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Research Article

An Algebraic Approach of Topological Indices Connected with Finite Quasigroups

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In mathematical chemistry, the algebraic polynomial serves as essential for calculating the most accurate expressions of distance-based, degree-distance-based, and degree-based topological indices. The chemical reactivity of molecules, which includes their tendency to engage in particular chemical processes or go through particular reactions, can be predicted using topological indices. Considerable effort has been put into examining the many topological descriptors of simple graphs using ring structures and well-known groups instead of nonassociative algebras, quasigroups, and loops. Both finite quasigroups and loops are the generalizations of groups. In this article, we calculate topological descriptors and some well-known polynomials, M-polynomial, Hosoya's polynomial, Schultz's polynomial, and modified Schultz polynomial of finite relatively prime graphs of most orders connected with two classes of quasigroups and go through their graphical aspects.

1. Introduction and Basic Definitions

A subfield of theoretical chemistry known as "chemical graph theory" studies chemical structures and reactions using the concepts of graph theory. A mathematical framework known as graph theory allows molecules to be represented as graphs, with chemical bonds acting as edges and atoms acting as vertices. The representational form utilized in mathematical models of chemical molecules is called a *molecular graph*. Many topological and structural properties of these molecules are investigated using concepts from graph theory. For instance, the degree and number of edges among the vertices of a chemical compound—a physical entity—can be used to predict the compound's boiling point. Thus, it is evident that when a chemical problem is described mathematically, the topology of the molecule structure plays a critical role in defining the favorable properties of the matching molecular structure. Between 1975 and 2023, several academics employed algebraic structures, rings, and groups to address problems related to graph theory. The author showed that the maximal prime order of the nontrivial subgroup of the finite abelian group is the vertex independence number of the intersecting graph connected to the abelian group [1]. But eight years after the publication of this paper, several mathematicians proposed a novel idea for figuring out a finite simple graph's vertex independence number using the vertex degrees [2]. In 1990, the author used the class of finite groups whose Cayley's graphs are planar to characterize well-known groups, quasi-Frobenius groups, and linked components of finite simple graphs, whose nodes were the noncentral conjugacy classes of the group investigated by [3, 4]. A thorough analysis has been done on degree-based topological descriptors of two distinct graphynes and minimum transmission, depending on certain parameters, in two-mode networks [5, 6]. In addition, Zaman et al. and Mondal et al. [7, 8] have provided a complete computation of topological indices related with regression models and two particular ring structures; however, the researchers are still in the process of discovering certain well-known descriptors and polynomials associated with nonassociative algebras.

From here, we use $\Xi = (\Lambda_1, \Lambda_2)$ for the undirected, simple, and finite graph, in which the set of edges is Λ_2 and the nodes are Λ_1. The distance, a positive integer, between any two distinct vertices τ_1 and τ_2 of Λ_1 can be denoted by $d(\tau_1, \tau_2)$, and d^τ is the degree of any vertex $\tau \in \Lambda_1$ in Ξ. Regarding degree-based topological indices, the M-polynomial has a similar function in calculating closed expressions of multiple degree-based topological indices [9]. The following is the definition of a graph $\Xi' sM$-polynomial associated with polynomial ring $\mathbb{R}[\theta_1, \theta_2]$ (see [10]):

$$M(\Xi; \theta_1, \theta_2) = \sum_{\lambda_1 \le \lambda_2} M_{(\lambda_1, \lambda_2)}(\Xi) \theta_1^{\lambda_1} \theta_2^{\lambda_2}, \quad (1)$$

where $M_{(\lambda_1, \lambda_2)}(\Xi)$ is the overall number of edges $\tau_1\tau_2 \in \Lambda_2(\Xi)$ such that $\{d^{\tau_1}, d^{\tau_2}\} = \{\lambda_1, \lambda_2\}$. In this article, we use $M(\Xi)$ instead of $M(\Xi; \theta_1, \theta_2)$. In reality, a topological index is a mapping from the set of real numbers to a class of isomorphic finite simple graphs [8]. For a graph Ξ, any degree-based topological index can be written as follows:

$$I(\Xi) = \sum_{\tau_1\tau_2 \in \Lambda_2(\Xi)} g(d^{\tau_1}, d^{\tau_2}), \quad (2)$$

where $g = g(\theta_1, \theta_2)$ represents a function that has been specifically selected for potential chemical applications [11]. The result shown above can also be expressed as

$$I(\Xi) = \sum_{\lambda_1 \le \lambda_2} M_{(\lambda_1, \lambda_2)}(\Xi) g(\lambda_1, \lambda_2). \quad (3)$$

Zagreb indices were first developed by Gutman and Trinajstić in 1972. The following defines the first Zagreb index according to [12]:

$$M_1(\Xi) = \sum_{\tau \in \Lambda_1(\Xi)} (d^\tau)^2. \quad (4)$$

Here is how the second Zagreb index is described:

$$M_2(\Xi) = \sum_{\tau_1\tau_2 \in \Lambda_2(\Xi)} d^{\tau_1} d^{\tau_2}. \quad (5)$$

In 2015, Furtula and Gutman introduced the term forgotten topological index, which is expressed as follows [13]:

$$F(\Xi) = \sum_{\tau_1\tau_2 \in \Lambda_2(\Xi)} \left[(d^{\tau_1})^2 + (d^{\tau_2})^2 \right]. \quad (6)$$

The second modified Zagreb index was introduced by Nikolić et al. in 2003, and it is defined as follows [14]:

$$M_2^m(\Xi) = \sum_{\tau_1\tau_2 \in \Lambda_2(\Xi)} \frac{1}{d^{\tau_1} d^{\tau_2}}. \quad (7)$$

Hu et al. put forward the generalized Randić index, which has been extensively researched in the chemistry and mathematics [15]. The following is a definition of the generalized Randić index:

$$R_\alpha(\Xi) = \sum_{\tau_1\tau_2 \in \Lambda_2(\Xi)} (d^{\tau_1} d^{\tau_2})^\alpha. \quad (8)$$

$\forall \alpha \in \mathbb{R}$. The definition of the inverse Randić index is as follows:

$$RR_\alpha(\Xi) = \sum_{\tau_1\tau_2 \in \Lambda_2(\Xi)} \frac{1}{(d^{\tau_1} d^{\tau_2})^\alpha}. \quad (9)$$

$\forall \alpha \in \mathbb{R}$. A connected graph's symmetric division deg index with the following definition was given by Vukicević in 2010 (see [16]).

$$\mathrm{SDD}(\Xi) = \sum_{\tau_1\tau_2 \in \Lambda_2(\Xi)} \left\{ \frac{\min\{d^{\tau_1}, d^{\tau_2}\}}{\max\{d^{\tau_1}, d^{\tau_2}\}} + \frac{\max\{d^{\tau_1}, d^{\tau_2}\}}{\min\{d^{\tau_1}, d^{\tau_2}\}} \right\}. \quad (10)$$

In 1987, Fajtlowicz introduced the concept of a graph's harmonic index, which is in [17]. It is described by

$$H(\Xi) = \sum_{\tau_1\tau_2 \in \Lambda_2(\Xi)} \frac{2}{d^{\tau_1} + d^{\tau_2}}. \quad (11)$$

The following is how Balaban introduced the inverse sum indeg index in 1982 (see [18]):

$$I(\Xi) = \sum_{\tau_1\tau_2 \in \Lambda_2(\Xi)} \frac{d^{\tau_1} d^{\tau_2}}{d^{\tau_1} + d^{\tau_2}}. \quad (12)$$

Furtula et al. presented the augmented Zagreb index, which can be summed up as follows [19]:

$$A(\Xi) = \sum_{\tau_1\tau_2 \in \Lambda_2(\Xi)} \left\{ \frac{d^{\tau_1} d^{\tau_2}}{d^{\tau_1} + d^{\tau_2} - 2} \right\}^3. \quad (13)$$

For a graph Ξ, the distance-based Wiener index is defined as follows:

$$W(\Xi) = \sum_{\{\tau_1, \tau_2\} \subseteq \Lambda_1(\Xi)} d(\tau_1, \tau_2). \quad (14)$$

The Hosoya polynomial, with derivatives at 1 yield the Wiener index, is one basic polynomial in the domain of distance-based topological indices [20]. The following formula represents the Hosoya polynomial of a graph Ξ related with $\mathbb{R}[\theta]$:

$$H(\Xi, \theta) = \sum_{i=0}^{D} d(\Xi, i) \theta^i, \quad (15)$$

where $d(\Xi, i)$ is the total number of node pairings in Ξ with a distance of i between them and $D = \max\{d(\tau_1, \tau_2): \tau_1, \tau_2 \in \Lambda_1(\Xi)\}$. The Wiener index is obtained as follows using the Hosoya polynomial's first derivative at $\theta = 1$:

$$W(\Xi) = \left.\frac{d(H(\Xi, \theta))}{d\theta}\right|_{\theta=1}. \tag{16}$$

For the study of structure-property interactions of molecules, Randić gave the hyper-Wiener index in 1993, in [21]. It is a different distance-based index and is expressed as follows:

$$\mathrm{WW}(\Xi) = \frac{1}{2} \sum_{\{\tau_1,\tau_2\} \subseteq \Lambda_1(\Xi)} \left(d(\tau_1, \tau_2) + d(\tau_1, \tau_2)^2\right). \tag{17}$$

Hyper-Wiener index can also be obtained with the help of the Hosoya polynomial according to Cash et al. [22].

$$\mathrm{WW}(\Xi) = \left.\frac{d(H(\Xi, \theta))}{d\theta}\right|_{\theta=1} + \left.\frac{1}{2}\frac{d^2(H(\Xi, \theta))}{d\theta^2}\right|_{\theta=1}. \tag{18}$$

A topological index that combines distance and degree was first introduced by Schultz in 1989 and is known as the Schultz index [23]. Graph Ξ's Schultz index can be obtained as

$$\mathrm{Sc}(\Xi) = \sum_{\{\tau_1,\tau_2\} \subseteq \Lambda_1(\Xi)} (d^{\tau_1} + d^{\tau_2}) d(\tau_1, \tau_2). \tag{19}$$

Afterward, the modified Shultz index, a degree-distance-based index with the following definition, was published in 1997 by Klavžar and Gutman [24].

$$\mathrm{Sc}^*(\Xi) = \sum_{\{\tau_1,\tau_2\} \subseteq \Lambda_1(\Xi)} (d^{\tau_1} d^{\tau_2}) d(\tau_1, \tau_2). \tag{20}$$

For a graph Ξ, the Schultz polynomial in integral domain $\mathbb{R}[\theta]$ can be calculated by

$$\mathrm{Sc}(\Xi, \theta) = \sum_{\{\tau_1,\tau_2\} \subseteq \Lambda_1(\Xi)} (d^{\tau_1} + d^{\tau_2}) \theta^{d(\tau_1,\tau_2)}. \tag{21}$$

A graph Ξ's modified Schultz polynomial in ring $\mathbb{R}[\theta]$ is written as follows:

$$\mathrm{Sc}^*(\Xi, \theta) = \sum_{\{\tau_1,\tau_2\} \subseteq \Lambda_1(\Xi)} (d^{\tau_1} d^{\tau_2}) \theta^{d(\tau_1,\tau_2)}. \tag{22}$$

The following are the relationships that connect the Shultz, modified Shultz indices, and the related polynomials:

$$\begin{aligned} \mathrm{Sc}(\Xi) &= \left.\frac{d(Sc(\Xi, \theta))}{d\theta}\right|_{\theta=1}, \\ \mathrm{Sc}^*(\Xi) &= \left.\frac{d(Sc^*(\Xi, \theta))}{d\theta}\right|_{\theta=1}. \end{aligned} \tag{23}$$

Under the assumption of three groupoids, $(\mathfrak{L}, \cdot)$, $(\mathfrak{L}, \backslash)$, and $(\mathfrak{L}, /)$, and the identities

$$\begin{aligned} \tau_1(\tau_1 \backslash \tau_2) = \tau_2, \tau_1 \backslash (\tau_1\tau_2) = \tau_2, \\ (\tau_1\tau_2)/\tau_2 = \tau_1, (\tau_1/\tau_2)\tau_2 = \tau_1, \end{aligned} \tag{24}$$

a mathematical system $(\mathfrak{L}, \cdot, \backslash, \text{and } /)$ is known as a *quasigroup* $\forall \tau_1, \tau_2 \in \mathfrak{L}$ (see [25]). A quasigroup which satisfies the identity law, $\tau e = e\tau = \tau \forall \tau \in \mathfrak{L}$ and for unique $e \in \mathfrak{L}$, is called *Loop*. If $|\mathfrak{L}|$ is some power of prime number p, then $\mathfrak{L}$ is called *p-loop* (see [26]).

