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Research Article

Forbidden Restrictions and the Existence of $P_{\geq 2}$-Factor and $P_{\geq 3}$-Factor

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The existence of factor and fractional factor in network graph in various settings has raised much attention from both mathematicians and computer scientists. It implies the availability of data transmission and network segmentation in certain special settings. In our paper, we consider $P_{\geq 2}$-factor and $P_{\geq 3}$-factor which are two special cases of general $\mathscr{H}$-factor. Specifically, we study the existence of these two kinds of path factor when some subgraphs are forbidden, and several conclusions on the factor-deleted graph, factor critical-covered graph, and factor uniform graph are given with regards to network parameters. Furthermore, we show that these bounds are best in some sense.

1. Introduction

All graphs considered in this work are finite simple graphs. Let $G=(V(G),E(G))$ be a graph, $N_G(v)$ be the neighborhood of vertex v, and $d_G(v)=|N_G(v)|$. Let $\omega(G)$ be the number of connected components in G and $i(G)=|\{v\in V(G)\colon d_G(v)=0\}|$. For the commonly used notations and terminologies, please refer to book [1] by Bondy and Mutry.

Let $n\geq 2$ and $P_{\geq n}$ be a path with at least n vertices. A $P_{\geq n}$-factor is a spanning subgraph of G such that each component is isomorphic to $P_{\geq n}$. A graph G is a $(P_{\geq n}, m)$-factor-deleted graph if removing any m edges from G, the resting subgraph still admits $P_{\geq n}$-factor. For $P_{\geq 2}$-factor, Akiyama et al. [2] demonstrated the following characteristic for its existence.

Lemma 1. *A graph G permits a $P_{\geq 2}$ -factor if and only if $2|X|\geq i(G-X)$ established for arbitrary vertex subset X of G.*

More recent results on graph factors in various settings can be referred to Gao et al. [3, 4], Wang and Zhang and Zhou et al. [5–10], and Zhu et al. [11, 12].

A graph R is factor-critical if deleting any vertex v, the resulting subgraph has a perfect matching. A graph G is called a sun if it is isomorphic to K_1, K_2, or the corona of a factor-critical graph, and the last class of sun is a big sun. Let $\mathrm{sun}(G)$ be the number of sun components of G. Kaneko [13] and Kano et al. [14] revealed that sun components can describe the existence of $P_{\geq 3}$-factor, i.e, a graph G admits a $P_{\geq 3}$-factor if and only if $2|S|\geq \mathrm{sun}(G-S)$ for any vertex subset S of G.

Zhang and Zhou [15] introduced the concept of $P_{\geq n}$-factor-covered graph, i.e., a graph G is $P_{\geq n}$-factor covered if for any edge e, there is a $P_{\geq n}$-factor containing e. Moreover, they obtained the following two conclusions for $P_{\geq n}$-factor-covered graph when $n=2$ or 3.

Lemma 2 (Zhang and Zhou [15]). *A connected graph G is a $P_{\geq 2}$-factor-covered graph if and only if*

$$i(G-S)\leq 2|S|-\varepsilon_1(S), \tag{1}$$

for any vertex subset S of G, where

$$\varepsilon_1(S)=\begin{cases}2, & \text{if } S \text{ is not an independent set},\\ 1, & S \text{ is independent, and there exists a} \\ & \text{nontrivial component of } G-S,\\ 0, & \text{otherwise}.\end{cases} \quad (2)$$

Lemma 3 (Zhang and Zhou [15]). *Assume G as a connected graph. Then, G is a $P_{\geq 3}$-factor-covered graph if and only if*

$$sun(G-S)\leq 2|S|-\varepsilon_2(S), \quad (3)$$

for any $S\subseteq V(G)$, where

$$\varepsilon_2(S)=\begin{cases}2, & \text{if } S \text{ is not an independent set},\\ 1, & S \text{ is independent and there exists a} \\ & \text{nonsun component of } G-S,\\ 0, & \text{otherwise}.\end{cases} \quad (4)$$

The concept of factor-covered graph can be further extended to factor-critical-covered graph. A graph G is $(P_{\geq n}, k)$-factor-critical covered if deleted any k vertices from G, and the resting subgraph is still a $P_{\geq n}$-factor-covered graph.

In computer data communication networks, there are three main indices to test the robustness and vulnerability of networks, and also, there are some variables of these parameters.

(i) Chvátal [16] firstly introduced toughness where $t(G) = +\infty$ if G is complete; otherwise

$$t(G)=\min\left\{\frac{|S|}{\omega(G-S)}\,|\,\omega(G-S)\geq 2\right\}. \quad (5)$$

Enomoto et al. [17] introduced a variant of toughness by revising the denominator to $\omega(G-S)-1$ and denoted it by $\tau(G)$. That is to say, $\tau(G)=+\infty$ if G is a complete graph; and

$$\tau(G)=\min\left\{\frac{|S|}{\omega(G-S)-1}\,|\,\omega(G-S)\geq 2\right\}, \quad (6)$$

for noncomplete graph.

(ii) Isolated toughness was introduced by Yang et al. [18] as follows: if G is a complete graph, then $I(G) = +\infty$; elsewise

$$I(G)=\min\left\{\frac{|S|}{i(G-S)}\,|\,S\subset V(G), i(G-S)\geq 2\right\}. \quad (7)$$

Similar to τ, Zhang and Liu [19] introduced a variant of isolated toughness by revising the denominator to $i(G-S)-1$, denoted by $I'(G)$: $I'(G)=+\infty$ for a complete graph G, and

$$I'(G)=\min\left\{\frac{|S|}{i(G-S)-1}\,|\,S\subset V(G), i(G-S)\geq 2\right\}, \quad (8)$$

for others.

(iii) Binding number is defined by Woodall [20] which is formulated by

$$\mathrm{bind}(G)=\min\left\{\frac{|N_G(X)|}{|X|}\,|\,\varnothing\neq X\subseteq V(G), N_G(X)\neq V(G)\right\}. \quad (9)$$

The main contributions of this article are three folded: (1) the relationships between $(P_{\geq 2}, m)$-factor-deleted graph and the above three parameters are studied; (2) toughness conditions for $(P_{\geq 2}, k)$-factor-critical covered and $(P_{\geq 3}, k)$-factor-critical covered graph are given; (3) toughness bounds for a graph to be $P_{\geq 2}$-factor uniform graph and $P_{\geq 3}$-factor uniform graph are determined. The main conclusions and detailed proofs are manifested in the next section, and then, in the third section, we present the sharpness of these bounds.

2. Main Results and Proofs

The purpose of this section is to present the main theorems and detailed proofs.

2.1. Bounds for $(P_{\geq 2}, m)$-Factor-Deleted Graphs

Theorem 4. *Let m be a positive integer and G be an $(m+1)$-edge-connected graph. If $t(G)>m/m+1$ (resp. $\tau(G)>1$) then G is a $(P_{\geq 2}, m)$-factor-deleted graph.*

Proof. For a complete graph G, the result follows from edge connectivity. Assume that G is not complete, and clearly we have $|V(G)|\geq m+2$. □

For arbitrary edge subset $E'=\{e_1,\cdots,e_m\}$ with m edges, let $G'=G-E'$, and we have $V(G')=V(G)$ and $E(G')=E(G)-E'$. We verify the theorem by means of proving that G' admits $P_{\geq 2}$-factor. In contrast, we assume G' has no $P_{\geq 2}$-factor, and hence, in view of Lemma 1, there is a subset S of $V(G')$ satisfying

$$i\left(G'-S\right)\geq 2|S|+1. \quad (10)$$

If $|S|=0$, then $i(G')\geq 1$ by (1) which contradicts to G is $(m+1)$-edge-connected and $|V(G)|\geq m+2$. Therefore, we infer $|S|\geq 1$ and $i(G'-S)\geq 2|S|+1\geq 3$. Deleting one edge from $G-S$, the number of its components adds most 1, thus $\omega(G'-S)=\omega(G-E-S)\leq \omega(G-S)+m$.

We divide $E'=\{e_i\}_{i=1}^m$ into three classes E_1', E_2', and E_3'.

If $e_i \in E'$ is a unique edge in K_2 which is a component in $G-S$, then $e_i \in E_1'$.

If $e_i \in E'$ and $e_i \in E(G-S)$, one of end vertex of e_i (say v_i) meets $d_{G-S}(v_i) \geq 2$, then $e_i \in E_2'$.

Otherwise, $e_i \in E'$ and at least one of its end vertices in S, then $e_i \in E_3'$.

We have $|E_1'| + |E_2'| \leq m$ and $|E_1'|, |E_2'| \in \{0,\cdots,m\}$. Select one vertex in each edge in E_2' with larger degree in $G-S$ and denote X by the set of these vertices. Thus, $|X| \leq |E_2'|$.

According to

$$\begin{aligned} \frac{m}{m+1} < t(G) &\leq \frac{|S|}{\omega(G-S)} \leq \frac{|S|}{\omega\left(G'-S\right)-m} \\ &\leq \frac{|S|}{i\left(G'-S\right)-m} \leq \frac{|S|}{2|S|+1-m}, \end{aligned} \tag{11}$$

or accordingly

$$\begin{aligned} 1 < \tau(G) &\leq \frac{|S|}{\omega(G-S)-1} \leq \frac{|S|}{\omega\left(G'-S\right)-m-1} \\ &\leq \frac{|S|}{i\left(G'-S\right)-m-1} \leq \frac{|S|}{2|S|-m}, \end{aligned} \tag{12}$$

we get $|S| \in \{1,\cdots,m-1\}$.

For $t(G)$, we have

$$\begin{aligned} \frac{m}{m+1} < t(G) &\leq \frac{|S\cup X|}{\omega(G-S\cup X)} = \frac{|S|+|X|}{\omega\left(G'-S\cup X\right)-\left|E_1'\right|} \\ &\leq \frac{|S|+|X|}{\omega\left(G'-S\right)-\left|E_1'\right|} \leq \frac{|S|+\left|E_2'\right|}{i\left(G'-S\right)-\left|E_1'\right|} \\ &\leq \frac{|S|+m-\left|E_1'\right|}{i\left(G'-S\right)-\left|E_1'\right|} \leq \frac{|S|+m-\left|E_1'\right|}{2|S|+1-\left|E_1'\right|}. \end{aligned} \tag{13}$$

Let $f(|E_1'|) = (|S| + m - |E_1'|)/(2|S| + 1 - |E_1'|)$ be a function with regard to $|E_1'|$. We have

$$f'\left(\left|E_1'\right|\right) = \frac{\left(2|S|+1-\left|E_1'\right|\right)\left(|S|+m-\left|E_1'\right|\right)'-\left(2|S|+1-\left|E_1'\right|\right)'\left(|S|+m-\left|E_1'\right|\right)}{\left(2|S|+1-\left|E_1'\right|\right)^2} = \frac{m-1-|S|}{\left(2|S|+1-\left|E_1'\right|\right)^2} \geq 0. \tag{14}$$

Hence, $f(|E_1'|)$ is a monotonically increasing function and $\max\{f(|E_1'|)\} = f(m)$. We get

$$\begin{aligned} \frac{m}{m+1} < t(G) &\leq \frac{|S|}{2|S|+1-m} = \frac{1}{2} + \frac{m/2-1/2}{2|S|+1-m} \\ &\leq \frac{1}{2} + \frac{m/2-1/2}{2+1-m} = \frac{-m^2+4m-2}{2}, \end{aligned} \tag{15}$$

which implies $m = 2$.

