Hindawi
Journal of Function Spaces
Volume 2023, Article ID 9965919, 11 pages
https://doi.org/10.1155/2023/9965919

Research Article

Boundedness of p-Adic Singular Integrals and Multilinear Commutator on Morrey-Herz Spaces

Yanlong Shi,[1] **Li Li**,[2] **and Zhonghua Shen**[1]

[1]*Zhejiang Pharmaceutical University, Ningbo, Zhejiang 315000, China*
[2]*School of Mathematics and Statistics, Ningbo University, Ningbo, Zhejiang 315000, China*

Correspondence should be addressed to Yanlong Shi; shiyan-long@hotmail.com

Received 10 July 2022; Revised 22 September 2022; Accepted 8 April 2023; Published 18 April 2023

Academic Editor: Alexander Meskhi

In this paper, we establish the boundedness of classical p-adic singular integrals on Morrey-Herz spaces, as well as the boundedness of multilinear commutator generated by p-adic singular integral operators and Lipschitz functions or by p-adic singular integral operators and λ-central BMO functions.

1. Introduction

During the last several decades, p-adic analysis has cemented its role in the field of mathematical physics [1–3]. That stimulates researchers to pay attention to harmonic analysis on the p-adic fields, especially about singular integral theory (for example, [4–6]). It is well known that singular integral operators are one of the main topics in the singular integral theory. Phillips and Taibleson [7] considered the classical singular integral operator of Calderón-Zygmund type on the p-adic fields.

Let $\Omega \in L^\infty(\mathbb{Q}_p^n)$ and satisfy

$$\Omega\left(p^j x\right) = \Omega(x) \quad \text{for all} \quad j \in \mathbb{Z}, \tag{1}$$

$$\int_{|x|_p=1} \Omega(x)dx = 0. \tag{2}$$

The p-adic singular integral operator is defined by

$$T(f)(x) = \lim_{z \longrightarrow -\infty} \int_{|x-y|_p>p^z} f(y) \frac{\Omega(x-y)}{|x-y|_p^n} dy \quad \text{for} \quad z \in \mathbb{Z}, \tag{3}$$

and the truncated singular integral operator is defined as

$$T_z(f)(x) = \int_{|x-y|_p>p^z} f(y) \frac{\Omega(x-y)}{|x-y|_p^n} dy \quad \text{for} \quad z \in \mathbb{Z}. \tag{4}$$

Here, $|\cdot|_p$ is non-Archimedean p-adic norm, $\mathbb{Q}_p^n$ consists of all points $x = (x_1, \cdots, x_n)$ for $n \in \mathbb{N}$, $x_j \in \mathbb{Q}_p (j = 1, \cdots, n)$ and $\mathbb{Q}_p$ is the field of p-adic numbers.

In 1975, Taibleson [8] proved that T and T_z are of type (q, q) for $1 < q < \infty$ and of weak type $(1, 1)$ when Ω satisfies the smoothness condition

$$\sup_{|y|_p=1} \sum_{j=1}^{\infty} \int_{|x|_p=1} \left|\Omega\left(x + p^j y\right) - \Omega(x)\right| dx < \infty. \tag{5}$$

In 2017, Wu et al. [4] introduced p-adic central Morrey spaces and established the boundedness of T and T_z on the p-adic central Morrey spaces. Furthermore, they obtained λ-central BMO estimates for the commutator of these singular integral operators on the p-adic central Morrey spaces. In 2019, Mo et al. [5] showed the boundedness of T and T_z on the p-adic generalized Morrey spaces as well as the boundedness of multilinear commutator defined by

$$T_z^{\mathbf{b}}(f)(x) = \int_{|x-y|_p > p^z} \prod_{i=1}^{m} (b_i(x) - b_i(y)) f(y) \frac{\Omega(x-y)}{|x-y|_p^n} dy \quad \text{for } z \in \mathbb{Z}, \quad m \in \mathbb{N}. \tag{6}$$

In 2023, Ho [9] extended the main results in [5] to Morrey spaces on general local fields. Moreover, he also extended the boundedness of the singular integral operators on Lorentz-Morrey spaces and Lorentz-Karamata-Morrey spaces on the p-adic fields.

Comparing the above studies with the singular integral theory on Euclidean space $\mathbb{R}^n$, we find that there does not exist any results of these singular integral operators on Herz spaces ([10, 11]) and Morrey-Herz spaces ([12, 13]) on the p-adic fields. Thus, this paper is aimed at the boundedness of T and T_z on Herz spaces and Morrey-Herz spaces on the p-adic fields. Furthermore, we also study the boundedness of the commutator $T_z^{\mathbf{b}}$ when the symbol function b_i belongs to λ-central bounded mean oscillation spaces and when it belongs to Lipschitz spaces.

This paper is organized as follows. In Section 2, we provide a brief introduction to the p-adic fields, the definition of some function spaces on the p-adic fields and some desired lemmas. In Section 3, we present our main results. In Section 4, we devote to the proof of the main results. Throughout this paper, the letter C will be used to denote various constants. The various uses of the letter do not denote the same constant, though.

2. Preliminaries

For a prime number p, let $\mathbb{Q}_p$ be the field of p-adic numbers defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to non-Archimedean p-adic norm $|\cdot|_p$. This norm $|\cdot|_p$ is defined as follows: if $x = 0$, $|0|_p = 0$; if $x \neq 0$ is an arbitrary rational number with the unique representation $x = p^\gamma m/n$, where m and n are not divisible by p, $\gamma = \gamma(x) \in \mathbb{Z}$, then $|x|_p = p^{-\gamma}$. It is not hard to see that the norm satisfies the following properties:

$$\begin{aligned} &|x|_p \geq 0, \forall x \in \mathbb{Q}_p,\ |x|_p = 0 \Leftrightarrow x = 0\,; \\ &|xy|_p = |x|_p |y|_p, \forall x, y \in \mathbb{Q}_p\,; \end{aligned} \tag{7}$$

$|x+y|_p \leq \max(|x|_p, |y|_p), \forall x, y \in \mathbb{Q}_p$, and when $|x|_p \neq |y|_p$, we have $|x+y|_p = \max(|x|_p, |y|_p)$.

The p-adic norm of $\mathbb{Q}_p^n = \mathbb{Q} \times \mathbb{Q} \times \cdots \times \mathbb{Q}$ is defined by

$$|x|_p = \max_{1 \leq j \leq n} |x_j|_p, \quad x \in \mathbb{Q}_p^n. \tag{8}$$

Denote by

$$B_\gamma(a) = \left\{ x \in \mathbb{Q}_p^n : |x-a|_p \leq p^\gamma \right\}, \tag{9}$$

the ball of radius p^γ with center at $a \in \mathbb{Q}_p^n$ and by

$$S_\gamma(a) = B_\gamma(a) \setminus B_{\gamma-1}(a) = \left\{ x \in \mathbb{Q}_p^n : |x-a|_p = p^\gamma \right\}, \tag{10}$$

the sphere of radius p^γ with center at $a \in \mathbb{Q}_p^n$, where $\gamma \in \mathbb{Z}$. For the sake of convenience, we let $B_\gamma = B_\gamma(0)$, $S_\gamma = S_\gamma(0)$ and denote by χ_k the characteristic function of the sphere S_k, then it is easy to see that the equalities $x_0 + B_\gamma = B_\gamma(x_0)$, $x_0 + S_\gamma = S_\gamma(x_0)$, and $B_\gamma(a) = \bigcup_{k \leq \gamma} S_k(a)$ hold for any $x_0 \in \mathbb{Q}_p^n$.

Since the space $\mathbb{Q}_p^n$ is a locally compact commutative group under addition, there exists the Haar measure dx on the additive group of $\mathbb{Q}_p^n$ normalized by $\int_{B_0} dx = |B_0| = 1$, where $|B_0|$ denotes the Haar measure of a measurable set $B_0 \subset \mathbb{Q}_p^n$. Then through a simple calculation, we can obtain that

$$|B_\gamma(a)| = p^{n\gamma},\ |S_\gamma(a)| = p^{n\gamma}(1 - p^{-n}), \quad \forall a \in \mathbb{Q}_p^n. \tag{11}$$

On the p-adic fields $\mathbb{Q}_p^n$, for $r \in [1, +\infty)$, the Lebesgue spaces are denoted by $L^r(\mathbb{Q}_p^n)$ with the norm

$$\|f\|_{L^r(\mathbb{Q}_p^n)} = \left(\int_{\mathbb{Q}_p^n} |f(x)|^r dx \right)^{1/r} < \infty, \tag{12}$$

and the weak Lebesgue spaces are denoted by $WL^r(\mathbb{Q}_p^n)$ with the quasinorm

$$\|f\|_{WL^r(\mathbb{Q}_p^n)} = \sup_{\mu > 0} \mu \left| \left\{ x \in \mathbb{Q}_p^n : |f(x)| > \mu \right\} \right|^{1/r} < \infty. \tag{13}$$

We now recall the definitions of Herz spaces and Morrey-Herz spaces on $\mathbb{Q}_p^n$.

Definition 1 (see [14]). Let $\sigma \in \mathbb{R}$ and $0 < s, r \leq \infty$. The homogeneous Herz space $\dot{K}_r^{\sigma,s}(\mathbb{Q}_p^n)$ is defined by

$$\dot{K}_r^{\sigma,s}\left(\mathbb{Q}_p^n\right) = \left\{ f \in L_{loc}^r\left(\mathbb{Q}_p^n \setminus \{0\}\right) : \|f\|_{\dot{K}_r^{\sigma,s}(\mathbb{Q}_p^n)} < \infty \right\}, \tag{14}$$

where

$$\|f\|_{\dot{K}_r^{\sigma,s}(\mathbb{Q}_p^n)} = \left(\sum_{k=-\infty}^{\infty} p^{k\sigma s} \|f\chi_k\|_{L^r(\mathbb{Q}_p^n)}^s \right)^{1/s}, \tag{15}$$

for $s < \infty$, and the usual modifications should be made when $s = \infty$.

Definition 2 (see [13]). Let $\sigma \in \mathbb{R}$, $0 < s, r \leq \infty$. The weak homogeneous Herz space $W\dot{K}_r^{\sigma,s}(\mathbb{Q}_p^n)$ is defined by

$$W\dot{K}_r^{\sigma,s}\left(\mathbb{Q}_p^n\right) = \left\{ f \text{ is a measurable function on } \mathbb{Q}_p^n \text{ and } \|f\|_{W\dot{K}_r^{\sigma,s}(\mathbb{Q}_p^n)} < \infty \right\}, \tag{16}$$

where

$$\|f\|_{W\dot{K}_r^{\sigma,s}(\mathbb{Q}_p^n)} = \sup_{\mu>0} \mu \left(\sum_{k=-\infty}^{k=\infty} p^{k\sigma s} |\{x \in S_k : |f(x)| > \mu\}|^{s/r} \right)^{1/s}, \tag{17}$$

for $s < \infty$, and the usual modifications should be made when $s = \infty$.

Obviously, $\dot{K}_r^{0,r}(\mathbb{Q}_p^n) = L^r(\mathbb{Q}_p^n)$ and $W\dot{K}_r^{0,r}(\mathbb{Q}_p^n) = WL^r(\mathbb{Q}_p^n)$.

Definition 3 (see [14]). Let $\sigma \in \mathbb{R}$, $0 < s, r \le \infty$, and λ be a nonnegative real number. Then, the homogeneous Morrey-Herz space $M\dot{K}_{l,q}^{\sigma,\lambda}(\mathbb{Q}_p^n)$ is defined by

$$M\dot{K}_{s,r}^{\sigma,\lambda}\left(\mathbb{Q}_p^n\right) = \left\{ f \in L_{\text{loc}}^r\left(\mathbb{Q}_p^n \setminus \{0\}\right) : \|f\|_{M\dot{K}_{s,r}^{\sigma,\lambda}(\mathbb{Q}_p^n)} < \infty \right\}, \tag{18}$$

where

$$\|f\|_{M\dot{K}_{s,r}^{\sigma,\lambda}(\mathbb{Q}_p^n)} = \sup_{k_0 \in \mathbb{Z}} p^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} p^{k\sigma s} \|f\chi_k\|_{L^r(\mathbb{Q}_p^n)}^s \right)^{1/s}, \tag{19}$$

for $s < \infty$, and the usual modifications should be made when $s = \infty$.

