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Research Article

Locating Edge Domination Number of Some Classes of Claw-Free Cubic Graphs

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Let $G = (V; E)$ be a simple graph with vertex set V and edge set E. In a graph G, a subset of edges denoted by $\mathscr{M}$ is referred to as an edge-dominating set of G if every edge that is not in $\mathscr{M}$ is incident to at least one member of $\mathscr{M}$. A set $\mathscr{M} \subseteq E$ is the locating edge-dominating set if for every two edges $e_1, e_2 \in (E - \mathscr{M})$, the sets $N(e_1) \cap \mathscr{M}$ and $N(e_2) \cap \mathscr{M}$ are nonempty and different. The edge domination number $\gamma_L(G)$ of G is the minimum cardinality of all edge-dominating sets of G. The purpose of this study is to determine the locating edge domination number of certain types of claw-free cubic graphs.

1. Introduction

The domination number is a key metric in graph theory, which plays a crucial role in analyzing and comprehending a graph structure [1]. The initial investigations into the concept of the domination number can be credited to scholars such as Berge [2]. He examined numerous facets of this notion and its applications in graph theory. This metric finds utility in diverse domains, encompassing network architecture [3], facility placement [4], and network efficiency and security challenges [5]. Determining the domination number of a graph is frequently a pivotal stage in resolving optimization problems within graph theory [6]. Edge-dominating sets play a crucial role in a wide range of applications, including network architecture, where the primary objective is to achieve effective connectivity while minimizing the number of edges required. Mitchell and Hedetniemi are the ones who initially proposed the concept of an edge-dominating set [7].

The extension of domination properties to fuzzy graphs is also possible. The following are important properties related to domination in fuzzy extensions on graphs: fuzzy domination, fuzzy total domination, fuzzy edge domination, fuzzy locating domination, and fuzzy connected domination ([8, 9]). In a recent study by Sarwar et al., the researchers examined certain formulas determining the lower and upper bounds of dominating and double-dominating energy. The authors also introduced the concept of double-dominating energy of m-polar fuzzy digraphs [10]. A decision model utilizing m-polar fuzzy preference relations was proposed to address multicriteria decision-making challenges. Studying rough approximations of graphs and hypergraphs [11], distance measures in rough environments [12], and decision-making based on a color spectrum in rough environments provides an extension of domination features [13]. The computation of domination numbers has been conducted for certain categories of graphs, including sparse graphs [14], grid graphs [15], planar graphs [16], regular graphs [17], bipartite graphs [18], social networks, random geometric graphs, pseudofractal scale-free web graphs [19], wheel graphs [20], neural networks [21], and others [22].

Motivated by the above articles, the present study undertakes the task of calculating the edge domination in claw-free cubic graphs. Claw-free cubic graphs have several applications in different fields; in network design, they can be used as models for network design problems, such as designing communication or transportation networks, and their properties make them suitable for creating efficient and reliable network topologies. In VLSI (very large-scale integration) design, claw-free cubic graphs can be used to represent the layout of integrated circuits, and the absence of claws simplifies the layout design process and leads to more regular and efficient chip layouts. In algorithm development, algorithms for various problems can be tested and developed on claw-free cubic graphs due to their well-defined properties. In computational biology, they have been used in modeling biological networks and evolutionary relationships and provide a structured way to represent and analyze complex biological data. In grid computing and mesh topologies, they can serve as models for designing grid or mesh topologies, and these are important for interconnecting computing nodes efficiently. In scheduling and timetabling, claw-free cubic graphs have been applied to scheduling and timetabling problems, where the goal is to allocate resources or time slots efficiently. In planar graphs, these graphs are a special case of planar graphs, which have applications in geographic information systems, cartography, and layout problems in computer-aided design.

This paper is organized in the following way. To demonstrate our primary findings, we will begin by reviewing some fundamental concepts, definitions, and notations relevant to graph theory and combinatorics in Section 2. In Section 3, we will demonstrate the significance of our findings by referring to the concepts presented in Section 2. The paper's final section has some concluding thoughts regarding the entire study.

2. Preliminaries

For undetermined notations and terminologies, we refer the readers to read the book by Haynes et al. [23].