If $m = 2$, then $|S| = 1$ and $i(G'-S) \geq 2|S| + 1 = 3$. If $\omega(G-S) \geq 2$, then $2/3 = m/(m+1) < t(G) \leq |S|/\omega(G-S) \leq 1/2$, a contradiction. Hence, $G-S$ is a connected graph, and there are at least 3 isolated vertices after removing 2 edges from $G-S$. That is to say, $G = K_1 \vee P_3$ which contradicts to G is a 3-edge-connected graph.

For $\tau(G)$, we have

$$\begin{aligned} 1 < \tau(G) &\leq \frac{|S\cup X|}{\omega(G-S\cup X)-1} = \frac{|S|+|X|}{\omega\left(G'-S\cup X\right)-\left|E_1'\right|-1} \\ &\leq \frac{|S|+|X|}{\omega\left(G'-S\right)-\left|E_1'\right|-1} \leq \frac{|S|+\left|E_2'\right|}{i\left(G'-S\right)-\left|E_1'\right|-1} \\ &\leq \frac{|S|+m-\left|E_1'\right|}{i\left(G'-S\right)-\left|E_1'\right|-1} \leq \frac{|S|+m-\left|E_1'\right|}{2|S|+1-\left|E_1'\right|-1} \\ &= \frac{|S|+m-\left|E_1'\right|}{2|S|-\left|E_1'\right|}. \end{aligned} \tag{16}$$

Let $g(|E_1'|) = (|S| + m - |E_1'|)/(2|S| - |E_1'|)$ be a function with regard to $|E_1'|$. We obtain

$$g'\left(\left|E_1'\right|\right) = \frac{\left(2|S|-\left|E_1'\right|\right)\left(|S|+m-\left|E_1'\right|\right)'-\left(2|S|-\left|E_1'\right|\right)'\left(|S|+m-\left|E_1'\right|\right)}{\left(2|S|-\left|E_1'\right|\right)^2} = \frac{m-|S|}{\left(2|S|-\left|E_1'\right|\right)^2} > 0. \tag{17}$$

Hence, $g(|E_1'|)$ is a monotonically increasing function and $\max\{g(|E_1'|)\} = g(m)$. We get

$$\begin{aligned} 1 < \tau(G) &\le \frac{|S|}{2|S| - m} = \frac{1}{2} + \frac{m}{2(2|S| - m)} \\ &\le \frac{1}{2} + \frac{m}{2(2-m)} = \frac{1}{2-m}, \end{aligned} \tag{18}$$

which implies $m = 2$.

If $m = 2$, then $|S| = 1$ and $i(G' - S) \ge 2|S| + 1 = 3$. If $\omega(G - S) \ge 2$, then $1 < \tau(G) \le |S|/\omega(G-S) - 1 \le 1$, a contradiction. Hence, $G - S$ is a connected graph, and there are at least three isolated vertices after removing two edges from $G - S$. That is to say, $G = K_1 \vee P_3$ which contradicts to G that is a 3-edge-connected graph.

Hence, the proof of result is completed.

Theorem 5. *Let m be a positive integer and G be an $(m+1)$-edge-connected graph. If $I(G) > 2m/(m+1)$ (resp. $I'(G) > 2$), then, G is a $(P_{\ge 2}, m)$-factor-deleted graph.*

Proof. For a complete graph G, the result follows from edge connectivity. Assume that G is not complete, and clearly, we have $|V(G)| \ge m + 2$. □

For arbitrary edge subset $E' = \{e_1, \cdots, e_m\}$ with m edges, let $G' = G - E'$, and we have $V(G') = V(G)$ and $E(G') = E(G) - E'$. We check the correctness of Theorem 5 via proving G' permits $P_{\ge 2}$-factor. If not, we assume G' has no $P_{\ge 2}$-factor, and hence ,using Lemma 1, there is a subset S of $V(G')$ satisfying (1).

If $|S| = 0$, then $i(G') \ge 1$ by (1) which contradicts to G being $(m+1)$-edge-connected and $|V(G)| \ge m + 2$. Therefore, we infer $|S| \ge 1$ and $i(G' - S) \ge 2|S| + 1 \ge 3$. Deleting one edge from $G - S$, the number of its isolated vertices adds most 2; thus, $i(G' - S) = i(G - E - S) \le i(G - S) + 2m$.

We divide E' into three classes E_1', E_2', and E_3' as described in Theorem 4, and hence, $|E_1'| + |E_2'| \le m$ and $|E_1'|, |E_2'| \in \{0, \cdots, m\}$. Also, we use the same way to select vertex set X, and thus, $|X| \le |E_2'|$.

For $I(G)$, we have

$$\begin{aligned} \frac{2m}{m+1} < I(G) &\le \frac{|S \cup X|}{i(G - S \cup X)} = \frac{|S| + |X|}{i\left(G' - S \cup X\right) - 2|E_1'|} \\ &\le \frac{|S| + |X|}{i\left(G' - S\right) - 2|E_1'|} \end{aligned} \tag{19}$$

Reset $f(|E_1'|) = (|S| + m - |E_1'|)/(2|S| + 1 - 2|E_1'|)$ be a function with regard to $|E_1'|$. We acquire

$$f'\left(|E_1'|\right) = \frac{\left(2|S| + 1 - 2|E_1'|\right)\left(|S| + m - |E_1'|\right)' - \left(2|S| + 1 - 2|E_1'|\right)'\left(|S| + m - |E_1'|\right)}{\left(2|S| + 1 - 2|E_1'|\right)^2} = \frac{2m - 1}{\left(2|S| + 1 - |E_1'|\right)^2} > 0. \tag{20}$$

Hence, $f(|E_1'|)$ is a monotonically increasing function and $\max\{f(|E_1'|)\} = f(m)$. Thus, we get

$$\begin{aligned} \frac{2m}{m+1} < I(G) &\le \frac{|S|}{2|S| + 1 - 2m} = \frac{1}{2} + \frac{m - 1/2}{2|S| + 1 - m} \\ &\le \frac{1}{2} + \frac{m - 1/2}{2 + 1 - 2m} = \frac{1}{3 - 2m}, \end{aligned} \tag{21}$$

a contradiction.

For $I'(G)$, we have

$$\begin{aligned} 2 < I'(G) &\le \frac{|S \cup X|}{i(G - S \cup X) - 1} = \frac{|S| + |X|}{i\left(G' - S \cup X\right) - 2|E_1'| - 1} \\ &\le \frac{|S| + |X|}{i\left(G' - S\right) - 2|E_1'| - 1} \le \frac{|S| + m - |E_1'|}{2|S| + 1 - 2|E_1'| - 1} \\ &= \frac{|S| + m - |E_1'|}{2|S| - 2|E_1'|}. \end{aligned} \tag{22}$$

Reset $g(|E_1'|) = (|S| + m - |E_1'|)/(2|S| - 2|E_1'|)$ be a function with regard to $|E_1'|$. We acquire

$$g'\left(|E_1'|\right) = \frac{\left(2|S| - 2|E_1'|\right)\left(|S| + m - |E_1'|\right)' - \left(2|S| - 2|E_1'|\right)'\left(|S| + m - |E_1'|\right)}{\left(2|S| - 2|E_1'|\right)^2} = \frac{2m}{\left(2|S| - |E_1'|\right)^2} > 0. \tag{23}$$

Hence, $g(|E_1'|)$ is a monotonically increasing function and $\max\{g(|E_1'|)\} = f(m)$. Thus, we get

$$2 < I'(G) \leq \frac{|S|}{2|S| - 2m} = \frac{1}{2} + \frac{m}{2|S| - 2m} \tag{24}$$

a contradiction if $m \geq 2$.

Specially, if $m = 1$, then

$$\begin{aligned} 2 < I'(G) &\leq \frac{|S|}{i(G-S) - 1} \leq \frac{|S|}{i\left(G' - S\right) - 2m - 1} \\ &\leq \frac{|S|}{2|S| + 1 - 3} = \frac{|S|}{2|S| - 2}, \end{aligned} \tag{25}$$

which implies $|S| = 1$. In this case, $i(G' - S) \geq 3$ leads to $i(G - S) \geq i(G' - S) - 2m \geq 1$ which contradicts to G being a 2-edge-connected graph.

Hence, the proof of this result is completed.

Theorem 6. *Let m be a positive integer and G be an $(m+1)$-edge-connected graph. If $\mathrm{bind}(G) > 3/2$, then, G is a $(P_{\geq 2}, m)$-factor-deleted graph.*

Proof. For a complete graph G, the result follows from edge connectivity. Assume that G is not complete, and clearly, $|V(G)| \geq m + 2$. □

Let $G' = G - E'$ for arbitrary edge subset E' with m edges, and we have $V(G') = V(G)$ and $E(G') = E(G) - E'$. Assume that G' has no $P_{\geq 2}$-factor, and hence, in view of Lemma 1, there is a subset S of $V(G')$ satisfying (1).

If $|S| = 0$, then, $i(G') \geq 1$ by (1) which contradicts to G being $(m+1)$-edge-connected and $|V(G)| \geq m + 2$. Therefore, we infer $|S| \geq 1$ and $i(G' - S) \geq 2|S| + 1 \geq 3$. Deleting one edge from $G - S$, the number of its isolated components adds most 2, thus, $i(G' - S) = i(G - E - S) \leq i(G - S) + 2m$.