Definition 4 (see [13]). Let $\sigma \in \mathbb{R}$, $0 < s, r \le \infty$, and λ be a nonnegative real number. The weak homogeneous Morrey-Herz space $WM\dot{K}_{s,r}^{\sigma,\lambda}(\mathbb{Q}_p^n)$ is defined by

$$WM\dot{K}_{s,r}^{\sigma,\lambda}\left(\mathbb{Q}_p^n\right) = \left\{ f \text{ is a measurable function on } \mathbb{Q}_p^n \text{ and } \|f\|_{WM\dot{K}_{s,r}^{\sigma,\lambda}(\mathbb{Q}_p^n)} < \infty \right\}, \tag{20}$$

where

$$\|f\|_{WM\dot{K}_{s,r}^{\sigma,\lambda}(\mathbb{Q}_p^n)} = \sup_{\mu>0} \mu \sup_{k_0 \in \mathbb{Z}} p^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} p^{k\sigma s} |\{x \in S_k : |f(x)| > \mu\}|^{s/r} \right)^{1/s}, \tag{21}$$

for $s < \infty$, and the usual modifications should be made when $s = \infty$.

It is apparent that $M\dot{K}_{s,r}^{\sigma,0}(\mathbb{Q}_p^n) = \dot{K}_r^{\sigma,s}(\mathbb{Q}_p^n)$ and $WM\dot{K}_{s,r}^{\sigma,0}(\mathbb{Q}_p^n) = W\dot{K}_r^{\sigma,s}(\mathbb{Q}_p^n)$.

In order to study the boundedness of the commutator of singular integral operators, we will need Lipschitz functions and λ-central BMO functions.

Definition 5 (see [14]). Let δ be a positive real number. The Lipschitz space $\Lambda_\delta(\mathbb{Q}_p^n)$ is defined to be the space of all measurable functions $f \in \mathbb{Q}_p^n$, which satisfies

$$\|f\|_{\Lambda_\delta(\mathbb{Q}_p^n)} = \sup_{x,h \in \mathbb{Q}_p^n, h \ne 0} \frac{|f(x+h) - f(x)|}{|h|_p^\delta} < \infty. \tag{22}$$

Definition 6 (see [15]). Given $\lambda < 1/n$, $1 < q < \infty$, the λ-central BMO space $\text{CBMO}_{q,\lambda}(\mathbb{Q}_p^n)$ is defined as the set of all functions $f \in L_{\text{loc}}^r(\mathbb{Q}_p^n \setminus \{0\})$, which satisfies

$$\|f\|_{\text{CBMO}_{q,\lambda}(\mathbb{Q}_p^n)} = \sup_{k \in \mathbb{Z}} \left(\frac{1}{|B_k|^{1+q\lambda}} \int_{B_k} \left| f(x) - f_{B_k} \right|^q dx \right)^{1/q} < \infty, \tag{23}$$

where

$$f_{B_k} = \frac{1}{|B_k|} \int_{B_k} f(x) dx. \tag{24}$$

We notice that when $\lambda = 0$, the important particular case of λ-central BMO spaces $\text{CBMO}_{q,\lambda}(\mathbb{Q}_p^n)$ is $\text{CBMO}_q(\mathbb{Q}_p^n)$ defined in [16] and that there is a following result.

Lemma 7 (see [17]). *Suppose that $0 \le \lambda < 1/n$, $1 < q < \infty$, $b \in CBMO_{q,\lambda}(\mathbb{Q}_p^n)$ and $j, k \in \mathbb{Z}$. Then,*

$$\left| b_{B_k} - b_{B_j} \right| \le p^n |j-k| \|b\|_{CBMO_{q,\lambda}(\mathbb{Q}_p^n)} \max\left\{ |B_k|^\lambda, |B_j|^\lambda \right\}. \tag{25}$$

At last, we present two desired lemmas which will be used in the proof of our main results. The first is the boundedness of T_z on Lebesgue spaces, and the second is the boundedness on Morrey-Herz spaces of p-adic Riesz potential $\mathcal{I}_\alpha$ defined by

$$\mathcal{I}_\alpha(f)(x) = \frac{1}{\Gamma_n(\alpha)} \int_{\mathbb{Q}_p^n} \frac{f(y)}{|x-y|_p^{n-\alpha}} dy, \Gamma_n(\alpha) = \frac{1 - p^{\alpha - n}}{1 - p^{-\alpha}} \quad \text{with} \quad \alpha \in \mathbb{C} \setminus \{0\}. \tag{26}$$

Lemma 8 (see [8]). *Suppose that $\Omega \in L^\infty(\mathbb{Q}_p^n)$ satisfies the conditions (1), (2), and (5). Then there exists a constant $C > 0$ independent of f, $z \in \mathbb{Z}$, and $\mu > 0$ such that*

$$\begin{aligned} &\|T_z(f)\|_{L^q(\mathbb{Q}_p^n)} \le C\|f\|_{L^q(\mathbb{Q}_p^n)} \quad \text{for} \quad q > 1, \\ &\left| \left\{ x \in \mathbb{Q}_p^n : T_z(f)(x) \right| > \mu \right\} \right| \le \frac{C\|f\|_{L^1(\mathbb{Q}_p^n)}}{\mu}. \end{aligned} \tag{27}$$

Moreover, $T(f) = \lim_{z \longrightarrow -\infty} T_z(f)$ exists, T is bounded on $L^q(\mathbb{Q}_p^n)$ for $q > 1$, and T is of weak type $(1, 1)$.

Lemma 9 (see [13]). *Let α be a complex number with $0 < \text{Re}(\alpha) < n$, $1/r = 1/q - \text{Re}(\alpha)/n$, $1 < q < r < \infty$, $\lambda \ge 0$, $\lambda - n/q + \text{Re}(\alpha) < \sigma < n(1 - 1/q)$, and $0 < l \le \infty$. Then there exists* *a constant $C > 0$ such that*

$$\|\mathcal{I}_\alpha(f)\|_{M\dot{K}^{\sigma,\lambda}_{l,r}(\mathbb{Q}_p^n)} \le C\|f\|_{M\dot{K}^{\sigma,\lambda}_{l,q}(\mathbb{Q}_p^n)}. \quad (28)$$

3. Main Results

In this paper, our first main result is to address the boundedness of T and T_z on Morrey-Herz spaces on the p-adic fields.

Theorem 10. *Suppose that $\Omega \in L^\infty(\mathbb{Q}_p^n)$ satisfies conditions (1), (2), and (5). Let $0 < l \le \infty$, $1 \le q < \infty$, $\lambda \ge 0$, and $\lambda - n/q < \sigma < n(1 - 1/q)$, then there exists a constant $C > 0$ independent of f, $z \in \mathbb{Z}$, and $\mu > 0$ such that*

$$\begin{aligned} \|T_z(f)\|_{M\dot{K}^{\sigma,\lambda}_{l,q}(\mathbb{Q}_p^n)} &\le C\|f\|_{M\dot{K}^{\sigma,\lambda}_{l,q}(\mathbb{Q}_p^n)} \text{ for } q > 1, \\ \|T_z(f)\|_{WM\dot{K}^{\sigma,\lambda}_{l,q}(\mathbb{Q}_p^n)} &\le C\|f\|_{M\dot{K}^{\sigma,\lambda}_{l,q}(\mathbb{Q}_p^n)} \text{ for } q = 1. \end{aligned} \quad (29)$$

Moreover, $T(f) = \lim_{z\longrightarrow-\infty} T_z(f)$ exists, T is bounded on $M\dot{K}^{\sigma,\lambda}_{l,q}(\mathbb{Q}_p^n)$ for $q > 1$, and T is bounded from $M\dot{K}^{\sigma,\lambda}_{l,q}(\mathbb{Q}_p^n)$ to $WM\dot{K}^{\sigma,\lambda}_{l,q}(\mathbb{Q}_p^n)$ for $q = 1$.

By letting $\lambda = 0$ in Theorem 10, we will get its counterparts on p-adic Herz spaces.

Corollary 11. *Suppose that $\Omega \in L^\infty(\mathbb{Q}_p^n)$ satisfies conditions (1), (2), and (5). Let $0 < l \le \infty$, $1 \le q < \infty$, and $-n/q < \sigma < n(1 - 1/q)$, then there exists a constant $C > 0$ independent of f, $z \in \mathbb{Z}$, and $\mu > 0$ such that*

$$\begin{aligned} \|T_z(f)\|_{\dot{K}^{\sigma,l}_{q}(\mathbb{Q}_p^n)} &\le C\|f\|_{\dot{K}^{\sigma,l}_{q}(\mathbb{Q}_p^n)} \text{ for } q > 1, \\ \|T_z(f)\|_{W\dot{K}^{\sigma,l}_{q}(\mathbb{Q}_p^n)} &\le C\|f\|_{\dot{K}^{\sigma,l}_{q}(\mathbb{Q}_p^n)} \text{ for } q = 1. \end{aligned} \quad (30)$$

Moreover, $T(f) = \lim_{z\longrightarrow-\infty} T_z(f)$ exists, T is bounded on $\dot{K}^{\sigma,l}_{q}(\mathbb{Q}_p^n)$ for $q > 1$, and T is bounded from $\dot{K}^{\sigma,l}_{q}(\mathbb{Q}_p^n)$ to $W\dot{K}^{\sigma,l}_{q}(\mathbb{Q}_p^n)$ for $q = 1$.

For $T_z^{\mathbf{b}}$, when the symbol function b_i belongs to Lipschitz spaces, our results regarding boundedness can be stated as

Theorem 12. *Suppose that $\Omega \in L^\infty(\mathbb{Q}_p^n)$ satisfies conditions (1), (2), and (5). Let δ_i for $i = 1, 2, \cdots, m$ be a positive real number such that $0 < \delta = \sum_{i=1}^m \delta_i < n$, $\lambda \ge 0$, $0 < l \le \infty$, $1/r = 1/q - \delta/n$, $1 < q < r < \infty$, and $\lambda - n/q + \delta < \sigma < n(1 - 1/q)$. If $b_i \in \Lambda_{\delta_i}(\mathbb{Q}_p^n)$, then there exists a constant $C > 0$ independent of f and $z \in \mathbb{Z}$ such that*

$$\left\|T_z^{\mathbf{b}}(f)\right\|_{M\dot{K}^{\sigma,\lambda}_{l,r}(\mathbb{Q}_p^n)} \le C\prod_{i=1}^m \|b_i\|_{\Lambda_{\delta_i}(\mathbb{Q}_p^n)}\|f\|_{M\dot{K}^{\sigma,\lambda}_{l,q}(\mathbb{Q}_p^n)}. \quad (31)$$

Corollary 13. *Suppose that $\Omega \in L^\infty(\mathbb{Q}_p^n)$ satisfies conditions (1), (2), and (5). Let δ_i for $i = 1, 2, \cdots, m$ be a positive real number such that $0 < \delta = \sum_{i=1}^m \delta_i < n$, $0 < l \le \infty$, $1/r = 1/q - \delta/n$, $1 < q < r < \infty$, and $-n/q + \delta < \sigma < n(1 - 1/q)$. If $b_i \in \Lambda_{\delta_i}(\mathbb{Q}_p^n)$, then there exists a constant $C > 0$ independent of f and $z \in \mathbb{Z}$ such that*

$$\left\|T_z^{\mathbf{b}}(f)\right\|_{\dot{K}^{\sigma,l}_{r}(\mathbb{Q}_p^n)} \le C\prod_{i=1}^m \|b_i\|_{\Lambda_{\delta_i}(\mathbb{Q}_p^n)}\|f\|_{\dot{K}^{\sigma,l}_{q}(\mathbb{Q}_p^n)}. \quad (32)$$

When the symbol function b_i belongs to λ-central bounded mean oscillation spaces, we have the following theorem and corollary.