2. Motivation and Applications

The Wiener index initiated the path of topological indices in 1947, modeling the paraffin's temperatures at boiling point as follows [27]:

$$t^B = a_1 x + a_2 y + a_3, \tag{25}$$

where a_1, a_2, and a_3 are constants for a given isomeric group, t^B is the boiling point, and x and y are the Wiener index and polarity number, respectively. The quantitative structure-property relationships between boiling temperatures and hyper-Wiener index were found in a range of cyclic and acyclic alkanes [28]. The first and second Zagreb indices were demonstrated to be effective in the estimation of the total π-electron energy of molecule [29]. The linear combination of the forgotten topological index and the first Zagreb index yields a mathematical model of several physicochemical properties of alkanes with good accuracy [13]. They were proposed for the approximation of stretched carbon skeleton [12]. Randić observed the association between the Randić index and physicochemical parameters of alkane such as boiling temperature, enthalpy of formation, and surface area. Encoding molecular structure information with topological indices has a low processing cost and a high predictive potential. Additionally, these molecular descriptors provide information on easily recognized structural properties. The interaction between the algebraic and graph theoretical characteristics of the simple graph is the main area of study for graphs constructed from nonassociative finite algebra. Information in communication theory can be related to this. Therefore, it makes sense to calculate the finite quasigroups' topological indices for relatively prime graph.

Theorem 1 (Lagrange's theorem). *Let $\mathfrak{L}$ be the finite loop and τ be any element of $\mathfrak{L}$. Then, $|\tau|$ divides the order of $\mathfrak{L}$.*

Theorem 2 (fundamental theorem of arithmetic). *Any positive integer n can be written as a product of the powers of prime numbers.*

Theorem 3 (see [30]). *With the help of two finite groups C_2, cyclic group of order 2, and $\mathbb{Z}_{2\alpha}$, even order group of residue classes, the algebraic structure $(C_2 \times \mathbb{Z}_{2\alpha}, *)$ is a quasigroup,* *where ∝ is a positive integer. We can denote this class of quasigroups by Ω_1.*

Theorem 4 (see [31]). *Let $C_{\propto}$ and $\mathbb{Z}_2$ be a cyclic group of order ∝ containing an element of order greater than 2 and two-element group of residue classes, respectively. Then, the algebraic structure $(C_{\propto} \times \mathbb{Z}_2, *)$ is a quasigroup. We represent this class of quasigroups by Ω_2.*

The layout of this work consists of the following two sections: in the first section, we calculate topological indices of two classes given in [30, 31], and in the second section, there are some polynomials of relatively prime graphs associated with these quasigroups.

3. Topological Indices and Finite Quasigroups

Definition 5 (relatively prime graph). A finite simple graph $G_{\mathfrak{L}}^{\mathrm{RP}}$ is said to be relatively prime graph if and only if each element of $\mathfrak{L}$ is the vertex of $G_{\mathfrak{L}}^{\mathrm{RP}}$ and $(|\tau_1|, |\tau_2|) = 1$; i.e., orders of two distinct elements of $\mathfrak{L}$ are relatively prime.

Example 1. The following Table 1 and Figure 1 indicate quasigroup of order 12 and its relatively prime graph, respectively.

Table 1: A finite quasigroup.

*	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	3	4	5	6	7	8	9	10	11	12	1
3	3	4	5	6	7	8	9	10	11	12	1	2
4	4	5	6	7	8	9	10	11	12	1	2	3
5	5	6	7	8	9	10	11	12	1	2	3	4
6	6	7	8	9	10	11	12	1	2	3	4	5
7	7	8	9	10	11	12	1	2	3	4	5	6
8	8	9	10	11	12	1	2	3	4	5	6	7
9	9	10	11	12	1	2	3	4	5	6	7	8
10	10	11	12	1	2	3	4	5	6	7	8	9
11	11	12	1	2	3	4	5	6	7	8	9	10
12	12	1	2	3	4	5	6	7	8	9	10	11

* in the table shows a binary operation.

Theorem 6. *A relatively prime graph $G_{\mathfrak{L}}^{RP}$ is star if and only if $\mathfrak{L}$ is p-loop.*

Proof. Let $G_{\mathfrak{L}}^{\mathrm{RP}}$ be a star graph, since the order of the identity element of $\mathfrak{L}$ is one and it is relatively prime to the order of each nonidentity element of $\mathfrak{L}$. Moreover, any two nonidentity elements are not adjacent in $G_{\mathfrak{L}}^{\mathrm{RP}}$. It is only possible when order of loop $\mathfrak{L}$ is some power of prime number by Theorem 1. Other direction of the proof is just consequence of the Lagrange theorem. It completes the proof. □

Theorem 7. *A relatively prime graph $G_{\mathfrak{L}}^{RP}$ is always connected.*

Proof. Because the vertex associated with identity element is adjacent to each vertex so trivially, we can say relatively prime graph $G_{\mathfrak{L}}^{\mathrm{RP}}$ is connected. □

Theorem 8. *Let $G_{\mathfrak{L}}^{RP} = (\Lambda_1, \Lambda_2)$ be the relatively prime graph associated with $\mathfrak{L} \in \Omega_1$, where $\propto = 2^{\beta-2}$ and β is the positive integer greater than 1. Then, the degree-based topological indices are as follows:*

(1) $M_1(G_{\mathfrak{L}}^{RP}) = 2^{\beta} + 4^{\beta} - 2^{\beta+1}$

(2) $M_2(G_{\mathfrak{L}}^{RP}) = 4^{\beta} + 1 - 2^{\beta+1}$

(3) $F(G_{\mathfrak{L}}^{RP}) = (2^{\beta} - 1)[1 + (2^{\beta} - 1)^2]$

(4) $M_2^m(G_{\mathfrak{L}}^{RP}) = 1$

(5) $R_{\alpha}(G_{\mathfrak{L}}^{RP}) = (2^{\beta} - 1)^{\alpha+1}$

(6) $RR_{\alpha}(G_{\mathfrak{L}}^{RP}) = (2^{\beta} - 1)^{1-\alpha}$

(7) $SDD(G_{\mathfrak{L}}^{RP}) = 2 + 4^{\beta} - 2^{\beta+1}$

(8) $H(G_{\mathfrak{L}}^{RP}) = 2 - 2^{1-\beta}$

(9) $I(G_{\mathfrak{L}}^{RP}) = 2^{\beta} + 2^{-\beta} - 2$

(10) $A(G_{\mathfrak{L}}^{RP}) = (2^{\beta} - 1)^4/(2^{\beta} - 2)^3$

Proof. The following are the vertex and edge partitions of relatively prime graph $G_{\mathfrak{L}}^{\mathrm{RP}}$ with Equations (4)–(13) yield the required results.

$$\begin{aligned} \Lambda_1^{(1)} &= \{\tau \in \Lambda_1 | \deg(\tau) = 1\}, \\ \Lambda_1^{(2)} &= \left\{\tau \in \Lambda_1 \middle| \deg(\tau) = 2^{\beta} - 1\right\}, \\ \Lambda_2 &= \left(e = \tau_1\tau_2 \in \Lambda_2 \middle| \deg(\tau_1) = 1, \deg(\tau_2) = 2^{\beta} - 1\right), \end{aligned} \tag{26}$$

where $|\Lambda_1^{(1)}| = 2^{\beta} - 1$, $|\Lambda_1^{(2)}| = 1$, and $|\Lambda_2| = 2^{\beta} - 1$. □

Theorem 9. *Let $G_{\mathfrak{L}}^{RP} = (\Lambda_1, \Lambda_2)$ be the relatively prime graph associated with $\mathfrak{L} \in \Omega_1$, where $\propto = 2^{\beta-2}$ and β is the positive integer greater than 1. Then, the distance-based topological indices are as follows:*

(1) $W(G_{\mathfrak{L}}^{RP}) = 4^{\beta} - 2^{\beta+1} + 1$

(2) $WW(G_{\mathfrak{L}}^{RP}) = 3(2^{2\beta-1} - 2^{\beta-1}) - 2^{\beta+1} + 2$

Proof. Let H_1 and H_2 be two distance-based subsets of $\Lambda_1 \times \Lambda_1$ defined by

$$\begin{aligned} H_1 &= \{(\tau_1, \tau_2) \in \Lambda_1 \times \Lambda_1 | d(\tau_1, \tau_2) = 1, \tau_1 \neq \tau_2\}, \\ H_2 &= \{(\tau_1, \tau_2) \in \Lambda_1 \times \Lambda_1 | d(\tau_1, \tau_2) = 2, \tau_1 \neq \tau_2\}, \end{aligned} \tag{27}$$

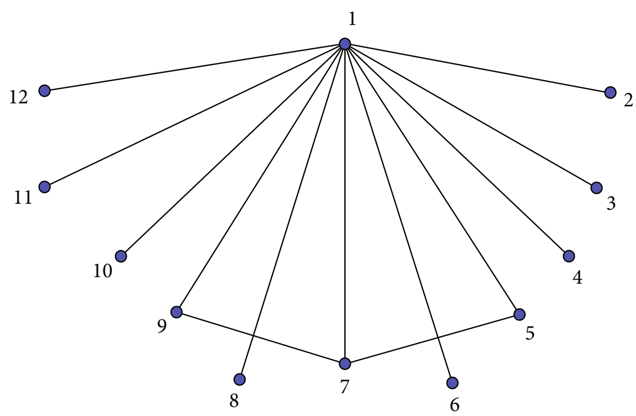

Figure 1: Relatively prime graph of order 12.

where cardinalities of H_1 and H_2 are $2^{\beta}-1$ and $2^{2\beta-1}-2^{\beta-1}-2^{\beta}+1$, respectively. It completes the proof with the help of Equations (14) and (17). □

Theorem 10. *Let $G_{\mathfrak{L}}^{RP}=(\Lambda_1,\Lambda_2)$ be the relatively prime graph associated with $\mathfrak{L}\in\Omega_1$, where $\propto=2^{\beta-2}$ and β is the positive integer greater than 1. Then, the degree-distance-based topological indices are as follows:*

(1) $Sc(G_{\mathfrak{L}}^{RP})=2^{2\beta}-2^{\beta}$

(2) $Sc^*(G_{\mathfrak{L}}^{RP})=2^{2\beta}+1-2^{\beta+1}$

Proof. Let K_1 and K_2 be two subsets of Λ_2 defined by

$$\begin{aligned}K_1&=\left\{\tau_1\tau_2\in\Lambda_2\middle|\deg(\tau_1)=1,\deg(\tau_2)=2^{\beta}-1\right\},\\K_2&=\{\tau_1\tau_2\in\Lambda_2|\deg(\tau_1)=1,\deg(\tau_2)=1\},\end{aligned}\tag{28}$$

where $|K_1|=2^{\beta}-1$ and $|K_2|=0$.

$$\begin{aligned}Sc\left(G_{\mathfrak{L}}^{RP}\right)&=\sum_{\{\tau_1,\tau_2\}\subseteq\Lambda_1\left(G_{\mathfrak{L}}^{RP}\right)}(d^{\tau_1}+d^{\tau_2})d(\tau_1,\tau_2)\\&=\left(2^{\beta}-1\right)\left[(1)+\left(2^{\beta}-1\right)\right](1)=\left(2^{\beta}-1\right)\left[1+2^{\beta}-1\right]\\&=2^{\beta}\left(2^{\beta}-1\right)=2^{2\beta}-2^{\beta},\\Sc^*\left(G_{\mathfrak{L}}^{RP}\right)&=\sum_{\{\tau_1,\tau_2\}\subseteq\Lambda_1\left(G_{\mathfrak{L}}^{RP}\right)}(d^{\tau_1}d^{\tau_2})d(\tau_1,\tau_2)\\&=\left(2^{\beta}-1\right)(1)\left(2^{\beta}-1\right)(1)=\left(2^{\beta}-1\right)^2=2^{2\beta}+1-2^{\beta+1}.\end{aligned}\tag{29}$$

It completes the proof. □

If $\propto=2$ and $\beta=3$, then Figure 2 indicates relatively prime graph of quasigroup $\mathfrak{L}$ in Ω_1.

