Wiley
Journal of Function Spaces
Volume 2024, Article ID 4829276, 11 pages
https://doi.org/10.1155/2024/4829276

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Research Article

Coefficient Bounds for q-Noshiro Starlike Functions in Conic Region

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Received 31 October 2023; Revised 29 March 2024; Accepted 22 April 2024; Published 28 May 2024

Academic Editor: H. Ozlem Guney

We present and examine a new family of analytic functions that can be described by a q-Ruscheweyh differential operator. We discuss several novel results, including coefficient inequalities and other noteworthy properties such as partial sums and radii of starlikeness. Moreover, coefficient estimates for the class of Janowski starlike functions associated with symmetric conic domains are also discussed.

1. Introduction

Assume that the set of all analytical functions in the open unit disc

$$\mathbb{U} = \{\varsigma \in \mathbb{C} : |\varsigma| < 1\} \tag{1}$$

is $h \in \Lambda$, and that $h \in \Lambda$ has the series representation of the form

$$h(\varsigma) = \varsigma + \sum_{m=2}^{\infty} a_m \varsigma^m \quad \varsigma \in \mathbb{U}. \tag{2}$$

A function $h \in \Lambda$ is said to be starlike class [1] of order ϱ, $(0 \le \varrho < 1)$ if

$$\Re\left(\frac{\varsigma h'(\varsigma)}{h(\varsigma)}\right) > \varrho, \varsigma \in \mathbb{U} \tag{3}$$

and are denoted as $S^*(\varrho)$.

Assume that h_1 and h_2 are two analytic functions in $\mathbb{U}$. Then, we say that the function h_2 is subordinate to the function h_1, and we write

$$h_2(\varsigma) \prec h_1(\varsigma) \quad (\varsigma \in \mathbb{U}), \tag{4}$$

if there exists a Schwarz function $\omega(\varsigma)$ with $\omega(0) = 0$ and $|\omega(\zeta)| < 1$, such that

$$h_2(\varsigma) = h_1(\omega(\varsigma)) \quad (\varsigma \in \mathbb{U}). \tag{5}$$

Assume that $h_1, h_2 \in \Lambda$ and h_1 be as in (2) and $h_2(\varsigma) = \varsigma + \sum_{m=2}^{\infty} b_m \varsigma^m \quad \varsigma \in \mathbb{U}$, then the Hadamard product $h_1(\varsigma) * h_2(\varsigma)$ is distinct as

$$h_1(\varsigma) * h_2(\varsigma) = \varsigma + \sum_{m=2}^{\infty} a_m b_m \varsigma^m. \tag{6}$$

1.1. Basic Results in q-Derivative. One of the pioneers in the use of q-calculus is Jackson [2, 3]. He presented the well-known q-derivative and q-integral in a systematic and structured manner. Since the early 1980s, geometric aspects of q-analysis have been thoroughly discussed and scrutinized, particularly in relation to quantum groups. This exploration has established a correlation between relevant frameworks and the realm of q-analysis. We recall some notation of q-calculus referred in [2–5]. The q-derivative operator is defined for values of $h \in \Lambda$ as follows:

$$\Delta_q h(\varsigma) = \frac{h(q\varsigma) - h(\varsigma)}{\varsigma(q-1)}, \varsigma \in \mathbb{U}. \tag{7}$$

Consider

$$\Delta_q \varsigma^m = [m, q]\varsigma^{m-1},$$
$$\Delta_q\left\{\sum_{m=1}^{\infty} a_m \varsigma^m\right\} = \sum_{m=1}^{\infty}[m, q]a_m\varsigma^{m-1}, \tag{8}$$

where $m \in \mathbb{N}$ and $\varsigma \in \mathbb{U}$. The subsequently

$$\lim_{q\longrightarrow 1}\frac{1-q^m}{1-q} = m \tag{9}$$

gives the q-analogue of m.

$$[m, q] = \frac{1-q^m}{1-q}. \tag{10}$$

We comprise the q-factorial, as follows:

$$[m, q]! = \begin{cases} [m, q][m-1, q]\cdots[1, q], & m = 1, 2, \cdots, \\ 1 & m = 0. \end{cases} \tag{11}$$

In works referenced as [6–8], the q-variant of the well-known Baskakov-Durrmeyer operator was proposed, with its foundations based on the q-beta function. The q-Picard integral operator and the q-Gauss–Weierstrass integral operator (as mentioned in [8–10]) are two novel q-speculations concerning complex operators. These operators underwent comprehensive investigation, with their geometric attributes meticulously analyzed within select subsets of holomorphic functions. As evident from [4, 11, 12], numerous operators are presently under scholarly scrutiny concerning their q-analogues. Citations [13–16] elucidate the q-symmetric derivative operator and its diverse applications. Bounds for coefficients of q-starlike functions in 2023 linked to the q-Bernoulli function are detailed in [17]. Recently, a new subclass of k-Janowski starlike functions, defined within the framework of the q-derivative operator and labeled as $k-\mathcal{ST}_{q,\eta}[E, F]$, has been detailed (refer to [18–20]). Resources discussing Janowski functions and associated content are contained within [21–23]. Special functions hold paramount importance across diverse sectors of applied sciences and mathematics. Several authors have investigated the geometric attributes of exceptionally distinct special functions, as evidenced in a range of studies (see [24–27]). Cotîrlă and Murugusundaramoorthy explored the consequences of Fekete-Szegö functional problems on Janowski-type starlike functions [28]. An in-depth examination of the relevant literature uncovers the incorporation of the q-variant of the well-known and heavily cited differential operator (for details, refer [4, 12, 29–35]) and the Ruscheweyh derivative operator [36].

We recall the q-Ruscheweyh derivative operator [37], also defined by Aldweby and Darus [12] (see also [20]).

For a function $h(\varsigma) \in \Lambda$, the q-Ruscheweyh differential operator is described by

$$Y_q^{\nu}h(\varsigma) = \varphi(q, \nu+1; \varsigma) * h(\varsigma) = \varsigma + \sum_{m=2}^{\infty}\psi_{m-1}a_m\varsigma^m, \varsigma \in \mathbb{U}, \quad \nu > -1, \tag{12}$$

where

$$\varphi(q, \nu+1; \varsigma) = \varsigma + \sum_{m=2}^{\infty}\psi_{m-1}a_m\varsigma^m, \tag{13}$$

$$\psi_{m-1} = \frac{\Gamma_q(\nu+n)}{[m-1, q]!\Gamma_q(\nu+1)} = \frac{[\nu+1, q]_{m-1}}{[m-1, q]!}, (\psi_0 = 1), \tag{14}$$

where Pochhammer symbol is represented by $[\nu+1, q]_{m-1}$ as below:

$$[\nu+1, q]_{m-1} = \begin{cases} 1, & m = 1, \\ [\nu+2, q][\nu+3, q]\cdots[\nu+m-1, q], & m = 2, 3, 4\cdots. \end{cases} \tag{15}$$