Theorem 14. *Suppose that $\Omega \in L^\infty(\mathbb{Q}_p^n)$ satisfies conditions (1), (2), and (5). Let $\lambda \ge 0$, $0 < l \le \infty$, $1/r = 1/q_1 + 1/q_2 + \cdots + 1/q_m + 1/q$ with $1 < q, q_1, \cdots, q_m < \infty$, $\nu = \nu_1 + \nu_2 + \cdots + \nu_m$ with $0 \le \nu_1, \nu_2, \cdots, \nu_m < 1/n$, and $\sigma_2 = \sigma_1 - n(1/q_1 + 1/q_2 + \cdots + 1/q_m + \nu)$ with $n\nu + \lambda - n/q < \sigma_1 < n(1 - 1/q) + \lambda$. If $b_i \in CBMO_{q_i,\nu_i}(\mathbb{Q}_p^n)$, then there exists a constant $C > 0$ independent of f and $z \in \mathbb{Z}$ such that*

$$\left\|T_z^{\mathbf{b}}(f)\right\|_{M\dot{K}^{\sigma_2,\lambda}_{l,r}(\mathbb{Q}_p^n)} \le C\prod_{i=1}^m \|b_i\|_{CBMO_{q_i,\nu_i}}\|f\|_{M\dot{K}^{\sigma_1,\lambda}_{l,q}(\mathbb{Q}_p^n)}. \quad (33)$$

Corollary 15. *Suppose that $\Omega \in L^\infty(\mathbb{Q}_p^n)$ satisfies conditions (1), (2), and (5). Let $0 < l \le \infty$, $1/r = 1/q_1 + 1/q_2 + \cdots + 1/q_m + 1/q$ with $1 < q, q_1, \cdots, q_m < \infty$, $\nu = \nu_1 + \nu_2 + \cdots + \nu_m$ with $0 \le \nu_1, \nu_2, \cdots, \nu_m < 1/n$, and $\sigma_2 = \sigma_1 - n(1/q_1 + 1/q_2 + \cdots + 1/q_m + \nu)$ with $n\nu - n/q < \sigma_1 < n(1 - 1/q)$. If $b_i \in CBMO_{q_i,\nu_i}(\mathbb{Q}_p^n)$, then there exists a constant $C > 0$ independent of f and $z \in \mathbb{Z}$ such that*

$$\left\|T_z^{\mathbf{b}}(f)\right\|_{\dot{K}^{\sigma_2,l}_{r}(\mathbb{Q}_p^n)} \le C\prod_{i=1}^m \|b_i\|_{CBMO_{q_i,\nu_i}}\|f\|_{\dot{K}^{\sigma_1,l}_{q}(\mathbb{Q}_p^n)}. \quad (34)$$

Let $\nu_1 = \cdots = \nu_m = 0$ in Theorem 14 and Corollary 15, we will obtain CBMO estimates of T_z^b.

Corollary 16. *Suppose that $\Omega \in L^\infty(\mathbb{Q}_p^n)$ satisfies conditions (1), (2), and (5). Let $\lambda \ge 0$, $0 < l \le \infty$, $1/r = 1/q_1 + 1/q_2 + \cdots + 1/q_m + 1/q$ with $1 < q, q_1, \cdots, q_m < \infty$, and $\sigma_2 = \sigma_1 - n(1/q_1 + 1/q_2 + \cdots + 1/q_m)$ with $\lambda - n/q < \sigma_1 < n(1 - 1/q) + \lambda$. If $b_i \in CBMO_{q_i}(\mathbb{Q}_p^n)$, then there exists a constant $C > 0$ independent of f and $z \in \mathbb{Z}$ such that*

$$\left\|T_z^{\mathbf{b}}(f)\right\|_{M\dot{K}^{\sigma_2,\lambda}_{l,r}(\mathbb{Q}_p^n)} \le C\prod_{i=1}^m \|b_i\|_{CBMO_{q_i}}\|f\|_{M\dot{K}^{\sigma_1,\lambda}_{l,q}(\mathbb{Q}_p^n)}. \quad (35)$$

Corollary 17. *Suppose that $\Omega \in L^\infty(\mathbb{Q}_p^n)$ satisfies conditions (1), (2), and (5). Let $0 < l \le \infty$, $1/r = 1/q_1 + 1/q_2 + \cdots + 1/q_m + 1/q$ with $1 < q, q_1, \cdots, q_m < \infty$, and $\sigma_2 = \sigma_1 - n(1/q_1 + 1/q_2 + \cdots + 1/q_m)$ with $-n/q < \sigma_1 < n(1 - 1/q)$. If $b_i \in CBMO_{q_i}(\mathbb{Q}_p^n)$, then there exists a constant $C > 0$ independent* *of f and $z \in \mathbb{Z}$ such that*

$$\left\| T_z^{\mathbf{b}}(f) \right\|_{\dot{K}_r^{\sigma_2,l}(\mathbb{Q}_p^n)} \le C \prod_{i=1}^{m} \|b_i\|_{CBMO_{q_i}} \|f\|_{\dot{K}_q^{\sigma_1,l}(\mathbb{Q}_p^n)}. \tag{36}$$

4. Proof of Main Results

Proof of Theorem 18. Suppose that $f \in M\dot{K}_{l,q}^{\sigma,\lambda}(\mathbb{Q}_p^n)$, we decompose f in the following way

$$f(x) = \sum_{j=-\infty}^{\infty} f(x)\chi_j(x) \equiv \sum_{j=-\infty}^{\infty} f_j(x), \quad j \in \mathbb{Z}. \tag{37}$$

(i) For the case of $q > 1$, by the decomposition of f, we get

$$\begin{aligned}
\|T_z(f)\|_{M\dot{K}_{l,q}^{\sigma,\lambda}(\mathbb{Q}_p^n)} &\le C \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} p^{k\sigma l} \|T_z(f)\chi_k\|_{L^q(\mathbb{Q}_p^n)}^l \right)^{1/l} \\
&\le C \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} p^{k\sigma l} \left(\sum_{j=-\infty}^{k-2} \left\| T_z(f_j)\chi_k \right\|_{L^q(\mathbb{Q}_p^n)} \right)^l \right\}^{1/l} \\
&\quad + C \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} p^{k\sigma l} \left(\sum_{j=k-1}^{k+1} \left\| T_z(f_j)\chi_k \right\|_{L^q(\mathbb{Q}_p^n)} \right)^l \right\}^{1/l} \\
&\quad + C \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} p^{k\sigma l} \left(\sum_{j=k+2}^{\infty} \left\| T_z(f_j)\chi_k \right\|_{L^q(\mathbb{Q}_p^n)} \right)^l \right\}^{1/l} \\
&\equiv C(\mathcal{N}_1 + \mathcal{N}_2 + \mathcal{N}_3).
\end{aligned} \tag{38}$$

For $\mathcal{N}_2$, by the L^q-boundedness of T_z, we obtain

$$\begin{aligned}
\mathcal{N}_2 &\le C \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} p^{k\sigma l} \left(\sum_{j=k-1}^{k+1} \left\| f_j \right\|_{L^q(\mathbb{Q}_p^n)} \right)^l \right\}^{1/l} \\
&\le C \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda} \left\{ \sum_{j=-\infty}^{k_0+1} \sum_{k=j-1}^{j+1} p^{(k-j)\sigma l} p^{j\sigma l} \left\| f_j \right\|_{L^q(\mathbb{Q}_p^n)}^l \right\}^{1/l} \\
&\le C \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda} \left\{ \sum_{j=-\infty}^{k_0+1} p^{j\sigma l} \left\| f_j \right\|_{L^q(\mathbb{Q}_p^n)}^l \left(\sum_{k=j-1}^{j+1} p^{(k-j)\sigma l} \right) \right\}^{1/l} \\
&\le C \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda} \left\{ \sum_{j=-\infty}^{k_0+1} p^{j\sigma l} \left\| f_j \right\|_{L^q(\mathbb{Q}_p^n)}^l \right\}^{1/l} \\
&\le \|f\|_{M\dot{K}_{l,q}^{\sigma,\lambda}(\mathbb{Q}_p^n)}.
\end{aligned} \tag{39}$$

For $\mathcal{N}_1$, note that $|x-y|_p = \max\{|x|_p, |y|_p\} = \max\{p^k, p^j\} = p^k$, where $x \in S_k$ and $y \in S_j$ with $j \le k-2$. Therefore, for $x \in S_k$, Hölder's inequality yields that

$$\begin{aligned}
\left| T_z(f\chi_j)(x) \right| &\le C \left| \int_{|y|_p > p^k} (f\chi_j)(x-y) \frac{\Omega(y)}{|y|_p^n} dy \right| \\
&\le C \left| \int_{|x-y|_p > p^k} (f\chi_j)(y) \frac{\Omega(x-y)}{|x-y|_p^n} dy \right| \\
&\le C \left| \int_{S_j} f(y) \frac{\Omega(x-y)}{|x-y|_p^n} dy \right| \\
&\le Cp^{-kn} \int_{S_j} |f(y)| dy \le Cp^{-kn} p^{jn/q'} \left\| f_j \right\|_{L^q(\mathbb{Q}_p^n)}.
\end{aligned} \tag{40}$$

Then, by Jensen's inequality, Hölder's inequality and the fact $\sigma < n(1 - 1/q)$, it follows that

$$\begin{aligned}
\mathcal{N}_1 &\le \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} p^{k\sigma l} \left(\sum_{j=-\infty}^{k-2} p^{-kn} p^{jn/q'} \left\| f_j \right\|_{L^q(\mathbb{Q}_p^n)} \|\chi_k\|_{L^q(\mathbb{Q}_p^n)} \right)^l \right\}^{1/l} \\
&\le \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} p^{k\sigma l} \left(\sum_{j=-\infty}^{k-2} p^{(j-k)n/q'} \left\| f_j \right\|_{L^q(\mathbb{Q}_p^n)} \right)^l \right\}^{1/l} \\
&\le \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} \left(\sum_{j=-\infty}^{k-2} p^{(k-j)(\sigma - n/q')} p^{j\sigma} \left\| f_j \right\|_{L^q(\mathbb{Q}_p^n)} \right)^l \right\}^{1/l} \\
&\le \begin{cases}
\sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda} \left\{ \sum_{j=-\infty}^{k_0-2} \sum_{k=j+2}^{k_0} p^{(k-j)l(\sigma - n/q')} p^{j\sigma l} \left\| f_j \right\|_{L^q(\mathbb{Q}_p^n)}^l \right\}^{1/l}, & \text{if } 0 < l \le 1 \\
\sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} \left(\sum_{j=-\infty}^{k-2} p^{(k-j)(\sigma - n/q')} p^{j\sigma l} \left\| f_j \right\|_{L^q(\mathbb{Q}_p^n)}^l \right) \times \left(\sum_{j=-\infty}^{k-2} p^{(k-j)[\sigma - n/q']} \right)^{l-1} \right\}^{1/l}, & \text{if } 1 < l < \infty \\
\sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda} \sup_{k\le k_0} \left\{ \sup_{j\le k-2} p^{j\sigma} \left\| f_j \right\|_{L^q(\mathbb{Q}_p^n)} \left(\sum_{j=-\infty}^{k-2} p^{(k-j)(\sigma - n/q')} \right) \right\}^{1/l}, & \text{if } l = \infty
\end{cases} \\
&\le \begin{cases}
\sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda} \left\{ \sum_{j=-\infty}^{k_0-2} p^{j\sigma l} \left\| f_j \right\|_{L^q(\mathbb{Q}_p^n)}^l \left(\sum_{k=j+2}^{\infty} p^{(k-j)l(\sigma - n/q')} \right) \right\}^{1/l}, & \text{if } 0 < l \le 1 \\
\sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda} \left\{ \sum_{j=-\infty}^{k_0-2} p^{j\sigma l} \left\| f_j \right\|_{L^q(\mathbb{Q}_p^n)}^l \left(\sum_{k=j+2}^{\infty} p^{(k-j)(\sigma - n/q')} \right) \right\}^{1/l}, & \text{if } 1 < l < \infty \le \|f\|_{M\dot{K}_{l,q}^{\sigma,\lambda}(\mathbb{Q}_p^n)}. \\
\sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda} \sup_{k\le k_0} \left\{ \sup_{j\le k-2} p^{j\sigma} \left\| f_j \right\|_{L^q(\mathbb{Q}_p^n)} \left(\sum_{k=j+2}^{\infty} p^{(k-j)(\sigma - n/q')} \right) \right\}^{1/l}, & \text{if } l = \infty
\end{cases}
\end{aligned} \tag{41}$$