Theorem 11. *Let $G_{\mathfrak{L}}^{RP}=(\Lambda_1,\Lambda_2)$ be the relatively prime graph associated with $\mathfrak{L}\in\Omega_1$, where $\propto=2^{\beta-2}p_1^{k_1}$, p_1 is an*

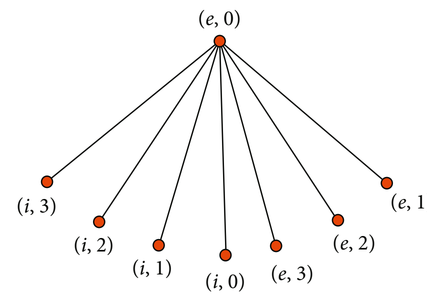

Figure 2: Relatively prime graph of order 8 associated with $(C_2\times\mathbb{Z}_4,*)$.

odd prime, k_1 is a natural number, and $\beta>1$ is a positive integer. Then, we have the following degree-based topological indices:

(1) $M_1(G_{\mathfrak{L}}^{RP})=(2^{2\beta}+2^{\beta}-2^{\beta+1}-1)p_1^{k_1}+(2^{\beta}+2^{2\beta}-1)p_1^{2k_1}-2^{2\beta}-2^{\beta}+2$

(2) $M_2(G_{\mathfrak{L}}^{RP})=(-2^{\beta}-3\times2^{2\beta}+1)p_1^{k_1}+(2^{2\beta+2}-2^{\beta+1})p_1^{2k_1}+2^{\beta+1}-1$

(3) $F(G_{\mathfrak{L}}^{RP})=(2^{3\beta}+2^{\beta+1}+2^{2\beta+2}+2^{\beta})p_1^{k_1}-(2^{\beta+2}+2^{\beta+1})p_1^{2k_1}+(2^{\beta}+1)p_1^{3k_1}-2^{3\beta}-2^{\beta+1}+1.$

(4) $M_2^m(G_{\mathfrak{L}}^{RP})=((2^{\beta}p_1^{k_1}-2^{\beta}-p_1^{k_1}+1)/2^{\beta}p_1^{k_1})+((p_1^{k_1}-1)/(2^{2\beta}p_1^{k_1}-2^{\beta}))+(2^{\beta-1}/(2^{\beta}p_1^{2k_1}-p_1^{k_1}))+((2^{\beta}p_1^{k_1}-2^{\beta}-p_1^{k_1}+1)/(2^{\beta}p_1^{k_1}-1))$

(5) $R_{\alpha}(G_{\mathfrak{L}}^{RP})=(2^{\beta}p_1^{k_1}-2^{\beta}-p_1^{k_1}+1)(2^{\beta}p_1^{k_1})^{\alpha}+(p_1^{k_1}-1)(2^{2\beta}p_1^{k_1}-2^{\beta})^{\alpha}+(2^{\beta}-1)(2^{\beta}p_1^{2k_1}-p_1^{k_1})^{\alpha}+(2^{\beta}p_1^{k_1}-2^{\beta}-p_1^{k_1}+1)(2^{\beta}p_1^{k_1}-1)^{\alpha}$

(6) $RR_{\alpha}(G_{\mathfrak{L}}^{RP})=((2^{\beta}p_1^{k_1}-2^{\beta}-p_1^{k_1}+1)/(2^{\beta}p_1^{k_1})^{\alpha})+((p_1^{k_1}-1)/(2^{2\beta}p_1^{k_1}-2^{\beta})^{\alpha})+((2^{\beta}-1)/(2^{\beta}p_1^{2k_1}-p_1^{k_1})^{\alpha})+((2^{\beta}p_1^{k_1}-2^{\beta}-p_1^{k_1}+1)/(2^{\beta}p_1^{k_1}-1)^{\alpha})$

(7) $SDD(G_{\mathfrak{L}}^{RP})=(2^{\beta}p_1^{k_1}-2^{\beta}-p_1^{k_1}+1)[(2^{2\beta}+p_1^{2k_1})/2^{\beta}p_1^{k_1}]+(p_1^{k_1}-1)[(2^{2\beta}+2^{2\beta}p_1^{2k_1}-2^{\beta+1}p_1^{k_1}+1)/(2^{2\beta}p_1^{k_1}-2^{\beta})]+(2^{\beta}-1)[(p_1^{2k_1}+2^{2\beta}p_1^{2k_1}-2^{\beta+1}p_1^{k_1}+1)/(2^{\beta}p_1^{2k_1}-p_1^{k_1})]+(2^{\beta}p_1^{k_1}-2^{\beta}-p_1^{k_1}+1)[(2+2^{2\beta}p_1^{2k_1}-2^{\beta+1}p_1^{2k_1})/(2^{\beta}p_1^{k_1}-1)]$

(8) $H(G_{\mathfrak{L}}^{RP})=((2^{\beta+1}p_1^{k_1}-2^{\beta+1}-2p_1^{k_1}+2)/(2^{\beta}+p_1^{k_1}))+((2p_1^{k_1}-2)/(2^{\beta}+2^{\beta}p_1^{k_1}-1))+((2^{\beta+1}-2)/(2^{\beta}p_1^{k_1}+p_1^{k_1}-1))+((2^{\beta+1}p_1^{k_1}-2^{\beta+1}-2p_1^{k_1}+2)/2^{\beta}p_1^{k_1})$

(9) $I(G_{\mathfrak{L}}^{RP})=(2^{\beta}-1)(p_1^{k_1}-1)[2^{\beta}p_1^{k_1}/(2^{\beta}+p_1^{k_1})]+(p_1^{k_1}-1)[2^{\beta}(2^{\beta}p_1^{k_1}-1)/(2^{\beta}+2^{\beta}p_1^{k_1}-1)]+(2^{\beta}-1)[p_1^{k_1}(2^{\beta}p_1^{k_1}-1)/(p_1^{k_1}+2^{\beta}p_1^{k_1}-1)]+(2^{\beta}-1)(p_1^{k_1}-1)[(2^{\beta}p_1^{k_1}-1)/2^{\beta}p_1^{k_1}]$

(10) $A(G_{\mathfrak{Q}}^{RP}) = (2^{\beta}p_1^{k_1} - 2^{\beta} - p_1^{k_1} + 1)[2^{\beta}p_1^{k_1}/(2^{\beta} + p_1^{k_1} - 2)]^3 + (p_1^{k_1} - 1)[2^{2\beta}p_1^{k_1}2^{\beta}/(2^{\beta} + 2^{\beta}p_1^{k_1} - 3)]^3 + (2^{\beta} - 1)[(2^{\beta}p_1^{2k_1} - p_1^{k_1})/(2^{\beta} + p_1^{k_1} + p_1^{k_1} - 3)]^3 + (2^{\beta}p_1^{k_1} - 2^{\beta} - p_1^{k_1} + 1)[(2^{\beta}p_1^{k_1} - 1)/(2^{\beta}p_1^{k_1} - 2)]^3$

Proof. The following are the partitions of Λ_1 and Λ_2:

$$\begin{aligned}
\Lambda_1^{(1)} &= \{\tau \in \Lambda_1 | \deg(\tau) = 1\},\\
\Lambda_1^{(2)} &= \left\{\tau \in \Lambda_1 \middle| \deg(\tau) = 2^{\beta}p_1^{k_1} - 1\right\},\\
\Lambda_1^{(3)} &= \left\{\tau \in \Lambda_1 \middle| \deg(\tau) = p_1^{k_1}\right\},\\
\Lambda_1^{(4)} &= \left\{\tau \in \Lambda_1 \middle| \deg(\tau) = 2^{\beta}\right\},\\
\Lambda_2^{(1)} &= \left\{e = \tau_1\tau_2 \in \Lambda_2 \middle| \deg(\tau_1) = 2^{\beta}p_1^{k_1} - 1, \deg(\tau_2) = 1\right\},\\
\Lambda_2^{(2)} &= \left\{e = \tau_1\tau_2 \in \Lambda_2 \middle| \deg(\tau_1) = 2^{\beta}p_1^{k_1} - 1, \deg(\tau_2) = p_1^{k_1}\right\},\\
\Lambda_2^{(3)} &= \left\{e = \tau_1\tau_2 \in \Lambda_2 \middle| \deg(\tau_1) = 2^{\beta}p_1^{k_1} - 1, \deg(\tau_2) = 2^{\beta}\right\},\\
\Lambda_2^{(4)} &= \left\{e = \tau_1\tau_2 \in \Lambda_2 \middle| \deg(\tau_1) = p_1^{k_1}, \deg(\tau_2) = 2^{\beta}\right\},
\end{aligned} \tag{30}$$

where $|\Lambda_1^{(1)}| = (2^{\beta} - 1)(p_1^{k_1} - 1)$, $|\Lambda_1^{(2)}| = 2^{\beta}p_1^{k_1}$, $|\Lambda_1^{(3)}| = 2^{\beta} - 1$, $|\Lambda_1^{(4)}| = p_1^{k_1} - 1$, $|\Lambda_2^{(1)}| = (2^{\beta} - 1)(p_1^{k_1} - 1)$, $|\Lambda_2^{(2)}| = 2^{\beta} - 1$, $|\Lambda_2^{(3)}| = p_1^{k_1} - 1$, and $|\Lambda_2^{(4)}| = (2^{\beta} - 1)(p_1^{k_1} - 1)$.

$$\begin{aligned}
M_2(G_{\mathfrak{Q}}^{RP}) &= \sum_{\tau_1\tau_2 \in \Lambda_2(G_{\mathfrak{Q}}^{RP})} d^{\tau_1}d^{\tau_2} = \sum_{\tau_1\tau_2 \in \Lambda_2^{(1)}(G_{\mathfrak{Q}}^{RP})} d^{\tau_1}d^{\tau_2}\\
&+ \sum_{\tau_1\tau_2 \in \Lambda_2^{(2)}(G_{\mathfrak{Q}}^{RP})} d^{\tau_1}d^{\tau_2} + \sum_{\tau_1\tau_2 \in \Lambda_2^{(3)}(G_{\mathfrak{Q}}^{RP})} d^{\tau_1}d^{\tau_2}\\
&+ \sum_{\tau_1\tau_2 \in \Lambda_2^{(4)}(G_{\mathfrak{Q}}^{RP})} d^{\tau_1}d^{\tau_2} = \left(-2^{\beta} - 3 \times 2^{2\beta} + 1\right)p_1^{k_1}\\
&+ \left(2^{2\beta+2} - 2^{\beta+1}\right)p_1^{2k_1} + 2^{\beta+1} - 1,\\
F(G_{\mathfrak{Q}}^{RP}) &= \sum_{\tau_1\tau_2 \in \Lambda_2(G_{\mathfrak{Q}}^{RP})} \left[(d^{\tau_1})^2 + (d^{\tau_2})^2\right] = \sum_{\tau_1\tau_2 \in \Lambda_2^{(1)}(G_{\mathfrak{Q}}^{RP})} \left[(d^{\tau_1})^2 + (d^{\tau_2})^2)\right]\\
&+ \sum_{\tau_1\tau_2 \in \Lambda_2^{(2)}(G_{\mathfrak{Q}}^{RP})} \left[(d^{\tau_1})^2 + (d^{\tau_2})^2\right] + \sum_{\tau_1\tau_2 \in \Lambda_2^{(3)}(G_{\mathfrak{Q}}^{RP})} \left[(d^{\tau_1})^2 + (d^{\tau_2})^2\right]\\
&+ \sum_{\tau_1\tau_2 \in \Lambda_2^{(4)}(G_{\mathfrak{Q}}^{RP})} \left[(d^{\tau_1})^2 + (d^{\tau_2})^2\right] = \left(2^{3\beta} + 2^{\beta+1} + 2^{2\beta+2} + 2^{\beta}\right)p_1^{k_1}\\
&- \left(2^{\beta+2} + 2^{\beta+1}\right)p_1^{2k_1} + \left(2^{\beta} + 1\right)p_1^{3k_1} - 2^{3\beta} - 2^{\beta+1} + 1.
\end{aligned} \tag{31}$$

Similarly, we can prove the other results. It completes the proof. □

To understand some of the following theorems, we introduce a mapping $T : S_{\text{odd}g} \longrightarrow A_t$ defined by

$$T(C_{2i+1}) = \begin{cases} 1 & \text{if} \quad \propto = \{3, 6, 12, 24, \cdots\} = C_{3,}\\ 6 & \text{if} \quad \propto = \{5,10,20,40, \cdots\} = C_5,\\ 15 & \text{if} \quad \propto = \{7,14,28,56, \cdots\} = C_7,\\ 28 & \text{if} \quad \propto = \{9,18,36,72, \cdots\} = C_9,\\ 45 & \text{if} \quad \propto = \{11,22,44,88, \cdots\} = C_{11},\\ 66 & \text{if} \quad \propto = \{13,26,52,104, \cdots\} = C_{13},\\ . \\ . \\ . \end{cases} \tag{32}$$

where $S_{\text{odd}g}$ and A_t are the sets of geometric sequences and alternate triangular numbers, respectively, with $T(C_{2i+1}) = T(2i - 1) \forall i \in \{1, 2, 3, \cdots\}$.