It is clear from (12) that

$$\begin{aligned} Y_q^0 h(\varsigma) &= h(\varsigma), \\ Y_q^1 h(\varsigma) &= \varsigma\Delta_q h(\varsigma), \\ Y_q^m h(\varsigma) &= \frac{\varsigma\Delta_q^m\left(\varsigma^{m-1}h(\varsigma)\right)}{[m, q]!}, (i \in \mathbb{N}), \\ \lim_{q\longrightarrow 1^-}\varphi(q, \nu+1; \varsigma) &= \frac{\varsigma}{(1-\varsigma)^{\nu+1}}, \\ \lim_{q\longrightarrow 1^-} Y_q^{\nu}h(\varsigma) &= h(\varsigma) * \frac{\varsigma}{(1-\varsigma)^{\nu+1}}. \end{aligned} \tag{16}$$

This demonstrates for $q \longrightarrow 1^-$, the q-Ruscheweyh differential operator reduces to the Ruscheweyh differential operator $D^{\delta}(h(\varsigma))$ [36]. The below expression is a clear and often-used derivation from (12).

$$\varsigma\Delta_q Y_q^{\nu}h(\varsigma) = \left(1 + \frac{[\nu, q]}{q^{\nu}}\right)Y_q^{\nu+1}h(\varsigma) - \frac{[\nu, q]}{q^{\nu}}Y_q^{\nu}h(\varsigma). \tag{17}$$

If $q \longrightarrow 1^-$, which is simplified to

$$\varsigma(Y^{\nu}h(\varsigma))' = (1+\nu)Y^{\nu+1}h(\varsigma) - \nu Y^{\nu}h(\varsigma). \tag{18}$$

Lately, making use of the q-Ruscheweyh differential operator, Zainab et al. [38] defined the following class.

Definition 1 (see [38]). The function $p(\varsigma) \in k - P_q[E, F]$, if and only if

$$p(\varsigma) \prec \frac{((3-q)+E(1+q))\tilde{p}_k(\varsigma)+((3-q)-E(1+q))}{((3-q)+F(1+q))\tilde{p}_k(\varsigma)+((3-q)-F(1+q))}, \tag{19}$$

where $-1 \leq F < E \leq 1, k \geq 0, q \in (0, 1)$ and

$$\tilde{p}_k(\varsigma) = \begin{cases} \frac{1+\varsigma}{1-\varsigma}, k=0, \\ 1+\frac{2}{\pi^2}\left(\log\frac{1+\sqrt{\varsigma}}{1-\sqrt{\varsigma}}\right)^2, k=1, \\ 1+\frac{2}{1-k^2}\sinh^2\left[\left(\frac{2}{\pi}\cos^{-1}k\right)\tan^{-1}h\sqrt{\varsigma}\right], 0<k<1, \\ 1+\frac{1}{k^2-1}\sin\left(\frac{\pi}{2Q(s)}\int_0^{u(\varsigma)/\sqrt{s}}\frac{1}{\sqrt{1-x^2}\sqrt{1-(sx)^2}}dx\right)+\frac{1}{k^2-1}, k>1. \end{cases} \tag{20}$$

The image of $\mathbb{U}$ is given by the function $\tilde{p}_k(\varsigma)$ as a conic region that is symmetric around the horizontal axis. Further details, see [39, 40]. If $\tilde{p}_k(\varsigma) = 1 + \delta_k(\varsigma) + \cdots$, then it is exhibited in [40] that from (20), one can have

$$\delta_k = \begin{cases} \frac{8\left(\cos^{-1}k\right)^2}{\pi^2\left(1-k^2\right)}, 0 \leq k < 1, \\ \frac{8}{\pi^2}, k=1, \\ \frac{\pi^2}{4\left(k^2-1\right)\sqrt{s}(1+s)Q^2(s)}, k>1. \end{cases} \tag{21}$$

The values of $p(\varsigma) \in k - P[E, F]$ are geometrically a part of q-symmetric conic domain $\Omega_{k,q}[E, F], -1 \leq F < E \leq 1, k \geq 0$ which is described by

$$\Omega_{k,q}[E,F] = \left\{g : \Re(\eta) > k|\eta-1|, \eta = \frac{((q-3)+F(1+q))g(\varsigma)+((3-q)-E(1+q))}{((3-q)+F(1+q))g(\varsigma)-((3-q)+E(1+q))}\right\}. \tag{22}$$

Motivated by q-analogue theory and the symmetric conic domain, we explore here the q-version of the Ruscheweyh differential operator by applying it to the Noshiro-type starlike functions that are associated with the symmetric conic domain. In this work, a novel family of k-Janowski starlike functions, formed by the generalized q-Ruscheweyh derivative operator in $\mathbb{U}$, is connected to the symmetric conic domain as in Definition 2.

Definition 2. For the function $h(\varsigma) \in \Lambda$ defined in $k - \mathcal{S}\mathcal{T}_{q,\eta}[E, F], -1 \leq F < E \leq 1, k \geq 0, \Leftrightarrow$

$$\Re\left[\frac{((q-3)+F(1+q))\left(\varsigma\Delta_q Y_q^\nu h(\varsigma)/(1-\eta)Y_q^\nu h(\varsigma)+\eta\varsigma\right)+((3-q)-E(1+q))}{((3-q)+F(1+q))\left(\varsigma\Delta_q Y_q^\nu h(\varsigma)/(1-\eta)Y_q^\nu h(\varsigma)+\eta\varsigma\right)-((3-q)+E(1+q))}\right]$$
$$> k\left|\frac{((q-3)+F(1+q))\left(\varsigma\Delta_q Y_q^\nu h(\varsigma)/(1-\eta)Y_q^\nu h(\varsigma)+\eta\varsigma\right)+((3-q)-E(1+q))}{((3-q)+F(1+q))\left(\varsigma\Delta_q Y_q^\nu h(\varsigma)/(1-\eta)Y_q^\nu h(\varsigma)+\eta\varsigma\right)-((3-q)+E(1+q))}-1\right|, \tag{23}$$

or equivalently

$$\frac{\varsigma\Delta_q Y_q^\nu h(\varsigma)}{(1-\eta)Y_q^\nu h(\varsigma)+\eta\varsigma} \in k - P[E, F]. \tag{24}$$

Fixing $\eta = 1$, we state the following new class Noshiro-type analytic functions denoted by $k - \mathcal{N}_{q,\eta}[E, F]$, which has not been studied in association with the q-derivative operator in a symmetric conic domain.