Now, let us turn to the estimates of $\mathcal{N}_3$. Note that $|x-y|_p = \max\{|x|_p, |y|_p\} = \max\{p^k, p^j\} = p^j$, where $x \in S_k$ and $y \in S_j$ with $j \ge k+2$. Therefore, for $x \in S_k$, we have

$$\left| T_z(f\chi_j)(x) \right| \le Cp^{-jn} \int_{S_j} |f(y)| dy \le Cp^{-jn/q} \left\| f_j \right\|_{L^q(\mathbb{Q}_p^n)}. \tag{42}$$

Thus,

$$\begin{aligned}
\mathcal{N}_3 &\le C \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} p^{k\sigma l} \left(\sum_{j=k+2}^{\infty} p^{(k-j)n/q} \left\| f_j \right\|_{L^q(\mathbb{Q}_p^n)} \right)^l \right\}^{1/l} \\
&\le C \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} \left(\sum_{j=k+2}^{k_0} p^{(k-j)(n/q+\sigma)} p^{j\sigma} \left\| f_j \right\|_{L^q(\mathbb{Q}_p^n)} \right)^l \right\}^{1/l} \\
&\quad + C \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} \left(\sum_{j=k_0+1}^{\infty} p^{(k-j)(n/q+\sigma)} p^{j\sigma} \left\| f_j \right\|_{L^q(\mathbb{Q}_p^n)} \right)^l \right\}^{1/l} \\
&\equiv \mathcal{J}_1 + \mathcal{J}_2.
\end{aligned} \tag{43}$$

For $\mathscr{J}_1$, using similar methods to the estimation of $\mathscr{N}_1$, by the fact $\sigma > \lambda - n/q \geq -n/q$, Jensen's inequality and Hölder's inequality, we obtain

$$\mathscr{J}_1 \leq \begin{cases} \sup\limits_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\left\{\sum\limits_{j=-\infty}^{k_0}\sum\limits_{k=-\infty}^{j-2} p^{(k-j)l(n/q+\sigma)} p^{j\sigma l}\left\|f_j\right\|_{L^q(\mathbb{Q}_p^n)}^l\right\}^{1/l}, \text{ if } 0<l\leq 1 \\ \sup\limits_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\left\{\sum\limits_{k=-\infty}^{k_0}\left(\sum\limits_{j=k+2}^{k_0} p^{(k-j)(n/q+\sigma)} p^{j\sigma l}\left\|f_j\right\|_{L^q(\mathbb{Q}_p^n)}^l\right)\times\left(\sum\limits_{k=j+2}^{k_0} p^{(k-j)(n/q+\sigma)}\right)^{l-1}\right\}^{1/l}, \text{ if } 1<l<\infty \\ \sup\limits_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\sup\limits_{k\leq k_0}\left\{\sup\limits_{j\leq k_0} p^{j\sigma}\left\|f_j\right\|_{L^q(\mathbb{Q}_p^n)}\left(\sum\limits_{j=k+2}^{k_0} p^{(k-j)(n/q+\sigma)}\right)\right\}^{1/l}, \text{ if } l=\infty \end{cases}$$
$$\leq \begin{cases} \sup\limits_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\left\{\sum\limits_{j=-\infty}^{k_0} p^{j\sigma l}\left\|f_j\right\|_{L^q(\mathbb{Q}_p^n)}^l\left(\sum\limits_{k=-\infty}^{j-2} p^{(k-j)l(n/q+\sigma)}\right)\right\}^{1/l}, \text{ if } 0<l\leq 1 \\ \sup\limits_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\left\{\sum\limits_{j=-\infty}^{k_0} p^{j\sigma l}\left\|f_j\right\|_{L^q(\mathbb{Q}_p^n)}^l\left(\sum\limits_{k=-\infty}^{j-2} p^{(k-j)(n/q+\sigma)}\right)\right\}^{1/l}, \text{ if } 1<l<\infty \leq \|f\|_{M\dot{K}_{l,q}^{\sigma,\lambda}(\mathbb{Q}_p^n)}. \\ \sup\limits_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\sup\limits_{k\leq k_0}\left\{\sup\limits_{j\leq k_0} p^{j\sigma}\left\|f_j\right\|_{L^q(\mathbb{Q}_p^n)}\left(\sum\limits_{k=-\infty}^{j-2} p^{(k-j)(n/q+\sigma)}\right)\right\}^{1/l}, \text{ if } l=\infty \end{cases} \tag{44}$$

For $\mathscr{J}_2$, one can see from the definition of Morrey-Herz spaces that

$$p^{j\sigma}\left\|f_j\right\|_{L^q} \leq C\left(\sum_{\ell=-\infty}^{j} p^{\ell\sigma l}\|f_\ell\|_{L^q(\mathbb{Q}_p^n)}^l\right)^{1/l} \leq Cp^{j\lambda}\|f\|_{M\dot{K}_{l,q}^{\sigma,\lambda}(\mathbb{Q}_p^n)} \text{ for } q\geq 1. \tag{45}$$

Here, by the estimates (45) and the fact $\sigma > \lambda - n/q \geq -n/q$, we obtain

$$\begin{aligned} \mathscr{J}_2 &\leq C\|f\|_{M\dot{K}_{l,q}^{\sigma,\lambda}(\mathbb{Q}_p^n)}\sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0}\left(\sum_{j=k_0+1}^{\infty} p^{(k-j)(n/q+\sigma)}p^{j\lambda}\right)^l\right\}^{1/l} \\ &\leq C\|f\|_{M\dot{K}_{l,q}^{\sigma,\lambda}(\mathbb{Q}_p^n)}\sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0}\left(\sum_{j=k_0+1}^{\infty} p^{(k-j)(n/q+\sigma)}p^{j\lambda}\right)^l\right\}^{1/l} \\ &\leq C\|f\|_{M\dot{K}_{l,q}^{\sigma,\lambda}(\mathbb{Q}_p^n)}\sup_{k_0\in\mathbb{Z}}\left\{\sum_{k=-\infty}^{k_0} p^{(k_0-k)(-n/q-\sigma)l}\left(\sum_{j=k_0+1}^{\infty} p^{(j-k_0)(\lambda-n/q-\sigma)}\right)^l\right\}^{1/l} \\ &\leq C\|f\|_{M\dot{K}_{l,q}^{\sigma,\lambda}(\mathbb{Q}_p^n)}. \end{aligned} \tag{46}$$

Therefore, we obtain

$$\mathscr{N}_3 \leq \mathscr{J}_1 + \mathscr{J}_2 \leq C\|f\|_{M\dot{K}_{l,q}^{\sigma,\lambda}(\mathbb{Q}_p^n)}. \tag{47}$$

At last, by the combination of the estimates of $\mathscr{N}_1$, $\mathscr{N}_2$, and $\mathscr{N}_3$, we can conclude that $\|T_z(f)\|_{M\dot{K}_{l,q}^{\sigma,\lambda}(\mathbb{Q}_p^n)} \leq C\|f\|_{M\dot{K}_{l,q}^{\sigma,\lambda}(\mathbb{Q}_p^n)}$.

(ii) For the case of $q = 1$, we will consider the weak type boundedness of $T_z(f)$. For any $\mu > 0$, the decomposition of f shows that

$$\begin{aligned} &\mu\sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} p^{k\sigma l}\left|\{x\in S_k : |T_z(f)(x)|>2\mu\}\right|^l\right\}^{1/l} \\ &= \mu\sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} p^{k\sigma l}\left|\left\{x\in S_k : \left|\sum_{j=-\infty}^{k+1} T_z\left(f_j\right)(x)\right|>\mu\right\}\right|^l\right\}^{1/l} \\ &\quad+\mu\sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} p^{k\sigma l}\left|\left\{x\in S_k : \left|\sum_{j=k+2}^{\infty} T_z\left(f_j\right)(x)\right|>\mu\right\}\right|^l\right\}^{1/l} \\ &\equiv C(\mathscr{M}_1+\mathscr{M}_2). \end{aligned} \tag{48}$$

For $\mathscr{M}_1$, applying the weak type property $(1,1)$ of $T_z(f)$ and the estimates (45), we can conclude that

$$\begin{aligned} \mathscr{M}_1 &\leq \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} p^{k\sigma l}\sum_{j=-\infty}^{k+1}\left\|f_j\right\|_{L^1(\mathbb{Q}_p^n)}^l\right\}^{1/l} \\ &\leq \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0}\sum_{j=-\infty}^{k+1} p^{(k-j)\sigma l}p^{j\sigma l}\left\|f_j\right\|_{L^1(\mathbb{Q}_p^n)}^l\right\}^{1/l} \\ &\leq \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0}\sum_{j=-\infty}^{k+1} p^{(k-j)\sigma l}p^{j\lambda l}\|f\|_{M\dot{K}_{l,1}^{\sigma,\lambda}(\mathbb{Q}_p^n)}^l\right\}^{1/l} \\ &\leq \|f\|_{M\dot{K}_{l,1}^{\sigma,\lambda}(\mathbb{Q}_p^n)}\sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} p^{k\lambda l}\sum_{j=-\infty}^{k+1} p^{(k-j)(\sigma-\lambda)l}\right\}^{1/l} \\ &\leq \|f\|_{M\dot{K}_{l,1}^{\sigma,\lambda}(\mathbb{Q}_p^n)}\sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} p^{k\lambda l}\right\}^{1/l} \leq \|f\|_{M\dot{K}_{l,1}^{\sigma,\lambda}(\mathbb{Q}_p^n)}, \end{aligned} \tag{49}$$

where the facts $\lambda - n < \sigma < 0$ and $\lambda \geq 0$ are used.

For $\mathscr{M}_2$, by the Chebyshev inequality, the estimates (42) and (45) and the fact $\lambda - n < \sigma < 0$, we have

$$\begin{aligned} \mathscr{M}_2 &= \mu\sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} p^{k\sigma l}\left|\left\{x\in S_k : \left|\sum_{j=k+2}^{\infty} T_z\left(f_j\right)(x)\right|>\mu\right\}\right|^l\right\}^{1/l} \\ &= \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} p^{k\sigma l}\left(\sum_{j=k+2}^{\infty}\int_{S_k}\left|T_z\left(f_j\right)(x)\right|dx\right)^l\right\}^{1/l} \\ &= \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} p^{k\sigma l}\left(\sum_{j=k+2}^{\infty} p^{(k-j)n}\left\|f_j\right\|_{L^1(\mathbb{Q}_p^n)}\right)^l\right\}^{1/l} \\ &= \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0}\left(\sum_{j=k+2}^{\infty} p^{(k-j)(n+\sigma)}p^{j\sigma}\left\|f_j\right\|_{L^1(\mathbb{Q}_p^n)}\right)^l\right\}^{1/l} \\ &= \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0}\left(\sum_{j=k+2}^{\infty} p^{(k-j)(n+\sigma)}p^{j\lambda}\|f\|_{M\dot{K}_{l,1}^{\sigma,\lambda}(\mathbb{Q}_p^n)}\right)^l\right\}^{1/l} \\ &= \|f\|_{M\dot{K}_{l,1}^{\sigma,\lambda}(\mathbb{Q}_p^n)}\sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} p^{k\lambda}\left(\sum_{j=k+2}^{\infty} p^{(k-j)(n+\sigma-\lambda)}\right)^l\right\}^{1/l} \\ &\leq \|f\|_{M\dot{K}_{l,1}^{\sigma,\lambda}(\mathbb{Q}_p^n)}\sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} p^{k\lambda l}\right\}^{1/l} \leq \|f\|_{M\dot{K}_{l,1}^{\sigma,\lambda}(\mathbb{Q}_p^n)}. \end{aligned} \tag{50}$$

Hence, by the combination of the estimates of $\mathscr{M}_1$ and $\mathscr{M}_2$, we can get the desired inequality $\|T_z\|_{WM\dot{K}^{\sigma,\lambda}_{l,1}(\mathbb{Q}_p^n)} \le C \|f\|_{M\dot{K}^{\sigma,\lambda}_{l,1}(\mathbb{Q}_p^n)}$.