Theorem 12. *Let $G_{\mathfrak{Q}}^{RP} = (\Lambda_1, \Lambda_2)$ be the relatively prime graph associated with $\mathfrak{Q} \in \Omega_1$, where $\propto = 2^{\beta-2}p_1^{k_1}$, p_1 is an odd prime, k_1 is a natural number, and $\beta > 1$ is a positive integer. Then, degree-distance-based topological indices are as follows:*

(1) $Sc(G_{\mathfrak{Q}}^{RP}) = (2^{\beta} - 2^{2\beta+1} - 2^{\beta+2} + 15)p_1^{k_1} + (2^{2\beta+2} - 3 \times 2^{2\beta+1} + 3)p_1^{2k_1} + 2^{\beta} + 2^{2\beta} + (2^{\beta+2} - 4)T(C_{2i+1}) - 6$

(2) $Sc^*(G_{\mathfrak{Q}}^{RP}) = (3 \times 2^{\beta+2} - 2^{2\beta+3} - 3 \times 2^{2\beta} - 4)p_1^{k_1} + (5 \times 2^{2\beta} - 3 \times 2^{\beta} + 2^{2\beta+2} - 2^{\beta+3} + 4)p_1^{2k_1} + (2^{2\beta} - 2^{\beta+1} - 2^{\beta} + 2)p_1^{3k_1} + 2^{\beta+1} + (2^{2\beta+1} - 2)T(C_{2i+1}) + 2^{2\beta+2} - 2^{\beta+2} - 2^{2\beta} + 2^{\beta} + 1$

Proof. Let $K_1, K_2, \cdots, K_9$ be the subsets of Λ_2 defined by

$$\begin{aligned}
K_1 &= \left\{\tau_1\tau_2 \in \Lambda_2 \middle| \deg(\tau_1) = 1, \deg(\tau_2) = 2^{\beta}p_1^{k_1} - 1\right\},\\
K_2 &= \left\{\tau_1\tau_2 \in \Lambda_2 \middle| \deg(\tau_1) = 2^{\beta}, \deg(\tau_2) = 2^{\beta}p_1^{k_1} - 1\right\},\\
K_3 &= \left\{\tau_1\tau_2 \in \Lambda_2 \middle| \deg(\tau_1) = p_1^{k_1}, \deg(\tau_2) = 2^{\beta}p_1^{k_1} - 1\right\},\\
K_4 &= \left\{\tau_1\tau_2 \in \Lambda_2 \middle| \deg(\tau_1) = 2^{\beta}, \deg(\tau_2) = p_1^{k_1}\right\},\\
K_5 &= \{\tau_1\tau_2 \in \Lambda_2 | \deg(\tau_1) = 1, \deg(\tau_2) = 1\},\\
K_6 &= \left\{\tau_1\tau_2 \in \Lambda_2 \middle| \deg(\tau_1) = 2^{\beta}, \deg(\tau_2) = 1\right\},\\
K_7 &= \left\{\tau_1\tau_2 \in \Lambda_2 \middle| \deg(\tau_1) = 2^{\beta}, \deg(\tau_2) = 2^{\beta}\right\},\\
K_8 &= \left\{\tau_1\tau_2 \in \Lambda_2 \middle| \deg(\tau_1) = p_1^{k_1}, \deg(\tau_2) = 1\right\},\\
K_9 &= \left\{\tau_1\tau_2 \in \Lambda_2 \middle| \deg(\tau_1) = p_1^{k_1}, \deg(\tau_2) = p_1^{k_1}\right\},
\end{aligned} \tag{33}$$

where $|K_1| = (2^{\beta} - 1)(p_1^{k_1} - 1)$, $|K_2| = p_1^{k_1} - 1$, $|K_3| = 2^{\beta} - 1$, $|K_4| = (2^{\beta} - 1)(p_1^{k_1} - 1)$, $|K_5| = 2^{2\beta-1}p_1^{2k_1} - 2^{\beta-1}p_1^{k_1} - 2^{\beta+1}p_1^{k_1} + 2^{\beta} + p_1^{k_1} - (2^{\beta} - 1)(p_1^{k_1} - 1)^2 - (2^{\beta} - 1)^2(p_1^{k_1} - 1) - (2^{\beta} - 1)(2^{\beta-1} - 1) - T(C_{2i+1})$, $|K_6| = (2^{\beta} - 1)(p_1^{k_1} - 1)^2$, $|K_7| = T(C_{2i+1})$, $|K_8| = (2^{\beta} - 1)^2(p_1^{k_1} - 1)$, and $|K_9| = (2^{\beta} - 1)(2^{\beta-1} - 1)$. The proof is complete by Equations (21) and (22). □

Numerical values $\propto = 3$, $\beta = 4$, $p_1 = 3$, $k_1 = 1$, and $T(C_{2i+1})$ represent the relatively prime graph (see Figure 3 to support Theorem 12).

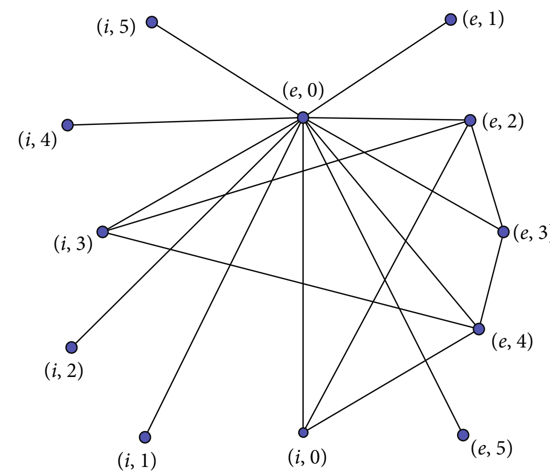

FIGURE 3: Relatively prime graph of order 12 associated with $(C_2 \times \mathbb{Z}_6, *)$.

Theorem 13. *Let $G_{\mathfrak{L}}^{RP} = (\Lambda_1, \Lambda_2)$ be the relatively prime graph associated with $\mathfrak{L} \in \Omega_2$, where $\propto = 2^{\beta-1}p_1^{k_1}$, p_1 is an odd prime, k_1 is a natural number, and $\beta \geq 1$ is a positive integer. Then, we have the following degree-based topological indices:*

(1) $M_1(G_{\mathfrak{L}}^{RP}) = (2^{\beta} - 1 - 2^{\beta+1} + 2^{2\beta-2})p_1^{k_1} + (2^{2\beta} + 2^{\beta-1} - 1)p_1^{2k_1} - 2^{2\beta-2} - 2$

(2) $M_2(G_{\mathfrak{L}}^{RP}) = (5 \times 2^{\beta+1} - 2^{2\beta} - 2^{2\beta-1} - 2^{\beta+2} + 2)p_1^{k_1} + (2^{2\beta+1} - 2^{\beta+1} - 5 \times 2^{\beta-1} + 2^{2\beta-1})p_1^{2k_1} + 2^{\beta-1} + 1$

(3) $F(G_{\mathfrak{L}}^{RP}) = (2^{\beta+3} - 2^{2\beta+1} + 2^{\beta+2} + 2^{3\beta-2} + 4)p_1^{k_1} + (2^{2\beta} - 2^{\beta+2} - 2^{2\beta+1} + 2^{\beta-1} + 2^{3\beta-1} - 2^{\beta} + 4)p_1^{2k_1} + (2^{3\beta} + 2^{\beta} - 5)p_1^{3k_1} + 2^{\beta-1} + 2^{2\beta} - 2^{3\beta-2} - 8$

(4) $M_2^m(G_{\mathfrak{L}}^{RP}) = (((2^{\beta} - 1)p_1^{k_1} - 3)/(2^{\beta}p_1^{k_1} - 1)) + ((2^{\beta-1} - 1)/p_1^{k_1}(2^{\beta}p_1^{k_1} - 1)) + ((p_1^{k_1} - 1)/2^{\beta-1}(2^{\beta}p_1^{k_1} - 1)) + (((p_1^{k_1} - 1)(2^{\beta} - 5))/2^{\beta-1}p_1^{k_1})$

(5) $R_{\alpha}(G_{\mathfrak{L}}^{RP}) = [(2^{\beta} - 1)p_1^{k_1} - 3](2^{\beta}p_1^{k_1} - 1)^{\alpha} + p_1^{k_1}(2^{\beta-1} - 1)(2^{\beta}p_1^{k_1} - 1)^{\alpha} + 2^{\beta-1}(p_1^{k_1} - 1)(2^{\beta}p_1^{k_1} - 1)^{\alpha} + 2^{\alpha\beta-\alpha}p_1^{k_1}(2^{\beta} - 5)(p_1^{k_1} - 1)$

(6) $RR_{\alpha}(G_{\mathfrak{L}}^{RP}) = [(2^{\beta} - 1)p_1^{k_1} - 3](2^{\beta}p_1^{k_1} - 1)^{\alpha} + p_1^{k_1}(2^{\beta-1} - 1)(2^{\beta}p_1^{k_1} - 1)^{\alpha} + 2^{\beta-1}(p_1^{k_1} - 1)(2^{\beta}p_1^{k_1} - 1)^{\alpha} + 2^{\alpha\beta-\alpha}p_1^{k_1}(2^{\beta} - 5)(p_1^{k_1} - 1)$

(7) $SDD(G_{\mathfrak{L}}^{RP}) = [(2^{\beta} - 1)p_1^{k_1} - 3][(1/(2^{\beta}p_1^{k_1} - 1)) + 2^{\beta}p_1^{k_1} - 1] + (2^{\beta-1} - 1)[(p_1^{k_1}/(2^{\beta}p_1^{k_1} - 1)) + ((2^{\beta}p_1^{k_1} - 1)/p_1^{k_1})] + (p_1^{k_1} - 1)[(2^{\beta-1}/(2^{\beta}p_1^{k_1} - 1)) + ((2^{\beta}p_1^{k_1} - 1)/2^{\beta-1})] + (2^{\beta} - 5)(p_1^{k_1} - 1)[(p_1^{k_1}/2^{\beta-1}) + (2^{\beta-1}/p_1^{k_1})] + (2^{\beta-1} - 1)(2/p_1^{k_1})$

(8) $H(G_{\mathfrak{L}}^{RP}) = [(2^{\beta} - 1)p_1^{k_1} - 3][2/2^{\beta}p_1^{k_1}] + (2^{\beta-1} - 1)[2/(p_1^{k_1} + 2^{\beta}p_1^{k_1} - 1)] + (p_1^{k_1} - 1)[2/(2^{\beta-1} + 2^{\beta}p_1^{k_1} - 1)] + (2^{\beta} - 5)(p_1^{k_1} - 1)[2/(2^{\beta-1} + p_1^{k_1})]$

(9) $I(G_{\mathfrak{L}}^{RP}) = [(2^{\beta} - 1)p_1^{k_1} - 3][(2^{\beta}p_1^{k_1} - 1)/2^{\beta}p_1^{k_1}] + (2^{\beta-1} - 1)[p_1^{k_1}(2^{\beta}p_1^{k_1} - 1)/(p_1^{k_1} + 2^{\beta}p_1^{k_1} - 1)] + (p_1^{k_1} - 1)[2^{\beta-1}(2^{\beta}p_1^{k_1} - 1)/(2^{\beta-1} + 2^{\beta}p_1^{k_1} - 1)] + (2^{\beta} - 5)(p_1^{k_1} - 1)[2^{\beta-1}p_1^{k_1}/(2^{\beta-1} + p_1^{k_1})]$