Note that there are at least 3 isolated vertices after removing m edges from $G - S$. Also, since $\delta(G) \geq \lambda(G) \geq m + 1$, we get $|S| \geq m + 1 - m/i(G' - S) \geq m + 1 - m/(2|S| + 1)$, i.e., $m \leq (2|S| + 1)(|S| - 1)/2|S|$. Let X be the vertex set of these isolated vertices in $G' - S$. If $N_G(X) \neq V(G)$, we acquire

$$\begin{aligned} \frac{3}{2} < \mathrm{bind}(G) &\leq \frac{|N_G(X)|}{|X|} \leq \frac{|S| + 2m}{i\left(G' - S\right)} \leq \frac{|S| + 2m}{2|S| + 1} \\ &\leq \frac{|S| + 2((2|S| + 1)(|S| - 1)/2|S|)}{2|S| + 1} \\ &= \frac{3}{2} - \frac{1}{|S|} - \frac{1}{2(2|S| + 1)} < \frac{3}{2}, \end{aligned} \tag{26}$$

a contradiction.

Now, we consider $N_G(X) \neq V(G)$. If there is a vertex v in $G - S$ meeting $d_{G-S}(v) = 1$, then, set $uv \in E(G - S)$ and $u \in X$ since $N_G(X) \neq V(G)$. We yield

$$\begin{aligned} \frac{3}{2} < \mathrm{bind}(G) &\leq \frac{|N_G(X - \{u\})|}{|X - \{u\}|} \leq \frac{|S| + 2m - 1}{i\left(G' - S\right) - 1} \\ &\leq \frac{|S| + 2m - 1}{2|S| + 1 - 1} = \frac{|S| + 2m - 1}{2|S|} \\ &\leq \frac{|S| + 2((2|S| + 1)(|S| - 1)/2|S|) - 1}{2|S|} \\ &= \frac{3|S|^2 - 2|S| - 1}{2|S|^2} < \frac{3}{2}, \end{aligned} \tag{27}$$

a contradiction.

If each vertex in X has a degree at least 2 in $G - S$, then, we can get the contradiction similar to what discussed above.

Hence, the proof of result is completed.

2.2. Toughness Conditions for $(P_{\geq 2}, k)$-Factor-Critical Covered and $(P_{\geq 3}, k)$-Factor-Critical Covered Graph

Theorem 7. *Let $k \in N \cup \{0\}$ and G be a graph with $\kappa(G) \geq k + 1$. If $(G) > (k+2)/3$ (resp.$\tau(G) > (k+2)/2$), then, G is a $(P_{\geq 2}, k)$-factor critical covered graph.*

Proof. If G is complete, the result follows from $\delta(G) \geq \kappa(G) \geq k + 1$. In what follows, we consider noncomplete graph. □

For any $U \subseteq V(G)$ with $|U| = k$, set $G' = G - U$. To demonstrate G is $(P_{\geq 2}, k)$-factor critical covered, it is enough to prove G' is $P_{\geq 2}$-factor covered. Otherwise, suppose G' is not $P_{\geq 2}$-factor covered; then, according to Lemma 2, there is a vertex subset S of G' such that

$$i\left(G' - S\right) \geq 2|S| - \varepsilon_1(S) + 1. \tag{28}$$

The following discussion is divided into three cases in terms of the value of $|S|$.

Case 1. $|S| = 0$.

In this case, $\varepsilon_1(S) = 0$ and $i(G') \geq 1$ by (2), which contradicts to $\delta(G) \geq \kappa(G) \geq k + 1$.

Case 2. $|S| = 1$.

We consider the following two subcases.

Case 3. $G' - S$ has no nontrivial component.

We infer $\varepsilon_1(S) = 0$ and $i(G' - S) \geq 2|S| - \varepsilon_1(S) + 1 = 3$. By means of the definition of toughness, we deduce

$$\frac{k+2}{3} < t(G) \leq \frac{|U \cup S|}{\omega(G - U \cup S)} \leq \frac{k+1}{3}, \tag{29}$$

or

$$\frac{k+2}{2} < \tau(G) \le \frac{|U \cup S|}{\omega(G - U \cup S) - 1} \le \frac{k+1}{2}, \quad (30)$$

a contradiction.

Case 4. $G' - S$ has a nontrivial component.

We yield $\varepsilon_1(S) = 1$, $i(G' - S) \ge 2|S| - \varepsilon_1(S) + 1 = 2$, and $\omega(G' - S) \ge 3$. Using the definition of toughness, we have

$$\frac{k+2}{3} < t(G) \le \frac{|U \cup S|}{\omega(G - U \cup S)} \le \frac{k+1}{3}, \quad (31)$$

or

$$\frac{k+2}{2} < \tau(G) \le \frac{|U \cup S|}{\omega(G - U \cup S) - 1} \le \frac{k+1}{2}, \quad (32)$$

a contradiction.

Case 5. $|S| \ge 2$.

We acquire $\varepsilon_1(S) \le 2$ and $i(G' - S) \ge 2|S| - \varepsilon_1(S) + 1 \ge 3$. In light of the definition of toughness, we obtain

$$\begin{aligned} \frac{k+2}{3} < t(G) &\le \frac{|U \cup S|}{\omega(G - U \cup S)} = \frac{k+|S|}{\omega\left(G' - S\right)} \\ &\le \frac{k+|S|}{i\left(G' - S\right)} \le \frac{k+|S|}{2|S| - \varepsilon_1(S) + 1} \le \frac{k+|S|}{2|S| - 1} \\ &= \frac{1}{2} + \frac{k+1/2}{2|S|-1} \le \frac{1}{2} + \frac{k+1/2}{2 \times 2 - 1} = \frac{k+2}{3}, \end{aligned} \quad (33)$$

or

$$\begin{aligned} \frac{k+2}{2} < \tau(G) &\le \frac{|U \cup S|}{\omega(G - U \cup S) - 1} = \frac{k+|S|}{\omega\left(G' - S\right) - 1} \\ &\le \frac{k+|S|}{i\left(G' - S\right) - 1} \le \frac{k+|S|}{2|S| - \varepsilon_1(S) + 1 - 1} \\ &\le \frac{k+|S|}{2|S| - 2} = \frac{1}{2} + \frac{k+1}{2|S|-2} \le \frac{1}{2} + \frac{k+1}{2 \times 2 - 2} = \frac{k+2}{2}, \end{aligned} \quad (34)$$

a contradiction.

Therefore, the result follows.

Theorem 8. *Let $k \in N \cup \{0\}$ and G be a graph with $\kappa(G) \ge k+1$ and $|V(G)| \ge k+3$. If $(G) > (k+2)/3$ (resp.$\tau(G) > (k+2)/2$), then, G is a $(P_{\ge 3}, k)$-factor critical covered graph.*

Proof. If G is a complete graph, then, the result follows from $|V(G)| \ge k+3$. We only consider noncomplete graph in what follows.

For any $U \subseteq V(G)$ with k vertices, let $G' = G - U$, and we aim to prove G' is $P_{\ge 3}$-factor covered. On the contrary, G is not a $P_{\ge 3}$-factor covered graph, and then, by Lemma 3, there is a subset S of $V(G')$ meeting

$$\text{sun}\left(G' - S\right) \ge 2|S| - \varepsilon_2(S) + 1. \quad (35)$$

□

The following discussion is divided into three cases by means of the value of $|S|$.

Case 1. $|S| = 0$.

In this case, we summarize $\varepsilon_2(S) = 0$ and $\text{sun}(G') \ge 1$ by (3). Using $\kappa(G) \ge k+1$ and $|U| = k$, we get $\text{sun}(G') = \omega(G') = 1$. Since $|V(G)| \ge k+3$, we confirm that G' is a big sun. Let R be the factor-critical graph of G' with $|V(R)| \ge 3$ and $v \in V(R)$ be a vertex in R. Using the definition of toughness, we obtain

$$\begin{aligned} \frac{k+2}{3} < t(G) &\le \frac{|U \cup (V(R) - \{v\})|}{\omega(G - U \cup (V(R) - \{v\}))} \\ &= \frac{k + |V(R)| - 1}{|V(R)|} = 1 + \frac{k-1}{|V(R)|} \\ &\le 1 + \frac{k-1}{3} = \frac{k+2}{3}, \end{aligned} \quad (36)$$

or

$$\begin{aligned} \frac{k+2}{2} < \tau(G) &\le \frac{|U \cup (V(R) - \{v\})|}{\omega(G - U \cup (V(R) - \{v\})) - 1} \\ &= \frac{k + |V(R)| - 1}{|V(R)| - 1} = 1 + \frac{k}{|V(R)| - 1} \\ &\le 1 + \frac{k}{3-1} = \frac{k+2}{2}, \end{aligned} \quad (37)$$

a contradiction.

Case 2. $|S| = 1$.

If there is a nonsun component of $G' - S$, we have $\varepsilon_2(S) = 1$, $\text{sun}(G' - S) \ge 2|S| - \varepsilon_2(S) + 1 = 2$ by (3), and $\omega(G' - S) \ge \text{sun}(G' - S) + 1$. Directly using the definition of toughness, we yield

$$\begin{aligned} \frac{k+2}{3} < t(G) &\le \frac{|U \cup S|}{\omega(G - U \cup S)} = \frac{|U| + |S|}{\omega\left(G' - S\right)} \\ &\le \frac{k+1}{\text{sun}\left(G' - S\right) + 1} \le \frac{k+1}{2|S| - \varepsilon_2(S) + 1 + 1} = \frac{k+1}{3}, \end{aligned} \quad (38)$$

or

$$\frac{k+2}{2} < \tau(G) \le \frac{|U \cup S|}{\omega(G - U \cup S) - 1} = \frac{|U| + |S|}{\omega\left(G' - S\right) - 1} \le \frac{k+1}{\operatorname{sun}\left(G' - S\right) + 1 - 1} \le \frac{k+1}{2|S| - \varepsilon_2(S) + 1} = \frac{k+1}{2}, \tag{39}$$

a contradiction.

If there is no nonsun component of $G' - S$, we get $\varepsilon_2(S) = 0$, $\operatorname{sun}(G' - S) \ge 2|S| - \varepsilon_2(S) + 1 = 3$ by (3), and $\omega(G' - S) = \operatorname{sun}(G' - S)$. In light of the definition of toughness, we infer

$$\frac{k+2}{3} < t(G) \le \frac{|U \cup S|}{\omega(G - U \cup S)} = \frac{|U| + |S|}{\omega\left(G' - S\right)} = \frac{k+1}{\operatorname{sun}\left(G' - S\right)} \le \frac{k+1}{2|S| - \varepsilon_2(S) + 1} = \frac{k+1}{3}, \tag{40}$$

or

$$\frac{k+2}{2} < \tau(G) \le \frac{|U \cup S|}{\omega(G - U \cup S) - 1} = \frac{|U| + |S|}{\omega\left(G' - S\right) - 1} = \frac{k+1}{\operatorname{sun}\left(G' - S\right) - 1} \le \frac{k+1}{2|S| - \varepsilon_2(S) + 1 - 1} = \frac{k+1}{2}, \tag{41}$$

a contradiction.