Example 3. For the function $h(\varsigma) \in \Lambda$ defined in $k - \mathcal{N}_{q,\eta}[E, F], -1 \leq F < E \leq 1, k \geq 0, \Leftrightarrow$

$$\Re\left[\frac{((q-3)+F(1+q))\Delta_q Y_q^\nu h(\varsigma)+((3-q)-E(1+q))}{((3-q)+F(1+q))\Delta_q Y_q^\nu h(\varsigma)-((3-q)+E(1+q))}\right]$$
$$> k\left|\frac{((q-3)+F(1+q))\Delta_q Y_q^\nu h(\varsigma)+((3-q)-E(1+q))}{((3-q)+F(1+q))\Delta_q Y_q^\nu h(\varsigma)-((3-q)+E(1+q))}-1\right|, \tag{25}$$

or equivalently

$$\Delta_q Y_q^\nu h(\varsigma) \in k - P[E, F]. \tag{26}$$

Remark 4. For function $h \in \Lambda$ and $k \geq 0, -1 \leq F < E \leq 1$, by fixing $\eta = 0$, it is observe that

$$k - \mathcal{S}\mathcal{T}_{q,0}^\nu[E, F] \equiv k - \mathcal{S}\mathcal{T}_q^\nu[E, F] \tag{27}$$

and has been studied and discussed extensively in [20].

Remark 5. By $\nu = 0$, we note in [20] further elaboration on the aforementioned class of starlike functions, and it is noteworthy that Noor and Malik [26] delineated the class of the Janowski k-starlike function. Additionally, Srivastava et al. [29] established the limit as q approaches $\lim_{q\longrightarrow 1^-}(k - \mathcal{S}\mathcal{T}_{q,\eta}[E, F]) = k - ST[E, F]$ the class of q - SF and $0 - \mathcal{S}\mathcal{T}_{q,\eta}[E, F] = S_q^*[E, F]$.

In the following sections, we discuss coefficient inequalities, partial sums results, and radii of starlikeness for $h \in k - \mathcal{S}\mathcal{T}_{q,\eta}[E, F]$.

2. Coefficient Bounds

Lemma 6 (see [22]). *Let $h(\varsigma) = 1 + \sum_{m=1}^{\infty} v_m \varsigma^m$ be subordinate to $H(\varsigma) = 1 + \sum_{m=1}^{\infty} V_m \varsigma^m . H(\varsigma)$ is univalent in $\mathbb{U}$ and $H(\mathbb{U})$ is convex, then*

$$|v_m| \le |V_1|, m \ge 1. \tag{28}$$

The following theorem provides a sufficient condition for functions to be in $k-\mathcal{ST}_{q,\eta}^{\nu}[E,F]$.

Theorem 7. *A function $h \in \Lambda$ with the form (2) will belong to the class $k-\mathcal{ST}_{q,\eta}^{\nu}[E,F], -1 \le F < E \le 1, k \ge 0$, provided that it fulfills the following condition:*

$$\sum_{m=2}^{\infty} \frac{\mathcal{G}_m}{(1+q)|F-E|} |a_m| < 1, \tag{29}$$

where

$$\mathcal{G}_m = \{2q(3-q)(k+1)([m-1,q]+\eta) + |((3-q) + F(1+q))[m,q] - ((3-q)+E(1+q))(1-\eta)|\}\psi_{m-1}. \tag{30}$$

Here, ψ_{m-1} is defined in (13).

Proof. Considering that (29) is true, it is sufficient to demonstrate that

$$k\left|\frac{((q-3)+F(1+q))\left(\varsigma \Delta q Y q \nu h \varsigma/(1-\eta) Y_q^{\nu} h(\varsigma) + \eta\varsigma\right) + ((3-q)-E(1+q))}{((3-q)+F(1+q))\left(\varsigma \Delta_q Y_q^{\nu} h(\varsigma)/(1-\eta) Y_q^{\nu} h(\varsigma) + \eta\varsigma\right) - ((3-q)+E(1+q))} - 1\right|,$$
$$-\Re\left[\frac{((q-3)+F(1+q))\left(\varsigma \Delta_q Y_q^{\nu} h(\varsigma)/(1-\eta) Y_q^{\nu} h(\varsigma) + \eta\varsigma\right) + ((3-q)-E(1+q))}{((3-q)+F(1+q))\left(\varsigma \Delta_q Y_q^{\nu} h(\varsigma)/(1-\eta) Y_q^{\nu} h(\varsigma) + \eta\varsigma\right) - ((3-q)+E(1+q))} - 1\right] < 1. \tag{31}$$

□

For our convenience, we assume

$$\begin{aligned}
& k\left|\frac{((q-3)+F(1+q))\left(\varsigma \Delta_q Y_q^{\nu} h(\varsigma)/(1-\eta) Y_q^{\nu} h(\varsigma) + \eta\varsigma\right) + ((3-q)-E(1+q))}{((3-q)+F(1+q))\left(\varsigma \Delta_q Y_q^{\nu} h(\varsigma)/(1-\eta) Y_q^{\nu} h(\varsigma) + \eta\varsigma\right) - ((3-q)+E(1+q))} - 1\right|, \\
& -\Re\left[\frac{((q-3)+F(1+q))\left(\varsigma \Delta_q Y_q^{\nu} h(\varsigma)/(1-\eta) Y_q^{\nu} h(\varsigma) + \eta\varsigma\right) + ((3-q)-E(1+q))}{((3-q)+F(1+q))\left(\varsigma \Delta_q Y_q^{\nu} h(\varsigma)/(1-\eta) Y_q^{\nu} h(\varsigma) + \eta\varsigma\right) - ((3-q)+E(1+q))} - 1\right], \\
& \le (k+1)\left|\frac{((q-3)+F(1+q))\varsigma \Delta_q Y_q^{\nu} h(\varsigma) + ((3-q)-E(1+q))(1-\eta) Y_q^{\nu} h(\varsigma) + \eta\varsigma}{((3-q)+F(1+q))\varsigma \Delta_q Y_q^{\nu} h(\varsigma) - ((3-q)+E(1+q))(1-\eta) Y_q^{\nu} h(\varsigma) + \eta\varsigma} - 1\right|, \\
& = 2(3-q)(k+1)\left|\frac{\left((1-\eta) Y_q^{\nu} h(\varsigma) + \eta\varsigma\right) - \varsigma \Delta_q Y_q^{\nu} h(\varsigma)}{((3-q)+F(1+q))\varsigma \Delta_q Y_q^{\nu} h(\varsigma) - ((3-q)+E(1+q))(1-\eta) Y_q^{\nu} h(\varsigma) + \eta\varsigma} - 1\right|, \\
& = 2(3-q)(k+1)\left|\frac{\sum_{m=2}^{\infty}((1-\eta)-[m,q])\psi_{m-1} a_m \varsigma^m}{\varsigma(F-E)(1+q) + \sum_{m=2}^{\infty}\begin{pmatrix}((3-q)+F(1+q))[m,q] \\ -((3-q)+E(1+q))(1-\eta)\end{pmatrix}\psi_{m-1} a_m \varsigma^m}\right|, \\
& = 2(3-q)(k+1)\left|\frac{-q\sum_{m=2}^{\infty}((1-\eta)-[m,q])\psi_{m-1} a_m \varsigma^m}{\varsigma(F-E)(1+q) + \sum_{m=2}^{\infty}\begin{pmatrix}((3-q)+F(1+q))[m,q] \\ -((3-q)+E(1+q))(1-\eta)\end{pmatrix}\psi_{m-1} a_m \varsigma^m}\right|, \\
& \le \frac{2q(3-q)(k+1)\sum_{m=2}^{\infty}([m-1,q]+\eta)\psi_{m-1}|a_m|}{|F-E|(1+q) - \sum_{m=2}^{\infty}|((3-q)+F(1+q))[m,q] - ((3-q)+E(1+q))(1-\eta)|\psi_{m-1}|a_m|}, \\
& = \frac{2q(3-q)(k+1)\sum_{m=2}^{\infty}([m-1,q]+\eta)|a_m|}{|F-E|(1+q)(1/\psi_{m-1}) - \sum_{m=2}^{\infty}|((3-q)+F(1+q))[m,q] - ((3-q)+E(1+q))(1-\eta)||a_m|}.
\end{aligned} \tag{32}$$