(10) $A(G_{\mathfrak{L}}^{RP}) = [(2^{\beta} - 1)p_1^{k_1} - 3][(2^{\beta}p_1^{k_1} - 1)/(2^{\beta}p_1^{k_1} - 2)]^3 + (2^{\beta-1} - 1)[p_1^{k_1}(2^{\beta}p_1^{k_1} - 1)/(p_1^{k_1} + 2^{\beta}p_1^{k_1} - 3)]^3 + (p_1^{k_1} - 1)[2^{\beta-1}(2^{\beta}p_1^{k_1} - 1)/(2^{\beta-1} + 2^{\beta}p_1^{k_1} - 3)]^3 + (2^{\beta} - 5)(p_1^{k_1} - 1)[2^{\beta-1}p_1^{k_1}/(2^{\beta-1} + p_1^{k_1} - 2)]^3$

Proof. The following are the partitions of Λ_1 and Λ_2:

$$\begin{aligned}
\Lambda_1^{(1)} &= \{\tau \in \Lambda_1 | \deg(\tau) = 1\},\\
\Lambda_1^{(2)} &= \left\{\tau \in \Lambda_1 \middle| \deg(\tau) = 2^{\beta}p_1^{k_1} - 1\right\},\\
\Lambda_1^{(3)} &= \left\{\tau \in \Lambda_1 \middle| \deg(\tau) = p_1^{k_1}\right\},\\
\Lambda_1^{(4)} &= \left\{\tau \in \Lambda_1 \middle| \deg(\tau) = 2^{\beta-1}\right\},\\
\Lambda_2^{(1)} &= \left\{\tau_1\tau_2 \in \Lambda_2 \middle| \deg(\tau_1) = 1, \deg(\tau_2) = 2^{\beta}p_1^{k_1} - 1\right\},\\
\Lambda_2^{(2)} &= \left\{\tau_1\tau_2 \in \Lambda_2 \middle| \deg(\tau_1) = 2^{\beta}p_1^{k_1} - 1, \deg(\tau_2) = p_1^{k_1}\right\},\\
\Lambda_2^{(3)} &= \left\{\tau_1\tau_2 \in \Lambda_2 \middle| \deg(\tau_1) = 2^{\beta}p_1^{k_1} - 1, \deg(\tau_2) = 2^{\beta-1}\right\},\\
\Lambda_2^{(4)} &= \left\{\tau_1\tau_2 \in \Lambda_2 \middle| \deg(\tau_1) = p_1^{k_1}, \deg(\tau_2) = 2^{\beta-1}\right\},
\end{aligned} \tag{34}$$

with cardinalities $|\Lambda_1^{(1)}| = (2^{\beta} - 1)p_1^{k_1} - 3$, $|\Lambda_1^{(2)}| = 1$, $|\Lambda_1^{(3)}| = 2^{\beta-1} - 1$, $|\Lambda_1^{(4)}| = p_1^{k_1} - 1$, $|\Lambda_2| = 2^{\beta}p_1^{k_1}$, $|\Lambda_2^{(1)}| = (2^{\beta} - 1)p_1^{k_1} - 3$, $|\Lambda_2^{(2)}| = 2^{\beta-1} - 1$, $|\Lambda_2^{(3)}| = p_1^{k_1} - 1$, and $|\Lambda_2^{(4)}| = (2^{\beta} - 5)(p_1^{k_1} - 1)$. □

Theorem 14. *Let $G_{\mathfrak{L}}^{RP} = (\Lambda_1, \Lambda_2)$ be the relatively prime graph associated with $\mathfrak{L} \in \Omega_2$, where $\propto = 2^{\beta-1}p_1^{k_1}$, p_1 is an odd prime,*

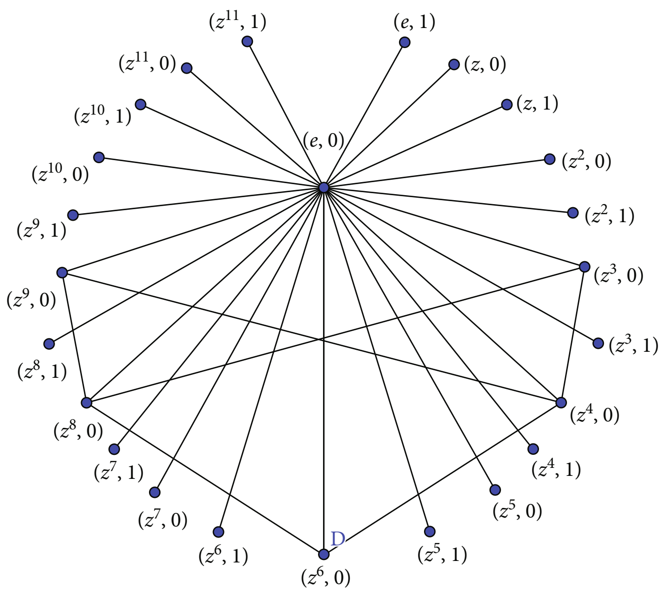

FIGURE 4: Relatively prime graph of order 24 associated with $(C_{12} \times \mathbb{Z}_2, *)$.

k_1 is a natural number, and $\beta \geq 1$ is a positive integer. Then, distance-based topological indices are as follows:

(1) $W(G_{\mathfrak{L}}^{RP}) = (2^{\beta+1} - 2^{\beta} - 2^{\beta+2} + 5)p_1^{k_1} + 2^{2\beta}p_1^{2k_1} - 2^{\beta} + 2^{\beta+1} - 4$

(2) $WW(G_{\mathfrak{L}}^{RP}) = -(3 \times 2^{\beta-1} + 2^{\beta+2} - 10)p_1^{k_1} + 3 \times 2^{2\beta-1} p_1^{2k_1} + 2^{\beta+1} - 8$

Proof. Let H_1 and H_2 be two distance-based subsets of $\Lambda_1 \times \Lambda_1$ defined by

$$\begin{aligned} H_1 &= \{(\tau_1, \tau_2) \in \Lambda_1 \times \Lambda_1 | d(\tau_1, \tau_2) = 1, \tau_1 \neq \tau_2\}, \\ H_2 &= \{(\tau_1, \tau_2) \in \Lambda_1 \times \Lambda_1 | d(\tau_1, \tau_2) = 2, \tau_1 \neq \tau_2\}, \end{aligned} \tag{35}$$

with $|H_1| = 2^{\beta+1}p_1^{k_1} - 2^{\beta} - 5p_1^{k_1} + 4$ and $|H_2| = 2^{2\beta-1}p_1^{2k_1} - 2^{\beta-1}p_1^{k_1} - 2^{\beta+1}p_1^{k_1} + 2^{\beta} + 5p_1^{k_1} - 4$. □

Theorem 15. *Let $G_{\mathfrak{L}}^{RP} = (\Lambda_1, \Lambda_2)$ be the relatively prime graph associated with $\mathfrak{L} \in \Omega_2$, where $\propto = 2^{\beta-1}p_1^{k_1}$, p_1 is an odd prime, k_1 is a natural number, and $\beta \geq 1$ is a positive integer. Then, degree-distance-based topological indices are as follows:*

(1) $Sc(G_{\mathfrak{L}}^{RP}) = (2^{2\beta} - 5 \times 2^{\beta} + 3 \times 2^{\beta+1} - 2^{\beta+3} - 2^{2\beta+2} + 1) p_1^{k_1} + (2^{2\beta+1} + 2^{\beta+2} + 2^{3\beta-1} - 1)p_1^{2k_1} - 2^{\beta+4} - 2^{2\beta} + 2^{\beta+1} - 4$

(2) $Sc^*(G_{\mathfrak{L}}^{RP}) = (2^{2\beta-1} - 2^{\beta+1} + 2)T(C_{2i+1}) + (2^{\beta+1} - 11 \times 2^{\beta} - 2^{2\beta} - 2^{2\beta+2} + 8)p_1^{k_1} + (2^{2\beta} - 2^{\beta} + 2^{\beta+1} + 2^{3\beta-1} + 4) p_1^{2k_1} + 2^{\beta-1} - 2^{2\beta-1} - 7 \times 2^{\beta} + 3$

Proof. Let $K_1, K_2, \cdots, K_9$ be the subsets of Λ_2 defined by

$$\begin{aligned} K_1 &= \left\{\tau_1\tau_2 \in \Lambda_2 \middle| \deg(\tau_1) = 2^{\beta}p_1^{k_1} - 1, \deg(\tau_2) = 1\right\}, \\ K_2 &= \left\{\tau_1\tau_2 \in \Lambda_2 \middle| \deg(\tau_1) = 2^{\beta}p_1^{k_1} - 1, \deg(\tau_2) = p_1^{k_1}\right\}, \\ K_3 &= \left\{\tau_1\tau_2 \in \Lambda_2 \middle| \deg(\tau_1) = 2^{\beta}p_1^{k_1} - 1, \deg(\tau_2) = 2^{\beta-1}\right\}, \\ K_4 &= \left\{\tau_1\tau_2 \in \Lambda_2 \middle| \deg(\tau_1) = p_1^{k_1}, \deg(\tau_2) = 2^{\beta-1}\right\}, \\ K_5 &= \{\tau_1\tau_2 \in \Lambda_2 | \deg(\tau_1) = 1, \deg(\tau_2) = 1\}, \\ K_6 &= \left\{\tau_1\tau_2 \in \Lambda_2 \middle| \deg(\tau_1) = 1, \deg(\tau_2) = p_1^{k_1}\right\}, \\ K_7 &= \left\{\tau_1\tau_2 \in \Lambda_2 \middle| \deg(\tau_1) = 1, \deg(\tau_2) = 2^{\beta-1}\right\}, \\ K_8 &= \left\{\tau_1\tau_2 \in \Lambda_2 \middle| \deg(\tau_1) = p_1^{k_1}, \deg(\tau_2) = p_1^{k_1}, \right. \\ K_9 &= \left\{\tau_1\tau_2 \in \Lambda_2 \middle| \deg(\tau_1) = 2^{\beta-1}, \deg(\tau_2) = 2^{\beta-1}, \right. \end{aligned} \tag{36}$$

where $|K_1| = (2^{\beta} - 1)p_1^{k_1} - 3$, $|K_2| = 2^{\beta-1} - 1$, $|K_3| = p_1^{k_1} - 1$, $|K_4| = (2^{\beta} - 5)(p_1^{k_1} - 1)$, $|K_5| = T(C_{2i+1})$, $|K_6| = 2^{\beta}p_1^{k_1} + 2^{\beta-1} p_1^{k_1} + 2^{\beta} + 2p_{1^{k1_1}} + 4$, $|K_7| = 2^{2\beta-1}p_{1^{2k_1}} - 2^{\beta+2}p_1^{k_1} - 2^{\beta-1} - 7p_1^{k_1} - 2T(C_{2i+1}) - 7$, $|K_8| = 2^{\beta-1} - 1$, and $|K_9| = T(C_{2i+1})$. □

Let $\mathfrak{L}$ be an element of the class Ω_2 with $\propto = 12$, $\beta = 3$, $p_1 = 3$, and $k_1 = 1$. Then, Figure 4 shows the relatively prime graph for Theorem 15.