Case 3. $|S| \ge 2$.

In this case, we acquire $\varepsilon_2(S) \le 2$ and $\operatorname{sun}(G' - S) \ge 2|S| - \varepsilon_2(S) + 1 \ge 3$ in terms of (3). We verify

$$\frac{k+2}{3} < t(G) \le \frac{|U \cup S|}{\omega(G - U \cup S)} = \frac{|U| + |S|}{\omega\left(G' - S\right)} \le \frac{k+|S|}{\operatorname{sun}\left(G' - S\right)} \le \frac{k+|S|}{2|S| - \varepsilon_2(S) + 1} \le \frac{k+|S|}{2|S| - 1} = \frac{1}{2} + \frac{k+1/2}{2|S| - 1} \le \frac{1}{2} + \frac{k+1/2}{2 \times 2 - 1} = \frac{k+2}{3}, \tag{42}$$

or

$$\frac{k+2}{2} < \tau(G) \le \frac{|U \cup S|}{\omega(G - U \cup S) - 1} = \frac{|U| + |S|}{\omega\left(G' - S\right) - 1} \le \frac{k+|S|}{\operatorname{sun}\left(G' - S\right) - 1} \le \frac{k+|S|}{2|S| - \varepsilon_2(S) + 1 - 1} \le \frac{k+|S|}{2|S| - 2} = \frac{1}{2} + \frac{k+1}{2|S| - 2} \le \frac{1}{2} + \frac{k+1}{2 \times 2 - 2} = \frac{k+2}{2}, \tag{43}$$

a contradiction.

Hence, Theorem 8 is verified.

Theorem 9. *Let $k \in N \cup \{0\}$ and G be a graph with $\kappa(G) \ge k + 1$. If $(G) > (k+1)/2(resp.I'(G) > k+1)$, then, G is a $(P_{\ge 2}, k)$-factor critical covered graph.*

Proof. If G is complete, we check the theorem using $\delta(G) \ge \kappa(G) \ge k + 1$. Hence, we only consider noncomplete graph in the following contents. □

For any $U \subseteq V(G)$ with $|U| = k$, set $G' = G - U$. To demonstrate G that is $(P_{\ge 2}, k)$-factor critical covered, it is enough to prove G' is $P_{\ge 2}$-factor covered. Otherwise, suppose G' is not $P_{\ge 2}$-factor covered; then, using Lemma 2, there is a vertex subset S of G' satisfying (2).

The following discussion is divided into three cases in terms of the value of $|S|$.

Case 1. $|S| = 0$.

In this case, we get contradiction as we discussed in Theorem 7.

Case 2. $|S| = 1$.

We consider the following two subcases.

Case 3. $G' - S$ has no nontrivial component.

We infer $\varepsilon_1(S) = 0$ and $i(G' - S) \ge 2|S| - \varepsilon_1(S) + 1 = 3$. By means of the definition of isolated toughness, we deduce

$$\frac{k+1}{2} < I(G) \le \frac{|U \cup S|}{i(G - U \cup S)} \le \frac{k+1}{3}, \tag{44}$$

or

$$k + 1 < I'(G) \le \frac{|U \cup S|}{i(G - U \cup S) - 1} \le \frac{k+1}{2}, \tag{45}$$

a contradiction.

Case 4. $G' - S$ has nontrivial component.

We yield $\varepsilon_1(S)=1$ and $i(G'-S)\ge 2|S|-\varepsilon_1(S)+1=2$. Using the definition of isolated toughness, we have

$$\frac{k+1}{2}<I(G)\le\frac{|U\cup S|}{i(G-U\cup S)}\le\frac{k+1}{2},\tag{46}$$

or

$$k+1<I'(G)\le\frac{|U\cup S|}{i(G-U\cup S)-1}\le k+1,\tag{47}$$

a contradiction.

Case 5. $|S|\ge 2$.

We acquire $\varepsilon_1(S)\le 2$ and $i(G'-S)\ge 2|S|-\varepsilon_1(S)+1\ge 3$. We can get the contradiction using the similar derivation to Theorem 7.

Therefore, we get the desired result.

Theorem 10. *Let $k\in N$ and G be a graph with $\kappa(G)\ge k+1$ and $|V(G)|\ge k+3$. If $I(G)>(k+3)/2$ (resp. $I'(G)>k+3$), then, G is a $(P_{\ge 3},k)$-factor critical covered graph.*

Proof. If G is a complete graph, the result is hold from $|V(G)|\ge k+3$. We only discuss noncomplete graph in the following context.

For any $U\subseteq V(G)$ with k vertices, let $G'=G-U$, and we aim to prove G' is $P_{\ge 3}$-factor covered. On the contrary, G is not a $P_{\ge 3}$-factor covered graph; then, using Lemma 3, there is a subset S of $V(G')$ satisfying (3).

The following discussion is divided into three cases according to how many elements in S. □

Case 1. $|S|=0$.

In this case, similar to what's discussed in Theorem 8, we have $\varepsilon_2(S)=0$ and $\mathrm{sun}(G')=\omega(G')=1$, and G' is a big sun. Let R be the factor-critical of G' with $|V(R)|\ge 3$. Using the definition of $I(G)$, we obtain

$$\begin{aligned}\frac{k+3}{2}<I(G)&\le\frac{|U\cup V(R)|}{i(G-U\cup V(R))}=\frac{k+|V(R)|}{|V(R)|}\\&=1+\frac{k}{|V(R)|}\le 1+\frac{k}{3}=\frac{k+3}{3},\end{aligned}\tag{48}$$

or

$$\begin{aligned}k+3<I'(G)&\le\frac{|U\cup V(R)|}{i(G-U\cup V(R))-1}\\&=\frac{k+|V(R)|}{|V(R)|-1}=1+\frac{k+1}{|V(R)|-1}\\&\le 1+\frac{k+1}{3-1}=\frac{k+3}{2},\end{aligned}\tag{49}$$

a contradiction.

Case 2. $|S|=1$.

We have $\varepsilon_2(S)\le 1$. Suppose that there are K_1's, bK_2's, and c big sun components $H_1,\cdots,H_c$ with $|V(H_i)|\ge 6$ in $G'-S$. Hence, $a+b+c=\mathrm{sun}(G'-S)\ge 2|S|-\varepsilon_2(S)+1\ge 2$ by (3). We select one vertex from each K_2 and choose vertex set of factor-critical subgraph of every big sun and then denote X by the vertex set of all these selected vertices. We infer $|X|=b+\sum_{i=1}^{c}|V(H_i)|/2$ and $i(G-U\cup S\cup X)\ge 2$. In terms of the definition of isolated toughness, we yield

$$\begin{aligned}\frac{k+3}{2}<I(G)&\le\frac{|U\cup S\cup X|}{i(G-U\cup S\cup X)}=\frac{|U|+|S|+|X|}{i\left(G'-S\cup X\right)}\\&\le\frac{k+1+b+\sum_{i=1}^{c}(|V(H_i)|/2)}{a+b+\sum_{i=1}^{c}(|V(H_i)|/2)}.\end{aligned}\tag{50}$$

It implies

$$\begin{aligned}2k+2&>(k+3)a+(k+1)b+(k+1)\sum_{i=1}^{c}\frac{|V(H_i)|}{2}\\&\ge(k+3)a+(k+1)b+(3k+3)c\\&\ge(k+1)(a+b+c)\ge 2k+2,\end{aligned}\tag{51}$$

a contradiction.

For $I'(G)$, we have

$$\begin{aligned}k+3<I'(G)&\le\frac{|U\cup S\cup X|}{i(G-U\cup S\cup X)-1}\\&=\frac{|U|+|S|+|X|}{i\left(G'-S\cup X\right)-1}\le\frac{k+1+b+\sum_{i=1}^{c}(|V(H_i)|/2)}{a+b+\sum_{i=1}^{c}(|V(H_i)|/2)-1}.\end{aligned}\tag{52}$$

It implies

$$\begin{aligned}2k+4&>(k+3)a+(k+2)b+(k+2)\sum_{i=1}^{c}\frac{|V(H_i)|}{2}\\&\ge(k+3)a+(k+2)b+(3k+6)c\\&\ge(k+2)(a+b+c)\ge 2k+4,\end{aligned}\tag{53}$$

a contradiction.

Case 3. $|S|\ge 2$.

In this case, we acquire $\varepsilon_2(S)\le 2$ and $a+b+c=\mathrm{sun}(G'-S)\ge 2|S|-\varepsilon_2(S)+1\ge 3$ in terms of (3). Let X be vertex subset defined as Case 2. We verify

$$\begin{aligned}\frac{k+3}{2}<I(G)&\le\frac{|U\cup S\cup X|}{i(G-U\cup S\cup X)}=\frac{|U|+|S|+|X|}{i\left(G'-S\cup X\right)}\\&\le\frac{k+|S|+b+\sum_{i=1}^{c}(|V(H_i)|/2)}{a+b+\sum_{i=1}^{c}(|V(H_i)|/2)},\end{aligned}\tag{54}$$

that is,

$$\begin{aligned}2k+2|S| &> (3+k)a+(k+1)b+(k+1)\sum_{i=1}^{c}\frac{|V(H_i)|}{2}\\ &\geq (3+k)a+(k+1)b+(3k+3)c\\ &\geq (k+1)(a+b+c)\geq (k+1)(2|S|-\varepsilon_2(S)+1)\\ &\geq (k+1)(2|S|-1).\end{aligned} \tag{55}$$

It is implies that $|S|<(3k+1)/2k\leq 2$ since $k\geq 1$, a contradiction.

For $I'(G)$, we confirm

$$\begin{aligned}k+3<I'(G) &\leq \frac{|U\cup S\cup X|}{i(G-U\cup S\cup X)-1}=\frac{|U|+|S|+|X|}{i\left(G'-S\cup X\right)-1}\\ &\leq \frac{k+|S|+b+\sum_{i=1}^{c}(|V(H_i)|/2)}{a+b+\sum_{i=1}^{c}(|V(H_i)|/2)-1},\end{aligned} \tag{56}$$

which means,

$$\begin{aligned}2k+3+|S| &> (3+k)a+(k+2)b+(k+2)\sum_{i=1}^{c}\frac{|V(H_i)|}{2}\\ &\geq (3+k)a+(k+2)b+(3k+6)c\\ &\geq (k+2)(a+b+c)\geq (k+2)(2|S|-\varepsilon_2(S)+1)\\ &\geq (k+2)(2|S|-1).\end{aligned} \tag{57}$$

It implies that $|S|<(3k+5)/(2k+3)\leq 2$, a contradiction. Hence, Theorem 10 is verified.