Let $G = (V; E)$ be a simple graph with vertex set V and edge set E. The neighborhood or open neighborhood of a vertex v (or edge e) of the graph G is represented by the variable $N(v)$ (or, resp., $N(e)$), and it is the set of all neighbors of the vertex v (or edge e), respectively. In a graph G, a subset of edges denoted by $\mathscr{M}$ is referred to as an edge-dominating set of G if every edge that is not in $\mathscr{M}$ is incident to at least one member of $\mathscr{M}$. A set $\mathscr{M} \subseteq E$ is the locating edge-dominating set if for every two edges $e_1, e_2 \in (E - \mathscr{M})$, the sets $N(e_1) \cap \mathscr{M}$ and $N(e_2) \cap \mathscr{M}$ are nonempty and different. The edge domination number $\gamma_L(G)$ of G is the minimum cardinality of all edge-dominating sets of G [20]. Slater was the first person to present and investigate the idea of positioning, often known as locating dominating set [24–26].

As a generalization of line graphs, claw-free graphs were first investigated and analyzed. The complete bipartite graph $K_{1,3}$ is a tree usually called the claw graph. A claw-free graph is a graph that does not have a claw as an induced subgraph. The construction of claw-free perfect graphs was the first topic that Chavatal and Sbihi investigated [27]. Later on, Chudnovsky and Seymour show how the claw-free graph can be constructed in the most general way [28]. Cubic graphs are connected graphs having the property that each vertex has a degree of exactly three [29]. A standard problem in structure enumeration is the production of cubic graphs, which can be thought of as a benchmark. Around the close of the 19th century, de Vries and Martinetti provided one of the first comprehensive lists of cubic-connected graphs when he provided a list of all graphs that are cubic with a maximum of 10 vertices [30]. They may be used to represent three-dimensional objects, which will enable you to find a dimension that is missing or investigate the impact that changes made to one or more dimensions have.

Here, we introduce claw-free graphs in a different and generalized manner. A complete graph K_4 with the removal of one edge is called diamond [31], and it is denoted by D. A *string of diamonds* L_N is a maximal sequence of diamonds $D_1, D_2, \cdots, D_N$ in which a vertex of degree two of D_i is joined by an edge to a vertex of degree two of $D_{i+1} (1 \leq i \leq N-1)$, and joining the pendent vertex of D_1 and D_N to a vertex of K_3 (see Figure 1). Furthermore, if G is a connected-claw-free simple cubic graph via every vertex in a diamond, then G is often referred to as a *ring of diamonds* (see Figure 2).

At this stage, we are going to look at two infinite families G and H. Let G_K be a graph that is obtained in the following manner for $k \geq 2$. Consider two copies of the path P_{2k}, each with the vertex sequences $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$. For each $n \in 1, 2, \cdots, 2k$, join a_{n-1} to b_n, b_{n-1} to a_n, a_1 to b_1, and a_n to b_n to complete the graph's structure. See for illustration, Figure 3(a). Consider H_K represent the graph formed from G_K for $K \geq 2$ through combining $a_1 a_n$ and $b_1 b_n$ and deleting $a_1 b_1$ and $a_n b_n$. See, for illustration, Figure 3(b). It has been brought to our attention that both G_K and H_K are cubic graphs of order $4K$.

3. Main Results

In this section, we will compute the locating edge-dominating set and the locating edge domination number for several kinds of claw-free cubic graphs.

Theorem 1. *Let L_N be a string of diamonds (see Figure 1). Then, locating edge-dominating number of L_N is $\gamma_L(L_N) = 2N + 4$.*