Therefore, we complete the proof of Theorem 18. □

Proof of Theorem 19. For any $x \in \mathbb{Q}_p^n$, since $\Omega \in L^\infty(\mathbb{Q}_p^n)$ and $b_i \in \Lambda_{\delta_i}(\mathbb{Q}_p^n)$ for $i = 1, 2, \cdots, m$, it is not difficult to see that

$$\left|T_z^{\mathbf{b}}(f)(x)\right| \le \int_{|x-y|_p>p^z} \prod_{i=1}^m |b_i(x) - b_i(y)||f(y)| \frac{|\Omega(x-y)|}{|x-y|_p^n} dy \le C \prod_{i=1}^m \|b_i\|_{\Lambda_{\delta_i}} \int_{\mathbb{Q}_p^n} \frac{|f(y)|}{|x-y|_p^{n-\delta}} dy \le C \prod_{i=1}^m \|b_i\|_{\Lambda_{\delta_i}} \mathscr{I}_\alpha^\delta(|f|)(x). \tag{51}$$

By Lemma 9, it is obvious that the commutator $T_z^{\mathbf{b}}(f)$ is bounded from $M\dot{K}^{\sigma_1,\lambda}_{l,q}(\mathbb{Q}_p^n)$ to $M\dot{K}^{\sigma_2,\lambda}_{l,r}(\mathbb{Q}_p^n)$ for all $z \in \mathbb{Z}$. □

Proof of Theorem 20. Similar to the proof of Theorem 18, let $f \in M\dot{K}^{\sigma_1,\lambda}_{l,q}(\mathbb{Q}_p^n)$ and decompose f into

$$f(x) = \sum_{j=-\infty}^{\infty} f(x)\chi_j(x) \equiv \sum_{j=-\infty}^{\infty} f_j(x), \quad j \in \mathbb{Z}. \tag{52}$$

(i) When $m = 1$, denote $T_z^{\mathbf{b}}(f)$ by $T_z^{b_1}(f)$, we consider

$$\begin{aligned}\left\|T_\alpha^{b_1}(f)\right\|_{M\dot{K}^{\sigma_2,\lambda}_{l,r}(\mathbb{Q}_p^n)} &\le C \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} p^{k\sigma_2 l} \left\|T_z^{b_1}(f)\chi_k\right\|^l_{L^r(\mathbb{Q}_p^n)}\right)^{1/l} \\ &\le C \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda} \left\{\sum_{k=-\infty}^{k_0} p^{k\sigma_2 l} \left(\sum_{j=-\infty}^{k-2} \left\|T_z^{b_1}(f_j)\chi_k\right\|_{L^r(\mathbb{Q}_p^n)}\right)^l\right\}^{1/l} \\ &\quad + C \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda} \left\{\sum_{k=-\infty}^{k_0} p^{k\sigma_2 l} \left(\sum_{j=k-1}^{k+1} \left\|T_z^{b_1}(f_j)\chi_k\right\|_{L^r(\mathbb{Q}_p^n)}\right)^l\right\}^{1/l} \\ &\quad + C \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda} \left\{\sum_{k=-\infty}^{k_0} p^{k\sigma_2 l} \left(\sum_{j=k+2}^{\infty} \left\|T_z^{b_1}(f_j)\chi_k\right\|_{L^r(\mathbb{Q}_p^n)}\right)^l\right\}^{1/l} \\ &\equiv C(\mathscr{E}_1 + \mathscr{E}_2 + \mathscr{E}_3).\end{aligned} \tag{53}$$

Let us first estimate $\mathscr{E}_2$, note that

$$T_z^{b_1}(f_j)\chi_k = \left(b_1 - b_{1B_k}\right) T_z(f_j)\chi_k - T_z\left(\left(b_1 - b_{1B_k}\right) f_j\right)\chi_k. \tag{54}$$

It is easy to see $1/r = 1/q + 1/q_1$ with $q, q_1 > 1$ implies $r > 1$. Applying the fact $\sigma_2 = \sigma_1 - n(1/q_1 + \nu_1)$, Hölder's inequality, Minkowski's inequality, the L^q-boundedness of T_z, and the L^r-boundedness of T_z, we have

$$\begin{aligned}\left\|T_z^{b_1}(f_j)\chi_k\right\|_{L^r(\mathbb{Q}_p^n)} &\le \left\|\left(b_1 - b_{1B_k}\right) T_z(f_j)\chi_k\right\|_{L^r(\mathbb{Q}_p^n)} + \left\|T_z\left(\left(b_1 - b_{1B_k}\right) f_j\right)\chi_k\right\|_{L^r(\mathbb{Q}_p^n)} \\ &\le \left(\int_{B_k} \left|b_1(x) - b_{1B_k}\right|^r \left|T_z(f_j)(x)\right|^r dx\right)^{1/r} + \left(\int_{B_k} \left|T_z\left(\left(b_1 - b_{1B_k}\right) f_j\right)(x)\right|^r dx\right)^{1/r} \\ &\le C\left(\int_{B_k} \left|b_1(x) - b_{1B_k}\right|^{q_1} dx\right)^{1/q_1} \left(\int_{B_k} \left|T_z(f_j)(x)\right|^q dx\right)^{1/q} + C\left(\int_{B_k} \left|\left(b_1(x) - b_{1B_k}\right) f_j(x)\right|^r dx\right)^{1/r} \\ &\le C|B_k|^{1/q_1+\nu_1} \|b_1\|_{CBMO_{q_1,\nu_1}} \left\|f_j\right\|_{L^q(\mathbb{Q}_p^n)} + C\left(\int_{B_k} \left|b_1(x) - b_{1B_k}\right|^{q_1} dx\right)^{1/q_1} \left(\int_{B_k} \left|f_j(x)\right|^q dx\right)^{1/q} \\ &\le C|B_k|^{1/q_1+\nu_1} \|b_1\|_{CBMO_{q_1,\nu_1}} \left\|f_j\right\|_{L^q(\mathbb{Q}_p^n)} \\ &\le C p^{kn(1/q_1+\nu_1)} \|b_1\|_{CBMO_{q_1,\nu_1}} \left\|f_j\right\|_{L^q(\mathbb{Q}_p^n)} \\ &\le C p^{k(\sigma_1-\sigma_2)} \|b_1\|_{CBMO_{q_1,\nu_1}} \left\|f_j\right\|_{L^q(\mathbb{Q}_p^n)}.\end{aligned} \tag{55}$$

Therefore, we get

$$\begin{aligned}\mathscr{E}_2 &\le C\|b_1\|_{CBMO_{q_1,\nu_1}} \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda} \left\{\sum_{k=-\infty}^{k_0} p^{k\sigma_2 l} \left(\sum_{j=k-1}^{k+1} p^{k(\sigma_1-\sigma_2)} \left\|f_j\right\|_{L^q(\mathbb{Q}_p^n)}\right)^l\right\}^{1/l} \\ &\le C\|b_1\|_{CBMO_{q_1,\nu_1}} \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda} \left\{\sum_{k=-\infty}^{k_0} \sum_{j=k-1}^{k+1} p^{k\sigma_1 l} \left\|f_j\right\|^l_{L^q(\mathbb{Q}_p^n)}\right\}^{1/l} \\ &\le C\|b_1\|_{CBMO_{q_1,\nu_1}} \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda} \left\{\sum_{j=-\infty}^{k_0+1} p^{j\sigma_1 l} \left\|f_j\right\|^l_{L^q(\mathbb{Q}_p^n)} \left(\sum_{k=j-1}^{j+1} p^{(k-j)\sigma_1 l}\right)\right\}^{1/l} \\ &\le C\|b_1\|_{CBMO_{q_1,\nu_1}} \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda} \left\{\sum_{j=-\infty}^{k_0+1} p^{j\sigma_1 l} \left\|f_j\right\|^l_{L^q(\mathbb{Q}_p^n)}\right\}^{1/l} \\ &\le C\|b_1\|_{CBMO_{q_1,\nu_1}} \|f\|_{M\dot{K}^{\sigma_1,\lambda}_{l,q}(\mathbb{Q}_p^n)}.\end{aligned} \tag{56}$$

Now, let us turn to the estimation of $\mathscr{E}_1$ and $\mathscr{E}_3$. If $x \in S_k$, we can easily deduce that

$$\begin{aligned}\left|T_z^{b_1}(f\chi_j)(x)\right| &\le C\left|\int_{|y|_p>p^z} \frac{(b_1(x) - b_1(y))\Omega(x-y) f_j(y)}{|x-y|_p^n} dy\right| \\ &\le C\left|\int_{S_j} \frac{(b_1(x) - b_1(y))\Omega(x-y) f(y)}{|x-y|_p^n} dy\right| \\ &\le C p^{-\max\{k,j\}n} \left[\int_{S_j} \left|b_1(x) - b_{1B_j}\right| |f(y)| dy + \int_{S_j} \left|b_1(y) - b_{1B_j}\right| |f(y)| dy\right].\end{aligned} \tag{57}$$

Then, using the facts $1/r = 1/q_1 + 1/q$, $\sigma_2 = \sigma_1 - n(1/q_1 + \nu_1)$, and $|j-k| \ge 2$, Lemma 7 and Hölder's inequality, we can get

