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Research Article

On Certain Analogues of Noor Integral Operators Associated with Fractional Integrals

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In this paper, we employ a q-Noor integral operator to perform a q-analogue of certain fractional integral operator defined on an open unit disc. Then, we make use of the Hadamard convolution product to discuss several related results. Also, we derive a class of convex functions by utilizing the q-fractional integral operator and apply the inspired presented theory of the differential subordination, to geometrically explore the most popular differential subordination properties of the aforementioned operator. In addition, we discuss an exciting inclusion for the given convex class of functions. Over and above, we investigate the q-fractional integral operator and obtain some applications for the differential subordination.

1. Introduction

The theory of quantum calculus and its applications has been applied in several branches of mathematics, engineering sciences, and physics. Hence, many researchers have used q-calculus to study discrete dynamical systems, discrete stochastic processes, q-deformed super algebras, q-transform analysis, and so on. In literature, differentiation and integration of function are formulated by using the quantum theory of calculus (or q-calculus) [1–3]. The Jackson q-calculus is also involved in various areas of science including fractional q-calculus, optimal control, nonlinear integrodifferential equations, q-difference, and q-integral equations [4–7]. Ismail et al. [8] are the first to employ the theory of q-calculus for investigating the geometric function theory. Srivastava in [9] points out some comprehensive reviews and applications in the geometric function theory of q-calculus and discusses many important applications of starlike functions. Arif et al. in [10, 11] derive some properties of multivalent functions by using q-calculus. Authors in [12] discuss q-calculus and the Salagean operator to obtain differential subordination results. Aouf and Mostafa [13] used differential subordination to define a new subclass of analytic functions with q-analogue fractional differential operator. Mahmood and Sokół [14] apply properties of the Ruscheweyh q-differential operator for a subclass of analytic functions and study some of its applications. Kanas and Raducanu [15] investigate q-analogues of the Ruscheweyh operator by using the Hadamard product; see, for some details, [14, 16] and [11, 17].

Let f be a real or complex value and D be unit disc $D=\{z,|z|<1\}$, $0<q<1$. Then, the q-difference operator is defined by [1]

$$D_q f(z)=\frac{f(z)-f(qz)}{z-qz}, z\in D. \tag{1}$$

The q-differentiation rules may be wrote as

$$D_q[f(z)g(z)]=g(z)D_q f(z)+f(qz)D_q g(z), \tag{2}$$

$$D_q\frac{f(z)}{g(z)}=\frac{g(z)D_q f(z)-f(z)D_q g(z)}{g(z)g(qz)}. \tag{3}$$

Let $\mathcal{A}[a, n]$ consist of analytic functions in the unit disc $D = \{z, |z| < 1\}$ of the form $f(z) = a + z^n + \sum_{k=2}^{\infty} a_{k+n} z^{k+n}$. For $a = 0$ and $n = 1$, we use $\mathcal{A} = \mathcal{A}[0, 1]$. Therefore, the function $f \in \mathcal{A}$ has the expansion of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{4}$$

Note that every function $f \in \mathcal{A}$ is normalized by $f(0) = 0$ and $f'(0) = 1$. The class of univalent functions in $\mathcal{A}$ is denoted by S. In particular, S^* is the class of starlike functions, CV is the class of convex functions, and K is the class of close-to-convex function [18, 19]. Recently, authors in [20] used the convolution to introduce three new subclasses of starlike functions, convex functions, and close-to-convex functions with the novel Borel distribution operator.

Let $f \in \mathcal{A}$ be given by (4) and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$. Then, the convolution of f and g is denoted by $f * g$, which is a function in $\mathcal{A}$ given by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k, z \in D. \tag{5}$$

We say that the function f is subordinate to g in D and write $f(z) \prec g(z)$ for $z \in D$ if there exists a Schwartz analytic function w in D such that $w(0) = 0$ and $|w(z)| < 1 (z \in D)$ and $f(z) = g(w(z))$ [21]. In particular, if the function g is univalent in D, then $f(z) \prec g(z)$ if and only if

$$\begin{aligned} f(0) &= g(0), \\ f(D) &\subseteq g(D). \end{aligned} \tag{6}$$

Ma and Minda [22] studied the class of starlike and convex functions by using the principle of differential subordination. Those differential subordinations provide interesting results when they are used to study new sets of univalent functions [23–25].

Making use of equations (4) and (1), we can easily obtain that

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1}, k \in \mathbb{N}, z \in D, \tag{7}$$

where

$$[k]_q = \frac{1 - q^k}{1 - q} = 1 + \sum_{i=1}^{k-1} q^i, [0]_q = 0. \tag{8}$$

It is clear that $[1]_q = 1$. For $k \in \mathbb{Z}^+$, the q-factorial is given by [3]

$$[k]_q! = \begin{cases} 1, & k = 0, \\ [1]_q [2]_q \cdots [k]_q, & k \in \mathbb{N}. \end{cases} \tag{9}$$

In addition, with $t > 0$, the q-Pochhammer symbol has the form [3]

$$[t]_{q,k} = \left([t]_q\right)_k = \begin{cases} 1, & k = 0, \\ [t]_q [t+1]_q \cdots [t+k-1]_q, & k \in \mathbb{N}. \end{cases} \tag{10}$$

Note that $[t]_{q,k} = [t]_k$ when $q \longrightarrow 1^-$.

For $t > 0$, the q-analogue of the gamma function is presented as

$$\begin{aligned} \Gamma_q(t+1) &= [t]_q \Gamma_q(t), \\ \Gamma_q(1) &= 1. \end{aligned} \tag{11}$$

Now, we define a q-analogue Noor integral operator $I_q^n : \mathcal{A} \longrightarrow \mathcal{A}$ as follows:

$$I_q^n f(z) = F_q^n(z) * f(z), \tag{12}$$

where F_q^n is defined by the relation

$$\begin{aligned} F_q^n(z) * h(z) &= \frac{z}{(1 - qz)(1 - z)}, (z \in D), \\ h(z) &= z + \sum_{k=2}^{\infty} \frac{[n+1]_{q,k-1}}{[k-1]_q} z^k, (z \in D). \end{aligned} \tag{13}$$

Hence, f is of the form (4). Therefore,

$$I_q^n f(z) = z + \sum_{k=2}^{\infty} \frac{[k-1]_{q,2}}{[n+1]_{q,k-1}} a_k z^k, (z \in D). \tag{14}$$

This, by taking $q \longrightarrow 1^-$, shows that the operator I_q^n defined in (14) reduces to the familiar Noor integral operator of [26, 27].

