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Research Article

Common Fixed Point Theorems on S-Metric Spaces for Integral Type Contractions Involving Rational Terms and Application to Fractional Integral Equation

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It has been shown that the findings of d-metric spaces may be deduced from S-metric spaces by considering $d(\varpi, \varkappa) = \Lambda(\varpi, \varpi, \varkappa)$. In this study, no such concepts that translate to the outcomes of metric spaces are considered. We establish standard fixed point theorems for integral type contractions involving rational terms in the context of complete S-metric spaces and discuss their implications. We also provide examples to illustrate the work. This paper's findings generalize and expand a number of previously published conclusions. In addition, the abstract conclusions are supported by an application of the Riemann-Liouville calculus to a fractional integral problem and a supportive numerical example.

1. Introduction

The development of nonlinear analysis has been largely influenced by fixed point theory. The Banach contraction principle is generally acknowledged to be one of the most useful theorems in nonlinear analysis [1]. Its relevance derives from its wide applicability to a multitude of mathematical fields. Numerous generalizations exist for this Banach contraction concept. In contrast, a variety of generalizations of metric spaces have been accomplished, one of which is a S-metric space. As a generalization of a metric, Sedghi et al. [2] established the notion of a S-metric in 2012.

Definition 1 (see [2]). Let $\Xi \neq \varnothing$ be a set and let $\Lambda : \Xi^3 \longrightarrow [0,\infty)$ be a function satisfying the following conditions for all $\varpi, \varkappa, \vartheta, \varsigma \in \Xi$:

(S1) $\Lambda(\varpi, \varkappa, \vartheta) = 0$ if and only if $\varpi = \varkappa = \vartheta$

(S2) $\Lambda(\varpi, \varkappa, \vartheta) \leq \Lambda(\varpi, \varpi, \varsigma) + \Lambda(\varkappa, \varkappa, \varsigma) + \Lambda(\vartheta, \vartheta, \varsigma)$

Then, the function Λ is called an S-metric on Ξ, and the pair (Ξ, Λ) is called an S-metric space (in short SMS).

The following instances are easily verifiable as SMS.

Example 1 (see [2]). Let $\Xi = \mathbb{R}^n$ and $\|.\|$ be a norm on Ξ; then, $\Lambda(\varpi, \varkappa, \vartheta) = \|\varkappa + \vartheta - 2\varpi\| + \|\varkappa - \vartheta\|$ is a SMS.

Example 2 (see [3]). Let $\Xi \neq \emptyset$ be a set and d be an ordinary metric on Ξ. Then, $\Lambda(\varpi, \varkappa, \vartheta) = d(\varpi, \vartheta) + d(\varkappa, \vartheta)$ for all ϖ, $\varkappa, \vartheta \in \Xi$ is a SMS.

For a $\Xi \neq \emptyset$ set and $j = \{1, 2, 3, 4\}$, denote

$$\begin{aligned} \mathrm{Fix}(\mathfrak{I}_j) &= \{\varpi \in \Xi : \mathfrak{I}_j \varpi = \varpi\}, \\ \mathrm{CFix}(\mathfrak{I}_1, \mathfrak{I}_2) &= \{\varpi : \mathfrak{I}_1 \varpi = \varpi = \mathfrak{I}_2\}, \\ \mathrm{CFix}(\mathfrak{I}_1, \mathfrak{I}_2, \mathfrak{I}_3, \mathfrak{I}_4) &= \{\varpi : \varpi = \mathfrak{I}_1 \varpi = \mathfrak{I}_2 \varpi = \mathfrak{I}_3 \varpi = \mathfrak{I}_4 \varpi\}. \end{aligned} \tag{1}$$

Sedghi et al. [2] extended the well-known Banach's contraction concept to a complete SMS. The assertion is as follows.

Theorem 2 (see [2]). *Let (Ξ, Λ) be a complete SMS and let $\mathfrak{I} : \Xi \longrightarrow \Xi$ be a self-mapping of Ξ such that*

$$\Lambda(\mathfrak{I}\varpi, \mathfrak{I}\varkappa, \mathfrak{I}\vartheta) \leq \alpha \Lambda(\varpi, \varkappa, \vartheta), \tag{2}$$

for all $\varpi, \varkappa, \vartheta \in \Xi$, where $\alpha \in (0, 1)$ is a constant. Then, $\mathfrak{I}$ has a unique fixed point in Ξ.