$$\begin{aligned}
\left\|T_z^{b_1}\left(f_j\right)\chi_k\right\|_{L^r\left(\mathbb{Q}_p^n\right)} &\le Cp^{-\max\{k,j\}n}\left\{\int_{S_k}\left(\int_{S_j}|b_1(x)-b_1(y)||f(y)|dy\right)^r dx\right\}^{1/r}\\
&\le Cp^{-\max\{k,j\}n}\left\{\int_{S_k}\left|b_1(x)-b_{1B_j}\right|^r\left(\int_{S_j}|f(y)|dy\right)^r dx\right\}^{1/r}\\
&\quad + Cp^{-\max\{k,j\}n}\left\{\int_{S_k}\left(\int_{S_j}\left|b_1(y)-b_{1B_j}\right||f(y)|dy\right)^r dx\right\}^{1/r}\\
&\le Cp^{-\max\{k,j\}n}\left\|f_j\right\|_{L^1\left(\mathbb{Q}_p^n\right)}\left(\int_{S_k}\left|b_1(x)-b_{1B_j}\right|^r dx\right)^{1/r}\\
&\quad + Cp^{-\max\{k,j\}n}|S_k|^{1/r}\int_{S_j}\left|b_1(y)-b_{1B_j}\right||f(y)|dy\\
&\le Cp^{-\max\{k,j\}n}\left|S_j\right|^{1-1/q}\left\|f_j\right\|_{L^q\left(\mathbb{Q}_p^n\right)}|S_k|^{1/r-1/q_1}\left(\int_{S_k}\left|b_1(x)-b_{1B_j}\right|^{q_1}dx\right)^{1/q_1}\\
&\quad + Cp^{-\max\{k,j\}n}|S_k|^{1/r}\left|S_j\right|^{1-1/q-1/q_1}\left\|f_j\right\|_{L^q\left(\mathbb{Q}_p^n\right)}\left(\int_{S_j}\left|b_1(y)-b_{1B_j}\right|^{q_1}dy\right)^{1/q_1}\\
&\le Cp^{-\max\{k,j\}n}\left|S_j\right|^{1-1/q}\left\|f_j\right\|_{L^q\left(\mathbb{Q}_p^n\right)}|S_k|^{1/r-1/q_1}\\
&\quad \times\left[\left(\int_{S_k}\left|b_1(x)-b_{1B_k}\right|^{q_1}dx\right)^{1/q_1}+|B_k|^{1/q_1}\left|b_{1B_k}-b_{1B_j}\right|\right]\\
&\quad + Cp^{-\max\{k,j\}n}|S_k|^{1/r}\left|S_j\right|^{1-1/q-1/q_1}\left|B_j\right|^{1/q_1+\nu_1}\left\|f_j\right\|_{L^q\left(\mathbb{Q}_p^n\right)}\|b_1\|_{CBMO_{q_1,\nu_1}}\\
&\le Cp^{-\max\{k,j\}n}\left|S_j\right|^{1-1/q}|S_k|^{1/r-1/q_1}\left\|f_j\right\|_{L^q\left(\mathbb{Q}_p^n\right)}\|b_1\|_{CBMO_{q_1,\nu_1}}\\
&\quad \times\left[|B_k|^{1/q_1+\nu_1}+|B_k|^{1/q_1}p^n|j-k|\max\left\{|B_k|^{\nu_1},\left|B_j\right|^{\nu_1}\right\}\right]\\
&\quad + Cp^{-\max\{k,j\}n}|S_k|^{1/r}\left|S_j\right|^{1-1/q-1/q_1}\left|B_j\right|^{1/q_1+\nu_1}\left\|f_j\right\|_{L^q\left(\mathbb{Q}_p^n\right)}\|b_1\|_{CBMO_{q_1,\nu_1}}\\
&\le Cp^{-\max\{k,j\}n}p^{jn(1-1/q)}p^{kn(1/r-1/q_1)}\|b_1\|_{CBMO_{q_1,\nu_1}}\left\|f_j\right\|_{L^q\left(\mathbb{Q}_p^n\right)}\\
&\quad \times\left[p^{kn(1/q_1+\nu_1)}+p^{kn/q_1}p^n|j-k|p^{\max\{k,j\}n\nu_1}\right]\\
&\quad + Cp^{-\max\{k,j\}n}p^{jn(1-1/q)}p^{kn/r}p^{jn\nu_1}\|b_1\|_{CBMO_{q_1,\nu_1}}\left\|f_j\right\|_{L^q\left(\mathbb{Q}_p^n\right)}\\
&\le Cp^{-\max\{k,j\}n}p^{jn(1-1/q)}p^{kn/r}\|b_1\|_{CBMO_{q_1,\nu_1}}\left\|f_j\right\|_{L^q\left(\mathbb{Q}_p^n\right)}\\
&\quad \times\left[p^{kn\nu_1}+p^n|j-k|p^{\max\{k,j\}n\nu_1}\right]+Cp^{-\max\{k,j\}n}p^{jn(1-1/q)}p^{kn/r}p^{jn\nu_1}\|b_1\|_{CBMO_{q_1,\nu_1}}\left\|f_j\right\|_{L^q\left(\mathbb{Q}_p^n\right)}\\
&\le Cp^{-\max\{k,j\}n}p^{jn(1-1/q)}p^{kn/r}p^{\max\{k,j\}n\nu_1}|j-k|\|b_1\|_{CBMO_{q_1,\nu_1}}\left\|f_j\right\|_{L^q\left(\mathbb{Q}_p^n\right)}\\
&\le\begin{cases} C(k-j)p^{kn(\nu_1-1+1/r)}p^{jn(1-1/q)}\|b_1\|_{CBMO_{q_1,\nu_1}}\left\|f_j\right\|_{L^q\left(\mathbb{Q}_p^n\right)}, \text{if} & j\le k-2\\ C(j-k)p^{jn(\nu_1-1}p^{jn(1-1/q)}p^{kn/r}\|b_1\|_{CBMO_{q_1,\nu_1}}\left\|f_j\right\|_{L^q\left(\mathbb{Q}_p^n\right)}, \text{if} & j\ge k+2\end{cases}\\
&\le\begin{cases} C(k-j)p^{kn(\nu_1-1+1/q_1+1/q)}p^{jn(1-1/q)}\|b_1\|_{CBMO_{q_1,\nu_1}}\left\|f_j\right\|_{L^q\left(\mathbb{Q}_p^n\right)}, \text{if} & j\le k-2\\ C(j-k)p^{jn(\nu_1+1/q_1-1/r)}p^{kn/r}\|b_1\|_{CBMO_{q_1,\nu_1}}\left\|f_j\right\|_{L^q\left(\mathbb{Q}_p^n\right)}, \text{if} & j\ge k+2\end{cases}\\
&\le\begin{cases} C(k-j)p^{(k-j)[\sigma_1-n(1-1/q)]}p^{j\sigma_1}p^{-k\sigma_2}\|b_1\|_{CBMO_{q_1,\nu_1}}\left\|f_j\right\|_{L^q\left(\mathbb{Q}_p^n\right)}, \text{if} & j\le k-2\\ C(j-k)p^{(k-j)(\sigma_1+n/q-n\nu_1)}p^{j\sigma_1}p^{-k\sigma_2}\|b_1\|_{CBMO_{q_1,\nu_1}}\left\|f_j\right\|_{L^q\left(\mathbb{Q}_p^n\right)}, \text{if} & j\ge k+2.\end{cases}
\end{aligned} \tag{58}$$

Thus, the fact $n\nu_1+\lambda-n/q<\sigma_1<n(1-1/q)+\lambda$ and (45) imply

$$\begin{aligned}
\mathscr{E}_1 &\le C\|b_1\|_{CBMO_{q_1,\nu_1}}\\
&\quad\times\sup_{k_0\in\mathbb{Z}}p^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0}p^{k\sigma_2 l}\left(\sum_{j=-\infty}^{k-2}(k-j)p^{(k-j)\left[\sigma_1-n\left(1-\frac{1}{q}\right)\right]}p^{j\sigma_1}p^{-k\sigma_2}\left\|f_j\right\|_{L^q\left(\mathbb{Q}_p^n\right)}\right)^l\right\}^{1/l}\\
&\le C\|b_1\|_{CBMO_{q_1,\nu_1}}\times\sup_{k_0\in\mathbb{Z}}p^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0}\left(\sum_{j=-\infty}^{k-2}(k-j)p^{(k-j)\left[\sigma_1-n\left(1-\frac{1}{q}\right)\right]}p^{j\sigma_1}\left\|f_j\right\|_{L^q\left(\mathbb{Q}_p^n\right)}\right)^l\right\}^{1/l}\\
&\le C\|b_1\|_{CBMO_{q_1,\nu_1}}\times\sup_{k_0\in\mathbb{Z}}p^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0}\left(\sum_{j=-\infty}^{k-2}(k-j)p^{(k-j)\left[\sigma_1-n\left(1-\frac{1}{q}\right)\right]}p^{j\lambda}\|f\|_{M\dot{K}_{l,q}^{\sigma_1,\lambda}\left(\mathbb{Q}_p^n\right)}\right)^l\right\}^{1/l}\\
&\le C\|b_1\|_{CBMO_{q_1,\nu_1}}\|f\|_{M\dot{K}_{l,q}^{\sigma_1,\lambda}\left(\mathbb{Q}_p^n\right)}\times\sup_{k_0\in\mathbb{Z}}p^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0}p^{k\lambda l}\left(\sum_{j=-\infty}^{k-2}(k-j)p^{(k-j)\left[\sigma_1-n\left(1-\frac{1}{q}\right)-\lambda\right]}\right)^l\right\}^{1/l}\\
&\le C\|b_1\|_{CBMO_{q_1,\nu_1}}\|f\|_{M\dot{K}_{l,q}^{\sigma_1,\lambda}\left(\mathbb{Q}_p^n\right)}\sup_{k_0\in\mathbb{Z}}p^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0}p^{k\lambda l}\right\}^{1/l}\\
&\le C\|b_1\|_{CBMO_{q_1,\nu_1}}\|f\|_{M\dot{K}_{l,q}^{\sigma_1,\lambda}\left(\mathbb{Q}_p^n\right)},
\end{aligned} \tag{59}$$

$$\begin{aligned}
\mathscr{E}_3 &\le C\|b_1\|_{CBMO_{q_1,\nu_1}}\\
&\quad\times\sup_{k_0\in\mathbb{Z}}p^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0}\left(\sum_{j=k+2}^{\infty}(j-k)p^{(k-j)k(\sigma_1+n/q-n\nu_1)}p^{j\sigma_1}\left\|f_j\right\|_{L^q\left(\mathbb{Q}_p^n\right)}\right)^l\right\}^{1/l}\\
&\le C\|b_1\|_{CBMO_{q_1,\nu_1}}\times\sup_{k_0\in\mathbb{Z}}p^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0}\left(\sum_{j=k+2}^{\infty}(j-k)p^{(k-j)k(\sigma_1+n/q-n\nu_1)}p^{j\lambda}\|f\|_{M\dot{K}_{l,q}^{\sigma_1,\lambda}\left(\mathbb{Q}_p^n\right)}\right)^l\right\}^{1/l}\\
&\le C\|b_1\|_{CBMO_{q_1,\nu_1}}\|f\|_{M\dot{K}_{l,q}^{\sigma_1,\lambda}\left(\mathbb{Q}_p^n\right)}\times\sup_{k_0\in\mathbb{Z}}p^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0}p^{k\lambda l}\left(\sum_{j=k+2}^{\infty}(j-k)p^{(k-j)k(\sigma_1+n/q-n\nu_1-\lambda)}\right)^l\right\}^{1/l}\\
&\le C\|b_1\|_{CBMO_{q_1,\nu_1}}\|f\|_{M\dot{K}_{l,q}^{\sigma_1,\lambda}\left(\mathbb{Q}_p^n\right)}\sup_{k_0\in\mathbb{Z}}p^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0}p^{k\lambda l}\right\}^{1/l}\le C\|b_1\|_{CBMO_{q_1,\nu_1}}\|f\|_{M\dot{K}_{l,q}^{\sigma_1,\lambda}\left(\mathbb{Q}_p^n\right)}.
\end{aligned} \tag{60}$$

Combining the estimates of $\mathscr{E}_1$, $\mathscr{E}_2$, and $\mathscr{E}_3$, we complete the proof for Theorem 12 in the case of $m=1$.

(ii) Now, we consider the case of $m\ge 2$. In order to simplify the proving process, for positive integer m and $1\le i\le m$, we denote by C_i^m the family of all finite subsets $\theta=\{\theta_1,\theta_2,\cdots,\theta_m\}$ of $\{1,2,\cdots,m\}$ of i different elements, let $\theta^c=\{1,2,\cdots,m\}\setminus\theta$ for any $\theta\in C_i^m$. For $\mathbf{b}=(b_1,b_2,\cdots,b_m)$, let $\mathbf{b}_\theta=(b_{\theta_1},b_{\theta_2},\cdots,b_{\theta_i})$, $b_\theta=b_{\theta_1}b_{\theta_2}\cdots b_{\theta_i}$, and b_{iB_k} denote the integral average of the function b_i over the set B_k, then

$$\begin{aligned}
\left(b(x)-b_{B_k}\right)_\theta &= \left(b_{\theta_1}(x)-b_{\theta_1 B_k}\right)\cdots\left(b_{\theta_i}(x)-b_{\theta_i B_k}\right),\\
\left(b_{B_j}-b_{B_k}\right)_\theta &= \left(b_{\theta_1 B_j}-b_{\theta_1 B_k}\right)\cdots\left(b_{\theta_i B_j}-b_{\theta_i B_k}\right),\\
\|\mathbf{b}_\theta\|_{CBMO_{\bar{q},\bar{\nu}}} &= \left\|b_{\theta_1}\right\|_{CBMO_{\bar{q}_1,\bar{\nu}_1}}\cdots\left\|b_{\theta_i}\right\|_{CBMO_{\bar{q}_m,\bar{\nu}_m}},
\end{aligned} \tag{61}$$

where $1/\bar{q}=1/\bar{q}_1+\cdots+1/\bar{q}_m$ and $\bar{\nu}=\bar{\nu}_1+\cdots+\bar{\nu}_m$.