Now, by using the idea of Cho and Aouf [28], we introduce the q-fractional Riemann-Liouville integral of order λ $(\lambda > 0)$ as follows.

Definition 1 (see [28]). The q-fractional Riemann-Liouville integral of order λ $(\lambda > 0)$ is defined for a function f by

$$D_{q,z}^{-\lambda} f(z) = \frac{1}{\Gamma_q(\lambda)} \int_0^z \frac{f(t)}{(1 - qt)^{1-\lambda}} d_q t, z \in D, \tag{15}$$

where f is an analytic function in D.

Many other useful studies are introduced in the field of analytic functions including the fractional integral operator and its applications (see [29–31]).

In this paper, we introduce the q-fractional integral operator by using the q-Noor integral operator. This operator is based on the q-fractional Riemann-Liouville integral of order λ $(\lambda > 0)$. Also, by using a newly defined q-fractional integral operator, we introduce a subclass $S_n^q(\lambda, \delta)$ of analytic functions and prove that $S_n^q(\lambda, \delta)$ is a convex set.

Furthermore, several exciting subordination results of the q-fractional integral operator are obtained.

2. Preliminary Lemmas

Lemma 2 ([19], Theorem 8.9, p. 254). *Let $f \in A$ and h be a convex function. If $f \prec h$, then*

$$f * g \prec h * g, \tag{16}$$

for all $g \in K$.

Lemma 3 ([32], Theorem 10, p. 259). *Let f_i, $i = 1, 2$, be the analytic function in D from the following form:*

$$f_i(z) = 1 + b_{i1}z + b_{i2}z^2 + \cdots, (i = 1, 2, z \in D). \tag{17}$$

If $\mathrm{Re}\, f_i(z) > \beta_i$, $0 \le \beta_i < 1$, then the function $f_1 * f_2$ is an analytic function that satisfies the inequality

$$\mathrm{Re}\,(f_1 * f_2) > 1 - 2(1 - \beta_1)(1 - \beta_2). \tag{18}$$

Lemma 4 ([33], Lemma 2, p. 2, [34]). *Let f be the analytic function in D from the following form:*

$$f(z) = 1 + b_1 z + b_2 z^2 + \cdots, (z \in D). \tag{19}$$

If $\mathrm{Re}\, f(z) > \beta$, $0 \le \beta < 1$, then

$$\mathrm{Re}\,(f(z)) > 2\beta - 1 + \frac{2(1-\beta)}{1+|z|}, (z \in D). \tag{20}$$

3. Definition and Coefficient Bounds

We introduce a q-fractional integral of the operator $I_q^n f(z)$.

Definition 5. Let $0 < q \le 1$, $\lambda \ge 0$, and $n \in \mathbb{N}$. The q-fractional integral of the operator $I_q^n f$ is defined by the following:

$$\begin{aligned} D_{q,z}^{-\lambda} I_q^n f(z) &= \frac{1}{\Gamma_q(\lambda)} \int_0^z \frac{I_q^n f(t)}{(1-qt)^{1-\lambda}} d_q t \\ &= \frac{1}{\Gamma_q(\lambda)} \int_0^z \frac{t}{(1-qt)^{1-\lambda}} d_q t + \frac{1}{\Gamma_q(\lambda)} \sum_{k=2}^{\infty} \frac{[k-1]_{q,2}}{[n+1]_{q,k-1}} \\ &\quad \cdot a_k \int_0^z \frac{t^k}{(1-qt)^{1-\lambda}} d_q t = \frac{1}{\Gamma_q(\lambda)} \frac{\Gamma_q(2)}{\Gamma_q(2+\lambda)} z^{1+\lambda} \\ &\quad + \frac{1}{\Gamma_q(\lambda)} \sum_{k=2}^{\infty} \frac{[k-1]_{q,2}}{[n+1]_{q,k-1}} \frac{\Gamma_q(k+1)}{\Gamma_q(k+\lambda+1)} a_k z^{k+\lambda}. \end{aligned} \tag{21}$$

We note that $D_{q,z}^{-\lambda} I_q^n f(z) \in \mathscr{A}[0, \lambda+1]$ and $D_{q,z}^{0} I_q^n f(z) = I_q^n f(z)$.

Now, by taking $q \longrightarrow 1^-$, we have

$$D_z^{-\lambda} I^n f(z) = \frac{1}{(\lambda)_2 \Gamma^2(\lambda)} z^{1+\lambda} + \frac{1}{\Gamma^2(\lambda)} \sum_{k=2}^{\infty} \frac{[k-1]_2}{[n+1]_{k-1}} \frac{k!}{(\lambda)_{k+1}} a_k z^{k+\lambda}. \tag{22}$$

Now, we study the subclass of analytic function by using the new operator.

Definition 6. Let the function $f \in \mathscr{A}$, $0 < q < 1$, $\delta \in [0, 1)$, and $\lambda \ge 0$. We define the subclass $S_n^q(\lambda, \delta)$ of functions f which satisfy the inequality

$$\mathrm{Re}\left\{ D_q\left(D_{q,z}^{-\lambda} I_q^n f(z) \right) \right\} > \delta. \tag{23}$$

By allowing $q \longrightarrow 1^-$ in Definition 6, the class $S_n^q(\lambda, \delta)$ is denoted by $S_n(\lambda, \delta)$.

Example 1. In this example, we show that the set $S_n^q(0, \delta)$ is nonempty.

Since $\mathrm{Re}\,((1 + (1-2\delta)z)/(1-z)) > 0$, we can find a function $f(z)$ such that

$$D_q\left(D_{q,z}^{0} I_q^n f(z) \right) = \frac{1 + (1-2\delta)z}{1-z}. \tag{24}$$

By using assertion (21) for $\lambda = 0$, we have

$$D_{q,z}^{0} I_q^n f(z) = z + \sum_{k=2}^{\infty} \frac{[k-1]_{q,2}}{[n+1]_{q,k-1}} a_k z^k. \tag{25}$$

So, we get

$$D_q\left(D_{q,z}^{0} I_q^n f(z) \right) = 1 + \sum_{k=1}^{\infty} \frac{[k]_{q,2}}{[n+1]_{q,k}} [k+1]_q a_{k+1} z^k. \tag{26}$$

From (24) and (26) and using the series expansion of $(1 + (1-2\delta)z)/(1-z)$, we obtain that

$$\frac{[k]_{q,2}}{[n+1]_{q,k}} [k+1]_q a_{k+1} = 2(1-\delta), k = 1, 2, 3, \cdots. \tag{27}$$

This, indeed, implies that

$$a_{k+1} = \frac{[n+1]_{q,k}}{2(1-\delta)[k]_{q,2}[k+1]_q}, k = 1, 2, 3, \cdots. \tag{28}$$

Therefore, we obtain that the function

$$f(z) = z + \sum_{k=2}^{\infty} \frac{[n+1]_{q,k-1}}{2(1-\delta)[k-1]_{q,2}[k]_q} z^k, \tag{29}$$

which is a member of the set $S_n^q(0, \delta)$.