If the final term exists, it can be bounded from above by 1.

$$\begin{aligned} &2q(3-q)(k+1)\sum_{m=2}^{\infty}([m-1,q]+\eta)|a_m| \\ &< |F-E|(1+q)\frac{1}{\psi_{m-1}} - \sum_{m=2}^{\infty}|((3-q)+F(1+q))[m,q] \\ &\quad -((3-q)+E(1+q))(1-\eta)||a_m|, \end{aligned} \tag{33}$$

which reduces to

$$\begin{aligned} &\sum_{m=2}^{\infty}\{2q(3-q)(k+1)([m-1,q]+\eta) \\ &\quad +|((3-q)+F(1+q))[m,q]-((3-q) \\ &\quad +E(1+q))(1-\eta)|\}\psi_{m-1}|a_m| < |F-E|(1+q). \end{aligned} \tag{34}$$

The proof is complete.

Corollary 8. *For a function, $h \in \Lambda$ holds the form (2) under the class $k-\mathcal{ST}_{q,\eta}^{\nu}[E,F], -1 \le F < E \le 1, k \ge 0$ if it fulfills the criterion*

$$\begin{aligned} &\sum_{m=2}^{\infty}\{2q(3-q)(k+1)([m-1,q])+|((3-q) \\ &\quad +F(1+q))[m,q]|\}|a_m| \le |F-E|(1+q). \end{aligned} \tag{35}$$

Theorem 9. *Let $h(\varsigma) \in k-\mathcal{ST}_{q,\eta}^{\nu}[E,F]$, $-1 \le F < E \le 1, k \ge 0$ holds the form (2), then*

$$|a_m| \le \prod_{j=0}^{m-2}\frac{\left|(E-F)\delta_k\psi_j - 2(qD[j,q]+\eta)\psi_j\right|}{2(q[j+1,q]+\eta)\psi_{j+1}}, m \ge 2, \tag{36}$$

where ψ_{m-1} is defined in (13).

Proof. Since $h(\varsigma) \in k-\mathcal{ST}_{q,\eta}^{\nu}[E,F]$, we have

$$\frac{\varsigma\Delta_q Y_q^{\nu}h(\varsigma)}{(1-\eta)Y_q^{\nu}h(\varsigma)+\eta\varsigma} = p(\varsigma), \tag{37}$$

where

$$p(\varsigma) \prec \frac{((3-q)+E(1+q))\tilde{p}_k(\varsigma)+((3-q)-E(1+q))}{((3-q)+F(1+q))\tilde{p}_k(\varsigma)+((3-q)-F(1+q))}. \tag{38}$$

□

If $\tilde{p}_k(\varsigma) = 1+\delta_k\varsigma+\cdots$, then

$$\begin{aligned} &\frac{((3-q)+E(1+q))\tilde{p}_k(\varsigma)+((3-q)-E(1+q))}{((3-q)+F(1+q))\tilde{p}_k(\varsigma)+((3-q)-F(1+q))} \\ &= 1+\frac{1}{4}(E-F)(q+1)\delta_k \\ &\quad +\frac{1}{4}\left[\left(-\frac{1}{4}Eq-\frac{1}{4}E+\frac{1}{4}Fq+\frac{1}{4}F\right)((F+1)(1+q)+2-2q)\right]\delta_k^2+\cdots. \end{aligned} \tag{39}$$

Now, if $p(\varsigma) = 1+\sum_{m=1}^{\infty}p^m\varsigma^m$, then by (28) and (39), we arrived

$$|p_m| \le \frac{1}{4}(E-F)(q+1)|\delta_k|, m \ge 1. \tag{40}$$

Now, from (37), we get

$$\varsigma\Delta_q Y_q^{\nu}h(\varsigma) = p(\varsigma)\left((1-\eta)Y_q^{\nu}h(\varsigma)+\eta\varsigma\right), \tag{41}$$

and using $p(\varsigma) = 1+\sum_{m=1}^{\infty}p_m\varsigma^m$, we arrived

$$\begin{aligned} &\varsigma+\sum_{m=2}^{\infty}[m,q]\psi_{m-1}a_m\varsigma^m = \left(1+\sum_{m=1}^{\infty}p_m\varsigma^m\right)\left((1-\eta)\left(\varsigma+\sum_{m=2}^{\infty}\psi_{m-1}a_m\varsigma^m\right)+\eta\varsigma\right), \\ &\sum_{m=1}^{\infty}[m,q]\psi_{m-1}a_m\varsigma^m = \left(1+\sum_{m=1}^{\infty}p_m\varsigma^m\right)\left((1-\eta)\sum_{m=1}^{\infty}\psi_{m-1}a_m\varsigma^m\right), p_0 = 1. \end{aligned} \tag{42}$$

By simple computation, we get

$$\sum_{m=1}^{\infty}([m,q]-(1-\eta))\psi_{m-1}a_m\varsigma^m = \left(\sum_{m=1}^{\infty}p_m\varsigma^m\right)\left((1-\eta)\sum_{m=1}^{\infty}\psi_{m-1}a_m\varsigma^m\right). \tag{43}$$

Using the Cauchy product, we arrived

$$\sum_{m=1}^{\infty}(q[m-1,q]+\eta)\psi_{m-1}a_m\varsigma^m = (1-\eta)\sum_{m=1}^{\infty}\sum_{j=1}^{m-1}\psi_{j-1}a_jp_{m-j}\varsigma^m. \tag{44}$$

Comparing the coefficients of ς^m, we arrived

$$(q[m-1,q]+\eta)\psi_{m-1}a_m = (1-\eta)\sum_{j=1}^{m-1}\psi_{j-1}a_jp_{m-j}, \tag{45}$$

which reduce that

$$a_m = \frac{(1-\eta)}{(q[m-1,q]+\eta)\psi_{m-1}}\sum_{j=1}^{m-1}\psi_{j-1}a_jp_{m-j}. \tag{46}$$

By (40), we have

$$|a_m| \le \frac{|\delta_k|(E-F)(q+1)(1-\eta)}{4(q[m-1,q]+\eta)\psi_{m-1}}\sum_{j=1}^{m-1}\psi_{j-1}|a_j|. \tag{47}$$