4. Algebraic Approach of Topological Indices with Graphical Representations

Theorem 16. *Let $G_{\mathfrak{L}}^{RP} = (\Lambda_1, \Lambda_2)$ be the relatively prime graph associated with $\mathfrak{L} \in \Omega_1$, where $\propto = 2^{\beta-2}$ and β is the positive integer greater than 1. Then, we have the following*

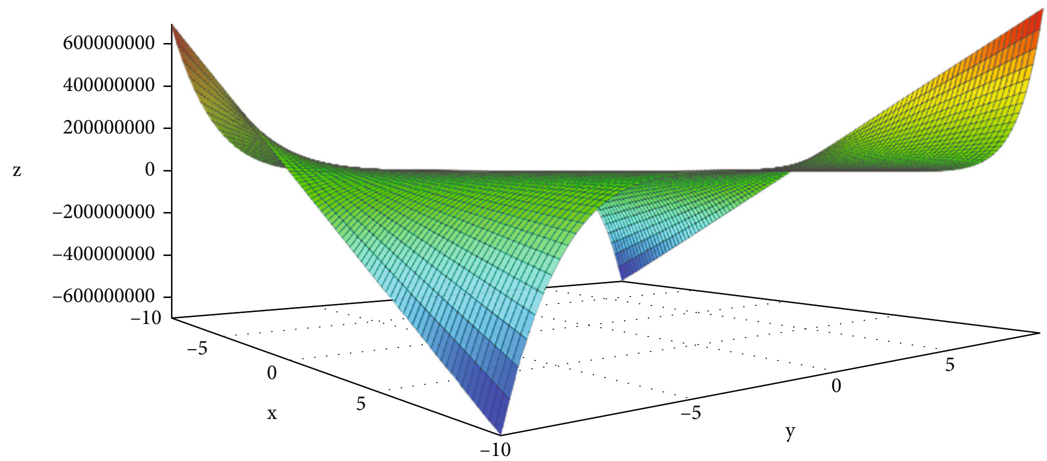

FIGURE 5: Graphical representation of M-polynomial associated with $(C_2 \times \mathbb{Z}_4, *)$.

polynomials of one and two variables in the integral domains $\mathbb{R}[\theta]$ *and* $\mathbb{R}[\theta_1, \theta_2]$:

(1) $M(G_{\mathfrak{L}}^{RP}; \theta_1, \theta_2) = (2^\beta - 1)\theta_1\theta_2^{2^\beta - 1}$

(2) $H(G_{\mathfrak{L}}^{RP}, \theta) = 2^\beta + (2^\beta - 1)\theta + (2^{2\beta-1} - 2^{\beta-1} - 2^\beta + 1)\theta^2$

(3) $Sc(G_{\mathfrak{L}}^{RP}, \theta) = 2^\beta(2^\beta - 1)\theta + (2^{2\beta} - 2^\beta - 2^{\beta+1} + 2)\theta^2$

(4) $Sc^*(G_{\mathfrak{L}}^{RP}, \theta) = (2^{2\beta} + 1 - 2^{\beta+1})\theta + (2^{2\beta-1} - 2^{\beta-1} - 2^\beta + 1)\theta^2$

Proof.

$$
\begin{aligned}
M(G_{\mathfrak{L}}^{RP}; \theta_1, \theta_2) &= \sum_{1 \le \lambda_1 \le \lambda_2 \le 2^\beta - 1} M_{(\lambda_1, \lambda_2)}(G_{\mathfrak{L}}^{RP})\theta_1^{\lambda_1}\theta_2^{\lambda_2} = \left(2^\beta - 1\right)\theta_1\theta_2^{2^\beta - 1}, \\
H(G_{\mathfrak{L}}^{RP}, \theta) &= \sum_{i=0}^{D} d(G_{\mathfrak{L}}^{RP}, i)\theta^i = 2^\beta + \left(2^\beta - 1\right)\theta + \left(2^{2\beta-1} - 2^{\beta-1} - 2^\beta + 1\right)\theta^2, \\
\mathrm{Sc}(G_{\mathfrak{L}}^{RP}, \theta) &= \sum_{\{\tau_1, \tau_2\} \subseteq \Lambda_1(G_{\mathfrak{L}}^{RP})} (d^{\tau_1} + d^{\tau_2})\theta^{d(\tau_1, \tau_2)} \\
&= \left(2^\beta - 1\right)\left(1 + \left(2^\beta - 1\right)\right)\theta + \left(2^{2\beta-1} - 2^{\beta-1} - 2^\beta + 1\right)(1+1)\theta^2 \\
&= 2^\beta\left(2^\beta - 1\right)\theta + \left(2^{2\beta} - 2^\beta - 2^{\beta+1} + 2\right)\theta^2, \\
\mathrm{Sc}^*(G_{\mathfrak{L}}^{RP}, \theta) &= \sum_{\{\tau_1, \tau_2\} \subseteq \Lambda_1(G_{\mathfrak{L}}^{RP})} (d^{\tau_1} d^{\tau_2})\theta^{d(\tau_1, \tau_2)} \\
&= \left(2^\beta - 1\right)\left((1)\left(2^\beta - 1\right)\right)\theta + \left(2^{2\beta-1} - 2^{\beta-1} - 2^\beta + 1\right)[(1)(1)]\theta^2 \\
&= \left(2^{2\beta} + 1 - 2^{\beta+1}\right)\theta + \left(2^{2\beta-1} - 2^{\beta-1} - 2^\beta + 1\right)\theta^2.
\end{aligned}
\tag{37}
$$

It completes the required proof. □

Example 2. If $\propto = 2$ and $\beta = 3$, then the graphical representation (Figure 5) indicates the surface of M-polynomial for quasigroup $\mathfrak{L}$ in Ω_1.

Theorem 17. *Let* $G_{\mathfrak{L}}^{RP} = (\Lambda_1, \Lambda_2)$ *be the relatively prime graph associated with* $\mathfrak{L} \in \Omega_1$, *where* $\propto = 2^{\beta-2}p_1^{k_1}$, p_1 *is an odd prime,* k_1 *is a natural number, and* $\beta > 1$ *is a positive integer. Then, polynomials in* $\mathbb{R}[\theta]$ *and* $\mathbb{R}[\theta_1, \theta_2]$ *are as follows:*

(1) $M(G_{\mathfrak{L}}^{RP}; \theta_1, \theta_2) = A\theta_1^{2^\beta}\theta_2^{p_1^{k_1}} + B\theta_1^{2^\beta}\theta_2^{2^\beta p_1^{k_1} - 1} + C\theta_1^{p_1^{k_1}}\theta_2^{2^\beta p_1^{k_1} - 1} + D\theta_1\theta_2^{2^\beta p_1^{k_1} - 1}$, *where*

$$
\begin{aligned}
A &= 2^\beta p_1^{k_1} - 2^\beta - p_1^{k_1} + 1, \\
B &= p_1^{k_1} - 1, \\
C &= 2^\beta - 1, \\
D &= 2^\beta p_1^{k_1} - 2^\beta - p_1^{k_1} + 1.
\end{aligned}
\tag{38}
$$

(2) $H(G_{\mathfrak{L}}^{RP}, \theta) = A\theta + B\theta^2$, *where*

$$
\begin{aligned}
A &= \left(2^\beta - 1\right)\left(p_1^{k_1} - 1\right), \\
B &= \left(-2^{\beta+1} - 2^{\beta-1} + 1\right)p_1^{k_1} + 2^{2\beta-1}p_1^{2k_1} + 2^\beta.
\end{aligned}
\tag{39}
$$

(3) $Sc(G_{\mathfrak{L}}^{RP}, \theta) = A\theta + B\theta^2$, *where*

$$
\begin{aligned}
A &= \left(2^{2\beta} - 2^\beta + 2^{2\beta} - 1\right)p_1^{k_1} + \left(2^\beta - 1\right)p_1^{2k_1} - 2^{2\beta} - 2^\beta, \\
B &= \left(2^{2\beta+1} - 3 \times 2^\beta - 2^{\beta+2} - 2^{2\beta+1} + 2^{\beta+2} + 1\right)p_1^{k_1} \\
&\quad + \left(2^{2\beta+1} - 2^{\beta+1}\right)p_1^{2k_1} + \left(2^{\beta+1} - 1\right)T(C_{2i+1}) \\
&\quad + 2^{\beta+3} + 2^{\beta+1} - 2^{2\beta-1} + 2^{\beta-1}.
\end{aligned}
\tag{40}
$$

(4) $Sc^*(G_{\mathfrak{L}}^{RP}, \theta) = A\theta + B\theta^2$, *where*

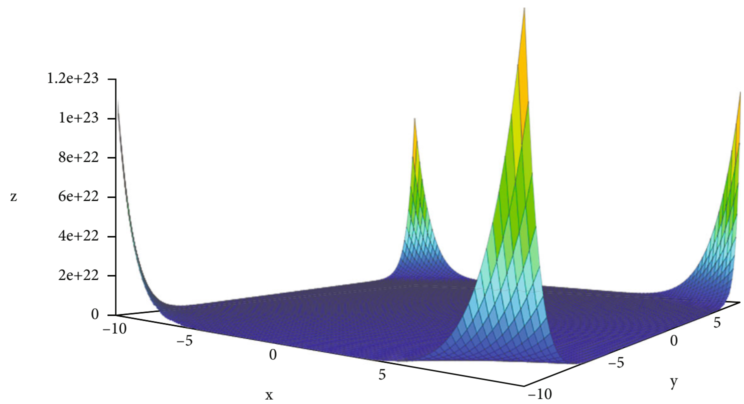

Figure 6: Graphical representation of M-polynomial associated with $(C_2 \times \mathbb{Z}_6, *)$.

$$
\begin{aligned}
A &= -\left(3 \times 2^{2\beta} + 2^{\beta}\right)p_1^{k_1} + \left(2^{2\beta+2} - 3 \times 2^{\beta}\right)p_1^{2k_1} + 2^{\beta+1} - 1,\\
B &= \left(3 \times 2^{\beta+1} - 2^{\beta-1} - 2^{2\beta+2} - 2\right)p_1^{k_1}\\
&\quad + \left(2^{2\beta-1} - 2^{\beta+1} - 3 \times 2^{\beta}2^{2\beta} - 2^{\beta-1} + 3\right)p_1^{2k_1}\\
&\quad + \left(2^{\beta+1} - 1\right)T(C_{2i+1}) + 2^{2\beta} + 2^{2\beta} - 2^{\beta+1}\\
&\quad - 2^{2\beta-1} + 2^{\beta-1} + 1.
\end{aligned} \tag{41}
$$

Proof. The partitions of Theorem 11, Corollary 18, and Theorem 12 with Equations (1), (15), (21), and (22) give the required results. □

Corollary 18. *Let $G^{RP}_{\mathfrak{L}} = (\Lambda_1, \Lambda_2)$ be the relatively prime graph associated with $\mathfrak{L} \in \Omega_1$, where $\propto = 2^{\beta-2}p_1^{k_1}$, p_1 is an odd prime, k_1 is a natural number, and $\beta > 1$ is a positive integer. Then, distance-based topological indices are as follows:*

(1) $W(G^{RP}_{\mathfrak{L}}) = (-2^{\beta+2} + 1)p_1^{k_1} + 2^{2\beta}p_1^{2k_1} + 2^{\beta+1} - 2^{\beta} + 1$

(2) $WW(G^{RP}_{\mathfrak{L}}) = (2^{\beta} - 3 \times 2^{\beta-1} - 3 \times 2^{\beta+1} + 2)p_1^{k_1} + 3 \times 2^{2\beta-1}p_1^{2k_1} + 2^{\beta+1} + 1$

Proof. Polynomials are studied to facilitate the calculations of topological indices. So Equations (16) and (17) give the required Wiener index $W(G^{RP}_{\mathfrak{L}})$ and hyper-Wiener index $WW(G^{RP}_{\mathfrak{L}})$, where the Hosoya polynomial is $A\theta + B\theta^2$ given in Theorem 17. □

Example 3. Numerical values $\propto = 3$, $\beta = 4$, $p_1 = 3$, $k_1 = 1$, and $T(C_{2i+1})$ represent the graph of M-polynomial (see Figure 6).