Theorem 7. *Let $0 < q < 1$, $\delta \in [0, 1)$, and $\lambda > 0$. Then, the class $S_n^q(\lambda, \delta)$ is convex.*

Proof. Consider two functions f_1 and f_2 from $S_n^q(\lambda, \delta)$ which are given in the form

$$f_1(z) = z + \sum_{k=2}^{\infty} a_{1k} z^k,$$
$$f_2(z) = z + \sum_{k=2}^{\infty} a_{2k} z^k. \tag{30}$$

It is sufficient to show that the function

$$h(z) = \alpha_1 f_1 + \alpha_2 f_2 \tag{31}$$

with $\alpha_1, \alpha_2 > 0, \alpha_1 + \alpha_2 = 1$ belongs to $S_n^q(\lambda, \delta)$. Since $h(z) = z + \sum_{k=2}^{\infty}(\alpha_1 a_{1k} + \alpha_2 a_{2k}) z^k$, we get

$$D_{q,z}^{-\lambda} I_q^n h(z) = \frac{1}{\Gamma_q(\lambda)} \frac{\Gamma_q(2)}{\Gamma_q(2+\lambda)} z^{1+\lambda} + \frac{1}{\Gamma_q(\lambda)} \sum_{k=2}^{\infty} \frac{[k-1]_{q,2}}{[n+1]_{q,k-1}} \frac{\Gamma_q(k+1)}{\Gamma_q(k+\lambda+1)} (\alpha_1 a_{1k} + \alpha_2 a_{2k}) z^{k+\lambda}. \tag{32}$$

By using the q-difference operator, we obtain

$$D_q\left(D_{q,z}^{-\lambda} I_q^n h(z)\right) = \frac{1}{\Gamma_q(\lambda)} \frac{\Gamma_q(2)}{\Gamma_q(2+\lambda)} [1+\lambda]_q z^{\lambda} + \frac{1}{\Gamma_q(\lambda)} \sum_{k=2}^{\infty} \frac{[k-1]_{q,2}}{[n+1]_{q,k-1}} \frac{\Gamma_q(k+1)}{\Gamma_q(k+\lambda+1)} \cdot [k+\lambda]_q (\alpha_1 a_{1k} + \alpha_2 a_{2k}) z^{k+\lambda-1}. \tag{33}$$

So, we get

$$\begin{aligned} &\mathrm{Re}\left\{D_q\left(D_{q,z}^{-\lambda} I_q^n h(z)\right)\right\} \\ &= \mathrm{Re}\left\{\frac{1}{\Gamma_q(\lambda)} \frac{\Gamma_q(2)}{\Gamma_q(2+\lambda)} [1+\lambda]_q z^{\lambda}\right\} \\ &\quad + \alpha_1 \,\mathrm{Re}\left\{\frac{1}{\Gamma_q(\lambda)} \sum_{k=2}^{\infty} \frac{[k-1]_{q,2}}{[n+1]_{q,k-1}} \frac{\Gamma_q(k+1)}{\Gamma_q(k+\lambda+1)} [k+\lambda]_q a_{1k} z^{k+\lambda-1}\right\} \\ &\quad + \alpha_2 \,\mathrm{Re}\left\{\frac{1}{\Gamma_q(\lambda)} \sum_{k=2}^{\infty} \frac{[k-1]_{q,2}}{[n+1]_{q,k-1}} \frac{\Gamma_q(k+1)}{\Gamma_q(k+\lambda+1)} [k+\lambda]_q a_{2k} z^{k+\lambda-1}\right\}, \end{aligned} \tag{34}$$

which implies

$$\begin{aligned} \mathrm{Re}\left\{D_q\left(D_{q,z}^{-\lambda} I_q^n h(z)\right)\right\} &= \alpha_1 \,\mathrm{Re}\left\{D_q\left(D_{q,z}^{-\lambda} I_q^n f_1(z)\right)\right\} \\ &\quad + \alpha_2 \,\mathrm{Re}\left\{D_q\left(D_{q,z}^{-\lambda} I_q^n f_2(z)\right)\right\} \\ &\geq \alpha_1 \delta + \alpha_2 \delta = \delta. \end{aligned} \tag{35}$$

Thus, the desired results are obtained. □

Taking $q \longrightarrow 1^-$ into Theorem 7 leads to the following corollary.

Corollary 8. *Let $\delta \in [0, 1)$ and $\lambda > 0$. Then, the class $S_n(\lambda, \delta)$ is convex.*

Lemma 9. *Let $0 < q < 1$, $h, s \in S_q^n(\lambda, \delta)$, $h(z) = s(z) + (q^c/[c]_q) z D_q s(z)$, and $c \in \mathbb{N}$ such that*

$$\Lambda_q^c(z) = z + \sum_{k=2}^{\infty} \frac{q^c + [c]_q}{q^c [k]_q + [c]_q} z^k \tag{36}$$

be a convex function. If $r, s \in \mathscr{A}$ with $r(0) = s(0) = 0$ and

$$r(z) + \frac{q^c z}{[c]_q} D_q r(z) \prec h(z), \tag{37}$$

then

$$s(z) = \frac{[c]_q}{z^c} \int_0^z t^{c-1} h(t) d_q t, \tag{38}$$

$$r(z) \prec s(z). \tag{39}$$

Proof. The differential equation

$$h(z) = s(z) + \frac{q^c}{[c]_q} z D_q s(z) \tag{40}$$

has a unique solution (38). Since $r, s \in \mathscr{A}$, then, in view of (4), we have

$$r(z) = z + \sum_{k=2}^{\infty} r_k z^k,$$
$$s(z) = z + \sum_{k=2}^{\infty} s_k z^k. \tag{41}$$

So, the differential subordination (37) can be rewritten in the form

$$\sum_{k=1}^{\infty} \left(\frac{q^c [k]_q}{[c]_q} + 1\right) r_k z^k \prec \sum_{k=1}^{\infty} \left(\frac{q^c [k]_q}{[c]_q} + 1\right) s_k z^k. \tag{42}$$