Note that $k\neq 0$ in Theorem 10. From Zhou et al. [21], we know that G is a $P_{\geq 3}$-factor covered graph if $I(G)>5/3$, and 5/3 is tight.

2.3. Toughness Conditions for Factor Uniform Graph. A graph G is a $P_{\geq n}$-factor uniform graph if for any two edges e_1 and e_2, G admits a $P_{\geq n}$-factor including e_1 and excluding e_2. Zhou and Sun [?] studied the binding number condition for $P_{\geq 2}$-factor uniform graph and $P_{\geq 3}$-factor uniform graph. In this section, we research on other two parameters: toughness and isolated toughness. The idea to prove the following results is based on the observation that G is $P_{\geq n}$-factor uniform if $G-e$ is $P_{\geq n}$-covered for any $e\in E(G)$.

Theorem 11. *Let G be a 2-edge-connected graph. If $(G)>1$ (resp.$\tau(G)>2$), then, G is a $P_{\geq 2}$-factor uniform graph.*

Proof. For any $e=uv$, $G'=G-e$ is connected since G is 2-edge-connected graph. To confirm Theorem 11, we need to verify that G' is $P_{\geq 2}$-factor covered. If not, we assume that G' is not $P_{\geq 2}$-factor covered. Using Lemma 2, there is a vertex subset S of G' satisfying

$$i\left(G'-S\right)\geq 2|S|-\varepsilon_1(S)+1. \tag{58}$$

Furthermore, we have $i(G-S)\leq i(G'-S)\leq i(G-S)+2$. We consider three cases according to the value of $|S|$. □

Case 1. If $|S|=0$.

We obtain $i(G')\geq 1$ which contradicts $\lambda(G)\geq 2$.

Case 2. If $|S|=1$.

Then, $\varepsilon_1(S)\leq 1$ and $i(G'-S)\geq 2|S|-\varepsilon_1(S)+1\geq 2$. If $i(G-S)\geq 2$, then

$$1<t(G)\leq \frac{|S|}{\omega(G-S)}\leq \frac{|S|}{i(G-S)}\leq \frac{1}{2}, \tag{59}$$

or

$$2<\tau(G)\leq \frac{|S|}{\omega(G-S)-1}\leq \frac{|S|}{i(G-S)-1}\leq 1, \tag{60}$$

a contradiction.

If $i(G-S)=1$, then, $e=uv\in E(G-S)$ and $\omega(G-S)\geq 2$. We infer

$$1<t(G)\leq \frac{|S|}{\omega(G-S)}\leq \frac{1}{2}, \tag{61}$$

or

$$2<\tau(G)\leq \frac{|S|}{\omega(G-S)-1}\leq 1, \tag{62}$$

a contradiction.

If $i(G-S)=0$, then, K_2 is a component in $G-S$ and $e=uv\in E(K_2)$. If there is another component in $G-S$ except K_2, then, $\omega(G-S)\geq 2$, and we get the contradiction similar to the derivation above. If $\omega(G-S)=1$, then, $G\cong K_3$ since G is 2-edge-connected graph. Special for K_3, we yield $t(K_3)=\tau(K_3)=+\infty$, $G'=P_3$ which is a $P_{\geq 2}$-factor covered graph. Hence, K_3 satisfies the condition of theorem which is a $P_{\geq 2}$-factor uniform graph.

Case 3. If $|S|\geq 2$.

Then, $\varepsilon_1(S)\leq 2$, $i(G'-S)\geq 2|S|-\varepsilon_1(S)+1\geq 3$ and $i(G-S)\geq i(G'-S)-2\geq 1$.

Notice that if $i(G-S)\neq i(G'-S)$, then, $e\in E(G-S)$ and $\omega(G-S)\geq i(G-S)+1\geq i(G'-S)-2+1=i(G'-S)-1$. If $i(G-S)=i(G'-S)$, then, $\omega(G-S)\geq i(G-S)=i(G'-S)$. Combining the above two cases, we have $\omega(G-S)\geq i(G'-S)-1$.

If $i(G-S)\geq 2$, then

$$\begin{aligned}1<t(G)&\leq\frac{|S|}{\omega(G-S)}\leq\frac{|S|}{i\left(G'-S\right)-1}\\&\leq\frac{|S|}{2|S|-\varepsilon_1(S)+1-1}\leq\frac{|S|}{2|S|-2},\end{aligned}\tag{63}$$

or

$$\begin{aligned}2<\tau(G)&\leq\frac{|S|}{\omega(G-S)-1}\leq\frac{|S|}{i\left(G'-S\right)-1-1}\\&\leq\frac{|S|}{2|S|-\varepsilon_1(S)+1-2}\leq\frac{|S|}{2|S|-3}.\end{aligned}\tag{64}$$

It implies $|S|<2$, a contradiction.

If $i(G-S)=1$, then, using the fact that $i(G'-S)\geq 3$, we confirm that K_1 and K_2 are components in $G-S$, $e=uv\in E(K_2)$, and $i(G'-S)=i(G-S)+2=3$. We acquire

$$\begin{aligned}1<t(G)&\leq\frac{|S|}{\omega(G-S)}\leq\frac{|S|}{i(G-S)+1}=\frac{|S|}{i\left(G'-S\right)-1}\\&\leq\frac{|S|}{2|S|-\varepsilon_1(S)+1-1}\leq\frac{|S|}{2|S|-2},\end{aligned}\tag{65}$$

or

$$\begin{aligned}2<\tau(G)&\leq\frac{|S|}{\omega(G-S)-1}\leq\frac{|S|}{i(G-S)+1-1}\\&=\frac{|S|}{i\left(G'-S\right)-2}\leq\frac{|S|}{2|S|-\varepsilon_1(S)+1-2}\leq\frac{|S|}{2|S|-3}.\end{aligned}\tag{66}$$

Again, in both situation we get $|S|<2$, which leads to a contradiction.

Theorem 12. *Let G be a 2-edge-connected graph. If $(G)>1$ (resp.$\tau(G)>2$), then, G is a $P_{\geq 3}$-factor uniform graph.*

Proof. For any $e=uv\in E(G)$, $G'=G-e$ is connected, and we only need to prove that G' is $P_{\geq 3}$-factor covered. On the contrary, G' is not $P_{\geq 3}$-factor covered, and we can find a subset S of $V(G')$ such that

$$\operatorname{sun}\left(G'-S\right)\geq 2|S|-\varepsilon_2(S)+1.\tag{67}$$

The following discussion is divided into three cases according to the value of $|S|$. □

Case 1. $|S|=0$.

Then, $\varepsilon_2(S)=0$ and $\operatorname{sun}(G')\geq 1$ by (67). It implies sun $(G')=1$, and G' is a big sun with at least six vertices. Moreover, G is a graph constructed by adding an edge in a big sun. Let R be the factor-critical of G' and $x\in V(R)$. We have

$$1<t(G)\leq\frac{|V(R)\setminus\{x\}|}{\omega(G-V(R)\setminus\{x\})}\leq\frac{|R|-1}{|R|-1}=1,\tag{68}$$

or

$$\begin{aligned}2<\tau(G)&\leq\frac{|V(R)\setminus\{x\}|}{\omega(G-V(R)\setminus\{x\})-1}\\&\leq\frac{|R|-1}{|R|-2}=1+\frac{1}{|R|-2}\leq 1+\frac{1}{3-2}=2,\end{aligned}\tag{69}$$

a contradiction.

Case 2. $|S|=1$.

Then, $\varepsilon_2(S)\leq 1$ and $\operatorname{sun}(G'-S)\geq 2$ by (67). If $\omega(G-S)\geq 2$, then

$$1<t(G)\leq\frac{|S|}{\omega(G-S)}\leq\frac{1}{2},\tag{70}$$

or

$$2<\tau(G)\leq\frac{|S|}{\omega(G-S)-1}\leq 1,\tag{71}$$

a contradiction. If $\omega(G-S)=1$, then, $e\in E(G-S)$, and it produces two sun components after deleting e from $G-S$. If $G-S$ isomorphic to K_2, then, $G\cong K_3$ which is a $P_{\geq 3}$-factor uniform graph. Otherwise, $|V(G-S)|\geq 3$, and there are at least two vertices having degree 1 in $G-S$. Let $xy\in E(G-S)$ such that $d_{G-S}(x)=1$. We acquire $1<t(G)\leq|S\cup\{y\}|/\omega(G-S\cup\{y\})\leq 1$ or $2<\tau(G)\leq|S\cup\{y\}|/\omega(G-S\cup\{y\})-1\leq 2$, a contradiction.

Case 3. $|S|\geq 2$.

In this case, $\varepsilon_2(S)\leq 2$, $\operatorname{sun}(G'-S)\geq 3$ by (67), $\operatorname{sun}(G-S)\geq\operatorname{sun}(G'-S)-2\geq 1$, and $\omega(G-S)\geq 2$. If $\operatorname{sun}(G-S)=\operatorname{sun}(G'-S)$ or $\operatorname{sun}(G-S)=\operatorname{sun}(G'-S)-1$, we deduce

$$\begin{aligned}1<t(G)&\leq\frac{|S|}{\omega(G-S)}\leq\frac{|S|}{\operatorname{sun}(G-S)}\\&\leq\frac{|S|}{\operatorname{sun}\left(G'-S\right)-1}\leq\frac{|S|}{2|S|-\varepsilon_2(S)+1-1}\\&\leq\frac{|S|}{2|S|-2}=\frac{1}{2}+\frac{1}{2|S|-2}\leq\frac{1}{2}+\frac{1}{2\times 2-2}=1,\end{aligned}\tag{72}$$

or

$$\begin{aligned}2<\tau(G) &\leq \frac{|S|}{\omega(G-S)-1} \leq \frac{|S|}{\operatorname{sun}(G-S)-1}\\ &\leq \frac{|S|}{\operatorname{sun}\left(G'-S\right)-1-1} \leq \frac{|S|}{2|S|-\varepsilon_2(S)+1-2}\\ &\leq \frac{|S|}{2|S|-3} = \frac{1}{2}+\frac{3}{2(2|S|-3)} \leq \frac{1}{2}+\frac{3}{2(2\times 2-3)} = 2,\end{aligned} \tag{73}$$

a contradiction.