Now, we prove

$$\frac{|\delta_k|(E-F)(q+1)(1-\eta)}{4(q[m-1,q]+\eta)\psi_{m-1}}\sum_{j=1}^{m-1}\psi_{j-1}|a_j| \leq \prod_{j=0}^{m-2}\frac{\left|(E-F)(q+1)\delta_k\psi_j-4(Dq[j,q]+\eta)\psi_j\right|}{4(q[j+1,q]+\eta)\psi_{j+1}} \tag{48}$$

by the induction method. For $m=2$, from (47) and using (36), we get

$$\begin{aligned}|a_2| &\leq \frac{|\delta_k|(E-F)(q+1)(1-\eta)}{4(q[1,q]+\eta)\psi_1}\sum_{j=1}^{2-1}\psi_{j-1}|a_j|,\\ &\leq \frac{|\delta_k|(E-F)(q+1)(1-\eta)}{4(q[1,q]+\eta)\psi_1}, \psi_0=1.\end{aligned} \tag{49}$$

Taking $m=3$, from (47), we arrived

$$\begin{aligned}|a_3| &\leq \frac{|\delta_k|(E-F)(q+1)(1-\eta)}{4(q[2,q]+\eta)\psi_2}\sum_{j=1}^{2}\psi_{j-1}|a_j|,\\ &= \frac{|\delta_k|(E-F)(q+1)(1-\eta)}{4(q[2,q]+\eta)\psi_2}(\psi_0|a_1|+\psi_1|a_2|),\\ &\leq \frac{|\delta_k|(E-F)(q+1)(1-\eta)}{4(q[2,q]+\eta)\psi_2}\\ &\quad\cdot\left(1+\frac{|\delta_k|(E-F)(q+1)(1-\eta)}{4(q[1,q]+\eta)\psi_1}\right).\end{aligned} \tag{50}$$

From (36), we have

$$\begin{aligned}|a_3| &\leq \prod_{j=0}^{1}\frac{\left|(E-F)(q+1)(1-\eta)\delta_k\psi_j-4(Dq[j,q]+\eta)\psi_j\right|}{4(q[j+1,q]+\eta)\psi_{j+1}},\\ &= \frac{(E-F)(q+1)(1-\eta)|\delta_k|}{4(q[1,q]+\eta)\psi_1}\left(\frac{(E-F)(q+1)|\delta_k|\psi_1+4(q[1,q]+\eta)\psi_1}{4(q[2,q]+\eta)\psi_2}\right),\\ &= \frac{|\delta_k|(q+1)(E-F)(1-\eta)}{4(q[2,q]+\eta)\psi_2}\left(1+\frac{(q+1)(E-F)|\delta_k|}{4(q[1,q]+\eta)}\right).\end{aligned} \tag{51}$$

Assume that the hypothesis is valid for $m=m+1$. From (47), we have

$$|a_m| \leq \frac{|\delta_k|(E-F)(q+1)(1-\eta)}{4(q[m-1,q]+\eta)\psi_{m-1}}\sum_{j=1}^{m-1}\psi_{j-1}|a_j|. \tag{52}$$

From (36), we get

$$|a_m| \leq \prod_{j=0}^{m-2}\frac{\left|(E-F)(q+1)(1-\eta)\delta_k\psi_j-4(Dq[j,q]+\eta)\psi_j\right|}{4(q[j+1,q]+\eta)\psi_{j+1}}. \tag{53}$$

Using induction principle,

$$\prod_{j=0}^{m-2}\frac{\left|(E-F)(q+1)(1-\eta)\delta_k\psi_j-4(Dq[j,q]+\eta)\psi_j\right|}{4(q[j+1,q]+\eta)\psi_{j+1}} \geq \frac{|\delta_k|(E-F)(q+1)(1-\eta)}{4(q[m-1,q]+\eta)\psi_{m-1}}\sum_{j=1}^{m-1}\psi_{j-1}|a_j|. \tag{54}$$

Multiplying the term on both sides by $(E-F)(q+1)(1-\eta)|\delta_k|\psi_{m-1}+4(q[m-1,q]+\eta)\psi_{m-1}/4(q[m-1,q]+\eta)\psi_m$, we arrived

$$\begin{aligned}&\prod_{j=0}^{m-2}\frac{\left|(E-F)(q+1)(1-\eta)\delta_k\psi_j-4(Dq[j,q]+\eta)\psi_j\right|}{4(q[j+1,q]+\eta)\psi_{j+1}}\\ &\geq \frac{|\delta_k|(E-F)(1-\eta)(q+1)\psi_{m-1}+4(q[m-1,q]+\eta)\psi_{m-1}}{4(q[m,q]+\eta)\psi_m}\\ &\quad\times\left(|\delta_k|\frac{(E-F)(q+1)(1-\eta)}{4([m-1,q]q+\eta)\psi_{m-1}}\right)\sum_{j=1}^{m-1}\psi_{j-1}|a_j|,\\ &= \frac{|\delta_k|(E-F)(q+1)(1-\eta)}{4(q[m,q]+\eta)\psi_m}\\ &\quad\cdot\left(\psi_{m-1}\frac{|\delta_k|(E-F)(q+1)(1-\eta)}{4(q[m-1,q]+\eta)\psi_{m-1}}\sum_{j=1}^{m-1}\psi_{j-1}|a_j|+\sum_{j=1}^{m-1}\psi_{j-1}|a_j|\right),\\ &\geq \frac{|\delta_k|(E-F)(q+1)(1-\eta)}{4(q[m,q]+\eta)\psi_m}\left(\psi_{m-1}|a_m|+\sum_{j=1}^{m-1}\psi_{j-1}|a_j|\right),\\ &= \frac{|\delta_k|(E-F)(q+1)(1-\eta)}{4(q[m,q]+\eta)\psi_m}\sum_{j=1}^{m}\psi_{j-1}|a_j|.\end{aligned} \tag{55}$$

That is,

$$\frac{|\delta_k|(E-F)(1-\eta)}{2(q[m-1,q]+\eta)\psi_{m-1}}\sum_{j=1}^{m-1}\psi_{j-1}|a_j| \leq \prod_{j=0}^{m-2}\frac{\left|(E-F)(1-\eta)\delta_k\psi_j-2(Dq[j,q]+\eta)\psi_j\right|}{2(q[j+1,q]+\eta)\psi_{j+1}}. \tag{56}$$

Hence, the result is valid for $m=m+1$. As a result, it is concluded by the induction principle that (36) is true for every $m\geq 2$.

3. Bounds of the Partial Sums

Discussing the result of Silvia [16] and Silverman [27], we analyze the division of a function represented as (2) by the series of cumulative partial terms $h_m(\varsigma)=\varsigma+\sum_{j=2}^{m}a_j\varsigma^j$. For the functions $h(\varsigma)\in\Lambda$ of the class $k-\mathcal{ST}_{q,\eta}^{\nu}[E,F]$, we analyze the lower bounds for $h(\varsigma)/h_m(\varsigma)'$, $h'(\varsigma)/h_m'(\varsigma)'$, $h_m(\varsigma)/h(\varsigma)$, and $h_m'(\varsigma)/h(\varsigma)$.