Proof. The string of diamonds L_N has $V(L_N) = \{x_i, y_i, z_i; 1 \leq i \leq 2\} \cup \{a_i, b_i, c_i, d_i; 1 \leq i \leq n\}$ and $E(L_N) = \{a_i b_i, b_i c_i, c_i d_i, a_i d_i; 1 \leq i \leq n\} \cup \{x_i y_i, y_i z_i, x_i z_i; 1 \leq i \leq 2\} \cup \{a_1 z_1, c_n z_n\}$ vertex and edge sets. The cardinality of the vertex set is $|V(L_N)| = 4N + 6$, while the cardinality of the edge set is $|E(L_N)| = 6N + 7$. In order to demonstrate the minimum cardinality required for the locating edge-dominating set, i.e., $\gamma_L(L_N) = 2N + 4$ with $N \geq 1$. Take edge set of L_N, i.e., $E(L_N) = \{a_i b_i, b_i c_i, c_i d_i, a_i d_i; 1 \leq i \leq n\} \cup \{x_i y_i, y_i z_i, x_i z_i; 1 \leq i \leq 2\} \cup \{a_1 z_1, c_n z_n\}$. Choose a dominating set $\mathscr{M}$ from edge set that is $\mathscr{M} = \{x_i y_i, x_i z_i; 1 \leq i \leq 2\} \cup \{a_i b_i, b_i c_i; 1 \leq i \leq n\}$. As we observed, the cardinality of locating edge-dominating set is $|\mathscr{M}| = 2N + 4$. Subtracting the dominating set from edge set, we get $E - \mathscr{M} = \{y_i z_i; 1 \leq i \leq 2\} \cup \{a_i d_i, b_i d_i, c_i d_i; 1 \leq i \leq n\} \cup \{c_i a_{i+1}; 1 \leq i \leq n-1\}$. The result of putting $\mathscr{M}$ and $E - \mathscr{M}$ together is as follows:

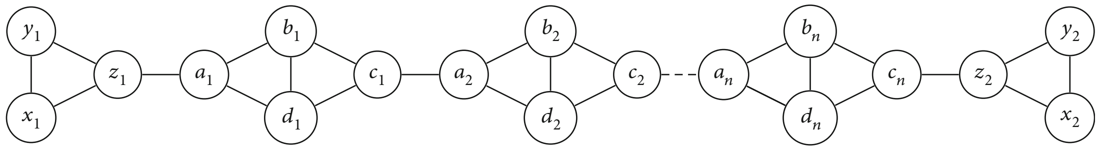


Figure 1: A string of diamonds L_N.

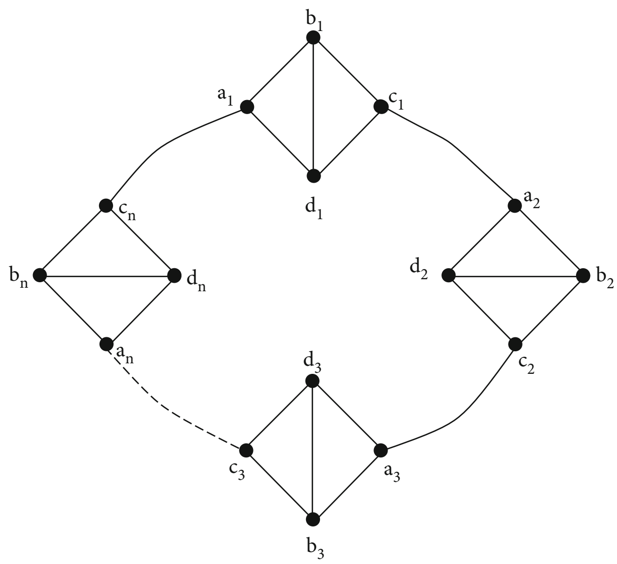
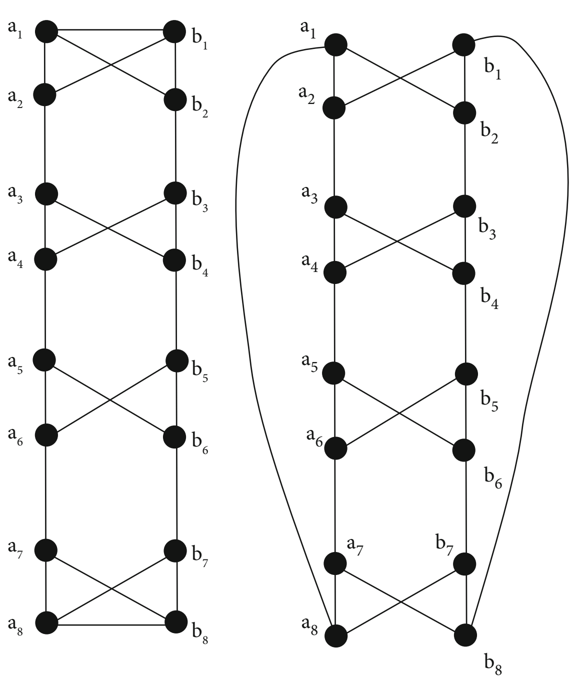


Figure 2: A ring of diamonds R_N.