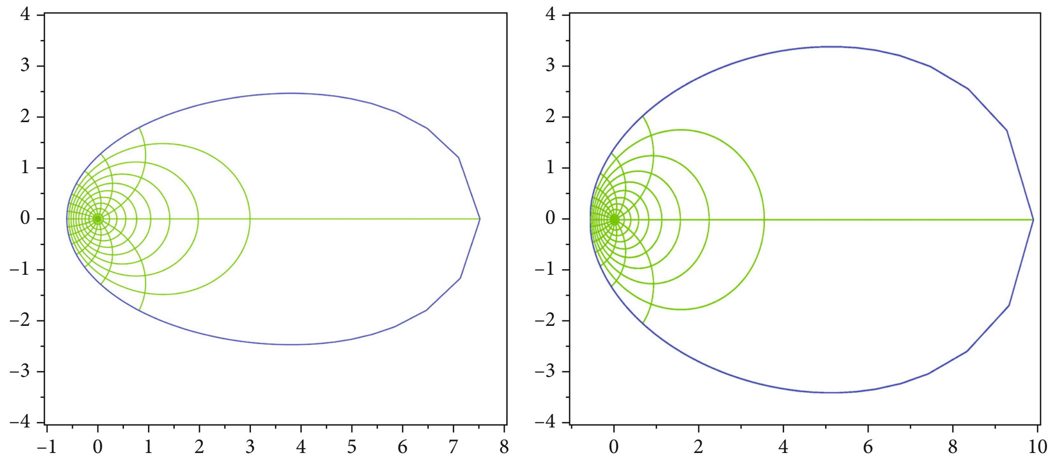

Figure 1: $\Lambda^1(z) = (-2/z)(z + \ln(1-z))$ and $\Lambda^2(z) = (-3/z^2)(z + (z^2/2) + \ln(1-z))$.

Applying Lemma 2 gives

$$\Lambda_q^c(z) * \sum_{k=1}^{\infty} \left(\frac{q^c [k]_q}{[c]_q} + 1 \right) r_k z^k \prec \Lambda_q^c(z) * \sum_{k=1}^{\infty} \left(\frac{q^c [k]_q}{[c]_q} + 1 \right) s_k z^k. \tag{43}$$

This proves (39). The proof of Lemma 9 is completed. □

Note that when $q \longrightarrow 1$, then $\Lambda_q^c(z) \longrightarrow \Lambda^c(z)$. For example, we have

$$\begin{aligned} \Lambda^1(z) &= \frac{-2}{z}(z + \ln(1-z)), \\ \Lambda^2(z) &= \frac{-3}{z^2}\left(z + \frac{z^2}{2} + \ln(1-z)\right), \\ \Lambda^3(z) &= \frac{-4}{z^3}\left(z + \frac{z^2}{2} + \frac{z^3}{3} + \ln(1-z)\right), \\ \Lambda^4(z) &= \frac{-5}{z^4}\left(z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \ln(1-z)\right). \end{aligned} \tag{44}$$

Remark 10. Plots of the suggested functions $\Lambda^1(z)$, $\Lambda^2(z)$, $\Lambda^3(z)$, and $\Lambda^4(z)$ in the unit disc D are illustrated in Figures 1 and 2. These plots show that these suggested functions are convex in the unit disc D.

An analytic function r in (37) is said to be a solution of the differential subordination. The analytic function φ is a dominant of the solution of the differential subordination (37), if $r \prec \varphi$ for all r satisfying (37).

A dominant $\widehat{\varphi}$ is said to be the best dominant of (37) if it satisfies $\widehat{\varphi} \prec \varphi$ for all dominants φ of (37). The best dominant is unique up to the rotation of D.

Theorem 11. *Let $0 < q < 1$, $r, h \in S_n^q(\lambda, \delta)$ with $h(z) = s(z) + (q^c z/[c]_q) D_q s(z)$, and $c \in \mathbb{N}$ such that $\Lambda_q^c(z)$ defined in (36) is a convex function. If $z^c F(z) = [c]_q \int_0^z t^{c-1} f(t) d_q t$, $z \in D$, the following differential subordination*

$$D_{q,z}^{-\lambda} I_q^n f(z) \prec h(z) \tag{45}$$

implies

$$D_{q,z}^{-\lambda} I_q^n F(z) \prec s(z). \tag{46}$$

Proof. In view of the definition F, we have $D_q(z^c F(z)) = [c]_q z^{c-1} h(z)$, $z \in D$. Since the q-derivative rule in (2) holds, we get

$$F(z) + \frac{q^c}{[c]_q} z D_q F(z) = h(z), z \in D. \tag{47}$$

Now, by using the $D_{q,z}^{-\lambda} I_q^n$, we for $z \in D$ can obtain the differential equation as follows:

$$D_{q,z}^{-\lambda} I_q^n (F(z)) + \frac{q^c}{[c]_q} z D_q \left(D_{q,z}^{-\lambda} I_q^n (F(z)) \right) = D_{q,z}^{-\lambda} I_q^n (f(z)), z \in D. \tag{48}$$

From (45) and (48), we have

$$D_{q,z}^{-\lambda} I_q^n (F(z)) + \frac{q^c}{[c]_q} z D_q \left(D_{q,z}^{-\lambda} I_q^n (F(z)) \right) \prec h(z), z \in D. \tag{49}$$

By using the notation

$$r(z) = D_{q,z}^{-\lambda} I_q^n F(z), z \in D, \tag{50}$$

we obtain $r(z) \in \mathcal{A}$. This implies that

$$r(z) + \frac{q^c}{[c]_q} z D_q r(z) \prec s(z) + \frac{q^c}{[c]_q} z D_q s(z), z \in D. \tag{51}$$