In addition, Sedghi et al. [2] proved that the results of d -metric spaces can be derived from SMS results if we consider $d(\varpi, \varkappa) = \Lambda(\varpi, \varpi, \varkappa)$. In [4, 5], Saluja discussed fixed point (FP) and common fixed point (CFP) type of results on implicit type of contraction conditions to prove CFP on SMS. Hieu et al. [6] discussed FP on Ciric quasicontractions to derive the FP for maps on SMS. In [7–9], the authors discussed FP on different types of contraction conditions on SMS. In [10], Sedghi et al. discussed CFP for two pairs of mappings on SMS.

In recent years, different contractive circumstances have been investigated using fixed point theory. Indeed, integral type contraction is among them. In 2002, Branciari [11] analyzed the existence of FP for mapping defined on a complete metric space satisfying a general contractive condition of integral type (see Theorem 2.1 of [11]). Following Branciari's [11] finding, other studies have been conducted on generalizing integral type contractive conditions for various contractive mappings that meet a variety of known features (see [12, 13]). Rhoades [14] has done comparable work, extending the finding of Branciari [11] by substituting the following condition for the contractive condition of integral type (3) of Theorem 2.1 in [11]:

$$\int_0^{d(\mathfrak{I}\varpi, \mathfrak{I}\varkappa)} \hbar(\wp) d\wp \leq k \int_0^{\max \{d(\varpi,\varkappa), d(\varpi,\mathfrak{I}\varpi), d(\varkappa,\mathfrak{I}\varkappa), (d(\varpi,\mathfrak{I}\varkappa)+d(\varkappa,\mathfrak{I}\varpi))/2\}} \hbar(\wp) d\wp, \tag{3}$$

for each $k \in [0, 1)$ and $\varpi, \varkappa \in \Xi$, where $\hbar : [0,\infty) \longrightarrow [0,\infty)$ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0, \infty)$, nonnegative, and such that for each $\varepsilon > 0$,

$$\int_0^{\varepsilon} \hbar(\wp) d\wp > 0. \tag{4}$$

In the framework of SMS, Rahman et al. [15] produced a CFP result of the Altman integral type for two pairs of self-mappings and provided an example to support the conclusion. Recently, Özgür and Taş [16] and Saluja [17] investigated novel integral type contractive conditions on S-metric spaces, establishing certain FP theorems for various integral type contractive conditions and providing examples to illustrate their findings. They also discovered a solution to the Fredholm integral equation. Rashwan and Hammad [18] have recently proven some CFP theorems for noncommutative mappings satisfying a general contractive condition of integral type involving rational terms in the context of complete metric spaces, and they provide examples to back up their findings.

Motivated by [2, 16, 18], we demonstrate CFP results for contractive conditions of integral type using rational expressions in the context of complete S-metric spaces (CSMS) in any a way that it can not convert to the results of metric spaces. To justify the work, examples are given in support. This paper's findings extend and generalize a number of previously published findings. In addition, for practise, we recommend the fractional integral equation expressed in terms of the Riemann-Liouville calculus. Using our new fixed point theorems, we establish the necessary criteria for obtaining a unique solution. In addition, a numerical example is given in the next section.

2. Preliminaries

Now, we will review some fundamental definitions, attributes, and auxiliary results pertaining to SMS.

Definition 3 (see [2]). Let (Ξ, Λ) be a SMS.

(1) A sequence $\{\varpi_n\}$ in Ξ converges to $\varpi \in \Xi$ if and only if $\lim_{n\longrightarrow\infty} \Lambda(\varpi_n, \varpi_n, \varpi) \longrightarrow 0$. We denote this by $\lim_{n\longrightarrow\infty} \varpi_n = \varpi$ or $\varpi_n \longrightarrow \varpi$ as $n \longrightarrow \infty$

(2) A sequence $\{\varpi_n\}$ in Ξ is called a Cauchy sequence if $\lim_{n\longrightarrow\infty} \Lambda(\varpi_n, \varpi_n, \varpi_m) \longrightarrow 0$

(3) The SMS (Ξ, Λ) is called complete if every Cauchy sequence in Ξ is convergent in Ξ

Definition 4. Let $\Xi \neq \emptyset$ be a set and let $\mathfrak{I}_1, \mathfrak{I}_2 : \Xi \longrightarrow \Xi$ be two self-mappings of Ξ. Then,

(i) a point $\varpi \in \Xi$ is called a fixed point of mapping $\mathfrak{I}_1$ if $\mathfrak{I}_1 \varpi = \varpi$