(a) (b)

Figure 3: Cubic graphs: (a) G_4 and (b) H_4.

$$
\begin{aligned}
N(y_1z_1) \cap \mathscr{M} &= \{x_1y_1, x_1z_1\},\\
N(y_2z_2) \cap \mathscr{M} &= \{x_2y_2, x_2z_2\},\\
N(z_1a_1) \cap \mathscr{M} &= \{x_1z_1, a_1b_1\},\\
N(c_nz_2) \cap \mathscr{M} &= \{b_nc_n, x_2z_2\},\\
N(a_id_i\,;1 \le i \le n) \cap \mathscr{M} &= \{a_ib_i\,;1 \le i \le n\},\\
N(b_id_i\,;1 \le i \le n) \cap \mathscr{M} &= \{a_ib_i \cup b_ic_i\,;1 \le i \le n\},\\
N(c_id_i\,;1 \le i \le n) \cap \mathscr{M} &= \{b_ic_i\,;1 \le i \le n\},\\
N(c_ia_{i+1}\,;1 \le i \le n-1) \cap \mathscr{M} &= \{b_ic_i \cup b_{i+1}c_{i+1}\,;1 \le i \le n-1\}.
\end{aligned}
\tag{1}
$$

From the above expressions, it is simple to see that they are all nonempty and distinct. It indicates that the dominating set $\mathscr{M}$ dominates all edges in L_N, and it satisfied the condition, i.e., for $e_1, e_2 \in (E - \mathscr{M})$, the sets $N(e_1) \cap \mathscr{M}$ and $N(e_2) \cap \mathscr{M}$ are nonempty and different.

Now, we will prove that the edge set $\mathscr{M}$ is a minimum locating edge-dominating set. For that, we will take the edge-dominating set $\mathscr{M}$, i.e., $\mathscr{M} = \{x_iy_i, x_iz_i\,;1 \le i \le 2\} \cup \{a_ib_i, b_ic_i\,;1 \le i \le n\}$. To prove the minimum cardinality, we will remove any edge from dominating set. For example, let us remove the edge $\{x_1y_1\}$ from the set $\mathscr{M}$. Now, consider the new dominating set $\mathscr{M}' = \{x_2y_2, x_iz_i\,;1 \le i \le 2\} \cup \{a_ib_i, b_ic_i\,;1 \le i \le n\}$. Subtracting the new dominating set $\mathscr{M}'$ from the edge set E, i.e., $E - \mathscr{M}' = \{x_iy_i\} \cup \{y_iz_i\,;1 \le i \le 2\} \cup \{a_ib_i, b_id_i\,;1 \le i \le n\} \cup \{c_ia_{i+1}\,;1 \le i \le n\}$. The result of putting $E - \mathscr{M}'$ and $\mathscr{M}'$ together is as follows:

$$
\begin{aligned}
N(x_1y_1) \cap \mathscr{M} &= \{x_1z_1\},\\
N(y_1z_1) \cap \mathscr{M} &= \{x_1z_1\}.
\end{aligned}
\tag{2}
$$

From the above expressions, it is easy to see that for e_1, $e_2 \in (E - \mathscr{M}')$, the sets $N(e_1) \cap \mathscr{M}$ and $N(e_2) \cap \mathscr{M}$ are the same, which is a contradiction itself. So, $\mathscr{M}$ is the minimum locating edge-dominating set, and the locating edge-dominating number of L_N is $\gamma_L(L_N) = 2N + 4$. □

Theorem 2. *Let R_N be a ring of diamonds. Then, locating edge domination number for R_N is $\gamma_L(R_N) = 2N$.*