Theorem 10. *For any* $h(\varsigma) \in k - \mathcal{ST}^{\nu}_{q,\eta}[E, F]$, *then*

$$\Re\left\{\frac{h(\varsigma)}{h_m(\varsigma)}\right\} \ge 1 - \frac{\ell}{\mathcal{G}_{m+1}}, \tag{57}$$

where $\mathcal{G}_{m+1}$ *is stated by (30) and* $\ell = (1+q)|F-E|$. *The function at its utmost extremity*

$$h(\varsigma) = \varsigma + \frac{\ell}{\mathcal{G}_{m+1}}\varsigma^{m+1} \tag{58}$$

provides the exact result.

Proof. Consider the function $w(\varsigma)$,

$$\begin{aligned} w(\varsigma) &= \frac{\mathcal{G}_{m+1}}{\ell}\left[\frac{h(\varsigma)}{h_m(\varsigma)} - \left(1 - \frac{\ell}{\mathcal{G}_{m+1}}\right)\right], \\ &= \frac{\mathcal{G}_{m+1}}{\ell}\frac{h(\varsigma)}{h_m(\varsigma)} - \frac{\mathcal{G}_{m+1}}{\ell} + 1, \end{aligned} \tag{59}$$

which reduces to

$$\begin{aligned} w(\varsigma) &= \frac{\mathcal{G}_{m+1}\left(1 + \sum_{j=2}^{\infty} a_j \varsigma^{j-1}\right)}{\ell\left(1 + \sum_{j=2}^{m} a_j \varsigma^{j-1}\right)} - \frac{\mathcal{G}_{m+1}}{\ell} + 1, \\ &= \frac{1 + \sum_{j=2}^{m} a_j \varsigma^{j-1} + (\mathcal{G}_{m+1}/\ell)\sum_{j=m+1}^{\infty} a_j \varsigma^{j-1}}{1 + \sum_{j=2}^{m} a_j \varsigma^{j-1}}. \end{aligned} \tag{60}$$

Using this, one may have

$$\left|\frac{w(\varsigma)-1}{w(\varsigma)+1}\right| \le \frac{(\mathcal{G}_{m+1}/\ell)\sum_{j=m+1}^{\infty}|a_j|}{2 - 2\sum_{j=2}^{m}|a_j| - (\mathcal{G}_{m+1}/\ell)\sum_{j=m+1}^{\infty}|a_j|}. \tag{61}$$

Now,

$$\left|\frac{w(\varsigma)-1}{w(\varsigma)+1}\right| \le 1, \tag{62}$$

if

$$\sum_{j=2}^{m}|a_j| + \frac{\mathcal{G}_{m+1}}{\ell}\sum_{j=m+1}^{\infty}|a_j| \le 1. \tag{63}$$

It is adequate to prove that the upper bound of $\sum_{j=2}^{\infty}(\mathcal{G}_j/\ell)|a_j|$, on the left side of (63), if

$$\sum_{j=2}^{m}|a_j| + \frac{\mathcal{G}_{m+1}}{\ell}\sum_{j=m+1}^{\infty}|a_j| \le \sum_{j=2}^{\infty}\frac{\mathcal{G}_j}{\ell}|a_j|, \tag{64}$$

which prompts the accompanying expression

$$\sum_{j=2}^{m}\frac{\mathcal{G}_j - \ell}{\ell}|a_j| + \sum_{j=m+1}^{\infty}\left(\frac{\mathcal{G}_j - \mathcal{G}_{m+1}}{\ell}\right)|a_j| \ge 0. \tag{65}$$

To verify the sharpness, the function arrived by (58) when $\varsigma = re^{i\pi/m}$.

$$\frac{h(\varsigma)}{h_m(\varsigma)} = 1 + \frac{\ell}{\mathcal{G}_{m+1}}\varsigma^m, = 1 + \frac{\ell}{\mathcal{G}_{m+1}}r^m e^{i\pi}, = 1 - \frac{\ell r^m}{\mathcal{G}_{m+1}}, = \frac{\mathcal{G}_{m+1} - \ell}{\mathcal{G}_{m+1}} \quad \text{when} \quad r \longrightarrow 1. \tag{66}$$

□

Theorem 11. *For any* $h(\varsigma) \in k - \mathcal{ST}^{\nu}_{q,\eta}[E, F]$, *then*

$$\Re\left\{\frac{h_m(\varsigma)}{h(\varsigma)}\right\} \ge \frac{\mathcal{G}_{m+1}}{\mathcal{G}_{m+1} + \ell}, \tag{67}$$

where $\mathcal{G}_{m+1}$ *is stated by (30) and* $\ell = (1+q)|F-E|$. *The bound (67) is suitable for the function, explained in (58).*

Proof. Again, we define a function $w(\varsigma)$ as

$$w(\varsigma) = \frac{\mathcal{G}_{m+1} + \ell}{\ell}\left[\frac{h_m(\varsigma)}{h(\varsigma)} - \frac{\mathcal{G}_{m+1}}{\mathcal{G}_{m+1} + \ell}\right], = \frac{(\mathcal{G}_{m+1} + \ell)h_m(\varsigma)}{\ell h(\varsigma)} - \frac{\mathcal{G}_{m+1}}{\ell}. \tag{68}$$

This suggests that

$$\begin{aligned} w(\varsigma) &= \frac{(\mathcal{G}_{m+1} + \ell)\left(1 + \sum_{j=2}^{m} a_j \varsigma^{j-1}\right)}{\ell\left(1 + \sum_{j=2}^{\infty} a_j \varsigma^{j-1}\right)} - \frac{\mathcal{G}_{m+1}}{\ell}, \\ &= \frac{1 + \sum_{j=2}^{m} a_j \varsigma^{j-1} - (\mathcal{G}_{m+1}/\ell)\sum_{j=m+1}^{\infty} a_j \varsigma^{j-1}}{1 + \sum_{j=2}^{\infty} a_j \varsigma^{j-1}}. \end{aligned} \tag{69}$$

This drives us to the following:

$$\frac{w(\varsigma)-1}{w(\varsigma)+1} = \frac{-(1 + (\mathcal{G}_{m+1}/\ell))\sum_{j=m+1}^{\infty} a_j \varsigma^{j-1}}{2 + 2\sum_{j=2}^{m} a_j \varsigma^{j-1} + (1 - (\mathcal{G}_{m+1}/\ell))\sum_{j=m+1}^{\infty} a_j \varsigma^{j-1}}, \tag{70}$$

which simplifies that

$$\left|\frac{w(\varsigma)-1}{w(\varsigma)+1}\right| \le \frac{(1 + (\mathcal{G}_{m+1}/\ell))\sum_{j=m+1}^{\infty}|a_j|}{2 - 2\sum_{j=2}^{m}|a_j| - (1 - (\mathcal{G}_{m+1}/\ell))\sum_{j=m+1}^{\infty}|a_j|}. \tag{71}$$