Now, applying Lemma 9, we obtain

$$r(z) \prec s(z), z \in D. \tag{52}$$

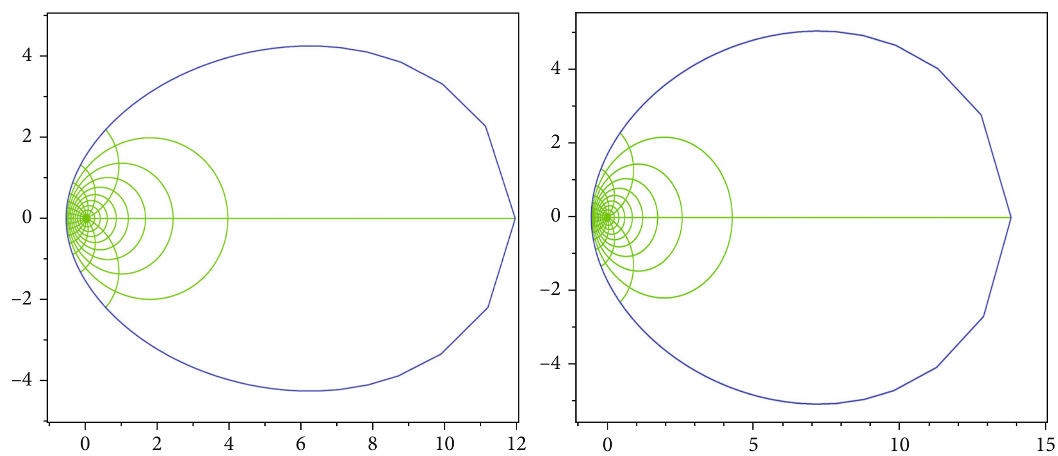

FIGURE 2: $\Lambda^3(z) = (-4/z^3)(z + (z^2/2) + (z^3/3) + \ln(1-z))$ and $\Lambda^4(z) = (-5/z^4)(z + (z^2/2) + (z^3/3) + (z^4/4) + \ln(1-z))$.

This means that the differential subordination (46) is established. This completes the proof of Theorem 11. □

By using Remark 10, we can obtain the following corollary.

Corollary 12. *Let $h \in S_n(\lambda, \delta)$ with $h(z) = s(z) + (z/c)s'(z)$. If $z^c F(z) = c\int_0^z t^{c-1} f(t)dt$, $z \in D$, the following differential subordination*

$$D_z^{-\lambda} I^n f(z) \prec h(z) \tag{53}$$

implies

$$D_z^{-\lambda *} I^n F(z) \prec s(z), \tag{54}$$

where $D_z^{-\lambda} I^n f$ is given by (22) and $c = 1, 2, 3, 4$.

In the next theorem, we derive an exciting inclusion for the class $S_n^q(\lambda, \delta)$.

Theorem 13. *Let $0 < q < 1, z \in D, \delta \in [0,1)$ and $h(z) = (1 + (2\delta - 1)z)/(1 + qz)$, $\delta \in [0,1)$, and $c \in \mathbb{N}$ such that $\Lambda_q^c(z)$, defined in (36), is a convex function. If $s(z) = I_q^c r(z) = ([c]_q/z^c)\int_0^z t^{c-1} h(t) d_q t$ is a convex function, then we have the following inclusion:*

$$I_q^c[S_n^q(\lambda, \delta)] \subseteq S_n^q(\lambda, \delta^*), \tag{55}$$

where $\delta^* = 1 + (2\delta - 1 - q)[c]_q \int_0^1 (t^c/(1 + qt)) d_q t$.

Proof. In view of Theorem 11, we obtain the following:

$$r(z) + \frac{q^c}{[c]_q} z D_q r(z) \prec h(z), \tag{56}$$

where $r(z) = D_{q,z}^{-\lambda} F(z)$, $z \in D$.

Now, by applying Lemma 9, we obtain $r(z) \prec s(z)$. Indeed,

$$D_{q,z}^{-\lambda} F(z) \prec s(z), \tag{57}$$

where

$$\begin{aligned} s(z) = \frac{[c]_q}{z^c}\int_0^z t^{c-1}\frac{1 + (2\delta - 1)z}{1 + qz} d_q t = 1 \\ + \frac{[c]_q(2\delta - 1 - q)}{z^c}\int_0^z \frac{t^c}{1 + qt} d_q t, z \in D. \end{aligned} \tag{58}$$

From the convexity of $s(z)$ and using the fact that $s(D)$ is symmetric with respect to the real axis, we have

$$\begin{aligned} \operatorname{Re}\left(D_{q,z}^{-\lambda} F(z)\right) \geq \min_{|z|=1}(z) = \operatorname{Re}(s(1)) \\ = 1 + [c]_q(2\delta - 1 - q)\int_0^1 \frac{t^c}{1 + qt} d_q t = \delta^*, z \in D. \end{aligned} \tag{59}$$

This completes the proof of Theorem 13. □

By using Remark 10, we can obtain the following corollaries.

Corollary 14. *Let $\delta \in [0,1)$ and $h(z) = (1 + (2\delta - 1)z)/(1 + z)$, $\delta \in [0,1)$. If $If(z) = (1/z)\int_0^z f(t)dt$, then we have the following inclusion:*

$$I[S_n(\lambda, \delta)] \subseteq S_n(\lambda, \delta^*), \tag{60}$$

where $\delta^* = 1 - 2(1 - \delta)(1 - \ln 2)$.

Corollary 15. *Let $\delta \in [0,1)$ and $h(z) = (1 + (2\delta - 1)z)/(1 + z)$, $\delta \in [0,1)$. If $I^2 f(z) = (2/z^2)\int_0^z t f(t)dt$, then we have the* *following inclusion:*

$$I^2[S_n(\lambda,\delta)] \subseteq S_n(\lambda,\delta^*), \tag{61}$$

where $\delta^* = 1 + 4(1-\delta)((1/2) - \ln 2)$.

Corollary 16. *Let* $\delta \in [0,1)$ *and* $h(z) = (1 + (2\delta - 1)z)/(1+z)$, $\delta \in [0,1)$. *If* $I^3 f(z) = (3/z^3)\int_0^z t^2 f(t)dt$, *then we have the following inclusion:*

$$I^3[S_n(\lambda,\delta)] \subseteq S_n(\lambda,\delta^*), \tag{62}$$

where $\delta^* = 1 - 8(1-\delta)((5/6) - \ln 2)$.

Corollary 17. *Let* $\delta \in [0,1)$ *and* $h(z) = (1 + (2\delta - 1)z)/(1+z)$, $\delta \in [0,1)$. *If* $I^4 f(z) = (4/z^4)\int_0^z t^3 f(t)dt$, *then we have the following inclusion:*

$$I^4[S_n(\lambda,\delta)] \subseteq S_n(\lambda,\delta^*), \tag{63}$$

where $\delta^* = 1 + 8(1-\delta)((7/12) - \ln 2)$.

