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Review Article

Inclusion Properties for Classes of *p*-Valent Functions

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Making use of a differential operator, which is defined here by means of the Hadamard product, we introduce classes of p-valent functions and investigate various important inclusion properties and characteristics for these classes. Also, a property preserving integrals is considered.

1. Introduction

Let $A(p)$ be the class of functions

$$E(\varkappa) = \varkappa^p + \sum_{k=p+1}^{\infty} a_k \varkappa^k, (p \in \mathbb{N} = \{1, 2, \cdots\}), \quad (1)$$

which are analytic and p-valent in $\mathbb{U} = \{\varkappa : |\varkappa| < 1\}$.

If E and G are analytic in $\mathbb{U}$, E is subordinate to G, $(E \prec G)$ if there exists an analytic function $\omega(0) = 0$ and $|\omega(\varkappa)| < 1$ such that $E(\varkappa) = G(\omega(\varkappa))$. Furthermore, if G is univalent in $\mathbb{U}$, then (see [1, 2])

$$E(\varkappa) \prec G(\varkappa) \Leftrightarrow E(0) = G(0), \quad E(\mathbb{U}) \subset G(\mathbb{U}). \quad (2)$$

For functions $E(\varkappa) \in A(p)$, given by (1) and $G(\varkappa) \in A(p)$ defined by

$$G(\varkappa) = \varkappa^p + \sum_{k=p+1}^{\infty} b_k \varkappa^k, \quad (3)$$

the Hadamard product of E and G is given by

$$(E * G)(\varkappa) = \varkappa^p + \sum_{k=p+1}^{\infty} b_k a_k \varkappa^k = (G * E)(\varkappa). \quad (4)$$

For $E(\varkappa) \in A(p)$, denote by $S_p^*(\zeta)$ and $K_p(\zeta)$ the classes of p-valently starlike and convex functions of order ζ and $0 \le \zeta < p$, respectively (see [3, 4]), satisfying

$$\Re\left\{\frac{\varkappa E'(\varkappa)}{E(\varkappa)}\right\} > \zeta, \quad (5)$$

$$\Re\left\{1 + \frac{\varkappa E''(\varkappa)}{E'(\varkappa)}\right\} > \zeta. \quad (6)$$

It follows from (5) and (6) that

$$E(\varkappa) \in K_p(\zeta) \Leftrightarrow \frac{\varkappa E'(\varkappa)}{p} \in S_p^*(\zeta). \quad (7)$$

See Goodman [5].

Also, denote by $C_p(\eta,\zeta)$ and $C_p^*(\eta,\zeta)$ the classes of p-valently close-to-convex and quasi-convex functions of order η and type ζ satisfying, respectively (see [6–8] (with $p=1$),

$$\Re\left\{\frac{\varkappa E'(\varkappa)}{G(\varkappa)}\right\}>\eta \quad \left(G\in S_p^*(\zeta), 0\le\eta,\zeta<p\right), \tag{8}$$

$$\Re\left\{\frac{\left(\varkappa E'(\varkappa)\right)'}{G'(\varkappa)}\right\}>\eta \quad \left(G\in K_p(\zeta), 0\le\eta,\zeta<p\right). \tag{9}$$

It follows from (8) and (9) that

$$E(\varkappa)\in C_p^*(\eta,\zeta)\Leftrightarrow\frac{\varkappa E'(\varkappa)}{p}\in C_p(\eta,\zeta). \tag{10}$$

Dziok and Srivastava [9] used the hypergeometric function (see Srivastava and Karlsson [10])

$$\Delta_{q,s}(\varkappa)=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\cdots(\alpha_q)_k}{(\beta_1)_k\cdots(\beta_s)_k(1)_k}\varkappa^k \tag{11}$$

and defined the linear operator

$$\begin{aligned}H_{p,q,s}(\alpha_1)E(\varkappa)&=\varkappa^p+\sum_{k=p+1}^{\infty}\frac{(\alpha_1)_{k-p}\cdots(\alpha_q)_{k-p}}{(\beta_1)_{k-p}\cdots(\beta_s)_{k-p}(1)_{k-p}}a_k\varkappa^k\\&=\varkappa^p+\sum_{k=p+1}^{\infty}\Gamma_k a_k\varkappa^k,\end{aligned} \tag{12}$$

where

$$(d)_k=\begin{cases}1, & \text{if } k=0,\\ d(d+1)\cdots(d+k-1), & \text{if } k\in\mathbb{N}.\end{cases} \tag{13}$$