Proof. The ring of diamonds R_N has $V(R_N) = \{a_i, b_i, c_i, d_i\,;1 \le i \le n\}$ and $E(R_N) = \{a_ib_i \cup b_ic_i \cup b_id_i \cup c_id_i \cup a_id_i\,;1 \le i \le n\} \cup \{c_ia_{i+1}\} \cup \{a_1c_n\}$ vertex and edge sets. The cardinality of the vertex set is $|V(R_N)| = 4N$, while the cardinality of the edge set is $|E(R_N)| = 6N$. In order to demonstrate the minimum cardinality required for the locating edge-dominating set, i.e., $\gamma_L(R_N) = 2N$ for $N \geq 2$. Take edge set of R_N, i.e., $E(R_N) = \{a_ib_i \cup b_ic_i \cup b_id_i \cup c_id_i \cup a_id_i\,;\, 1 \leq i \leq n\} \cup \{c_ia_{i+1}\,;\, 1 \leq i \leq n-1\} \cup \{a_1c_n\}$. Choose a dominating set $\mathscr{M}$ from the edge set that is $\mathscr{M} = \{a_ib_i \cup b_ic_i\,;\, 1 \leq i \leq n\}$. As we observed, the cardinality of locating edge-dominating set is $|\mathscr{M}| = 2N$. Subtracting the dominating set from edge set, we get $E - \mathscr{M} = \{a_id_i \cup b_id_i \cup c_id_i\,;\, 1 \leq i \leq n\} \cup \{c_ia_{i+1}\,;\, 1 \leq i \leq n-1\} \cup \{a_1c_n\}$. The result of putting $E - \mathscr{M}$ and $\mathscr{M}$ together is as follows:

$$\begin{aligned} N(a_id_i\,;\, 1 \leq i \leq n) \cap \mathscr{M} &= \{a_ib_i\,;\, 1 \leq i \leq n\},\\ N(b_id_i\,;\, 1 \leq i \leq n) \cap \mathscr{M} &= \{a_ib_i \cup b_ic_i\,;\, 1 \leq i \leq n\},\\ N(c_id_i\,;\, 1 \leq i \leq n) \cap \mathscr{M} &= \{b_ic_i\,;\, 1 \leq i \leq n\},\\ N(c_ia_{i+1}\,;\, 1 \leq i \leq n-1) \cap \mathscr{M} &= \{b_ic_i \cup a_{i+1}b_{i+1}\,;\, 1 \leq i \leq n-1\},\\ N(a_1c_n) \cap \mathscr{M} &= \{b_nc_n \cup a_1b_1\}. \end{aligned} \tag{3}$$

From the above expressions, it is simple to see that they are all nonempty and distinct. It indicates that the dominating set $\mathscr{M}$ dominates all edges in R_N, and it satisfied the condition, i.e., for $e_1, e_2 \in (E - \mathscr{M})$, the sets $N(e_1) \cap \mathscr{M}$ and $N(e_2) \cap \mathscr{M}$ are nonempty and different.

Now, we will prove that the edge set $\mathscr{M}$ is a minimum locating edge-dominating set. For that, we will take edge-dominating set $\mathscr{M}$, i.e., $\mathscr{M} = \{a_ib_i \cup b_ic_i\,;\, 1 \leq i \leq n\}$. To prove the minimum cardinality, we will remove any edge from dominating set. For example, let us remove the edge $\{a_1b_1\}$ from the set $\mathscr{M}$. Now, consider the new dominating set $\mathscr{M}' = \{a_ib_i\,;\, 2 \leq i \leq n\} \cup \{b_ic_i\,;\, 1 \leq i \leq n\}$. Subtracting the new dominating set $\mathscr{M}'$ from the edge set E, i.e., $E - \mathscr{M}' = \{a_1b_1\} \cup \{a_id_i \cup b_id_i \cup c_id_i\,;\, 1 \leq i \leq n\} \cup \{c_ia_{i+1}\,;\, 1 \leq i \leq n-1\} \cup \{a_1c_n\}$. The result of putting $E - \mathscr{M}'$ and $\mathscr{M}'$ together is as follows:

$$\begin{aligned} N(a_1b_1) \cap \mathscr{M}' &= \{b_1c_1\},\\ N(a_1d_1) \cap \mathscr{M}' &= \phi. \end{aligned} \tag{4}$$