Theorem 18. *Let* $0 < q < 1$, $r(z), h(z) \in S_n^q(\lambda,\delta)$, $z \in D$, *with* $h(z) = s(z) + (q^c z/[c]_q)D_q s(z)$, *and* $c \in \mathbb{N}$ *such that* $\Lambda_q^c(z)$, *defined in (36), is a convex function. If* $f \in \mathscr{A}$ *satisfies*

$$z^{1-c} D_q\left(D_{q,z}^{-\lambda} I_q^n f(z)\right) \prec r(z), \tag{64}$$

then we have the following result

$$\frac{[c]_q D_{q,z}^{-\lambda} I_q^n f(z)}{z^c} \prec s(z). \tag{65}$$

Proof. Denoted by $r(z) = ([c]_q D_{q,z}^{-\lambda} I_q^n f(z))/z^c$, we obtain

$$\frac{z^c}{[c]_q} r(z) = D_{q,z}^{-\lambda} I_q^n f(z). \tag{66}$$

By applying q-derivative and using rule (2), we derive

$$r(z) + \frac{q^c z}{[c]_q} D_q r(z) = z^{1-c} D_q\left[D_{q,z}^{-\lambda} I_q^n f(z)\right]. \tag{67}$$

By applying Lemma 9, we get

$$r(z) \prec s(z). \tag{68}$$

This implies the differential subordination (70). This completes the proof of Theorem 18. □

By using Remark 10, we can obtain the following corollary.

Corollary 19. *Let* $r(z), h(z) \in S_n(\lambda,\delta)$, $z \in D$, *with* $h(z) = s(z) + (z/c)s'(z)$. *If* $f \in \mathscr{A}$ *satisfies*

$$\frac{\left(D_z^{-\lambda} I^n f(z)\right)}{z^{c-1}} \prec r(z), \tag{69}$$

then we have the following result:

$$\frac{c D_z^{-\lambda} I^n f(z)}{z^c} \prec s(z), \tag{70}$$

where $c = 1, 2, 3, 4$.

4. Applications

In this section, we obtain some interesting applications involving the q-analogue differential subordination.

Theorem 20. *Let* $0 < q < 1$, $-1 \le B < A \le 1$, *and* $c \in \mathbb{N}$, *such that* $\Lambda_q^c(z)$, *defined in (36), is a convex function. If* $f \in \mathscr{A}$ *satisfies*

$$D_q\left[D_{q,z}^{1-c} I_q^n f(z)\right] \prec \frac{1}{\Gamma_q(c-1)} \frac{\Gamma_q(2)}{\Gamma_q(c)} z^{c-1} \frac{1+Az}{1+Bz}, \tag{71}$$

then we have the following result:

$$\operatorname{Re}\left\{\left(\frac{[c]_q D_{q,z}^{1-c} I_q^n f(z)}{z^c}\right)^{1/n}\right\} > \left(\frac{1}{\Gamma_q(c-1)} \frac{\Gamma_q(2)}{\Gamma_q(c)} \int_0^1 u^{c-1} \frac{1-Au}{1-Bu} d_q u\right)^{1/n}, n \ge 1. \tag{72}$$

Proof. Let $r(z) = ([c]_q D_{q,z}^{-\lambda} I_q^n f(z))/z^c, z \in D$. By putting $\lambda = c - 1$ in the assertions (67) and using (82), we obtain

$$r(z) + \frac{q^c z}{[c]_q} D_q r(z) \prec \frac{1}{\Gamma_q(c-1)} \frac{\Gamma_q(2)}{\Gamma_q(c)} \frac{1+Az}{1+Bz}. \tag{73}$$

By applying Theorem 18 and Lemma 9, we for $z \in D$ derive

$$r(z) \prec s(z) = \frac{1}{\Gamma_q(c-1)} \frac{\Gamma_q(2)}{\Gamma_q(c)} \frac{[c]_q}{z^c} \int_0^z t^{c-1} \frac{1+At}{1+Bt} d_q t. \tag{74}$$

By using the principle of differential subordination, we have

$$\frac{[c]_q D_{q,z}^{1-c} I_q^n f(z)}{z^c} = \frac{1}{\Gamma_q(c-1)} \frac{\Gamma_q(2)}{\Gamma_q(c)} [c]_q \int_0^1 u^{c-1} \frac{1+Auw(z)}{1+Buw(z)} d_q u. \tag{75}$$

Taking into account that $-1 \le B < A \le 1$, we write

$$\operatorname{Re}\left\{\frac{D_{q,z}^{1-c} I_q^n f(z)}{z^c}\right\} > \frac{1}{\Gamma_q(c-1)} \frac{\Gamma_q(2)}{\Gamma_q(c)} \int_0^1 u^{c-1} \frac{1-Au}{1-Bu} d_q u. \tag{76}$$

Since Re $(w^{1/n}) \geq (\mathrm{Re}\{w\})^{1/n}$, for Re $\{w\} > 0$ and $n \geq 1$, we establish inequality (83).

To prove the sharpness of (83), we, for $f \in \mathscr{A}, z \in D$, define

$$\frac{D_{q,z}^{1-c} I_q^n f(z)}{z^c} = \frac{1}{\Gamma_q(c-1)} \frac{\Gamma_q(2)}{\Gamma_q(c)} \int_0^1 u^{c-1} \frac{1+Auz}{1+Buz} d_q u. \quad (77)$$

For this function, we have

$$D_q\left[D_{q,z}^{1-c} I_q^n f(z)\right] = \frac{1}{\Gamma_q(c-1)} \frac{\Gamma_q(2)}{\Gamma_q(c)} z^{c-1} \frac{1+Az}{1+Bz}. \quad (78)$$

When $z \longrightarrow -1$, we get

$$\frac{D_{q,z}^{1-c} I_q^n f(z)}{z^c} \longrightarrow \frac{1}{\Gamma_q(c-1)} \frac{\Gamma_q(2)}{\Gamma_q(c)} \int_0^1 u^{c-1} \frac{1-Au}{1-Bu} d_q u. \quad (79)$$

This completes the proof of Theorem 20. □

We can obtain the following corollaries by using Remark 10.

Corollary 21. *Let $-1 \leq B < A \leq 1$. If $f \in \mathscr{A}$ satisfies*

$$\left(D_z^{-1} I^n f(z)\right) \prec z \frac{1+Az}{1+Bz}, \quad (80)$$

then we have the following result:

$$\mathrm{Re}\left\{\left(\frac{2D_z^{-1} I^n f(z)}{z^2}\right)^{1/n}\right\} > \left(\int_0^1 u \frac{1-Au}{1-Bu} du\right)^{1/n}, n \geq 1. \quad (81)$$

Corollary 22. *Let $-1 \leq B < A \leq 1$. If $f \in \mathscr{A}$ satisfies*

$$\left(D_z^{-3} I^n f(z)\right) \prec \frac{1}{12} z^3 \frac{1+Az}{1+Bz}, \quad (82)$$

then we have the following result:

$$\mathrm{Re}\left\{\left(\frac{4D_z^{-3} I^n f(z)}{z^4}\right)^{1/n}\right\} > \left(\frac{1}{12}\int_0^1 u^3 \frac{1-Au}{1-Bu} du\right)^{1/n}, n \geq 1. \quad (83)$$

