Golberg, A. S. Muleshkov, C. S. Chen, and A. H.-D. Cheng, "Polynomial particular solutions for certain partial differential operators," *Numerical Methods for Partial Differential Equations*, vol. 19, no. 1, pp. 112–133, 2003.

[25] A. S. Muleshkov and M. Golberg, "Particular solutions of the multi-Helmholtz-type equation," *Engineering Analysis with Boundary Elements*, vol. 31, no. 7, pp. 624–630, 2007.

[26] G. Yao, C. S. Chen, and C. C. Tsai, "A revisit on the derivation of the particular solution for the differential operator Δ^2-λ^2," *Advances in Applied Mathematics and Mechanics*, vol. 1, no. 6, pp. 750–768, 2009.

[27] C. S. Chen, S. W. Lee, and C. S. Huang, "Derivation of particular solutions using Chebyshev polynomial based functions," *International Journal of Computational Methods*, vol. 4, no. 1, pp. 15–32, 2007.

[28] K. E. Atkinson, “The numerical evaluation of particular solutions for Poisson’s equation,” *IMA Journal of Numerical Analysis*, vol. 5, no. 3, pp. 319–338, 1985.

[29] C. S. Chen, M. Ganesh, M. A. Golberg, and A. H.-D. Cheng, “Multilevel compact radial functions based computational schemes for some elliptic problems,” *Computers & Mathematics with Application*, vol. 43, no. 3-5, pp. 359–378, 2002.

[30] J. Shen, “Efficient spectral-Galerkin method I,” *SIAM Journal on Scientific Computing*, vol. 15, no. 6, pp. 1489–1505, 1994.

[31] Z. Zhang, “Superconvergence of a Chebyshev spectral collocation method,” *Journal of Scientific Computing*, vol. 34, no. 3, pp. 237–246, 2008.

[32] F. F. Dou, Y. S. Liu, and C. S. Chen, “The method of particular solutions for solving nonlinear Poisson problems,” *Computers & Mathematics with Applications*, vol. 77, no. 2, pp. 501–513, 2019.

[33] J. W. Zhou and D. P. Yang, “An improved a posteriori error estimate for the Galerkin spectral method in one dimension,” *Computers & Mathematics with Applications*, vol. 61, no. 2, pp. 334–340, 2011.

[34] I. Babuska, J. Osborn, and J. Pitkäranta, “Analysis of mixed methods using mesh dependent norms,” *Mathematics of Computation*, vol. 35, no. 152, pp. 1039–1062, 1980.

[35] A. Charbonneau, K. Dossou, and R. Pierre, “A residual-based a posteriori error estimator for the Ciarlet-Raviart formulation of the first biharmonic problem,” *Numerical Methods for Partial Differential Equations*, vol. 13, no. 1, pp. 93–111, 1997.

[36] G. Leoni, *A First Course in Sobolev Spaces*, AMS, Rhode Island, 2009.

[37] X. M. Xiang, *Numerical Analysis of Spectral Method*, Science Press, Beijing, 2000.

[38] B. Y. Guo and Z. Q. Wang, “Legendre-Gauss collocation methods for ordinary differential equations,” *Advances in Computational Mathematics*, vol. 30, no. 3, pp. 249–280, 2009.

[39] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin, 2001.