We write

$$\begin{aligned}
\left\|T_z^{\mathbf{b}}(f)\right\|_{M\dot{K}_{l,r}^{\sigma_2,\lambda}\left(\mathbb{Q}_p^n\right)} &\le C\sup_{k_0\in\mathbb{Z}}p^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0}p^{k\sigma_2 l}\left\|T_z^{\mathbf{b}}(f)\chi_k\right\|_{L^r\left(\mathbb{Q}_p^n\right)}^l\right)^{1/l}\\
&\le C\sup_{k_0\in\mathbb{Z}}p^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0}p^{k\sigma_2 l}\left(\sum_{j=-\infty}^{k-2}\left\|T_z^{\mathbf{b}}\left(f_j\right)\chi_k\right\|_{L^r\left(\mathbb{Q}_p^n\right)}\right)^l\right\}^{1/l}\\
&\quad + C\sup_{k_0\in\mathbb{Z}}p^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0}p^{k\sigma_2 l}\left(\sum_{j=k-1}^{k+1}\left\|T_z^{\mathbf{b}}\left(f_j\right)\chi_k\right\|_{L^r\left(\mathbb{Q}_p^n\right)}\right)^l\right\}^{1/l}\\
&\quad + C\sup_{k_0\in\mathbb{Z}}p^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0}p^{k\sigma_2 l}\left(\sum_{j=k+2}^{\infty}\left\|T_z^{\mathbf{b}}\left(f_j\right)\chi_k\right\|_{L^r\left(\mathbb{Q}_p^n\right)}\right)^l\right\}^{1/l}\\
&\equiv C(\mathscr{G}_1+\mathscr{G}_2+\mathscr{G}_3).
\end{aligned} \tag{62}$$

Let us first estimate $\mathscr{G}_2$, and notice that

$$\begin{aligned}
T_z^{\mathbf{b}}\left(f_j\right)(x) &= \int_{|y|_p>p^z}\frac{\Omega(x-y)f_j(y)}{|x-y|_p^n}\\
&\quad\cdot\prod_{i=1}^m\left[\left(b_i(x)-b_{iB_k}\right)+\left(b_i(y)-b_{iB_k}\right)\right]dy
\end{aligned}$$

$$
\begin{aligned}
&= \prod_{i=1}^{m}\left(b_i(x)-b_{iB_k}\right)T_z\left(f_j\right)(x)+(-1)^m T_z \\
&\cdot\left(\prod_{i=1}^{m}\left(b_i-b_{iB_k}\right)f_j\right)(x)+\sum_{i=1}^{m-1}\sum_{\theta\in C_i^m}(-1)^{m-i}(b(x) \\
&-b_{B_k})_\theta\int_{|y|_p>p^z}\frac{\Omega(x-y)f_j(y)}{|x-y|_p^{n-\alpha}}\prod_{i=1}^{m}\left(b(y)-b_{B_k}\right)_{\theta^c}dy \\
&= \prod_{i=1}^{m}\left(b_i(x)-b_{iB_k}\right)T_z\left(f_j\right)(x)+(-1)^m T_z \\
&\cdot\left(\prod_{i=1}^{m}\left(b_i-b_{iB_k}\right)f_j\right)(x) \\
&+\sum_{i=1}^{m-1}\sum_{\theta\in C_i^m}(-1)^{m-i}\left(b(x)-b_{B_k}\right)_\theta T_z\left(\left(b-b_{B_k}\right)_{\theta^c}f_j\right)(x) \\
&\equiv \mathscr{H}_1+\mathscr{H}_2+\mathscr{H}_3.
\end{aligned} \tag{63}
$$

For $\mathscr{H}_1$, applying Hölder's inequality, the boundedness of T_z on $L^q(\mathbb{Q}_p^n)$, $1/r=1/q_1+1/q_2+\cdots+1/q_m+1/q$, and $\sigma_2=\sigma_1-n(1/q_1+\cdots+1/q_m+\nu_1+\cdots+\nu_m)$, we obtain

$$
\begin{aligned}
&\left\|\prod_{i=1}^{m}\left(b_i-b_{iB_k}\right)T_z\left(f_j\right)\chi_k\right\|_{L^r(\mathbb{Q}_p^n)} \\
&\le C\left(\int_{B_k}\left|\prod_{i=1}^{m}\left(b_i(x)-b_{iB_k}\right)T_z\left(f_j\right)(x)\right|^r dx\right)^{1/r} \\
&\le C\prod_{i=1}^{m}\left(\int_{B_k}\left|b_i(x)-b_{iB_k}\right|^{q_i}dx\right)^{1/q_i}\left(\int_{B_k}\left|T_z\left(f_j\right)(x)\right|^q dx\right)^{1/q} \\
&\le C\prod_{i=1}^{m}|B_k|^{1/q_i+\nu_i}\|b_i\|_{\mathrm{CBMO}_{q_i,\nu_i}}\left\|f_j\right\|_{L^q(\mathbb{Q}_p^n)} \\
&\le C\prod_{i=1}^{m}p^{kn(1/q_i+\nu_i)}\|b_i\|_{\mathrm{CBMO}_{q_i,\nu_i}}\left\|f_j\right\|_{L^q(\mathbb{Q}_p^n)} \\
&\le Cp^{k(\sigma_1-\sigma_2)}\prod_{i=1}^{m}\|b_i\|_{\mathrm{CBMO}_{q_i,\nu_i}}\left\|f_j\right\|_{L^q(\mathbb{Q}_p^n)}.
\end{aligned} \tag{64}
$$

For $\mathscr{H}_2$, by Hölder's inequality, the boundedness of T_z on $L^r(\mathbb{Q}_p^n)$ for $r>1$ and the fact $\sigma_2=\sigma_1-n(1/q_1+\cdots+1/q_m+\nu_1+\cdots+\nu_m)$, we get

$$
\begin{aligned}
&\left\|(-1)^m T_z\left(\prod_{i=1}^{m}\left(b_i-b_{iB_k}\right)f_j\right)\chi_k\right\|_{L^r(\mathbb{Q}_p^n)} \\
&\le C\left(\int_{B_k}\left|\prod_{i=1}^{m}\left(b_i(x)-b_{iB_k}\right)f_j(x)\right|^r dx\right)^{1/r} \\
&\le C\prod_{i=1}^{m}\left(\int_{B_k}\left|b_i(x)-b_{iB_k}\right|^{q_i}dx\right)^{1/q_i}\left(\int_{B_k}\left|f_j(x)\right|^q dx\right)^{1/q} \\
&\le C\prod_{i=1}^{m}|B_k|^{1/q_i+\nu_i}\|b_i\|_{\mathrm{CBMO}_{q_i,\nu_i}}\left\|f_j\right\|_{L^q(\mathbb{Q}_p^n)} \\
&\le C\prod_{i=1}^{m}p^{kn(1/q_i+\nu_i)}\|b_i\|_{\mathrm{CBMO}_{q_i,\nu_i}}\left\|f_j\right\|_{L^q(\mathbb{Q}_p^n)} \\
&\le Cp^{k(\sigma_1-\sigma_2)}\prod_{i=1}^{m}\|b_i\|_{\mathrm{CBMO}_{q_i,\nu_i}}\left\|f_j\right\|_{L^q(\mathbb{Q}_p^n)}.
\end{aligned} \tag{65}
$$

For $\mathscr{H}_3$, let

$$
\begin{aligned}
&1/h=\sum_{\theta_i\in\theta}1/q_i,\ 1/z=\sum_{\theta_i\in\theta^c}1/q_i, \\
&1/\nu'=\sum_{\theta_i\in\theta}1/\nu_i,\ 1/\nu''=\sum_{\theta_i\in\theta}1/\nu_i,\ 1/\omega=1/z+1/q,
\end{aligned} \tag{66}
$$

then $1/r=1/h+1/\omega$. Using Hölder's inequality and the L^ω-boundedness of T_z, we have

$$
\begin{aligned}
&\left\|\sum_{i=1}^{m-1}\sum_{\theta\in C_i^m}(-1)^{m-i}\left(b-b_{B_k}\right)_\theta T_z\left(\left(b-b_{B_k}\right)_{\theta^c}f_j\right)\chi_k\right\|_{L^r(\mathbb{Q}_p^n)} \\
&\le C\sum_{i=1}^{m-1}\sum_{\theta\in C_i^m}\left(\int_{B_k}\left|\left(b(x)-b_{B_k}\right)_\theta\right|^h dx\right)^{1/h} \\
&\times\left(\int_{B_k}\left|T_z\left(\left(b-b_{B_k}\right)_{\theta^c}f_j\right)(x)\right|^\omega dx\right)^{1/\omega} \\
&\le C\sum_{i=1}^{m-1}\sum_{\theta\in C_i^m}\left(\int_{B_k}\left|\left(b(x)-b_{B_k}\right)_\theta\right|^h dx\right)^{1/h} \\
&\times\left(\int_{B_k}\left|\left(b(x)-b_{B_k}\right)_{\theta^c}\right|^z dx\right)^{1/z}\times\left(\int_{B_k}\left|f_j(x)\right|^q dx\right)^{1/q} \\
&\le C|B_k|^{1/h+\nu'}\|\mathbf{b}_\theta\|_{\mathrm{CBMO}_{h,\nu'}}|B_k|^{1/z+\nu''}\|\mathbf{b}_{\theta^c}\|_{\mathrm{CBMO}_{z,\nu''}}\left\|f_j\right\|_{L^q(\mathbb{Q}_p^n)} \\
&\le C\prod_{i=1}^{m}|B_k|^{1/q_i+\nu_i}\|b_i\|_{\mathrm{CBMO}_{q_i,\nu_i}}\left\|f_j\right\|_{L^q(\mathbb{Q}_p^n)} \\
&\le C\prod_{i=1}^{m}p^{kn(1/q_i+\nu_i)}\|b_i\|_{\mathrm{CBMO}_{q_i,\nu_i}}\left\|f_j\right\|_{L^q(\mathbb{Q}_p^n)} \\
&\le Cp^{k(\sigma_1-\sigma_2)}\prod_{i=1}^{m}\|b_i\|_{\mathrm{CBMO}_{q_i,\nu_i}}\left\|f_j\right\|_{L^q(\mathbb{Q}_p^n)}.
\end{aligned} \tag{67}
$$

Then, similar to the estimation of $\mathscr{E}_2$, it is not difficult for us to get

$$
\begin{aligned}
\mathscr{G}_2&\le C\prod_{i=1}^{m}\|b_i\|_{\mathrm{CBMO}_{q_i,\nu_i}}\sup_{k_0\in\mathbb{Z}}p^{-k_0\lambda} \\
&\cdot\left\{\sum_{k=-\infty}^{k_0}p^{k\sigma_2 l}\left(\sum_{j=k-1}^{k+1}p^{k(\sigma_1-\sigma_2)}\left\|f_j\right\|_{L^q(\mathbb{Q}_p^n)}\right)^l\right\}^{1/l} \\
&\le C\prod_{i=1}^{m}\|b_i\|_{\mathrm{CBMO}_{q_i,\nu_i}}\|f\|_{M\dot{K}_{l,q}^{\sigma_1,\lambda}(\mathbb{Q}_p^n)}.
\end{aligned} \tag{68}
$$

Next, we will estimate $\mathscr{G}_1$ and $\mathscr{G}_3$. Let

$$
1/\tau=\sum_{\theta_i\in\theta}1/q_i,\ 1/\tau'=\sum_{\theta_i\in\theta^c}1/q_i,\ 1/\vartheta=\sum_{\theta_i\in\theta}1/\nu_i,\ 1/\vartheta'=\sum_{\theta_i\in\theta}1/\nu_i. \tag{69}
$$