If $\operatorname{sun}(G-S)=\operatorname{sun}(G'-S)-2$, then, edge $e=uv$ belongs to a nonsun component W, while removing e will produce two sun components. It means at least one of u and v is a cut vertex of component W, and without loss of generality, we set u as a cut vertex in W. Hence, we get

$$\begin{aligned}1<t(G) &\leq \frac{|S\cup\{u\}|}{\omega(G-S\cup\{u\})} \leq \frac{|S\cup\{u\}|}{\omega(G-S)+1}\\ &\leq \frac{|S|+1}{\operatorname{sun}(G-S)+2} \leq \frac{|S|+1}{\operatorname{sun}\left(G'-S\right)-2+2}\\ &\leq \frac{|S|+1}{2|S|-\varepsilon_2(S)+1} \leq \frac{|S|+1}{2|S|-1} = \frac{1}{2}+\frac{3}{2(2|S|-1)}\\ &\leq \frac{1}{2}+\frac{3}{2(2\times 2-1)} = 1,\end{aligned} \tag{74}$$

or

$$\begin{aligned}2<\tau(G) &\leq \frac{|S\cup\{u\}|}{\omega(G-S\cup\{u\})-1} \leq \frac{|S\cup\{u\}|}{\omega(G-S)+1-1}\\ &\leq \frac{|S|+1}{\operatorname{sun}(G-S)+1} \leq \frac{|S|+1}{\operatorname{sun}\left(G'-S\right)-2+1}\\ &\leq \frac{|S|+1}{2|S|-\varepsilon_2(S)+1-1} \leq \frac{|S|+1}{2|S|-2}\\ &= \frac{1}{2}+\frac{1}{|S|-1} \leq \frac{1}{2}+\frac{1}{2-1} = \frac{3}{2},\end{aligned} \tag{75}$$

a contradiction.

Thus, the proof of Theorem 12 is completed.

Theorem 13. *Let G be a 2-edge-connected graph. If $(G)>(|V(G)|-2)/2$ (resp. $I'(G)>|V(G)|-2$), then, G is a $P_{\geq 2}$-factor uniform graph.*

Proof. Clearly, we have $|V(G)|\geq 3$. For any $e=uv$, $G'=G-e$ is connected since G is a 2-edge-connected graph. Similar as Theorem 11, we only need to verify that G' is $P_{\geq 2}$-factor covered. In contrast, suppose that G' is not $P_{\geq 2}$-factor covered. In terms of Lemma 2, there is a vertex subset S of G' that meets (58). Furthermore, $i(G'-S)\in\{i(G-S), i(G-S)+1, i(G-S)+2\}$.

We consider three cases in view of the value of $|S|$. □

Case 1. $|S|=0$.

We get $i(G')\geq 1$ which contradicts to $\lambda(G)\geq 2$.

Case 2. $|S|=1$.

Then, $\varepsilon_1(S)\leq 1$ and $i(G'-S)\geq 2|S|-\varepsilon_1(S)+1\geq 2$. If $i(G-S)\geq 2$, then

$$\frac{|V(G)|-2}{2}<I(G)\leq \frac{|S|}{i(G-S)}\leq \frac{1}{2}, \tag{76}$$

or

$$|V(G)|-2<I'(G)\leq \frac{|S|}{i(G-S)-1}\leq 1, \tag{77}$$

a contradiction.

If $i(G-S)=1$, then, $e=uv\in E(G-S)$ and assume $d_{G-S}(u)\geq d_{G-S}(v)=1$. We infer

$$\frac{|V(G)|-2}{2}<I(G)\leq \frac{|S\cup\{u\}|}{i(G-S\cup\{u\})}\leq 1, \tag{78}$$

or

$$|V(G)|-2<I'(G)\leq \frac{|S\cup\{u\}|}{i(G-S\cup\{u\})-1}\leq 2, \tag{79}$$

a contradiction.

If $i(G-S)=0$, then, K_2 is a component in $G-S$ and $e=uv\in E(K_2)$. If there is another component in $G-S$ except K_2, then denote this component by W. Select $w\in V(W)$ such that w has a minimum degree in $G-S$ among all vertices in W. Hence, $i(G-S\cup\{u\}\cup N_{G-S}(w))\geq 2$ and

$$\begin{aligned}\frac{|V(G)|-2}{2}<I(G) &\leq \frac{|S\cup\{u\}\cup N_{G-S}(w)|}{i(G-S\cup\{u\}\cup N_{G-S}(w))}\\ &\leq \frac{2+|V(W)|-1}{2} = \frac{1+|V(W)|}{2}\\ &\leq \frac{1+|V(G)|-3}{2} = \frac{|V(G)|-2}{2},\end{aligned} \tag{80}$$

or

$$\begin{aligned}|V(G)|-2<I'(G) &\leq \frac{|S\cup\{u\}\cup N_{G-S}(w)|}{i(G-S\cup\{u\}\cup N_{G-S}(w))-1}\\ &\leq \frac{2+|V(W)|-1}{2-1} = 1+|V(W)\\ &\leq 1+|V(G)|-3 = |V(G)|-2,\end{aligned} \tag{81}$$

a contradiction. If $\omega(G-S)=1$, then, G becomes K_3. As discussed in Theorem 11, K_3 meets the condition of Theorem 13 that is a $P_{\geq 2}$-factor uniform graph.

Case 3. $|S|\geq 2$.

Then, $\varepsilon_1(S) \leq 2$, $i(G' - S) \geq 2|S| - \varepsilon_1(S) + 1 \geq 3$ and $i(G - S) \geq i(G' - S) - 2 \geq 1$. We consider the following subcases in light of the value of $i(G - S)$.

Case 4. $i(G - S) \geq 2$.

If $i(G - S) = i(G' - S)$, then $|V(G)| \geq 4$,

$$\begin{aligned}1 &\leq \frac{|V(G)| - 2}{2} < I(G) \leq \frac{|S|}{i(G - S)} \leq \frac{|S|}{i\left(G' - S\right)} \\ &\leq \frac{|S|}{2|S| - \varepsilon_1(S) + 1} \leq \frac{|S|}{2|S| - 1} = \frac{1}{2} + \frac{1}{2(2|S| - 1)} \\ &\leq \frac{1}{2} + \frac{1}{2(2 \times 2 - 1)} = \frac{2}{3},\end{aligned} \tag{82}$$

or

$$\begin{aligned}2 &\leq |V(G)| - 2 < I'(G) \leq \frac{|S|}{i(G - S) - 1} \\ &\leq \frac{|S|}{i\left(G' - S\right) - 1} \leq \frac{|S|}{2|S| - \varepsilon_1(S) + 1 - 1} \\ &\leq \frac{|S|}{2|S| - 2} = \frac{1}{2} + \frac{1}{2|S| - 2} \leq \frac{1}{2} + \frac{1}{2 \times 2 - 2} = 1,\end{aligned} \tag{83}$$

a contradiction.

If $i(G - S) \neq i(G' - S)$, then, $|V(G)| \geq 6$,

$$\begin{aligned}2 &\leq \frac{|V(G)| - 2}{2} < I(G) \leq \frac{|S|}{i(G - S)} \leq \frac{|S|}{i\left(G' - S\right) - 2} \\ &\leq \frac{|S|}{2|S| - \varepsilon_1(S) + 1 - 2} \leq \frac{|S|}{2|S| - 3} \\ &= \frac{1}{2} + \frac{3}{2(2|S| - 3)} \leq \frac{1}{2} + \frac{3}{2(2 \times 2 - 3)} = 2,\end{aligned} \tag{84}$$

a contradiction. For $\tau(G)$, if $|S| \geq 3$, then

$$\begin{aligned}4 &\leq |V(G)| - 2 < I'(G) \leq \frac{|S|}{i(G - S) - 1} \\ &\leq \frac{|S|}{i\left(G' - S\right) - 2 - 1} \leq \frac{|S|}{2|S| - \varepsilon_1(S) + 1 - 3} \\ &\leq \frac{|S|}{2|S| - 4} = \frac{1}{2} + \frac{2}{2|S| - 4} \leq \frac{1}{2} + \frac{2}{2 \times 3 - 4} = \frac{3}{2},\end{aligned} \tag{85}$$

a contradiction. If $|S| = 2$, we can easily check that $4 \leq |V(G)| - 2 < I'(G) \leq |S|/i(G - S) - 1 \leq 2$, a contradiction.

Case 5. $i(G - S) = 1$.

Since $i(G' - S) \geq 3$, we confirm that K_1 and K_2 are components in $G - S$, $e = uv \in E(K_2)$ and $i(G' - S) = i(G - S) + 2 = 3$. Using $|V(G)| \geq 5$, we acquire

$$\begin{aligned}\frac{3}{2} &\leq \frac{|V(G)| - 2}{2} < I(G) \leq \frac{|S \cup \{u\}|}{i(G - S \cup \{u\})} \\ &= \frac{|S| + 1}{i(G - S) + 1} = \frac{|S| + 1}{i\left(G' - S\right) - 2 + 1} \\ &\leq \frac{|S| + 1}{2|S| - \varepsilon_1(S) + 1 - 1} \leq \frac{|S| + 1}{2|S| - 2} \\ &= \frac{1}{2} + \frac{2}{2|S| - 2} \leq \frac{1}{2} + \frac{2}{2 \times 2 - 2} = \frac{3}{2},\end{aligned} \tag{86}$$

or

$$\begin{aligned}3 &\leq |V(G)| - 2 < I'(G) \leq \frac{|S \cup \{u\}|}{i(G - S \cup \{u\}) - 1} \\ &= \frac{|S| + 1}{i(G - S) + 1 - 1} = \frac{|S| + 1}{i\left(G' - S\right) - 2} \\ &\leq \frac{|S| + 1}{2|S| - \varepsilon_1(S) + 1 - 2} \leq \frac{|S| + 1}{2|S| - 3} \\ &= \frac{1}{2} + \frac{5}{2(2|S| - 3)} \leq \frac{1}{2} + \frac{5}{2(2 \times 2 - 3)} = 3,\end{aligned} \tag{87}$$

a contradiction.

Thus, we confirm that Theorem 13 is established.

Theorem 14. *Let G be a 2-edge-connected graph. If $(G) > (|V(G)| - 2)/2$(resp.$I'(G) > |V(G)| - 2$), then, G is a $P_{\geq 3}$-factor uniform graph.*

Proof. For any $e = uv \in E(G)$, $G' = G - e$ is connected, and we only need to prove that G' is $P_{\geq 3}$-factor covered. On the contrary, G' is not $P_{\geq 3}$-factor covered. Then, there exists a subset S of $V(G')$ satisfying (67). □

Let a, b, c be the number of K_1 components, K_2 components, and big sun components in $G - S$, respectively. Let $H_1, \cdots, H_c$ be big sun components in $G - S$ with $|V(H_i)| \geq 6$. Choosing one vertex from each K_2 component in $G - S$ and let X be the set of these vertices. Set R_i as the factor-critical subgraph of H_i and $Y = \cup_{i=1}^{c} V(R_i)$. We have $|X| = b$, $|Y| = \sum_{i=1}^{c} |H_i|/2$ and $a + b + c = \mathrm{sun}(G - S) \geq \mathrm{sun}(G' - S) - 2$. The following discussion is divided into three cases according to the value of $|S|$.