Theorem 23. *Let $0 < q < 1$, $-1 \leq B < A \leq 1$, and $c \in \mathbb{N}$ such that $\Lambda_q^c(z), z \in D$, defined in (36), is a convex function. If $f \in \mathscr{A}$ satisfies*

$$D_q\left[D_{q,z}^{1-c} I_q^n f(z)\right] \prec \frac{1}{\Gamma_q(c-1)} \frac{\Gamma_q(2)}{\Gamma_q(c)} z^{c-1} \frac{1+qz}{1+z}, \quad (84)$$

then we have the following result:

$$\mathrm{Re}\left\{\left(\frac{D_{q,z}^{1-c} I_q^n f(z)}{z^c}\right)^{1/n}\right\} > \left(\frac{1}{\Gamma_q(c-1)} \frac{\Gamma_q(2)}{\Gamma_q(c)} \left(\frac{1}{[c]_q} + (q-1)\int_0^1 \frac{u^c}{1+u} d_q u\right)\right)^{1/n}, n \geq 1. \quad (85)$$

Proof. Similar to the proof of Theorem 20, for $r(z) = ([c]_q D_{q,z}^{1-c} I_q^n f(z))/z^c$, the differential subordination (92) is equivalent to

$$r(z) + \frac{q^c z}{[c]_q} D_q r(z) \prec \frac{1}{\Gamma_q(c-1)} \frac{\Gamma_q(2)}{\Gamma_q(c)} \frac{1+qz}{1+z}. \quad (86)$$

Therefore,

$$\begin{aligned}\mathrm{Re}\left\{\left(\frac{D_{q,z}^{1-c} I_q^n f(z)}{z^c}\right)^{1/n}\right\} &> \left(\frac{1}{\Gamma_q(c-1)} \frac{\Gamma_q(2)}{\Gamma_q(c)} \int_0^1 u^{c-1} \frac{1+qu}{1+u} d_q u\right)^{1/n} \\ &= \left(\frac{1}{\Gamma_q(c-1)} \frac{\Gamma_q(2)}{\Gamma_q(c)} \int_0^1 \left(u^{c-1} + \frac{(q-1)u^c}{1-u} d_q u\right)^{1/n}\right. \\ &= \left(\frac{1}{\Gamma_q(c-1)} \frac{\Gamma_q(2)}{\Gamma_q(c)} \left(\frac{1}{[c]_q} + (q-1)\int_0^1 \frac{u^c}{1+u} d_q u\right)\right)^{1/n}, n \geq 1.\end{aligned} \quad (87)$$

By using Remark 10, we can obtain the following corollaries. □

Corollary 24. *Let $-1 \leq B < A \leq 1$. If $f \in \mathscr{A}$ satisfies*

$$\left(D_z^{-1} I^n f(z)\right) \prec z, \quad (88)$$

then we have the following result:

$$\mathrm{Re}\left\{\left(\frac{D_z^{-1} I^n f(z)}{z^2}\right)^{1/n}\right\} > \left(\frac{1}{2}\right)^{1/n}, n \geq 1. \quad (89)$$

Corollary 25. *Let $-1 \leq B < A \leq 1$. If $f \in \mathscr{A}$ satisfies*

$$\left(D_z^{-2} I^n f(z)\right) \prec \frac{z^2}{2}, \quad (90)$$

then we have the following result:

$$\mathrm{Re}\left\{\left(\frac{D_z^{-2} I^n f(z)}{z^3}\right)^{1/n}\right\} > \left(\frac{1}{4}\right)^{1/n}, n \geq 1. \quad (91)$$

Corollary 26. *Let $-1 \leq B < A \leq 1$. If $f \in \mathscr{A}$ satisfies*

$$\left(D_z^{-3} I^n f(z)\right) \prec \frac{z^3}{12}, \quad (92)$$

then we have the following result:

$$\mathrm{Re}\left\{\left(\frac{D_z^{-3} I^n f(z)}{z^4}\right)^{1/n}\right\} > \left(\frac{1}{48}\right)^{1/n}, n \geq 1. \quad (93)$$

Theorem 27. *Let $0 < q < 1$, $-1 \le B_i < A_i \le 1$, $i = 1, 2$, and $c \in \mathbb{N}$ such that $\Lambda_q^c(z)$, defined in (36), is a convex function. If $f \in \mathcal{A}$ satisfies*

$$D_q\left[D_{q,z}^{1-c} I_q^n f_i(z)\right] \prec \frac{1}{\Gamma_q(c-1)} \frac{\Gamma_q(2)}{\Gamma_q(c)} z^{c-1} \frac{1 + A_i z}{1 + B_i z}, (i = 1, 2), \tag{94}$$

then we have the following result:

$$D_q\left[D_{q,z}^{1-c} I_q^n (f_1 * f_2)(z)\right] \prec \frac{1 + (1 - 2\zeta)z}{1 + z}, n \ge 1, \tag{95}$$

where

$$\zeta = \frac{1}{\Gamma_q(c-1)} \frac{\Gamma_q(2)}{\Gamma_q(c)} \left(1 - \frac{4(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} \left(1 + \int_0^1 \frac{u^{c-1}}{1 + u} d_q u\right)\right). \tag{96}$$

Proof. Let $r_i(z) = ([c]_q D_{q,z}^{1-c} I_q^n f_i(z))/z^c$, $i = 1, 2$. Then, by using the differential subordinations (67) for $\lambda = c - 1$ and (112), we infer

$$r_i(z) + \frac{q^c}{[c]_q} D_q r_i(z) \prec \frac{1}{\Gamma_q(c-1)} \frac{\Gamma_q(2)}{\Gamma_q(c)} \frac{1 + A_i z}{1 + B_i z}, (i = 1, 2). \tag{97}$$

By applying Theorem 18 and Lemma 9, we have

$$r_i(z) \prec s_i(z) = \frac{1}{\Gamma_q(c-1)} \frac{\Gamma_q(2)}{\Gamma_q(c)} \frac{[c]_q}{z^c} \int_0^z t^{c-1} \frac{1 + A_i t}{1 + B_i t} d_q t, (i = 1, 2). \tag{98}$$

Also, we define a function $h_i(z), i = 1, 2$ in the following form:

$$r_i(z) = \frac{[c]_q}{\Gamma_q(c-1)} \frac{\Gamma_q(2)}{\Gamma_q(c)} h_i(z), (i = 1, 2, z \in D). \tag{99}$$