Setting the function

$$D_{p,\lambda}^{n}E(\varkappa)=\varkappa^p+\sum_{k=p+1}^{\infty}\left(\frac{p+(k-p)\lambda}{p}\right)^n\varkappa^k \quad (n\in\mathbb{N}_0, p\in\mathbb{N}), \tag{14}$$

we define a function $D_{p,\lambda}^{*n}E(\varkappa)$ in terms of the Hadamard product (or convolution) by

$$\left(D_{p,\lambda}^{n}*D_{p,\lambda}^{*n}\right)(\varkappa)=\varkappa^p+\sum_{k=p+1}^{\infty}\Gamma_k\varkappa^k \quad (\varkappa\in\mathbb{U}). \tag{15}$$

Let

$$\begin{aligned}H_{p,\lambda}^{n}(\alpha_1)E(\varkappa)&=D_{p,\lambda}^{*n}*E(\varkappa)\\&=\varkappa^p+\sum_{k=p+1}^{\infty}\frac{(\alpha_1)_{k-p}\cdots(\alpha_q)_{k-p}}{(\beta_1)_{k-p}\cdots(\beta_s)_{k-p}(1)_{k-p}}\\&\quad\cdot\left(\frac{p}{p+(k-p)\lambda}\right)^n a_k\varkappa^k.\end{aligned} \tag{16}$$

From (16), it can be easy to verify that

$$\begin{aligned}\varkappa\left(H_{p,\lambda}^{n}(\alpha_1)E(\varkappa)\right)'&=\alpha_1 H_{p,\lambda}^{n}(\alpha_1+1)E(\varkappa)\\&\quad-(\alpha_1-p)H_{p,\lambda}^{n}(\alpha_1)E(\varkappa),\end{aligned} \tag{17}$$

$$\begin{aligned}\frac{\lambda}{p}\varkappa\left(H_{p,\lambda}^{n+1}(\alpha_1)E(\varkappa)\right)'&=H_{p,\lambda}^{n}(\alpha_1)E(\varkappa)\\&\quad-(1-\lambda)H_{p,\lambda}^{n+1}(\alpha_1)E(\varkappa).\end{aligned} \tag{18}$$

Using the operator $H_{p,\lambda}^{n}(\alpha_1)$, we introduce the subclasses.

$$S_{p,\lambda}^{*}(n,\alpha_1;\zeta)=\left\{E\in A(p): H_{p,\lambda}^{n}(\alpha_1)E(\varkappa)\in S_p^*(\zeta)\right\}, \tag{19}$$

$$K_{p,\lambda}(n,\alpha_1;\zeta)=\left\{E\in A(p): H_{p,\lambda}^{n}(\alpha_1)E(\varkappa)\in K_p(\zeta)\right\}, \tag{20}$$

$$C_{p,\lambda}(n,\alpha_1;\eta,\zeta)=\left\{E\in A(p): H_{p,\lambda}^{n}(\alpha_1)E(\varkappa)\in C_p(\eta,\zeta)\right\}, \tag{21}$$

$$C_{p,\lambda}^{*}(n,\alpha_1;\eta,\zeta)=\left\{E\in A(p): H_{p,\lambda}^{n}(\alpha_1)E(\varkappa)\in C_p^*(\eta,\zeta)\right\}. \tag{22}$$

We note that

$$E(\varkappa)\in K_{p,\lambda}(n,\alpha_1;\zeta)\Leftrightarrow\frac{\varkappa E'(\varkappa)}{p}\in S_{p,\lambda}^{*}(n,\alpha_1;\zeta), \tag{23}$$

$$E(\varkappa)\in C_{p,\lambda}^{*}(n,\alpha_1;\eta,\zeta)\Leftrightarrow\frac{\varkappa E'(\varkappa)}{p}\in C_{p,\lambda}(n,\alpha_1;\eta,\zeta). \tag{24}$$

2. Main Results

Unless otherwise mentioned, we assume that $n\in\mathbb{N}_0$, $\lambda, 0\le\zeta$, $\eta<p$ and $p\in\mathbb{N}$.