From the above expressions, it is easy to see that the set $N(a_1d_1) \cap \mathscr{M}$ is empty, which is a contradiction itself. So, $\mathscr{M}$ is the minimum locating edge-dominating set, and the locating edge-dominating number of R_N is $\gamma_L(R_N) = 2N$. □

Theorem 3. *Let G_k be a cubic graph. Then, locating edge domination number of G_k is $\gamma_L(G_k) = 2K$.*

Proof. The cubic graph G_K has $V(G_K) = \{a_i \cup b_i\,;\, 1 \leq i \leq 2n\}$ and $E(G_K) = \{a_ia_{i+1} \cup b_ib_{i+1}\,;\, 1 \leq i \leq 2n-1\} \cup \{b_ia_{i+1} \cup a_ib_{i+1}\,;\, 1 \leq i \leq 2n-1\} \cup \{a_1b_1\} \cup \{a_{2n}b_{2n}\}$ vertex and edge sets. The cardinality of the vertex set is $|V(G_k)| = 4K$, while the cardinality of the edge set is $|E(G_k)| = 6K$. In order to demonstrate the minimum cardinality required for the locating edge-dominating set, i.e., $\gamma_L(G_k) = 2K$ for $K \geq 1$. Take edge set of G_K, i.e., $E(G_K) = \{a_ia_{i+1} \cup b_ib_{i+1}\,;\, 1 \leq i \leq 2n-1\} \cup \{b_ia_{i+1} \cup a_ib_{i+1}\,;\, 1 \leq i \leq 2n-1\} \cup \{a_1b_1\} \cup \{a_{2n}b_{2n}\}$. Choose a dominating set $\mathscr{M}$ from the edge set that is $\mathscr{M} = \{a_2b_1\} \cup \{a_{2n}b_{2n-1}\} \cup \{a_ia_{i+1} \cup b_ib_{i+1}\}$, where i is even and $2 \leq i \leq 2n-2$. As we observed, the cardinality of locating edge-dominating set is $|\mathscr{M}| = 2K$. Subtracting the dominating set from edge set, we get $E - \mathscr{M} = \{a_ib_{i+1}, b_ib_{i+1}\,;\, 1 \leq i \leq 2n-1 \cup a_ia_{i+1}\,;\, 3 \leq i \leq 2n-1\} \cup \{b_ia_{i+1}\,;\, 3 \leq i \leq 2n-3\} \cup \{a_1b_1, a_{2n}b_{2n}\}$ (where i is odd). The result of putting $E - \mathscr{M}$ and $\mathscr{M}$ together is as follows:

$$\begin{aligned} N(a_1b_1) \cap \mathscr{M} &= \{a_2b_1, a_1a_2\},\\ N(a_1b_2) \cap \mathscr{M} &= \{b_2b_3\},\\ N(b_1b_2) \cap \mathscr{M} &= \{a_2b_1, b_2b_3\},\\ N(a_3a_4) \cap \mathscr{M} &= \{a_2a_3, a_4a_5\},\\ &\ \ \vdots\\ N(a_{2n}b_{2n}) \cap \mathscr{M} &= \{a_{2n}b_{2n-1}\}. \end{aligned} \tag{5}$$

From the above expressions, it is simple to see that they are all nonempty and distinct. It indicates that the dominating set $\mathscr{M}$ dominates all edges in G_K, and it satisfied the condition, i.e., for $e_1, e_2 \in (E - \mathscr{M})$, the sets $N(e_1) \cap \mathscr{M}$ and $N(e_2) \cap \mathscr{M}$ are nonempty and different.