Theorem 19. *Let $G^{RP}_{\mathfrak{L}} = (\Lambda_1, \Lambda_2)$ be the relatively prime graph associated with $\mathfrak{L} \in \Omega_2$, where $\propto = 2^{\beta-1}p_1^{k_1}$, p_1 is an odd prime, k_1 is a natural number, and $\beta \geq 1$ is a positive integer. Then, some well-known elements of $\mathbb{R}[\theta]$ and $\mathbb{R}[\theta_1, \theta_2]$ are as follows:*

(1) $M(G^{RP}_{\mathfrak{L}}; \theta_1, \theta_2) = A\theta_1\theta_2^{2^{\beta}p_1^{k_1}-1} + B\theta_1^{p_1^{k_1}}\theta_2^{2^{\beta}p_1^{k_1}-1} + C\theta_1^{2^{\beta-1}+p_1^{k_1}}\theta_2^{2^{\beta}p_1^{k_1}+2^{\beta-1}-1}$, *where*

$$
\begin{aligned}
A &= \left(2^{\beta} - 1\right)p_1^{k_1} - 3,\\
B &= 2^{\beta-1} - 1,\\
C &= \left(2^{\beta} - 5\right)\left(p_1^{k_1} - 1\right)^2.
\end{aligned} \tag{42}
$$

(2) $H(G^{RP}_{\mathfrak{L}}, \theta) = A\theta + B\theta^2$, *where*

$$
\begin{aligned}
A &= \left(2^{\beta+1} - 5\right)p_1^{k_1} - 2^{\beta} + 4,\\
B &= \left(-2^{\beta-1} - 2^{\beta+1} + 5\right)p_1^{k_1} + \left(2^{2\beta-1}\right)p_1^{2k_1} + 2^{\beta} - 4.
\end{aligned} \tag{43}
$$

(3) $Sc(G^{RP}_{\mathfrak{L}}, \theta) = A\theta + B\theta^2$, *where*

$$
\begin{aligned}
A &= 2T(C_{2i+1}) + \left(2^{2\beta} - 3 \times 2^{\beta-1} - 3 \times 2^{\beta+1} + 3\right)p_1^{k_1}\\
&\quad + \left(2^{\beta} + 2^{2\beta} - 5\right)p_1^{2k_1} - 2^{2\beta-1} + 3 \times 2^{\beta-1} + 2,\\
B &= -\left(2^{2\beta+1} + 2^{\beta+2} + 1\right)p_1^{k_1} + \left(2^{3\beta-2} + 2^{2\beta-1} + 2^{\beta-1} + 2^{\beta} + 2\right)\\
&\quad \cdot p_1^{2k_1} + 2^{\beta} - 2^{\beta+2} - 2^{2\beta-2} - 3.
\end{aligned} \tag{44}
$$

(4) $Sc^*(G^{RP}_{\mathfrak{L}}, \theta) = A\theta + B\theta^2$, *where*

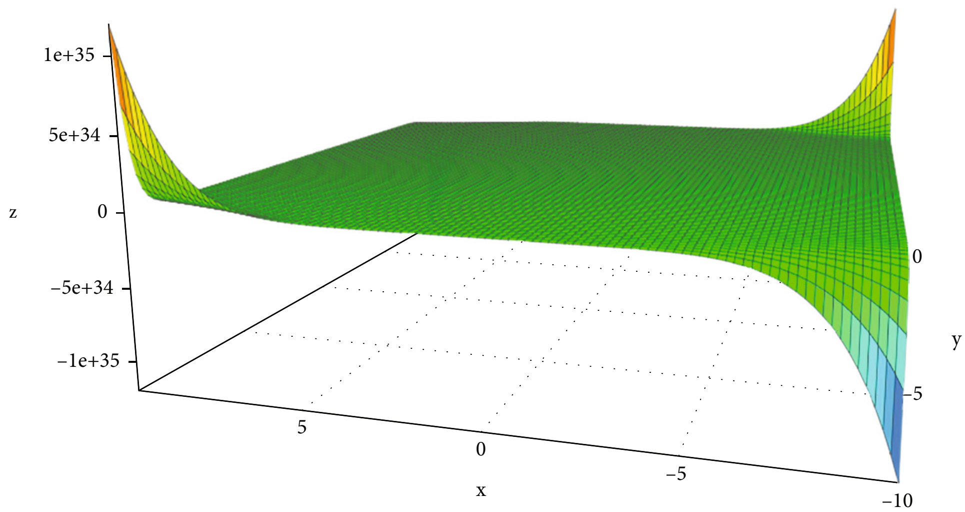

Figure 7: Graphical representation of M-polynomial associated with $(C_{12} \times \mathbb{Z}_2, *)$.

$$\begin{aligned} A &= \left(3 \times 2^{\beta-1} - 2^{\beta+2} - 2^{2\beta} + 2\right) p_1^{k_1} \\ &\quad + \left(2^{2\beta} - 2^{\beta+1} + 3 \times 2^{2\beta-1} - 5 \times 2^{\beta-1}\right) p_1^{2k_1} + 2^{\beta-1} + 3, \\ B &= \left(2^{2\beta-2} - 2^{\beta} + 1\right) T(C_{2i+1}) \\ &\quad + \left(2^{\beta} - 2^{2\beta+1} - 7 \times 2^{\beta-1} + 4\right) p_1^{k_1} \\ &\quad + \left(2^{\beta} + 2^{\beta-1} + 2^{3\beta-2} + 2^{\beta-1} + 2\right) \\ &\quad \cdot p_1^{2k_1} - 2^{2\beta-2} - 7 \times 2^{\beta-1}. \end{aligned} \tag{45}$$

Proof. The partitions of Theorems 13, 14, and 15 with Equations (1), (15), (21), and (22) constitute the required results. □

Example 4. Let $\mathfrak{L}$ be an element of the class Ω_2 with $\propto = 12$, $\beta = 3$, $p_1 = 3$, and $k_1 = 1$. Then, Figure 7 is the surface of M-polynomial.

Theorem 20. *Let $G_{\mathfrak{L}}^{RP} = (\Lambda_1, \Lambda_2)$ be the relatively prime graph associated with $\mathfrak{L} \in \Omega_2$, where $\propto = 2^{\beta-1} p_1^{k_1} p_2^{k_2}$, p_1 and p_2 are odd primes, k_1 and k_2 are natural numbers, and $\beta \geq 1$ is a positive integer. Then, M-polynomial of ring $\mathbb{R}[\theta_1, \theta_2]$ is given by*

$$\begin{aligned} M\left(G_{\mathfrak{L}}^{RP}; \theta_1, \theta_2\right) &= A\theta_1 \theta_2^{2^{\beta} p_1^{k_1} p_2^{k_2} - 1} + B\theta_1^{2^{\beta} p_1^{k_1} p_2^{k_2} - 1} \theta_2^{p_1^{k_1}} \\ &\quad + C\theta_1^{p_2^{k_2}} \theta_2^{2^{\beta} p_1^{k_1} p_2^{k_2} - 1} + D\theta_1^{p_1^{k_1}} \theta_2^{p_2^{k_2}}, \end{aligned} \tag{46}$$

where $A = 2^{\beta} p_1^{k_1} p_2^{k_2} - p_2^{k_2} - 2$, $B = p_2^{k_2} - 1$, $C = p_1^{k_1} - 1$, and $D = (p_1^{k_1} - 1)(p_2^{k_2} - 1)$.

Proof. The following are the partitions of Λ_1 and Λ_2:

$$\begin{aligned} \Lambda_1^{(1)} &= \{\tau \in \Lambda_1 | \deg(\tau) = 1\}, \\ \Lambda_1^{(2)} &= \left\{\tau \in \Lambda_1 \middle| \deg(\tau) = 2^{\beta} p_1^{k_1} p_2^{k_2} - 1\right\}, \\ \Lambda_1^{(3)} &= \left\{\tau \in \Lambda_1 \middle| \deg(\tau) = p_1^{k_1}\right\}, \\ \Lambda_1^{(4)} &= \left\{\tau \in \Lambda_1 \middle| \deg(\tau) = p_2^{k_2}\right\}, \\ \Lambda_2^{(1)} &= \left\{\tau_1\tau_2 \in \Lambda_2 \middle| \deg(\tau_1) = 1, \deg(\tau_2) = 2^{\beta} p_1^{k_1} p_2^{k_2} - 1\right\}, \\ \Lambda_2^{(2)} &= \left\{\tau_1\tau_2 \in \Lambda_2 \middle| \deg(\tau_1) = 2^{\beta} p_1^{k_1} p_2^{k_2} - 1, \deg(\tau_2) = p_1^{k_1}\right\}, \\ \Lambda_2^{(3)} &= \left\{\tau_1\tau_2 \in \Lambda_2 \middle| \deg(\tau_1) = 2^{\beta} p_1^{k_1} p_2^{k_2} - 1, \deg(\tau_2) = p_2^{k_2}\right\}, \\ \Lambda_2^{(4)} &= \left\{\tau_1\tau_2 \in \Lambda_2 \middle| \deg(\tau_1) = p_1^{k_1}, \deg(\tau_2) = p_2^{k_2}\right\}, \end{aligned} \tag{47}$$

where $|\Lambda_1^{(1)}| = 2^{\beta} p_1^{k_1} p_2^{k_2} - p_2^{k_2} - 2$, $|\Lambda_1^{(2)}| = 1$, $|\Lambda_1^{(3)}| = p_2^{k_2} - 1$, $|\Lambda_1^{(4)}| = p_1^{k_1} - 1$, $|\Lambda_2| = 2^{\beta} p_1^{k_1} p_1^{k_1}$, $|\Lambda_2^{(1)}| = 2^{\beta} p_1^{k_1} p_2^{k_2} - p_2^{k_2} - 2$, $|\Lambda_2^{(2)}| = p_2^{k_2} - 1$, $|\Lambda_2^{(3)}| = p_1^{k_1} - 1$, and $|\Lambda_2^{(4)}| = (p_1^{k_1} - 1)(p_2^{k_2} - 1)$. □

Theorem 21. *Let $G_{\mathfrak{L}}^{RP} = (\Lambda_1, \Lambda_2)$ be the relatively prime graph associated with $\mathfrak{L} \in \Omega_2$, where $\propto = 2^{\beta-1} p_1^{k_1} p_2^{k_2}$, p_1 and p_2 are odd primes, k_1 and k_2 are natural numbers, and $\beta \geq 1$ is a positive integer. Then, the following is the Hosoya polynomial in ring $\mathbb{R}[\theta]$:*

$$H\left(G_{\mathfrak{L}}^{RP}, \theta\right) = A\theta + B\theta^2, \tag{48}$$

where $A = 2^{\beta} p_1^{k_1} p_2^{k_2} + 2^{\beta} p_2^{k_2} - 2^{\beta} - 1$ and $B = [2^{2\beta-1} p_1^{k_1} p_2^{k_2} - 2^{\beta-1} - 2^{\beta} - 2^{\beta} p_1^{-k_1} + (2^{\beta} + 1) p_1^{-k_1} p_2^{-k_2}] p_1^{k_1} p_2^{k_2}$.

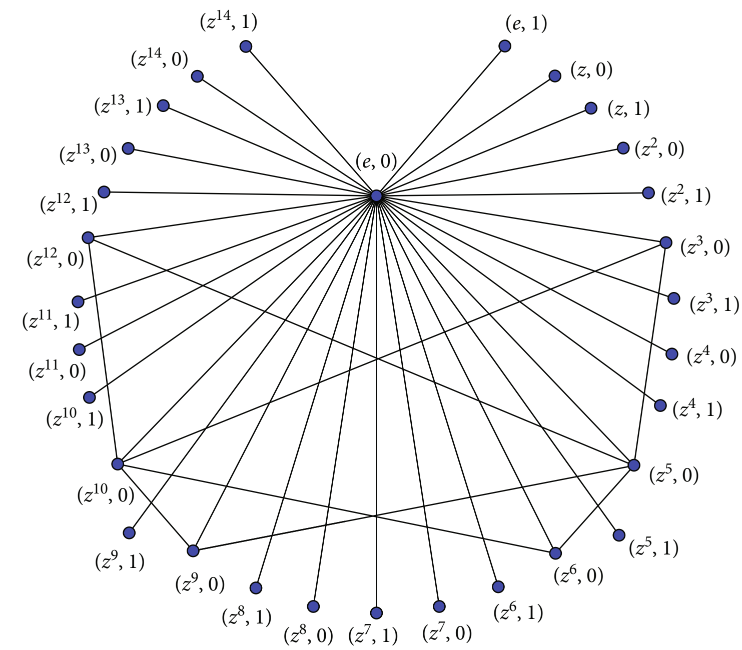

FIGURE 8: Relatively prime graph of order 30 associated with $(C_{15} \times \mathbb{Z}_2, *)$.