Now,

$$\left|\frac{w(\varsigma)-1}{w(\varsigma)+1}\right| \le 1, \tag{72}$$

if

$$\sum_{j=2}^{m}|a_j| + \sum_{j=m+1}^{\infty}|a_j| \le 1. \tag{73}$$

It is adequate to prove that the upper bound of $\sum_{j=2}^{\infty}(\mathcal{G}_j/\ell)|a_j|$, on the left side of (73), if

$$\sum_{j=2}^{m}|a_j| + \sum_{j=m+1}^{\infty}|a_j| \le \sum_{j=2}^{\infty}\frac{\mathcal{G}_j}{\ell}|a_j|, \tag{74}$$

which tends to the below form

$$\sum_{j=2}^{m}\left(\frac{\mathcal{G}_j}{\ell}-1\right)|a_j| + \sum_{j=m+1}^{\infty}\left(\frac{\mathcal{G}_j}{\ell}-1\right)|a_j| \ge 0. \tag{75}$$

That is,

$$\sum_{j=2}^{\infty}\left(\frac{\mathcal{G}_j}{\ell}-1\right)|a_j| \ge 0. \tag{76}$$

Hence, equality possesses the function $h(\varsigma)$, as represented in (58). □

Theorem 12. *For any $h(\varsigma)\in k-\mathcal{ST}_{q,\eta}^{\nu}[E,F]$, then*

$$\Re\left\{\frac{h'(\varsigma)}{h_m'(\varsigma)}\right\} \ge 1-\frac{\ell(m+1)}{\mathcal{G}_{m+1}}, \tag{77}$$

where $\mathcal{G}_{m+1}$ is stated by (30) and $\ell=(1+q)|F-E|$. The bound (77) is suitable for the given function, represented by (58).

Proof. Let us take a function $w(\varsigma)$.

$$w(\varsigma)=\frac{\mathcal{G}_{m+1}}{\ell(m+1)}\left[\frac{h'(\varsigma)}{h_m'(\varsigma)}-\frac{\mathcal{G}_{m+1}-\ell(m+1)}{g(m+1)}\right], \tag{78}$$

which becomes

$$\begin{aligned} w(\varsigma)&=\frac{\mathcal{G}_{m+1}\left(1+\sum_{j=2}^{m} ja_j\varsigma^{j-1}\right)}{\ell(m+1)\left(1+\sum_{j=2}^{m} ja_j\varsigma^{j-1}\right)}-\frac{\mathcal{G}_{m+1}-\ell(m+1)}{g(m+1)},\\ &=\frac{1+\sum_{j=2}^{m} ja_j\varsigma^{j-1}+(\mathcal{G}_{m+1}/\ell(m+1))\sum_{j=m+1}^{\infty} ja_j\varsigma^{j-1}}{1+\sum_{j=2}^{m} ja_j\varsigma^{j-1}}. \end{aligned} \tag{79}$$

It reduces us to

$$\frac{w(\varsigma)-1}{w(\varsigma)+1}=\frac{(\mathcal{G}_{m+1}/\ell(m+1))\sum_{j=m+1}^{\infty} ja_j\varsigma^{j-1}}{2+2\sum_{j=2}^{m} ja_j\varsigma^{j-1}+(\mathcal{G}_{m+1}/\ell(m+1))\sum_{j=m+1}^{\infty} ja_j\varsigma^{j-1}}, \tag{80}$$

which tends to

$$\left|\frac{w(\varsigma)-1}{w(\varsigma)+1}\right| \le \frac{(\mathcal{G}_{m+1}/\ell(m+1))\sum_{j=m+1}^{\infty} j|a_j|}{2-2\sum_{j=2}^{m} j|a_j|-(\mathcal{G}_{m+1}/\ell(m+1))\sum_{j=m+1}^{\infty} j|a_j|}. \tag{81}$$

Now,

$$\left|\frac{w(\varsigma)-1}{w(\varsigma)+1}\right| \le 1, \tag{82}$$

if

$$\sum_{j=2}^{m} j|a_j| + \frac{\mathcal{G}_{m+1}}{\ell(m+1)}\sum_{j=m+1}^{\infty} j|a_j| \le 1. \tag{83}$$

It is adequate to prove that the upper bound of $\sum_{j=2}^{\infty}(\mathcal{G}_j/\ell)|a_j|$, on the left side of (83), if

$$\sum_{j=2}^{m} j|a_j| + \frac{\mathcal{G}_{m+1}}{\ell(m+1)}\sum_{j=m+1}^{\infty} j|a_j| \le \sum_{j=2}^{\infty}\frac{\mathcal{G}_j}{\ell}|a_j|, \tag{84}$$

which tends to the below form

$$\sum_{j=2}^{m}\left(\frac{\mathcal{G}_j}{\ell}-j\right)|a_j| + \sum_{j=m+1}^{\infty}\left(\frac{\mathcal{G}_j}{\ell}-\frac{j\mathcal{G}_{m+1}}{\ell(m+1)}\right)|a_j| \ge 0. \tag{85}$$

□

Theorem 13. *If $h(\varsigma)\in k-\mathcal{ST}_{q,\eta}^{\nu}[E,F]$, then*

$$\Re\left\{\frac{h_m'(\varsigma)}{h'(\varsigma)}\right\} \ge \frac{\mathcal{G}_{m+1}}{\mathcal{G}_{m+1}+\ell(m+1)}, \tag{86}$$

where $\mathcal{G}_{m+1}$ is stated in (30) and $\ell=(1+q)|F-E|$. The bound (86) is suitable for the function represented by (58).

Proof. Let us consider a function $w(\varsigma)$ as below:

$$w(\varsigma)=\frac{\mathcal{G}_{m+1}+\ell(m+1)}{\ell(m+1)}\left[\frac{h_m'(\varsigma)}{h'(\varsigma)}-\frac{\mathcal{G}_{m+1}}{\mathcal{G}_{m+1}+\ell(m+1)}\right], \tag{87}$$

equivalently

$$\begin{aligned} w(\varsigma)&=\frac{(\mathcal{G}_{m+1}+\ell(m+1))\left(1+\sum_{j=2}^{\infty} ja_j\varsigma^{j-1}\right)}{\ell(m+1)\left(1+\sum_{j=2}^{m} ja_j\varsigma^{j-1}\right)}-\frac{\mathcal{G}_{m+1}}{\ell(m+1)},\\ &=\frac{1+\sum_{j=2}^{m} ja_j\varsigma^{j-1}-(\mathcal{G}_{m+1}/\ell(m+1))\sum_{j=m+1}^{\infty} ja_j\varsigma^{j-1}}{1+\sum_{j=2}^{\infty} ja_j\varsigma^{j-1}}. \end{aligned} \tag{88}$$