Now, we will prove that the edge set $\mathscr{M}$ is a minimum locating edge-dominating set. For that, we will take edge-dominating set $\mathscr{M}$, i.e., $\mathscr{M} = \{a_2b_1\} \cup \{a_{2n}b_{2n-1}\} \cup \{a_ia_{i+1} \cup b_ib_{i+1}\}$, where i is even and $2 \leq i \leq 2n-2$. To prove the minimum cardinality, we will remove any edge from dominating set. For example, let us remove the edge $\{a_2b_1\}$ from the set $\mathscr{M}$. Now, consider the new dominating set $\mathscr{M}' = \{a_{2n}b_{2n-1}\} \cup \{a_ia_{i+1} \cup b_ib_{i+1}\}$, where i is even and $2 \leq i \leq 2n-2$. Subtracting the new dominating set $\mathscr{M}'$ from the edge set E, we get $E - \mathscr{M}' = \{a_2b_1\} \cup \{a_ib_{i+1}, b_ib_{i+1}\,;\, 1 \leq i \leq 2n-1 \cup a_ia_{i+1}\,;\, 3 \leq i \leq 2n-1\} \cup \{b_ia_{i+1}\,;\, 3 \leq i \leq 2n-3\} \cup \{a_1b_1, a_{2n}b_{2n}\}$ (where i is odd). The result of putting $E - \mathscr{M}'$ and $\mathscr{M}'$ together is as follows:

$$\begin{aligned} N(a_2b_1) \cap \mathscr{M}' &= \{a_2a_3\},\\ N(a_1b_1) \cap \mathscr{M}' &= \phi,\\ N(a_1a_2) \cap \mathscr{M}' &= \{a_2a_3\}. \end{aligned} \tag{6}$$

From the above expressions, it is easy to see that for $e_1, e_2 \in (E - \mathscr{M}')$, the sets $N(e_1) \cap \mathscr{M}$ and $N(e_2) \cap \mathscr{M}$ are the same and empty, which is a contradiction itself. So, $\mathscr{M}$ is the minimum locating edge-dominating set, and the locating edge-dominating number of G_K is $\gamma_L(G_K) = 2K$. □

Theorem 4. *Let H_k be a cubic graph. Then, locating edge domination number of H_k is $\gamma_L(G_k) = 2K$.*

Proof. The cubic graph H_k has $V(H_K) = \{a_i \cup b_i\,;\, 1 \leq i \leq 2n\}$ and $E(H_K) = \{a_ia_{i+1} \cup b_ib_{i+1}\,;\, 1 \leq i \leq 2n-1\} \cup \{b_ia_{i+1} \cup a_ib_{i+1}\,;\, 1 \leq i \leq 2n-1\} \cup \{a_1a_{2n}\} \cup \{b_1b_{2n}\}$ vertex and edge sets. The cardinality of the vertex set is $|V(H_k)| = 4K$, while the cardinality of the edge set is $|E(H_k)| = 6K$. In order to demonstrate the minimum cardinality required for the locating edge-dominating set, i.e., $\gamma_L(H_k) = 2K$ for $K \geq 2$. Take edge set of H_K, i.e., $E(H_K) = \{a_i a_{i+1} \cup b_i b_{i+1}\,; 1 \leq i \leq 2n-1\} \cup \{b_i a_{i+1} \cup a_i b_{i+1}\,; 1 \leq i \leq 2n-1\} \cup \{a_1 a_{2n}\} \cup \{b_1 b_{2n}\}$. Choose a dominating set $\mathscr{M}$ from the edge set that is $\mathscr{M} = \{a_2 b_1\} \cup \{a_{2n} b_{2n-1}\} \cup \{a_i a_{i+1} \cup \{b_i b_{i+1}\}$, where i is even and $2 \leq i \leq 2n-2$. As we observed, the cardinality of locating edge-dominating set is $\gamma_L(H_k) = 2K$. Subtracting the dominating set from edge set, we get $E - \mathscr{M} = \{a_1 a_{2n}\} \cup \{a_i a_{i+1} \cup b_i b_{i+1} \cup a_i b_{i+1}\,; 1 \leq i \leq 2n-1\} \cup \{b_i a_{i+1}\,; 3 \leq i \leq 2n-3\} \cup \{b_1, b_{2n}\}$, where i is odd. The results of putting M and $E - \mathscr{M}$ together are as follows:

$$
\begin{aligned}
N(a_1 a_2) \cap \mathscr{M} &= \{a_2 b_1, a_2 a_3\}, \\
N(a_1 b_2) \cap \mathscr{M} &= \{b_2 b_3\}, \\
N(b_1 b_2) \cap \mathscr{M} &= \{a_2 b_1, b_2 b_3\}, \\
N(b_3 b_4) \cap \mathscr{M} &= \{b_2 b_3, b_4 b_5\}, \\
&\;\;. \\
&\;\;. \\
&\;\;. \\
N(a_1 a_{2n}) \cap \mathscr{M} &= \{a_{2n} b_{2n-1}\}, \\
N(b_1 b_{2n}) \cap \mathscr{M} &= \{a_2 b_1\}.
\end{aligned}
\tag{7}
$$