The following lemma due to Miller and Mocanu is required to prove the results.

Lemma 1 (see [11]). *Let $\varphi(\tau,\vartheta)$ be the complex function*

$$\varphi: D\longrightarrow\mathbb{C}, (D\subset\mathbb{C}\times\mathbb{C}), \tag{25}$$

$\mathbb{C}$ being the complex plane and let $\tau=\tau_1+i\tau_2$, $\vartheta=\vartheta_1+i\vartheta_2$. Suppose that $\varphi(\tau,\vartheta)$ satisfies the following conditions:

(i) $\varphi(\tau,\vartheta)$ is continuous in D

(ii) $(1,0) \in D$ *and* $\Re\{\varphi(1,0)\} > 0$

(iii) $\Re\{\varphi(i\tau_2, \vartheta_1)\} \le 0$ *for all* $(i\tau_2, \vartheta_1) \in D$ *and such that* $\vartheta_1 \le -(1+\tau_2^2)/2$

Let

$$h(\varkappa) = 1 + h_1\varkappa + h_2\varkappa^2 + \cdots \tag{26}$$

be regular in $\mathbb{U}$ *such that* $(h(\varkappa), \varkappa h'(\varkappa)) \in D$ *for all* $\varkappa \in \mathbb{U}$. *If*

$$\Re\left\{\varphi\left(h(\varkappa), \varkappa h'(\varkappa)\right)\right\} > 0, \tag{27}$$

then

$$\Re\{h(\varkappa)\} > 0. \tag{28}$$

Theorem 2. *Let* $E(\varkappa) \in A(p)$. *Then,*

$$S^*_{p,\lambda}(n, \alpha_1 + 1; \zeta) \subset S^*_{p,\lambda}(n, \alpha_1; \zeta). \tag{29}$$

Proof. Let $E(\varkappa) \in S^*_{p,\lambda}(n, \alpha_1 + 1; \zeta)$, and

$$\frac{\varkappa\left(H^n_{p,\lambda}(\alpha_1)E(\varkappa)\right)'}{H^n_{p,\lambda}(\alpha_1)E(\varkappa)} = \zeta + (p-\zeta)h(\varkappa), \tag{30}$$

where $h(x)$ given by (26) we have

$$\frac{H^n_{p,\lambda}(\alpha_1+1)E(\varkappa)}{H^n_{p,\lambda}(\alpha_1)E(\varkappa)} = \frac{1}{\alpha_1}\{(\alpha_1+\zeta) - p + (p-\zeta)h(\varkappa)\}. \tag{31}$$

Differentiating (31), we have

$$\frac{z\left(H^n_{p,\lambda}(\alpha_1+1)E(\varkappa)\right)'}{H^n_{p,\lambda}(\alpha_1+1)E(\varkappa)} = \frac{\varkappa\left(H^n_{p,\lambda}(\alpha_1)E(\varkappa)\right)'}{H^n_{p,\lambda}(\alpha_1)E(\varkappa)} + \frac{(p-\zeta)\varkappa h'(\varkappa)}{(\alpha_1+\zeta) - p + (p-\zeta)h(\varkappa)}, \tag{32}$$

which, in view of (30), leads to

$$\frac{\varkappa\left(H^n_{p,\lambda}(\alpha_1+1)E(\varkappa)\right)'}{H^n_{p,\lambda}(\alpha_1+1)E(\varkappa)} - \zeta = (p-\zeta)h(\varkappa) + \frac{(p-\zeta)\varkappa h'(\varkappa)}{(\alpha_1+\zeta) - p + (p-\zeta)h(\varkappa)}. \tag{33}$$