Proof. Let H_1 and H_2 be two distance-based subsets of $\Lambda_1 \times \Lambda_1$ defined by

$$\begin{aligned} H_1 &= \{(\tau_1, \tau_2) \in \Lambda_1 \times \Lambda_1 | d(\tau_1, \tau_2) = 1, \tau_1 \neq \tau_2\}, \\ H_2 &= \{(\tau_1, \tau_2) \in \Lambda_1 \times \Lambda_1 | d(\tau_1, \tau_2) = 2, \tau_1 \neq \tau_2\}, \end{aligned} \tag{49}$$

such that $|H_1| = 2^\beta p_1^{k_1} p_2^{k_2} + 2^\beta p_2^{k_2} - 2^\beta - 1$ and $|H_2| = 2^{2\beta-1} p_1^{2k_1} p_2^{2k_2} - 2^{\beta-1} p_1^{k_1} p_2^{k_2} - 2^\beta p_1^{k_1} p_2^{k_2} - 2^\beta p_2^{k_2} + 2^\beta + 1$. □

Theorem 22. *Let $G_{\mathfrak{L}}^{RP} = (\Lambda_1, \Lambda_2)$ be the relatively prime graph associated with $\mathfrak{L} \in \Omega_2$, where $\propto = 2^{\beta-1} p_1^{k_1} p_2^{k_2}$, p_1 and p_2 are odd primes, k_1 and k_2 are natural numbers, and $\beta \geq 1$ is a positive integer. Then, the Schultz polynomial of integral domain $\mathbb{R}[\theta]$ is*

$$Sc\left(G_{\mathfrak{L}}^{RP}, \theta\right) = A\theta + B\theta^2, \tag{50}$$

where

$$\begin{aligned} A = \Big(&2^{2\beta} p_1^{k_1} p_2^{k_2} - 2^{\beta+2} + 2^\beta p_1^{k_1} - p_2^{-k_2} - p_1^{-k_1} \\ &+ p_1^{k_1} - p_1^{k_1} p_2^{-k_2} - p_1^{-k_1} p_2^{k_2} + 2 p_1^{-k_1} p_2^{-k_2}\Big) p_1^{k_1} p_2^{k_2}, \\ B = \Big(&2^{2\beta-1} p_1^{k_1} p_2^{k_2} - 2^{\beta-1} - 2^{\beta+1} - 2^\beta p_1^{-k_1} + 2^\beta p_1^{-k_1} p_2^{-k_2} \\ &+ 2^{2\beta-1} p_1^{2k_1} p_2^{k_2} - 2^{\beta-1} p_1^{k_1} - 2^{\beta+1} p_1^{k_1} + 2^\beta p_2^{-k_2} - p_1^{k_1} - 5 p_2^{-k_2} \\ &+ 2^\beta p_2^{k_2} + p_2^{k_2} + p_1^{-k_1} p_2^{k_2} - 5 p_1^{-k_1} + 4 p_1^{-k_1} p_2^{-k_2}\Big) p_1^{k_1} p_2^{k_2}. \end{aligned} \tag{51}$$

Proof. Let $K_1, K_2, \cdots, K_9$ be the subsets of Λ_2 defined by

$$\begin{aligned} K_1 &= \left\{\tau_1\tau_2 \in \Lambda_2 \middle| \deg(\tau_1) = 2^\beta p_1^{k_1} p_2^{k_2} - 1, \deg(\tau_2) = 1\right\}, \\ K_2 &= \left\{\tau_1\tau_2 \in \Lambda_2 \middle| \deg(\tau_1) = 2^\beta p_1^{k_1} p_2^{k_2} - 1, \deg(\tau_2) = p_1^{k_1}\right\}, \\ K_3 &= \left\{\tau_1\tau_2 \in \Lambda_2 \middle| \deg(\tau_1) = 2^\beta p_1^{k_1} p_2^{k_2} - 1, \deg(\tau_2) = p_2^{k_2}\right\}, \\ K_4 &= \left\{\tau_1\tau_2 \in \Lambda_2 \middle| \deg(\tau_1) = p_1^{k_1}, \deg(\tau_2) = p_2^{k_2}\right\}, \\ K_5 &= \{\tau_1\tau_2 \in \Lambda_2 | \deg(\tau_1) = 1, \deg(\tau_2) = 1\}, \\ K_6 &= \left\{\tau_1\tau_2 \in \Lambda_2 \middle| \deg(\tau_1) = 1, \deg(\tau_2) = p_1^{k_1}\right\}, \\ K_7 &= \left\{\tau_1\tau_2 \in \Lambda_2 \middle| \deg(\tau_1) = 1, \deg(\tau_2) = p_2^{k_2}\right\}, \\ K_8 &= \left\{\tau_1\tau_2 \in \Lambda_2 \middle| \deg(\tau_1) = p_1^{k_1}, \deg(\tau_2) = p_1^{k_1}, \right. \\ K_9 &= \left\{\tau_1\tau_2 \in \Lambda_2 \middle| \deg(\tau_1) = p_2^{k_2}, \deg(\tau_2) = p_2^{k_2}, \right. \end{aligned} \tag{52}$$

with $|K_1| = 2^\beta p_1^{k_1} p_2^{k_2} - p_2^{k_2} - 2$, $|K_2| = p_2^{k_2} - 1$, $|K_3| = p_1^{k_1} - 1$, $|K_4| = (p_1^{k_1} - 1)(p_2^{k_2} - 1)$, $|K_5| = T(C_{2i+1})$, $|K_6| = 2^{2\beta-1} p_1^{2k_1} p_2^{2k2_2} - 2^{\beta-1} p_1^{k_1} p_2^{k_2} - 2^{\beta+1} p_1^{k_1} p_2^{k_2} - 2^\beta p_2^{k_2} + 2^\beta - 2T(C_{2i+1}) - p_1^{k_1} p_2^{k_2} - p_2^{k_2} + 4$, $|K_7| = 2^\beta p_1^{k_1} p_2^{k_2} + (p_1^{k_1} + 1)(p_2^{k_2} - 1)$, $|K_8| = T(C_{2i+1})$, and $|K_9| = p_1^{k_1} - 2$. □

Theorem 23. *Let $G_{\mathfrak{L}}^{RP} = (\Lambda_1, \Lambda_2)$ be the relatively prime graph associated with $\mathfrak{L} \in \Omega_2$, where $\propto = 2^{\beta-1} p_1^{k_1} p_2^{k_2}$, p_1 and p_2 are odd primes, k_1 and k_2 are natural numbers, and $\beta \geq$*

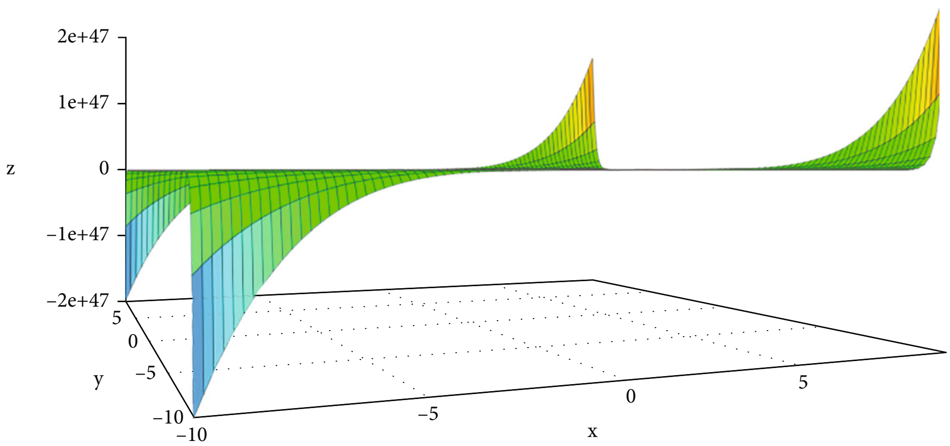

FIGURE 9: Graphical representation of M-polynomial associated with $(C_{15} \times \mathbb{Z}_2, *)$.

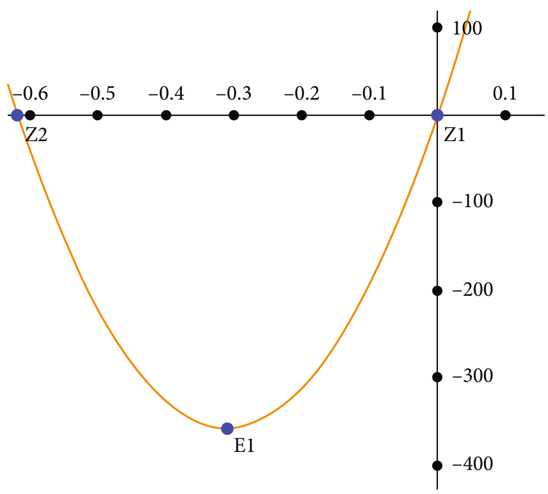

FIGURE 10: Graphical representation of modified Schultz polynomial associated with $(C_{15} \times \mathbb{Z}_2, *)$.

1 is a positive integer. Then, the following is the modified Schultz polynomial of integral domain $\mathbb{R}[\theta]$:

$$Sc^*\left(G_{\mathfrak{L}}^{RP}, \theta\right) = A\theta + B\theta^2, \tag{53}$$

where

$$\begin{aligned} A = &\Big(2^{2\beta} p_1^{k_1} p_2^{k_2} - 3 \times 2^{\beta} - 2^{\beta} p_2^{k_2} + 2 p_1^{-k_1} p_2^{-k_2} \\ &+ 2^{\beta+1} p_1^{k_1} p_2^{k_2} - 2^{\beta} p_1^{k_1} + p_2^{-k_2} - 1 - 2^{\beta} p_2^{k_2} \\ &+ p_1^{k_1} p_2^{k_2} - p_1^{k_1} - p_2^{k_2}\Big) p_1^{k_1} p_2^{k_2}, \\ B = &\left(1 - 2p_1^{k_1} + p_1^{2k_1}\right) T(C_{2i+1}) \\ &+ \Big(2^{2\beta-1} p_1^{2k_1} p_2^{k_2} - 2^{\beta-1} p_1^{k_1} - 2^{\beta+1} p_1^{k_1} - 2^{\beta} \\ &+ 2^{\beta} p_1^{k_1} p_2^{-k_2} - p_1^{k_1} - 2 + 4 p_1^{k_1} p_2^{-k_2} \\ &+ 2^{\beta} p_2^{k_2} + 2 p_2^{k_2} - p_1^{-k_1} p_2^{k_2} - p_1^{-k_1} p_2^{k_2}\Big) p_1^{k_1} p_2^{k_2}. \end{aligned} \tag{54}$$

Example 5. Let $\mathfrak{L}$ be an element of the class Ω_2 with $\propto = 15$, $\beta = 1$, $p_1 = 3$, $k_1 = 1$, $p_2 = 5$, and $k_2 = 1$. Then, Figures 8–10 give relatively prime graph, surface of M-polynomial, and parabolic curve of modified Schultz polynomial, respectively.

5. Conclusion and Future Directions

This paper is the portrayal of multidisciplinary research among algebra, graph theory, and chemical graph theory. We have calculated degree, distance, and degree-distance-based topological indices of relatively prime graphs associated with two classes of quasigroups $(C_2 \times \mathbb{Z}_{2\propto}, *)$ and $(C_{\propto} \times \mathbb{Z}_2, *)$. M-polynomials, Hosoya polynomials, Schultz polynomials, and modified Schultz polynomials corresponding to these two classes indicate the three dimensional graphical representations known as surfaces $(\theta_1, \theta_2, M(G_{\mathfrak{L}}^{\mathrm{RP}}))$, plane curves $(\theta, H(G_{\mathfrak{L}}^{\mathrm{RP}}))$, $(\theta, \mathrm{Sc}(G_{\mathfrak{L}}^{\mathrm{RP}}))$, and $(\theta, \mathrm{Sc}^*(G_{\mathfrak{L}}^{\mathrm{RP}}))$, respectively. In other words, we can say that it is an effort to understand the topology of a particular graph related with nonassociative algebras through algebraic polynomials.

The isotopy-isomorphy property of these quasigroups allows us to call them G-loops, as most of their structural properties are very similar to those of finite groups. So the graphical study of flexible quasigroups, Bol quasigroups, and alternative quasigroups is still new to learn. In time to come, it will be very interesting to study adjacency and Laplacian spectrum, vertex connectivity, edge connectivity, and algebraic connectivity of these graphs interrelated with nonassociative binary operations and their connected applications like data structure and Cheeger's inequality.

Data Availability

The work's supporting data are referenced throughout the text.

Conflicts of Interest

There are no conflicts of interest, according to the authors.

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