From the above expressions, it is simple to see that they are all nonempty and distinct. It indicates that the dominating set $\mathscr{M}$ dominates all edges in H_K, and it satisfied the condition, i.e., for $e_1, e_2 \in (E - \mathscr{M})$, the sets $N(e_1) \cap \mathscr{M}$ and $N(e_2) \cap \mathscr{M}$ are nonempty and different.

Now, we will prove that the edge set $\mathscr{M}$ is a minimum locating edge-dominating set. For that, we will take edge-dominating set $\mathscr{M}$, i.e., $\mathscr{M} = \{a_2 b_1\} \cup \{a_{2n} b_{2n-1}\} \cup \{a_i a_{i+1} \cup \{b_i b_{i+1}\,; \}$. To prove the minimum cardinality, we will remove any edge from dominating set. For example, let us remove the edge $\{a_2 b_1\}$ from the set $\mathscr{M}$. Now, consider the new dominating set $\mathscr{M}' = \{a_{2n} b_{2n-1}\} \cup \{a_i a_{i+1} \cup \{b_i b_{i+1}\}$, where i is even and $2 \leq i \leq 2n-2$. Subtracting the new dominating set $\mathscr{M}'$ from the edge set E, we get $E - \mathscr{M}' = \{a_2 b_1\} \cup \{a_1 a_{2n}\} \cup \{a_i a_{i+1} \cup b_i b_{i+1}\,; 1 \leq i \leq 2n-3\} \cup \{a_i a_{i+1}, b_i b_{i+1}\,; 1 \leq i \leq 2n-1\}$, where i is odd. The result of putting $E - \mathscr{M}'$ and $\mathscr{M}'$ together is as follows:

$$
\begin{aligned}
N(a_2 b_1) \cap \mathscr{M}' &= \{a_2 a_3\}, \\
N(a_1 a_2) \cap \mathscr{M}' &= \{a_2 a_3\}.
\end{aligned}
\tag{8}
$$

From the above expressions, it is easy to see that for e_1, $e_2 \in (E - \mathscr{M}')$, the sets $N(e_1) \cap \mathscr{M}$ and $N(e_2) \cap \mathscr{M}$ are the same, which is a contradiction itself. So, $\mathscr{M}$ is the minimum locating edge-dominating set, and the locating edge-dominating number of H_K is $\gamma_L(G_K) = 2K$. □

4. Conclusions

Claw-free cubic graphs are graphs with no induced subgraphs that are isomorphic to a claw. In this study, we used fundamental graph theory and combinatorics definitions to ascertain the edge-locating dominating set and the domination number for various classes of claw-free cubic graphs. We have discovered the locating edge-dominating set and the domination number for the cubic graphs G_k and H_k, as well as the string of diamonds L_N and the ring of diamonds R_N. The fact that the edge-dominating number for these graphs is less than almost half of their orders is highly interesting information. Because of their well-defined form and features, graphs are valuable in modeling, analysis, and problem-solving tools that may be applied across various domains. This current work is limited to edge-locating dominating set and domination number of various classes of simple claw-free cubic graphs, and its consequences may not be validated for the directed graphs. For future studies, it is recommended that researchers compute the fuzzy total domination, fuzzy edge domination, and fuzzy locating domination numbers for each of the several classes of claw-free cubic graphs.

Data Availability

All relevant data is given in the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

The first author, Muhammad Shoaib Sardar, conducted this research, conceptualized the idea of the research, investigated the problem, and wrote the original text of the manuscript. The second coauthor, Hamna Choudhry, reviewed, edited, and analyzed this paper. The third and last coauthor, Jia-Bao Liu, supervised this research and also studied the validation of the results.

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