Let

$$\varphi(\tau, \vartheta) = (p-\zeta)\tau + \frac{(p-\zeta)\vartheta}{(\alpha_1+\zeta) - p + (p-\zeta)\vartheta}, \tag{34}$$

with $h(\varkappa) = \tau = \tau_1 + i\tau_2, \varkappa h'(\varkappa) = \vartheta = \vartheta_1 + i\vartheta_2$. Then,

(i) $\varphi(\tau, \vartheta)$ is continuous in $D = \mathbb{C} \setminus \{((\alpha_1 - p) + \zeta)/(\zeta - p)\} \times \mathbb{C}$

(ii) $(1,0) \in D$ and $\Re\{\varphi(1,0)\} = p - \zeta$

(iii) $\Re\{\varphi(i\tau_2, \vartheta_1)\} \le 0$ for all $(i\tau_2, \vartheta_1) \in D$ and such that $\vartheta_1 \le -(1+\tau_2^2)/2$

$$\begin{aligned}\Re\{\varphi(i\tau_2, \vartheta_1)\} &= \Re\left\{\frac{(p-\zeta)\vartheta_1}{(\alpha_1 - p) + \zeta + (p-\zeta)i\tau_2}\right\} \\ &= \frac{(\alpha_1 - p + \zeta)(p-\zeta)\vartheta_1}{(\alpha_1 - p + \zeta)^2 + (p-\zeta)^2\tau_2^2} < 0,\end{aligned} \tag{35}$$

for $\vartheta_1 < 0$; therefore, the function $\varphi(\tau, \vartheta)$ satisfies the conditions in Lemma 1; thus, we have Re $\{h(\varkappa)\} > 0 (\varkappa \in \mathbb{U})$, that is, $f \in S^*_{p,\lambda}(n, \alpha_1; \zeta)$. □

Theorem 3. *For* $E(\varkappa) \in A(p)$, *we have*

$$K_{p,\lambda}(n, \alpha_1 + 1; \zeta) \subset K_{p,\lambda}(n, \alpha_1; \zeta). \tag{36}$$

Proof. Applying (23) and using Theorem 2, we have

$$\begin{aligned} E(\varkappa) \in K_{p,\lambda}(n, \alpha_1 + 1; \zeta) &\Leftrightarrow H^n_{p,\lambda}(\alpha_1+1)E(\varkappa) \in K_p(\zeta) \\ &\Leftrightarrow \frac{\varkappa\left(H^n_{p,\lambda}(\alpha_1+1)E(\varkappa)\right)'}{p} \in S^*_p(\zeta) \\ &\Rightarrow \frac{\varkappa\left(H^n_{p,\lambda}(\alpha_1)E(\varkappa)\right)'}{p} \in S^*_p(\zeta) \\ &\Leftrightarrow E(\varkappa) \in K_{p,\lambda}(n, \alpha_1; \zeta). \end{aligned} \tag{37}$$

□

Theorem 4. *For* $E(\varkappa) \in A(p)$, *we have*

$$C_{p,\lambda}(n, \alpha_1 + 1; \eta, \zeta) \subset C_{p,\lambda}(n, \alpha_1; \eta, \zeta)(\eta \ge 0, \zeta < p; p \in \mathbb{N}). \tag{38}$$

Proof. Let $E(\varkappa) \in C_{p,\lambda}(n, \alpha_1 + 1; \eta, \zeta)$; then, from (21), there exists a function $G(\varkappa) \in S^*_{p,\lambda}(n, \alpha_1 + 1; \zeta)$ such that

$$\Re\left\{\frac{\varkappa\left(H^n_{p,\lambda}(\alpha_1+1)E(\varkappa)\right)'}{H^n_{p,\lambda}(\alpha_1+1)G(\varkappa)}\right\} > \eta. \tag{39}$$