Case 1. $|S| = 0$.

Then, $\varepsilon_2(S) = 0$ and $\mathrm{sun}(G') \geq 1$ by (67). It implies $\mathrm{sun}(G') = 1$, G' is a big sun with at least six vertices, and $|V(G)| \geq 6$. Moreover, G is a graph constructed by adding an edge in a big sun. Let R be the factor-critical of G'. We obtain

$$
\begin{aligned}
2 \le \frac{|V(G)|-2}{2} < I(G) &\le \frac{|V(R)|}{i(G-V(R))} \\
&\le \frac{|R|}{|R|-1} = 1 + \frac{1}{|R|-1} \le 1 + \frac{1}{3-1} = \frac{3}{2},
\end{aligned} \tag{88}
$$

or

$$
\begin{aligned}
4 \le |V(G)| - 2 < I'(G) &\le \frac{|V(R)|}{i(G-V(R))-1} \\
&\le \frac{|R|}{|R|-2} = 1 + \frac{2}{|R|-2} \le 1 + \frac{2}{3-2} = 3,
\end{aligned} \tag{89}
$$

a contradiction.

Case 2. $|S| = 1$.

In this case, $\varepsilon_2(S) \le 1$, $\mathrm{sun}(G' - S) \ge 2$ by (67), and $a = 0$ since $|S| = 1$ and G is 2-edge-connected.

Case 3. $\mathrm{sun}(G - S) = \mathrm{sun}(G' - S)$.

We get $i(G - S \cup X \cup Y) = b + \sum_{i=1}^{c} |H_i|/2 \ge b + 3c \ge b + c = \mathrm{sun}(G - S) = \mathrm{sun}(G' - S) \ge 2$ and $|V(G)| \ge 4$ (if $|V(G)| = 3$, then $G \cong K_3$, $G - S$ isomorphic to K_2 which contradicts to $\mathrm{sun}(G - S) = \mathrm{sun}(G' - S) \ge 2$).

If $|V(G)| \ge 6$, using the definition of isolated toughness, we have

$$
\begin{aligned}
2 \le \frac{|V(G)|-2}{2} < I(G) &\le \frac{|S \cup X \cup Y|}{i(G - S \cup X \cup Y)} \\
&\le \frac{1 + b + \sum_{i=1}^{c}(|H_i|/2)}{b + \sum_{i=1}^{c}(|H_i|/2)},
\end{aligned} \tag{90}
$$

which implies $b + \sum_{i=1}^{c} |H_i|/2 < 1$, a contradiction. For $I'(G)$, we yield

$$
\begin{aligned}
4 \le |V(G)| - 2 < I'(G) &\le \frac{|S \cup X \cup Y|}{i(G - S \cup X \cup Y) - 1} \\
&\le \frac{1 + b + \sum_{i=1}^{c}(|H_i|/2)}{b + \sum_{i=1}^{c}(|H_i|/2) - 1},
\end{aligned} \tag{91}
$$

which implies $3b + 3\sum_{i=1}^{c} |H_i|/2 < 5$, contradicting to $b + c \ge 2$.

If $|V(G)| = 5$, then, $c = a = 0$ and

$$
\frac{3}{2} = \frac{|V(G)|-2}{2} < I(G) \le \frac{|S \cup X|}{i(G - S \cup X)} \le \frac{1+b}{b}, \tag{92}
$$

or

$$
3 = |V(G)| - 2 < I'(G) \le \frac{|S \cup X|}{i(G - S \cup X) - 1} \le \frac{1+b}{b-1}, \tag{93}
$$

which implies $b < 2$ which contradicts to $b = \mathrm{sun}(G - S) = \mathrm{sun}(G' - S) \ge 2$.

If $|V(G)| = 4$, then, $c = a = 0$ and $b = 1$ contradicting to $b = \mathrm{sun}(G - S) = \mathrm{sun}(G' - S) \ge 2$.

Case 4. $\mathrm{sun}(G - S) = \mathrm{sun}(G' - S) - 1$.

In this case, $\mathrm{sun}(G - S) \ge 1$ since $\mathrm{sun}(G' - S) \ge 2$.

Claim 1. If K_2 is one of components in $G - S$, then, $e \in E(K_2)$.

Proof. Suppose K_2 is a component in $G - S$ and $e \in E(K_2)$ is exactly a deleted edge, set $u \in V(K_2)$. If $G - S$ is isomorphic to K_2, then, G is isomorphic to K_3 which is clearly a $P_{\ge 3}$ -factor uniform graph. □

If there is a K_1 component in $G - S$, then, $|V(G)| \ge 4$ and

$$
1 \le \frac{|V(G)|-2}{2} < I(G) \le \frac{|S \cup \{u\}|}{i(G - S \cup \{u\})} \le \frac{2}{2} = 1, \tag{94}
$$

or

$$
2 \le |V(G)| - 2 < I'(G) \le \frac{|S \cup \{u\}|}{i(G - S \cup \{u\}) - 1} \le \frac{2}{2-1} = 2, \tag{95}
$$

a contradiction.

If there is another K_2 component or big sun component in $G - S$ (say W), then, there is a vertex x in W such that $d_{G-S}(x) = 1$ and assume $xy \in E(G - S)$. We have $|V(G)| \ge 5$ and

$$
\frac{3}{2} \le \frac{|V(G)|-2}{2} < I(G) \le \frac{|S \cup \{u, y\}|}{i(G - S \cup \{u, y\})} \le \frac{3}{2}, \tag{96}
$$

or

$$
3 \le |V(G)| - 2 < I'(G) \le \frac{|S \cup \{u, y\}|}{i(G - S \cup \{u, y\}) - 1} \le \frac{3}{2-1} = 3, \tag{97}
$$

a contradiction.

If there exists a nonsun component in $G - S$ (say M), the,n we select $x \in V(M)$ with its degree in $G - S$ as small as possible. We infer

$$
\begin{aligned}
\frac{|V(G)|-2}{2} < I(G) &\le \frac{|S \cup \{u\} \cup (V(M) - \{x\})|}{i(G - S \cup \{u\} \cup (V(M) - \{x\}))} \\
&\le \frac{|V(G)|-2}{2},
\end{aligned} \tag{98}
$$

or

$$
\begin{aligned}
|V(G)| - 2 < I'(G) &\le \frac{|S \cup \{u\} \cup (V(M) - \{x\})|}{i(G - S \cup \{u\} \cup (V(M) - \{x\})) - 1} \\
&\le |V(G)| - 2,
\end{aligned} \tag{99}
$$

a contradiction.

Hence, the claim is hold.

From Claim 1, we see that there is a nonsun component W in $G-S$ with $|V(W)| \geq 3$ (and hence, $|V(G)| \geq 5$), delete edge $e=uv$ from W, and then, it produces a new sun component in $G-S$. Thus, there is a vertex x in W with $d_{G-S}(x)=1$, and set $xy \in E(G-S)$. Note that $\mathrm{sun}(G-S) \geq 1$, if K_1 is a component in $G-S$, then, we yield

$$\frac{3}{2} \leq \frac{|V(G)|-2}{2} < I(G) \leq \frac{|S \cup \{y\}|}{i(G-S \cup \{y\})} \leq \frac{2}{2} = 1, \quad (100)$$

or

$$3 \leq |V(G)|-2 < I'(G) \leq \frac{|S \cup \{y\}|}{i(G-S \cup \{y\})-1} \leq \frac{2}{2-1} = 2, \quad (101)$$

a contradiction. If K_2 or a big sun is a component in $G-S$ (denote this sun component by M), then, there is a vertex x' in M with $d_{G-S}(x')=1$, and set $x'y' \in E(G-S)$. We acquire

$$\frac{3}{2} \leq \frac{|V(G)|-2}{2} < I(G) \leq \frac{\left|S \cup \left\{y, y'\right\}\right|}{i\left(G-S \cup \left\{y, y'\right\}\right)} \leq \frac{3}{2}, \quad (102)$$

or

$$3 \leq |V(G)|-2 < I'(G) \leq \frac{\left|S \cup \left\{y, y'\right\}\right|}{i\left(G-S \cup \left\{y, y'\right\}\right)-1} \leq \frac{3}{2-1} = 3, \quad (103)$$

a contradiction.

Case 3. $\mathrm{sun}(G-S) = \mathrm{sun}(G'-S)-2$.

In this case, there is a nonsun component W in $G-S$, and it produces two sun components after deleting $e=uv$ from W. Thus, there are at least two vertices $x, x' \in V(W)$ such that $d_{G-S}(x) = d_{G-S}(x') = 1$. Set $xy, x'y' \in E(W)$ and note that y and y' are allowed to be the same vertex (if $W \cong P_3$). If $W \cong P_3$, then, $y=y'$, $|V(G)| \geq 4$, and

$$1 \leq \frac{|V(G)|-2}{2} < I(G) \leq \frac{|S \cup \{y\}|}{i(G-S \cup \{y\})} \leq \frac{2}{2} = 1, \quad (104)$$

or

$$2 \leq |V(G)|-2 < I'(G) \leq \frac{|S \cup \{y\}|}{i(G-S \cup \{y\})-1} \leq \frac{2}{2-1} = 2, \quad (105)$$

a contradiction. Otherwise, $|V(G)| \geq 5$, and

$$\frac{3}{2} \leq \frac{|V(G)|-2}{2} < I(G) \leq \frac{\left|S \cup \left\{y, y'\right\}\right|}{i\left(G-S \cup \left\{y, y'\right\}\right)} \leq \frac{3}{2}, \quad (106)$$

or

$$3 \leq |V(G)|-2 < I'(G) \leq \frac{\left|S \cup \left\{y, y'\right\}\right|}{i\left(G-S \cup \left\{y, y'\right\}\right)-1} \leq \frac{3}{2-1} = 3, \quad (107)$$

a contradiction.

Case 4. $|S| \geq 2$.