(ii) a point $\varpi \in \Xi$ is common fixed point of $\mathfrak{I}_1$ and $\mathfrak{I}_2$ if $\mathfrak{I}_1 \varpi = \mathfrak{I}_2 \varpi = \varpi$

(iii) in [19], ϖ is called a coincidence point point of $\mathfrak{I}_1$ and $\mathfrak{I}_2$, if $\rho = \mathfrak{I}_1 \varpi = \mathfrak{I}_2 \varpi$ for some $\varpi \in \Xi$, and ρ is called a point of coincidence of $\mathfrak{I}_1$ and $\mathfrak{I}_2$

(iv) in [20], $\mathfrak{I}_1$ and $\mathfrak{I}_2$ are said to be commuting if $\mathfrak{I}_1 \mathfrak{I}_2 \varpi = \mathfrak{I}_2 \mathfrak{I}_1 \varpi$ for all $\varpi \in \Xi$

(v) in [21], $\mathfrak{I}_1$ and $\mathfrak{I}_2$ are said to be weakly compatible if they commute at their coincidence points, i.e., if $\mathfrak{I}_1\varpi = \mathfrak{I}_2\varpi$ for some $\varpi \in \Xi$ implies $\mathfrak{I}_1\mathfrak{I}_2\varpi = \mathfrak{I}_2\mathfrak{I}_1\varpi$

Proposition 5. (see [19]). *Let $\mathfrak{I}_1$ and $\mathfrak{I}_2$ be weakly compatible self-mappings on a set Ξ. If $\mathfrak{I}_1$ and $\mathfrak{I}_2$ have a unique point of coincidence $\rho = \mathfrak{I}_1\varpi = \mathfrak{I}_2\varpi$, then $\rho \in CFix(\mathfrak{I}_1, \mathfrak{I}_2)$ is unique.*

Lemma 6 (see [2], Lemma 2.5). *Let (Ξ, Λ) be a SMS. Then, we have $\Lambda(\varpi, \varpi, \varkappa) = \Lambda(\varkappa, \varkappa, \varpi)$ for all $\varpi, \varkappa \in \Xi$.*

Lemma 6 can be considered as a symmetry condition on an S-metric space.

Lemma 7 (see [2], Lemma 2.12). *Let (Ξ, Λ) be a SMS. If $\varpi_n \longrightarrow \varpi$ and $\varkappa_n \longrightarrow \varkappa$ as $n \longrightarrow \infty$, then $\Lambda(\varpi_n, \varpi_n, \varkappa_n) \longrightarrow \Lambda(\varpi, \varpi, \varkappa)$ as $n \longrightarrow \infty$.*

The link between a metric and a S-metric is demonstrated in the following lemma.

Lemma 8 (see [6]). *Let (Ξ, d) be a metric space. Then, the following properties are satisfied:*

(1) $\Lambda_d(\varpi, \varkappa) = d(\varpi, \vartheta) + d(\varkappa, \vartheta)$ for all $\varpi, \varkappa, \vartheta \in \Xi$ is a SMS

(2) $\varpi_n \longrightarrow \varpi$ in (Ξ, d) if and only if $\varpi_n \longrightarrow \varpi$ in (Ξ, Λ_d)

(3) $\{\varpi_n\}$ is Cauchy in (Ξ, d) if and only if $\{\varpi_n\}$ is Cauchy in (Ξ, Λ_d)

(4) (Ξ, d) is complete if and only if (Ξ, Λ_d) is complete

The function Λ_d described in Lemma 8 (1) is referred to as the S-metric created by the metric d. In [6, 9], there is an example of a S-metric that is not derived by any metric.

3. Main Results

Our first result is the following.

Theorem 9. *Let (Ξ, Λ) be a CSMS, and let $\mathfrak{I}_1, \mathfrak{I}_2 : \Xi \longrightarrow \Xi$ such that*

$$\int_0^{\Lambda(\mathfrak{I}_1\varpi,\mathfrak{I}_1\varkappa,\mathfrak{I}_2\vartheta)} \hbar(\wp)d\wp, \tag{5}$$