Put

$$\frac{\varkappa\left(H^n_{p,\lambda}(\alpha_1)E(\varkappa)\right)'}{H^n_{p,\lambda}(\alpha_1)G(\varkappa)} = \eta + (p-\eta)h(\varkappa), \tag{40}$$

where $h(\varkappa)$ is given by (26). Applying (17) in (40) differentiating the resulting equation and multiplying by $\varkappa$, we have

$$\alpha_1\varkappa\left(H_{p,\lambda}^n(\alpha_1+1)E(\varkappa)\right)' = \{\eta+(p-\eta)h(\varkappa)\}\varkappa\left(H_{p,\lambda}^n(\alpha_1)g(\varkappa)\right)' + (p-\eta)\varkappa h'(\varkappa)H_{p,\lambda}^n(\alpha_1)G(\varkappa) + (\alpha_1-p)\varkappa\left(H_{p,\lambda}^n(\alpha_1)E(\varkappa)\right)'. \quad (41)$$

Since $G\in S_{p,\lambda}^*(n,\alpha_1+1;\zeta)$, then by Theorem 2, we have $G\in S_{p,\lambda}^*(n,\alpha_1;\zeta)$. Let

$$\frac{\varkappa\left(H_{p,\lambda}^n(\alpha_1)G(\varkappa)\right)'}{H_{p,\lambda}^n(\alpha_1)G(\varkappa)} = \zeta+(p-\zeta)\widehat{H}(\varkappa), \quad (42)$$

where $\mathfrak{R}\{\widehat{H}(\varkappa)\}>0$. Applying (17) in (42), we have

$$\alpha_1\frac{H_{p,\lambda}^n(\alpha_1+1)G(\varkappa)}{H_{p,\lambda}^n(\alpha_1)G(\varkappa)} = \alpha_1-p+\zeta+(p-\zeta)\widehat{H}(\varkappa). \quad (43)$$

From (41) and (43), we have

$$\frac{\varkappa\left(H_{p,\lambda}^n(\alpha_1+1)E(\varkappa)\right)'}{H_{p,\lambda}^n(\alpha_1+1)G(\varkappa)} - \eta = (p-\eta)h(\varkappa) + \frac{(p-\eta)\varkappa h'(\varkappa)}{\alpha_1-p+\zeta+(p-\zeta)\widehat{H}(\varkappa)}. \quad (44)$$

Now, let $\varphi(\tau,\vartheta)=(p-\eta)\tau + (((p-\eta)\vartheta)/(\alpha_1-p+\zeta+(p-\zeta)\widehat{H}(\varkappa)))$, with $h(\varkappa)=\tau=\tau_1+i\tau_2$, $\varkappa h'(\varkappa)=\vartheta=\vartheta_1+i\vartheta$. Then,

(i) $\varphi(\tau,\vartheta)$ is continuous in $D=\mathbb{C}\setminus\{((\alpha_1-p)+\zeta)/(\zeta-p)\}\times\mathbb{C}$

(ii) $(1,0)\in D$ and $\mathfrak{R}\{\varphi(1,0)\}=p-\eta$

(iii) $\mathfrak{R}\{\varphi(i\tau_2,\vartheta_1)\}\le 0$ for all $(i\tau_2,\vartheta_1)\in D$ and such that $\vartheta_1\le -(1+\tau_2^2)/2$

$$\mathfrak{R}\{\varphi(i\tau_2,\vartheta_1)\} = \mathfrak{R}\left\{\frac{(p-\eta)[\alpha_1-p+\zeta+(p-\zeta)h_1(a,b)]\upsilon_1}{[\alpha_1-p+\zeta+(p-\zeta)h_1(a,b)]^2+[(p-\zeta)h_2(a,b)]^2}\right\}<0, \quad (45)$$

for $\vartheta_1<0$, where $\widehat{H}(\varkappa)=h_1(a,b)+ih_2(a,b)$, $h_1(a,b)$ and $h_2(a,b)$ being functions of a and b, and $\mathfrak{R}\{\widehat{H}(\varkappa)\}=h_1(a,b)>0$; thus, we have Re $\{h(\varkappa)\}>0$, that is, $E\in C_{p,\lambda}(n,\alpha_1;\eta,\zeta)$. □

Theorem 5. *For $E(\varkappa)\in A(p)$, we have*

$$C_{p,\lambda}^*(n,\alpha_1+1;\eta,\zeta)\subset C_{p,\lambda}^*(n,\alpha_1;\eta,\zeta)(\eta\ge 0,\alpha<p;p\in\mathbb{N}). \quad (46)$$