In this case, $\varepsilon_2(S) \leq 2$, $a+b+c = \mathrm{sun}(G'-S) \geq 3$ by (67), and $\mathrm{sun}(G-S) \geq \mathrm{sun}(G'-S)-2 \geq 1$. We have $|V(G)| \geq 5$,

$$\begin{aligned} \frac{3}{2} &\leq \frac{|V(G)|-2}{2} < I(G) \\ &\leq \frac{|S \cup X \cup Y|}{i(G-S \cup X \cup Y)} \leq \frac{|S| + b + \sum_{i=1}^{c}(|H_i|/2)}{b + \sum_{i=1}^{c}(|H_i|/2)}. \end{aligned} \quad (108)$$

Then, the rest proof process is consistent with the part of Theorem 4 and Theorem 5 in Gao et al. [22], and we will not repeat here.

Hence, the proof of Theorem 14 is finished.

3. Sharpness

In this section, we present some counterexamples to verify that the bounds of parameters in theorems in the second section are tight.

3.1. Sharpness of Theorem 4-Theorem 6. We manifest that (1) $\lambda(G) \geq m+1$ and $t(G) > m/(m+1)$ or $\tau(G) > 1$ in Theorem 4 cannot change to $\lambda(G) \geq m$ and $t(G) = m/(m+1)$ (or $\tau(G) = 1$); (2) $\lambda(G) \geq m+1$ and $I(G) > 2m/(m+1)$ or $I'(G) > 2$ in Theorem 5 cannot change to $\lambda(G) \geq m$ and $I(G) = 2m/(2m+1)$ (or $I'(G) = 1$); (3) $\lambda(G) \geq m+1$ and $\mathrm{bind}(G) > 3/2$ in Theorem 6 cannot change to $\lambda(G) \geq m$ and $\mathrm{bind}(G) = 3/2$.

Let $G = K_m \vee (mK_2 \cup K_1)$. Taking one vertex from each K_2 and denote X by the set of these vertices, we have

$$\begin{aligned} t(G) &= \frac{|V(K_m)|}{\omega(G-V(K_m))} = \frac{m}{m+1}, \\ \tau(G) &= \frac{|V(K_m)|}{\omega(G-V(K_m))-1} = \frac{m}{m+1-1} = 1, \\ I(G) &= \frac{|V(K_m) \cup X|}{i(G-V(K_m)-X)} = \frac{2m}{m+1}, \\ I'(G) &= \frac{|V(K_m) \cup X|}{i(G-V(K_m)-X)-1} = \frac{2m}{m+1-1} = 2, \\ bind(G) &= \frac{|N_G(V(mK_2))|}{|V(mK_2)|} = \frac{3m}{2m} = \frac{3}{2}. \end{aligned} \quad (109)$$

Set $E' = E(mK_2)$ and $G' = G - E' = K_m \vee ((2m+1)K_1)$. Then, $|E'| = m$, and by setting $S = K_m$, we have

$$i\left(G' - S\right) = 2m + 1 > 2m = 2|S|. \tag{110}$$

Thus, G' has no $P_{\geq 2}$-factor, and accordingly, G is not a $(P_{\geq 2}, m)$-factor-deleted graph.

3.2. Sharpness of Theorem 7 and Theorem 8. We show that the toughness bounds in Theorem 7 and Theorem 8 are best. Consider $G = K_{k+2} \vee (3K_1)$, and we have $\kappa(G) = k + 2$, $t(G) = (k+2)/3$ and $\tau(G) = (k+2)/2$. Set $U \subseteq V(G)$ with $|U| = k$, and let $G' = G - U = K_2 \vee (3K_1)$. Take $S = K_2$ in G', then, we have $\varepsilon_1(S) = \varepsilon_2(S) = 2$,

$$\begin{aligned} i\left(G' - S\right) &= 3 > 2 = 2|S| - \varepsilon_1(S), \\ \mathrm{sun}\left(G' - S\right) &= 3 > 2 = 2|S| - \varepsilon_2(S). \end{aligned} \tag{111}$$

Hence, according to Lemma 2, G' is not $P_{\geq 2}$-factor covered, and G is not a $(P_{\geq 2}, k)$-factor critical covered graph. Moreover, in terms of Lemma 3, G' is not $P_{\geq 3}$-factor covered, and G is not a $(P_{\geq 3}, k)$-factor critical covered graph.

3.3. Sharpness of Theorem 9. We depict that the isolated toughness bounds in Theorem 9 for a graph to be $(P_{\geq 2}, k)$-factor critical covered are best. Consider $G = K_{k+1} \vee (2K_1 \cup K_t)$ where t is enough large, and we have $\kappa(G) = k + 1$, $I(G) = (k + 1/2)(k+1)/2$ and $I'(G) = k + 1$. Set $U \subseteq V(G)$ with $|U| = k$, and let $G' = G - U = K_1 \vee (2K_1 \cup K_t)$. Set S as the first K_1 in G', then, we have $\varepsilon_1(S) = 1$ and

$$i\left(G' - S\right) = 2 > 1 = 2|S| - \varepsilon_1(S). \tag{112}$$

Hence, by means of Lemma 2, G' is not $P_{\geq 2}$-factor covered, and G is not a $(P_{\geq 2}, k)$-factor critical covered graph.

3.4. Sharpness of Theorem 10. The isolated toughness conditions in Theorem 10 are tight. Consider $G = K_{k+1} \vee (2K_2 \cup G'')$ where G'' is connected but not a sun. Set $U \subset V(K_{k+1})$ with $|U| = k$, $G' = G - U = K_1 \vee (2K_2 \cup G'')$, and $S = K_1$ in G'. Selecting one vertex from each K_2 in the $2K_2$ part and denoting X by the set of these two vertices, we confirm

$$\begin{aligned} I(G) &= \frac{|U \cup S \cup X|}{i(G - U \cup S \cup X)} = \frac{|U| + |S| + |X|}{i\left(G' - S \cup X\right)} \\ &= \frac{k+1+2}{2} = \frac{3+k}{2}, \\ I'(G) &= \frac{|U \cup S \cup X|}{i(G - U \cup S \cup X) - 1} \\ &= \frac{|U| + |S| + |X|}{i\left(G' - S \cup X\right) - 1} = \frac{k+1+2}{2-1} = k + 3. \end{aligned} \tag{113}$$

On the other hand, $\varepsilon_2(S) = 1$ since G'' is a nonsun component of $G' - S$ and

$$\mathrm{sun}\left(G' - S\right) = 2 > 1 = 2|S| - \varepsilon_2(S). \tag{114}$$

In view of Lemma 3, G' is not $P_{\geq 3}$-factor covered, and G is not a $(P_{\geq 3}, k)$-factor critical covered graph.

3.5. Sharpness of Theorem 11. The toughness bounds in Theorem 11 are tight. Consider $G = K_2 \vee (K_1 \cup K_2)$ which is 2-edge-connected graph with $t(G) = 1$ and $\tau(G) = 2$. Select $e \in E(K_1 \cup K_2)$ and set $G' = G - e = K_2 \vee (3K_1)$. Let $S = V(K_2) \subseteq V(G')$. We have $\varepsilon_1(S) = 2$ and

$$i\left(G' - S\right) = 3 > 2 = 2|S| - \varepsilon_1(S). \tag{115}$$

Therefore, by means of Lemma 2, G' is not $P_{\geq 2}$-factor covered, and G is not a $P_{\geq 2}$-factor uniform graph.

3.6. Sharpness of Theorem 12. The isolated toughness bounds in Theorem 12 are sharp. Consider $G = K_2 \vee (2K_2)$ which is a 2-edge-connected graph. We have $t(G) = 1$ and $\tau(G) = 2$. Let $e \in E(2K_2)$, $G' = G - e = K_2 \vee (K_2 \cup 2K_1)$, and S be the vertex set of first K_2 in G'. We infer $\varepsilon_2(S) = 2$ and

$$\mathrm{sun}\left(G' - S\right) = 3 > 2 = 2|S| - \varepsilon_2(S). \tag{116}$$

Hence, in terms of Lemma 3, G' is not $P_{\geq 3}$-factor covered, and G is not a $P_{\geq 3}$-factor uniform graph.

3.7. Sharpness of Theorem 13 and Theorem 14. To show the isolated toughness bounds in Theorem 13 and Theorem 14 that are sharp, we consider $G = K_1 \vee (K_2 \cup K_t)$ where t is a large number. Select one vertex from K_2 and $t - 1$ vertices from K_t and denote X by the vertex subset of these vertices. We have

$$\begin{aligned} I(G) &= \frac{|V(K_1) \cup X|}{i(G - V(K_1) \cup X)} \\ &= \frac{1+t}{2} = \frac{1 + |V(G)| - 3}{2} = \frac{|V(G)| - 2}{2}, \\ I'(G) &= \frac{|V(K_1) \cup X|}{i(G - V(K_1) \cup X) - 1} \\ &= \frac{1+t}{2-1} = 1 + |V(G)| - 3 = |V(G)| - 2. \end{aligned} \tag{117}$$

On the other hand, let $e \in E(K_2)$ and $G' = G - e = K_1 \vee (2K_1 \cup K_t)$. Let S be the vertex set of first K_1 in G', and then, we have $\varepsilon_1(S) = \varepsilon_2(S) = 1$,

$$\begin{aligned} i\left(G' - S\right) &= 2 > 1 = 2|S| - \varepsilon_1(S), \\ \mathit{sun}\left(G' - S\right) &= 2 > 1 = 2|S| - \varepsilon_2(S). \end{aligned} \tag{118}$$

Therefore, by means of Lemma 2, G' is not $P_{\geq 2}$-factor covered, and G is not a $P_{\geq 2}$-factor uniform graph. Also, in terms of Lemma 3, G' is not $P_{\geq 3}$-factor covered, and G is not a $P_{\geq 3}$-factor uniform graph.

4. Open Problems

The restrictions in factor critical graphs can be further extended to more general ones. For instance, a graph G is a $(P_{\geq n}, k, m)$-factor critical covered graph if removing any k vertices from G, the resting subgraph is still a $(P_{\geq n}, m)$-factor covered graph (that is, if for any $E \subseteq E(G)$ with $|E| = m$, G has a $P_{\geq n}$-factor containing all the edges in E, and then, G is called a $(P_{\geq n}, m)$-factor covered graph). The biggest obstacle to solve these problems is lacking of necessary and sufficient condition for $(P_{\geq n}, m)$-factor covered graph. Hence, as the first step, we need to expand the results on $P_{\geq 2}$-factor covered graph and $P_{\geq 3}$-factor covered graph determined by Zhang and Zhou [15] to necessary and sufficient condition of $(P_{\geq 2}, m)$-factor covered graph and $(P_{\geq 3}, m)$-factor covered graph. These problems are worthy of deep study in the future.

Data Availability

This work is a pure theoretical contribution, and no data are contained in the paper.

Conflicts of Interest

All authors declare no conflict of interests in publishing this work.

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