This yields

$$\begin{aligned} \frac{w(\varsigma)-1}{w(\varsigma)+1}&=\frac{\sum_{j=2}^{m} ja_j\varsigma^{j-1}-\sum_{j=2}^{\infty} ja_j\varsigma^{j-1}-(\mathcal{G}_{m+1}/\ell(m+1))\sum_{j=m+1}^{\infty} ja_j\varsigma^{j-1}}{2+\sum_{j=2}^{m} ja_j\varsigma^{j-1}+\sum_{j=2}^{\infty} ja_j\varsigma^{j-1}-(\mathcal{G}_{m+1}/\ell(m+1))\sum_{j=m+1}^{\infty} ja_j\varsigma^{j-1}},\\ &=\frac{-\sum_{j=m+1}^{\infty}(1+(\mathcal{G}_{m+1}/\ell(m+1)))ja_j\varsigma^{j-1}}{2+2\sum_{j=2}^{m} ja_j\varsigma^{j-1}+\sum_{j=m+1}^{\infty}(1-(\mathcal{G}_{m+1}/\ell(m+1)))ja_j\varsigma^{j-1}}. \end{aligned} \tag{89}$$

That is,

$$\left|\frac{w(\varsigma)-1}{w(\varsigma)+1}\right| \le \frac{(1+(\mathcal{G}_{m+1}/\ell(m+1)))\sum_{j=m+1}^{\infty} j|a_j|}{2-2\sum_{j=2}^{m} j|a_j|-(1-(\mathcal{G}_{m+1}/\ell(m+1)))\sum_{j=m+1}^{\infty} j|a_j|}. \quad (90)$$

Now,

$$\left|\frac{w(\varsigma)-1}{w(\varsigma)+1}\right| \le 1, \quad (91)$$

if

$$\sum_{j=2}^{m} j|a_j| + \sum_{j=m+1}^{\infty} j|a_j| \le 1. \quad (92)$$

The left part of (92) is bounded above by $\sum_{j=2}^{\infty}(\mathcal{G}_j/\ell)|a_j|$ if

$$\sum_{j=2}^{m} j|a_j| + \sum_{j=m+1}^{\infty} j|a_j| \le \sum_{j=2}^{\infty} \frac{\mathcal{G}_j}{\ell}|a_j|, \quad (93)$$

which reduced as

$$\sum_{j=2}^{m} j|a_j| + \sum_{j=m+1}^{\infty} j|a_j| \le \sum_{j=2}^{m} \frac{\mathcal{G}_j}{\ell}|a_j| + \sum_{j=m+1}^{\infty} \frac{\mathcal{G}_j}{\ell}|a_j|, \quad (94)$$

which tends to the below form:

$$\sum_{j=2}^{m}\left(\frac{\mathcal{G}_j}{\ell}-j\right)|a_j| + \sum_{j=m+1}^{\infty}\left(\frac{\mathcal{G}_j}{\ell}-j\right)|a_j| \ge 0. \quad (95)$$

Therefore,

$$\sum_{j=2}^{\infty}\left(\frac{\mathcal{G}_j}{\ell}-j\right)|a_j| \ge 0. \quad (96)$$

Radius of starlikeness for $f \in k-\mathcal{ST}_{q,\eta}^{\nu}[E,F]$.

In this section, we will discuss about the radius of starlikeness [17, 41, 42] for $k-\mathcal{ST}_{q,\eta}^{\nu}[E,F]$ of order ϱ. □

Theorem 14. *Let $h(\varsigma) \in k-\mathcal{ST}_{q,\eta}^{\nu}[E,F]$, is a starlike function containing the order of $\varrho \in [0,1)$ which is defined in $|\varsigma| < r = r_1(\varrho)$, where*

$$r_1(\varrho) = \left(\frac{\mathcal{G}_m(1-\varrho)\psi_{m-1}}{(m-\varrho)\ell}\right)^{1/m-1}, m = 2, 3, \cdots, \quad (97)$$

where $\mathcal{G}_m$ is stated in (30) and $\ell = (1+q)|F-E|$.

Proof. Let us consider a function $h(\varsigma) \in k-\mathcal{ST}_{q,\eta}^{\nu}[E,F]$ in the form (2). Then, which gives

$$\sum_{m=2}^{\infty} \frac{\mathcal{G}_m \psi_{m-1}}{\ell}|a_m| < 1, \quad (98)$$

where $\mathcal{G}_m$ is stated in (30) and $\ell = (1+q)|F-E|$. For $\varrho \in [0,1)$, and $h(\varsigma)$ is starlike of order ϱ, we know that

$$\left|\frac{\varsigma h'(\varsigma)}{h(\varsigma)}-1\right| < 1-\varrho. \quad (99)$$

Simple computation, we get

$$\sum_{m=2}^{\infty} \frac{m-\varrho}{1-\varrho}|a_m||\varsigma|^{m-1} < 1. \quad (100)$$

The above expression (100) is true if the condition holds

$$\sum_{m=2}^{\infty} \frac{m-\varrho}{1-\varrho}|a_m||\varsigma|^{m-1} < \sum_{m=2}^{\infty} \frac{\mathcal{G}_m \psi_{m-1}}{\ell}|a_m|. \quad (101)$$

Now, solving (101) for $|\varsigma|$, we obtain

$$|\varsigma|^{m-1} < \frac{\mathcal{G}_m(1-\varrho)\psi_{m-1}}{(m-\varrho)\ell}. \quad (102)$$

Setting $|\varsigma| = r(\varrho)$ in (102), we get

$$r(\varrho) = \left(\frac{\mathcal{G}_m(1-\varrho)\psi_{m-1}}{(m-\varrho)\ell}\right)^{1/m-1} \quad (103)$$

as required. □

4. Conclusion

We have concentrated the q-version of the popular Ruscheweyh differential operator and applied it to characterize and concentrate a new class $k-\mathcal{ST}_{q,\eta}^{\tau}[E,F]$ of q-starlike functions connected with the symmetric conic domain and covers certain coefficient inequalities for q-starlike functions including coefficient bounds and sufficient conditions. Utilizing similar relationships in extraordinary cases, the results related to partial sums for the functions of a class and the radius of starlikeness for $f \in k-\mathcal{ST}_{q,\eta}^{\tau}[E,F]$ are discussed which have not been examined to date. Moreover, by fixing $\eta = 1$, one can state the results discussed in Theorems 12 to 14 for the class $k-\mathcal{N}_q^{\tau}[E,F]$, as given in Example 3, which are also new and not yet discussed so far. Additionally by fixing $\eta = 0$, as in Remark 4, we have $k-\mathcal{ST}_{q,0}^{\tau}[E,F] \equiv k-\mathcal{ST}_q^{\tau}[E,F]$ which are acquired as a special case from our results, as demonstrated in [38]. Also, to motivate further research on the subject matter, we have chosen to draw the attention of interested readers toward a considerably large number of related recent publications (see, for example, [32–35]) and developments in the area of mathematical analysis. In conclusion, we choose to reiterate an important observation, which was presented in the recently-published review-cum-expository review chapter by Srivastava ([32], p. 340), who pointed out the fact that the results for the abovementioned or new q-analogues can easily (and possibly trivially) be translated into the corresponding results for the so-called $(p\,;q)$-analogues (with $0 < |q| < p \le 1$) by applying some obvious parametric and argument variations with the additional parameter p being redundant. Further, the study can be extended by using the q-Srivastava–Attiya operator discussed in [43].

Data Availability

There is no data used in the manuscript.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This study is self-supported.

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