Proof. Using (24), we can prove Theorem 5 as that making in Theorem 3. □

3. Inclusion Results for $F_{p,c}$

The generalized Libera integral operator $F_{p,c}$ (see [12]) is defined by

$$F_{p,c}E(\varkappa) = \frac{c+p}{z^c}\int_0^z t^{c-1}E(t)dt, = \varkappa^p + \sum_{k=p+1}^{\infty}\frac{c+p}{c+k}a_k\varkappa^k (c>-p;p\in\mathbb{N}), \quad (47)$$

which satisfies

$$\varkappa\left(H_{p,\lambda}^n(\alpha_1)F_{p,c}(E)(\varkappa)\right)' = (c+p)H_{p,\lambda}^n(\alpha_1)E(\varkappa) - cH_{p,\lambda}^n(\alpha_1)F_{p,c}(E)(\varkappa). \quad (48)$$

Theorem 6. *Let $c+p>0$ and $E\in S_{p,\lambda}^*(n,\alpha_1;\zeta)$; then, $F_{p,c}(E)(\varkappa)\in S_{p,\lambda}^*(n,\alpha_1;\zeta)$.*

Proof. Let $E\in S_{p,\lambda}^*(n,\alpha_1;\zeta)$ and put

$$\frac{\varkappa\left(H_{p,\lambda}^n(\alpha_1)F_{p,c}(E)(\varkappa)\right)'}{H_{p,\lambda}^n(\alpha_1)F_{p,c}(E)(\varkappa)} = \zeta+(p-\zeta)h(\varkappa), \quad (49)$$

where $h(\varkappa)$ is given by (26). Applying (48) in (49), we have

$$\frac{H_{p,\lambda}^n(\alpha_1)E(\varkappa)}{H_{p,\lambda}^n(\alpha_1)F_{p,c}(E)(\varkappa)} = \frac{1}{c+p}\{c+\zeta+(p-\zeta)h(\varkappa)\}. \quad (50)$$

Differentiating (50), we have

$$\frac{\varkappa\left(H_{p,\lambda}^n(\alpha_1)E(\varkappa)\right)'}{H_{p,\lambda}^n(\alpha_1)E(\varkappa)} = \frac{\varkappa\left(H_{p,\lambda}^n(\alpha_1)F_{p,c}(E)(\varkappa)\right)'}{H_{p,\lambda}^n(\alpha_1)F_{p,c}(E)(\varkappa)} + \frac{(p-\zeta)\varkappa h'(\varkappa)}{c+\zeta+(p-\zeta)h(\varkappa)}, \quad (51)$$

where in view of (49), we have

$$\frac{\varkappa\left(H_{p,\lambda}^n(\alpha_1)E(\varkappa)\right)'}{H_{p,\lambda}^n(\alpha_1)E(\varkappa)} = \zeta+(p-\zeta)h(\varkappa) + \frac{(p-\zeta)\varkappa h'(\varkappa)}{c+\zeta+(p-\zeta)h(\varkappa)}. \quad (52)$$

Let

$$\varphi(\tau,\vartheta) = (p-\zeta)\tau + \frac{(p-\zeta)\vartheta}{c+\zeta+(p-\zeta)\tau}, \quad (53)$$

with $h(\varkappa)=\tau=\tau_1+i\tau_2$, $\varkappa h'(\varkappa)=\vartheta=\vartheta_1+i\vartheta_2$. Then,

(i) $\varphi(\tau,\vartheta)$ is continuous in $D=\mathbb{C}\setminus\{(c+\zeta)/(\zeta-p)\}\times\mathbb{C}$