Therefore, by Minkowski's inequality, Hölder's inequality, and Lemma 7, it follows that

$$\begin{aligned}
&\left\|T_z^{\mathbf{b}}\left(f_j\right)\chi_k\right\|_{L^r(\mathbb{Q}_p^n)} \\
&\le Cp^{-\max\{k,j\}n}\left\{\int_{S_k}\left(\int_{S_j}\prod_{i=1}^m |b_i(x)-b_i(y)||f(y)|dy\right)^r dx\right\}^{1/r} \\
&\le Cp^{-\max\{k,j\}n}\int_{S_j}\left(\int_{S_k}\prod_{i=1}^m |b_i(x)-b_i(y)|^r dx\right)^{1/r}|f(y)|dy \\
&\le Cp^{-\max\{k,j\}n}\sum_{i=0}^m\sum_{\theta\in C_i^m}\left(\int_{S_k}\left|\left(b(x)-b_{B_k}\right)_\theta\right|^r dx\right)^{1/r} \\
&\quad\times\int_{S_j}\left|\left(b(y)-b_{B_k}\right)_{\theta^c}\right||f(y)|dy \\
&\le Cp^{-\max\{k,j\}n}\sum_{i=0}^m\sum_{\theta\in C_i^m}\left(\int_{S_k}\left|\left(b(x)-b_{B_k}\right)_\theta\right|^\tau dx\right)^{1/\tau}|B_k|^{1/r-1/\tau} \\
&\quad\times\left(\int_{S_j}\left|\left(b(y)-b_{B_k}\right)_{\theta^c}\right|^{\tau'}dy\right)^{1/\tau'}\left(\int_{S_j}|f(y)|^q\right)^{1/q}|B_j|^{1-1/\tau'-1/q} \\
&\le Cp^{-\max\{k,j\}n}\left\|f_j\right\|_{L^q(\mathbb{Q}_p^n)}\sum_{i=0}^m\sum_{\theta\in C_i^m}\left(\int_{S_k}\left|\left(b(x)-b_{B_k}\right)_\theta\right|^\tau dx\right)^{1/\tau}|B_k|^{1/r-1/\tau} \\
&\quad\times\left[\left(\int_{S_j}\left|\left(b(y)-b_{B_j}\right)_{\theta^c}\right|^{\tau'}dy\right)^{1/\tau'}+|B_j|^{1/\tau'}\left|\left(b_{B_k}-b_{B_j}\right)_{\theta^c}\right|\right]|B_j|^{1-1/\tau'-1/q} \\
&\le Cp^{-\max\{k,j\}n}p^{kn/r}p^{jn(1-1/q)}\left\|f_j\right\|_{L^q(\mathbb{Q}_p^n)}\sum_{i=0}^m\sum_{\theta\in C_i^m}p^{kn\vartheta}\|\mathbf{b}_\theta\|_{CBMO_{\tau,\vartheta}} \\
&\quad\times\|\mathbf{b}_{\theta^c}\|_{CBMO_{\tau',\vartheta'}}\left[p^{jn\vartheta'}+p^{mn}|j-k|^m p^{\max\{k,j\}n\vartheta'}\right] \\
&\le Cp^{-\max\{k,j\}n}p^{kn/r}p^{jn(1-1/q)}\left\|f_j\right\|_{L^q(\mathbb{Q}_p^n)} \\
&\quad\times|j-k|^m p^{\max\{k,j\}n\vartheta}\|\mathbf{b}_\theta\|_{CBMO_{\tau,\vartheta}}p^{\max\{k,j\}n\vartheta'}\|\mathbf{b}_{\theta^c}\|_{CBMO_{\tau',\vartheta'}} \\
&\le Cp^{-\max\{k,j\}n}p^{kn/r}p^{jn(1-1/q)}\left\|f_j\right\|_{L^q(\mathbb{Q}_p^n)}|j-k|^m\prod_{i=1}^m p^{\max\{k,j\}n\nu_i}\|b_i\|_{CBMO_{q_i,\nu_i}} \\
&\le Cp^{\max\{k,j\}n(\nu-1)}p^{kn/r}p^{jn(1-1/q)}|j-k|^m\prod_{i=1}^m\|b_i\|_{CBMO_{q_i,\nu_i}}\left\|f_j\right\|_{L^q(\mathbb{Q}_p^n)} \\
&\le\begin{cases} C(k-j)^m p^{(k-j)[\sigma_1-n(1-1/q)]}p^{j\sigma_1}p^{-k\sigma_2}\prod_{i=1}^m\|b_i\|_{CBMO_{q_i,\nu_i}}\left\|f_j\right\|_{L^q(\mathbb{Q}_p^n)}, & \text{if } j\le k-2; \\ C(j-k)^m p^{(k-j)(\sigma_1+n/q-n\nu)}p^{j\sigma_1}p^{-k\sigma_2}\prod_{i=1}^m\|b_i\|_{CBMO_{q_i,\nu_i}}\left\|f_j\right\|_{L^q(\mathbb{Q}_p^n)}, & \text{if } j\ge k+2. \end{cases}
\end{aligned} \tag{70}$$

Thus, similar to the estimation of $\mathscr{E}_1$ and $\mathscr{E}_3$, we get

$$\begin{aligned}
\mathscr{G}_1 &\le C\prod_{i=1}^m\|b_i\|_{CBMO_{q_i,\nu_i}} \\
&\quad\times\sup_{k_0\in\mathbb{Z}}p^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0}\left(\sum_{j=-\infty}^{k-2}(k-j)^m p^{(k-j)[\sigma_1-n(1-1/q)]}p^{j\sigma_1}\left\|f_j\right\|_{L^q(\mathbb{Q}_p^n)}\right)^l\right\}^{1/l} \\
&\le C\prod_{i=1}^m\|b_i\|_{\mathrm{CBMO}_{q_i,\nu_i}}\|f\|_{M\dot{K}_{l,q}^{\sigma_1,\lambda}(\mathbb{Q}_p^n)}\times\sup_{k_0\in\mathbb{Z}}p^{-k_0\lambda} \\
&\quad\cdot\left\{\sum_{k=-\infty}^{k_0}p^{k\lambda l}\left(\sum_{j=-\infty}^{k-2}(k-j)^m p^{(k-j)[\sigma_1-n(1-1/q)-\lambda]}\right)^l\right\}^{1/l} \\
&\le C\prod_{i=1}^m\|b_i\|_{\mathrm{CBMO}_{q_i,\nu_i}}\|f\|_{M\dot{K}_{l,q}^{\sigma_1,\lambda}(\mathbb{Q}_p^n)}. \\
\mathscr{G}_3 &\le C\prod_{i=1}^m\|b_i\|_{\mathrm{CBMO}_{q_i,\nu_i}} \\
&\quad\times\sup_{k_0\in\mathbb{Z}}p^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0}\left(\sum_{j=k+2}^{\infty}(j-k)^m p^{(k-j)(\sigma_1+n/q-n\nu)}p^{j\sigma_1}\left\|f_j\right\|_{L^q(\mathbb{Q}_p^n)}\right)^l\right\}^{1/l} \\
&\le C\prod_{i=1}^m\|b_i\|_{\mathrm{CBMO}_{q_i,\nu_i}}\|f\|_{M\dot{K}_{l,q}^{\sigma_1,\lambda}(\mathbb{Q}_p^n)}\times\sup_{k_0\in\mathbb{Z}}p^{-k_0\lambda} \\
&\quad\times\left\{\sum_{k=-\infty}^{k_0}p^{k\lambda l}\left(\sum_{j=k+2}^{\infty}(j-k)^m p^{(k-j)(\sigma_1+n/q-n\nu-\lambda)}\right)^l\right\}^{1/l} \\
&\le C\prod_{i=1}^m\|b_i\|_{\mathrm{CBMO}_{q_i,\nu_i}}\|f\|_{M\dot{K}_{l,q}^{\sigma_1,\lambda}(\mathbb{Q}_p^n)}.
\end{aligned} \tag{71}$$

Hence, by the combination of the estimates of $\mathscr{G}_1$, $\mathscr{G}_2$, and $\mathscr{G}_3$, we have finished the proof for Theorem 14 in the case of $m\ge 2$. This completes the proof of Theorem 14. □

5. Conclusion

We mainly focused on the boundedness of classical p-adic singular integrals on Morrey-Herz spaces. Besides, we also obtained the boundedness of the multilinear commutator generated by p-adic singular integral operators and Lipschitz functions or by p-adic singular integral operators and λ-central BMO functions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This research is partially supported by the Teacher Professional Development Program of Domestic University Visiting Scholar in Zhejiang Province under grant no. FX2022076, the National Nature Science Foundation of China under grant no. 11871177, the National Education Scientific Planning under grant no. BIA210166, and the Zhejiang Provincial Philosophy and Social Sciences Planning Project under grant no. 21NDJC134YB.

References

[1] V. S. Vladimirov, "Tables of integrals for complex-valued functions of p-adic arguments," *Proceedings of the Steklov Institute of Mathematics*, vol. 284, Supplement 2, pp. 1–59, 2014.

[2] V. S. Vladimirov, I. V. Volovich, and E. I. Zelenov, *p-Adic Analysis and Mathematical Physics*, World Scientific, Singapore, 1994.

[3] A. Khrennikov and M. Nilsson, *p-Adic Deterministic and Random Dynamical*, Kluwer Academic Publishers, Dordrecht, 2004.

[4] Q. Y. Wu, S. Z. Lu, and Z. W. Fu, "p-adic central function spaces and singular integral operators," *Chinese Journal of Contemporary Mathematics*, vol. 38, no. 1, pp. 73–90, 2017.

[5] H. X. Mo, Z. Han, L. Yang, and X. J. Wang, "p-adic singular integrals and their commutators in generalized Morrey spaces," *Journal of Inequalities and Applications*, vol. 2019, no. 1, article 65, p. 13, 2019.

[6] S. Ashraf and Q. Jahan, "A note on boundedness of singular integral operators on function spaces over local fields," 2022, https://arxiv.org/abs/2204.02941.

[7] K. Phillips and M. H. Taibleson, "Singular integrals in several variables over a local field," *Pacific Journal of Mathematics*, vol. 30, no. 1, pp. 209–231, 1969.

[8] M. H. Taibleson, *Fourier Analysis on Local Fields*, Princeton Univ Press, Princeton NJ, 1975.

[9] K. P. Ho, "Singular integral operators for rearrangement-invariant Morrey spaces on local fields," *Boletín de la Sociedad Matemática Mexicana*, vol. 29, no. 1, 2023.

[10] S. S. Volosivets, "Maximal function and Riesz potential on *p*-adic linear spaces," *p-Adic Numbers, Ultrametric Analysis, and Applications*, vol. 5, no. 3, pp. 226–234, 2013.

[11] S. S. Volosivets, "Generalized fractional integrals in p-adic Morrey and Herz spaces," *p-Adic Numbers, Ultrametric Analysis and Applications*, vol. 9, no. 1, pp. 53–61, 2017.

[12] D. V. Duong, "Hausdorff operator and commutator on weighted Morrey-Herz spaces on p-adic fields," *Analysis and Mathematical Physics*, vol. 11, no. 1, article 31, p. 26, 2021.

[13] Y. Shi, Y. Shi, and S. Chen, "*p*-adic Riesz potential and its commutators on Morrey-Herz spaces," *Journal of Function Spaces*, vol. 2022, Article ID 7227544, 11 pages, 2022.

[14] N. M. Chuong and D. V. Duong, "Weighted Hardy-Littlewood operators and commutators on p-adic functional spaces," *p-Adic Numbers, Ultrametric Analysis, and Applications*, vol. 5, no. 1, pp. 65–82, 2013.

[15] Q. Y. Wu and Z. W. Fu, "Hardy-Littlewood-Sobolev inequalities on *p*-adic central Morrey spaces," *Journal of Function Spaces*, vol. 2015, Article ID 419532, 7 pages, 2015.

[16] Z. W. Fu, Q. Y. Wu, and S. Z. Lu, "Sharp estimates of p-adic hardy and Hardy-Littlewood-Pólya operators," *Acta Mathematica Sinica, English Series*, vol. 29, no. 1, pp. 137–150, 2013.

[17] Q. Y. Wu, L. Mi, and Z. W. Fu, "Boundedness of *p*-adic Hardy operators and their commutatorson *p*-adic central Morrey and BMO spaces," *Journal of Function Spaces*, vol. 2013, Article ID 359193, 10 pages, 2013.