By using the differential subordination (112), we write

$$\operatorname{Re}(h_i(z)) > \frac{1 - A_i}{1 - B_i}, i = 1, 2. \tag{100}$$

Similar to the proof of Theorem 20, we derive

$$\begin{aligned} \frac{[c]_q D_{q,z}^{1-c} I_q^n f_i(z)}{z^c} &= \frac{1}{\Gamma_q(c-1)} \frac{\Gamma_q(2)}{\Gamma_q(c)} \frac{[c]_q}{z^c} \int_0^z t^{c-1} h_i(t) d_q t, \\ \frac{[c]_q D_{q,z}^{1-c} I_q^n (f_1 * f_2)(z)}{z^c} &= \frac{1}{\Gamma_q(c-1)} \frac{\Gamma_q(2)}{\Gamma_q(c)} \frac{[c]_q}{z^c} \int_0^z t^{c-1} h_0(t) d_q t, \end{aligned} \tag{101}$$

where

$$\begin{aligned} r_0(z) &= \frac{[c]_q D_{q,z}^{1-c} I_q^n (f_1 * f_2)(z)}{z^c} \\ &= \frac{1}{\Gamma_q(c-1)} \frac{\Gamma_q(2)}{\Gamma_q(c)} \frac{[c]_q}{z^c} \int_0^z t^{c-1} (h_1 * h_2)(t) d_q t. \end{aligned} \tag{102}$$

It is clear that

$$(h_1 * h_2)(z) = 1 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots. \tag{103}$$

Therefore, by applying Lemma 3, we have

$$\operatorname{Re}(h_1 * h_2) > 1 - 2(1 - \beta_1)(1 - \beta_2) = \beta. \tag{104}$$

By applying Lemma 4 and using the fact that $(2(1 - \beta))/(1 + u|z|) > (2(1 - \beta))/(1 + u)$, we calculate

$$\begin{aligned} \operatorname{Re}(r_0(z)) &= \frac{1}{\Gamma_q(c-1)} \frac{\Gamma_q(2)}{\Gamma_q(c)} [c]_q \int_0^1 u^{c-1} \operatorname{Re}(h_1 * h_2)(uz) d_q u \\ &\ge \frac{1}{\Gamma_q(c-1)} \frac{\Gamma_q(2)}{\Gamma_q(c)} [c]_q \int_0^1 u^{c-1} \left(2\beta - 1 + \frac{2(1 - \beta)}{1 + u|z|}\right) d_q u \\ &\ge \frac{1}{\Gamma_q(c-1)} \frac{\Gamma_q(2)}{\Gamma_q(c)} [c]_q \int_0^1 u^{c-1} \left(2\beta - 1 + \frac{2(1 - \beta)}{1 + u}\right) d_q u \\ &= \frac{1}{\Gamma_q(c-1)} \frac{\Gamma_q(2)}{\Gamma_q(c)} \left(2\beta - 1 + 2(1 - \beta) \int_0^1 \frac{u^{c-1}}{1 + u} d_q u\right). \end{aligned} \tag{105}$$

By using inequality (104), we derive

$$2\beta - 1 = 1 - 4(1 - \beta_1)(1 - \beta_2) = 1 - \frac{4(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)}, \tag{106}$$

$$2(1 - \beta) = 4(1 - \beta_1)(1 - \beta_2) = \frac{4(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)}. \tag{107}$$

By using assertions (106) and (107), we obtain

$$\begin{aligned} \operatorname{Re}(r_0(z)) &\ge \frac{1}{\Gamma_q(c-1)} \frac{\Gamma_q(2)}{\Gamma_q(c)} \left(1 - \frac{4(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} \right. \\ &\quad \left. + \frac{4(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} \int_0^1 \frac{u^{c-1}}{1 + u} d_q u\right) \\ &= \frac{1}{\Gamma_q(c-1)} \frac{\Gamma_q(2)}{\Gamma_q(c)} \left(1 - \frac{4(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} \right. \\ &\quad \left. \cdot \left(1 + \int_0^1 \frac{u^{c-1}}{1 + u} d_q u\right)\right). \end{aligned} \tag{108}$$

This completes the proof of Theorem 27. □

We can obtain the following corollaries by using Remark 10.

Corollary 28. *Let* $-1 \leq B_i < A_i \leq 1$, $i = 1, 2$. *If* $f \in \mathscr{A}$ *satisfies*

$$\left(D_z^{-1} I^n f_i(z)\right) \prec z \frac{1 + A_i z}{1 + B_i z}, (i = 1, 2), \tag{109}$$

then we have the following result:

$$\left(D_z^{-1} I^n (f_1 * f_2)(z)\right) \prec \frac{1 + (1 - 2\zeta)z}{1 + z}, n \geq 1, \tag{110}$$

where

$$\zeta = \left(1 - \frac{4(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)}\right)(2 - \ln 2). \tag{111}$$

Corollary 29. *Let* $-1 \leq B_i < A_i \leq 1$, $i = 1, 2$. *If* $f \in \mathscr{A}$ *satisfies*

$$\left(D_z^{-3} I^n f_i(z)\right) \prec \frac{z^3}{12} \frac{1 + A_i z}{1 + B_i z}, (i = 1, 2), \tag{112}$$

then we have the following result:

$$\left(D_z^{-3} I^n (f_1 * f_2)(z)\right) \prec \frac{1 + (1 - 2\zeta)z}{1 + z}, n \geq 1, \tag{113}$$

where

$$\zeta = \frac{1}{12}\left(1 - \frac{4(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)}\right)\left(\frac{11}{6} - \ln 2\right). \tag{114}$$

5. Conclusions

In this paper, the topics related to applications in the geometric function theory of q-calculus are presented. The proposed q-differential operator was applied to introduce a q-analogue of a fractional integral operator, and the geometric behavior of the operator is also investigated using the principle of differential subordination. Several interesting results of the q-analogue fractional integral operator are obtained here by following the differential subordination method. A new class of convex functions, $S_n^q(\lambda, \delta)$, are defined, and an inclusion for the class $S_n^q(\lambda, \delta)$ is obtained. The fractional integral operator $D_{q,z}^{-\lambda} I_q^n$ is defined on open unit disc D, and some properties of differential subordination are studied. Therefore, the results obtained in this research could be further used for writing the dual theory of differential subordination which is added to the study of the q-fractional integral operator.

Data Availability

There is no data availability statement to be declared.

Conflicts of Interest

The authors declare no competing interests regarding the publication of the article.

Authors' Contributions

All authors contributed equally and significantly in writing this paper. They read and approved the final manuscript.

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