(ii) $(1,0)\in D$ and $\mathfrak{R}\{\varphi(1,0)\}=p-\zeta$

(iii) $\mathfrak{R}\{\varphi(i\tau_2,\vartheta_1)\}\le 0$ for all $(i\tau_2,\vartheta_1)\in D$ and such that $\vartheta_1\le(1+\tau_2^2)/2$

$$\Re\{\varphi(i\tau_2, \vartheta_1)\} = \Re\left\{\frac{(p-\zeta)\vartheta_1}{c+\zeta+(p-\zeta)i\tau_2}\right\} = \frac{(c+\zeta)(p-\zeta)\vartheta_1}{(c+\zeta)^2+(p-\zeta)^2\tau_2^2} < 0, \tag{54}$$

for $\tau_1 < 0$; therefore, the function $\varphi(\tau, \vartheta)$ satisfies the conditions in Lemma 1, Re $\{h(\varkappa)\} > 0 (\varkappa \in \mathbb{U})$ and $E \in F_{p,c}S^*_{p,\lambda}(n, \alpha_1; \zeta)$. □

Theorem 7. *Let $c+p>0$ and $E \in K^*_{p,\lambda}(n, \alpha_1; \zeta)$; then, $F_{p,c}(E)(\varkappa) \in K^*_{p,\lambda}(n, \alpha_1; \zeta)$.*

Proof. Applying Theorem 6 and (23), we have

$$\begin{aligned} E(\varkappa) \in K^*_{p,\lambda}(n, \alpha_1; \zeta) &\Longleftrightarrow \frac{\varkappa E'}{p} \in S^*_{p,\lambda}(n, \alpha_1; \zeta) \\ &\Longleftrightarrow \frac{\varkappa}{p}\left(F_{p,c}(E)(\varkappa)\right)' \in S^*_{p,\lambda}(n, \alpha_1; \zeta) \\ &\Longleftrightarrow F_{p,c}(E)(\varkappa) \in K^*_{p,\lambda}(n, \alpha_1; \zeta). \end{aligned} \tag{55}$$

□

Theorem 8. *Let $c+p>0$ and $E \in C_{p,\lambda}(n, \alpha_1; \eta, \zeta)$; then, $F_{p,c}(E)(\varkappa) \in C_{p,\lambda}(n, \alpha_1; \eta, \zeta)$.*

Proof. Let $E \in C_{p,\lambda}(n, \alpha_1; \eta, \zeta)$; then, from (21), there exists a function $G(\varkappa) \in S^*_{p,\lambda}(n, \alpha_1; \zeta)$ such that

$$\Re\left\{\frac{\varkappa\left(H^n_{p,\lambda}(\alpha_1)E(\varkappa)\right)'}{H^n_{p,\lambda}(\alpha_1)G(\varkappa)}\right\} > \eta (\varkappa \in \mathbb{U}). \tag{56}$$

Put

$$\frac{\varkappa\left(H^n_{p,\lambda}(\alpha_1)F_{p,c}(E)(\varkappa)\right)'}{H^n_{p,\lambda}(\alpha_1)F_{p,c}(G)(\varkappa)} = \eta + (p-\eta)h(\varkappa), \tag{57}$$

where $h(\varkappa)$ is given by (26). Applying (48) in (57) differentiating the resulting equation with respect to $\varkappa$ and multiplying by $\varkappa$, we have

$$\begin{aligned} (c+p)\varkappa\left(H^n_{p,\lambda}(\alpha_1)(E)(\varkappa)\right)' &= \{\eta+(p-\eta)h(\varkappa)\}\varkappa\left(H^n_{p,\lambda}(\alpha_1)F_{p,c}(G)(\varkappa)\right)' \\ &+ (p-\eta)\varkappa h'(\varkappa)H^n_{p,\lambda}(\alpha_1)F_{p,c}(G)(\varkappa) \\ &+ c\varkappa\left(H^n_{p,\lambda}(\alpha_1)F_{p,c}E(\varkappa)\right)'. \end{aligned} \tag{58}$$

Since $G \in S^*_{p,\lambda}(n, \alpha_1; \zeta)$, then by Theorem 6, we have $F_{p,c}(G)(\varkappa) \in S^*_{p,\lambda}(n, \alpha_1; \zeta)$. Let

$$\frac{\varkappa\left(H^n_{p,\lambda}(\alpha_1)F_{p,c}(G)(\varkappa)\right)'}{H^n_{p,\lambda}(\alpha_1)F_{p,c}(G)(\varkappa)} = \zeta + (p-\zeta)\widehat{H}(\varkappa), \tag{59}$$

where $\Re\{\widehat{H}(\varkappa)\} > 0$. Applying (48) in (59), we have

$$(c+p)\frac{H^n_{p,\lambda}(\alpha_1)(G)(\varkappa)}{H^n_{p,\lambda}(\alpha_1)F_{p,c}(G)(\varkappa)} = c+\zeta+(p-\zeta)\widehat{H}(\varkappa). \tag{60}$$

From (58) and (60), we have

$$\frac{\varkappa\left(H^n_{p,\lambda}(\alpha_1)E(\varkappa)\right)'}{H^n_{p,\lambda}(\alpha_1)G(\varkappa)} - \eta = (p-\eta)h(\varkappa) + \frac{(p-\eta)\varkappa h'(\varkappa)}{c+\zeta+(p-\zeta)\widehat{H}(\varkappa)}. \tag{61}$$

Now, let

$$\varphi(\tau, \vartheta) = (p-\eta)\tau + \frac{(p-\eta)\vartheta}{c+\zeta+(p-\zeta)\widehat{H}(\varkappa)}. \tag{62}$$

It is easy to see that $\varphi(\tau, \vartheta)$ satisfies the conditions (i) and (ii) of Lemma 1 in $D = \mathbb{C} \setminus \{(c+\zeta)/(\zeta-p)\} \times \mathbb{C}$. For (iii), we have

$$\Re\{\varphi(i\tau_2, \vartheta_1)\} = \Re\left\{\frac{(p-\eta)[c+\zeta+(p-\zeta)h_1(a,b)]\vartheta_1}{[c+\zeta+(p-\zeta)h_1(a,b)]^2+[(p-\zeta)h_2(a,b)]^2}\right\} < 0, \tag{63}$$

for $\vartheta_1 < 0$, where $\widehat{H}(\varkappa) = h_1(a,b) + ih_2(a,b)$, $h_1(a,b)$ and $h_2(a,b)$ being functions of a and b, and $\Re\{\widehat{H}(\varkappa)\} = h_1(a,b) > 0$. Thus, we have Re $\{h(\varkappa)\} > 0$, that is, $F_{p,c}E \in C_{p,a}(n, \alpha_1; \eta, \zeta)$. □

Similarly, we can prove the following theorem.

Theorem 9. *Let $c+p>0$ and $E \in C^*_{p,\lambda}(n, \alpha_1; \eta, \zeta)$; then, $F_{p,c}E \in C^*_{p,\lambda}(n, \alpha_1; \eta, \zeta)$.*

Remark 10.

(1) Using (18) instead of (17) in Theorems 2–5, we have new inclusion results

(2) For special values of the parameters in (16), we obtain another new inclusion results for different classes

4. Conclusion

Using the hypergeometric function (see Srivastava and Karlsson [10]) and Hadamard product, we defined an operator for p-valent functions. This operator generalizes many other operators for special values of its parameters and has two recurrence relations and then defined four classes related to starlike, convex, close-to-convex, and quasi-to-convex p-valent functions. We used Miller and Mocanu lemma [11] for second differential inequalities to obtain inclusion relations for these classes and also for the generalized Libera integral operator.

5. Future Studies

The authors suggest to obtain the inclusion results for the classes using the following lemma according to Jack [13] instead of Lemma 1.

Jack's lemma [13] state that, if $\omega(\varkappa)$ is analytic function in $\mathbb{U}$, with $\omega(0) = 0$, $|\omega(\varkappa)|$ attains its maximum value on the circle $|\varkappa| = r < 1$ at a point $\varkappa_0 \in \mathbb{U}$ and $\xi \geq 1$; then,

$$\varkappa_0 \omega'(\varkappa_0) = \xi \omega \varkappa_0. \quad (64)$$

Data Availability

During the current study, the data are derived arithmetically.

Conflicts of Interest

The authors do not have any competing interests.

Authors' Contributions

The authors approve and read the article.

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