$$\begin{aligned}\leq k_1\int_0^{\Lambda(\varpi,\varkappa,\vartheta)}\hbar(\wp)d\wp+k_2\int_0^{\Lambda(\varpi,\varkappa,\mathfrak{I}_1\varpi)}\hbar(\wp)d\wp+k_3\int_0^{\Lambda(\vartheta,\varkappa,\mathfrak{I}_2\vartheta)}\hbar(\wp)d\wp+k_4\\ \cdot\int_0^{\Lambda(\vartheta,\varkappa,\mathfrak{I}_2\vartheta)[1+\Lambda(\varpi,\varkappa,\mathfrak{I}_1\varpi)]/[1+\Lambda(\varpi,\varkappa,\vartheta)]}\hbar(\wp)d\wp+k_5\\ \cdot\int_0^{\Lambda(\varpi,\varkappa,\mathfrak{I}_2\vartheta)[1+\Lambda(\varpi,\varkappa,\mathfrak{I}_1\varpi)]/[1+\Lambda(\varpi,\varkappa,\vartheta)]}\hbar(\wp)d\wp+k_6\\ \cdot\int_0^{\Lambda(\varpi,\varkappa,\mathfrak{I}_1\varpi)\Lambda(\vartheta,\varkappa,\mathfrak{I}_2\vartheta)/[2\Lambda(\varpi,\varkappa,\mathfrak{I}_2\vartheta)+\Lambda(\vartheta,\varkappa,\mathfrak{I}_1\varpi)+\Lambda(\varpi,\varkappa,\vartheta)]}\hbar(\wp)d\wp,\end{aligned} \tag{6}$$

for all $\varpi, \varkappa, \vartheta \in \Xi$, where k_1, k_2, k_3, k_4, k_5, and k_6 are nonnegative reals such that $k_1 + k_2 + 4k_3 + 4k_4 + 3k_5 + 2k_6 < 1$ and $\hbar$ is defined in (4). Then, $\varpi^* \in CFix(\mathfrak{I}_1, \mathfrak{I}_2)$ is a singleton set.

Proof. Let $\varpi_0 \in \Xi$ and the sequence $\{\varpi_n\}$ be defined as $\varpi_{2n+1} = \mathfrak{I}_1\varpi_{2n}$ and $\varpi_{2n+2} = \mathfrak{I}_2\varpi_{2n+1}$ for $n = 1, 2, \cdots$. Then, from inequality (6) for $\varpi = \varkappa = \varpi_{2n-1}$ and $\vartheta = \varpi_{2n}$ and using notion of SMS and Lemma 6, we have

$$\begin{aligned}\int_0^{\Lambda(\varpi_{2n},\varpi_{2n},\varpi_{2n+1})}\hbar(\wp)d\wp &= \int_0^{\Lambda(\mathfrak{I}_1\varpi_{2n-1},\mathfrak{I}_1\varpi_{2n-1},\mathfrak{I}_2\varpi_{2n})}\hbar(\wp)d\wp \leq k_1\int_0^{\Lambda(\varpi_{2n-1},\varpi_{2n-1},\varpi_{2n})}\hbar(\wp)d\wp+k_2\int_0^{\Lambda(\varpi_{2n-1},\varpi_{2n-1},\mathfrak{I}_1\varpi_{2n-1})}\hbar(\wp)d\wp+k_3\\ &\cdot\int_0^{\Lambda(\varpi_{2n},\varpi_{2n-1},\mathfrak{I}_2\varpi_{2n})}\hbar(\wp)d\wp+k_4\int_0^{\Lambda(\varpi_{2n},\varpi_{2n-1},\mathfrak{I}_2\varpi_{2n})[1+\Lambda(\varpi_{2n-1},\varpi_{2n-1},\mathfrak{I}_1\varpi_{2n-1})]/[1+\Lambda(\varpi_{2n-1},\varpi_{2n-1},\varpi_{2n})]}\hbar(\wp)d\wp+k_5\\ &\cdot\int_0^{\Lambda(\varpi_{2n-1},\varpi_{2n-1},\mathfrak{I}_2\varpi_{2n})[1+\Lambda(\varpi_{2n-1},\varpi_{2n-1},\mathfrak{I}_1\varpi_{2n-1})]/[1+\Lambda(\varpi_{2n-1},\varpi_{2n-1},\varpi_{2n})]}\hbar(\wp)d\wp+k_6\\ &\cdot\int_0^{\Lambda(\varpi_{2n-1},\varpi_{2n-1},\mathfrak{I}_1\varpi_{2n-1})\Lambda(\varpi_{2n},\varpi_{2n-1},\mathfrak{I}_2\varpi_{2n})/[2\Lambda(\varpi_{2n-1},\varpi_{2n-1},\mathfrak{I}_2\varpi_{2n})+\Lambda(\varpi_{2n},\varpi_{2n-1},\mathfrak{I}_1\varpi_{2n-1})+\Lambda(\varpi_{2n-1},\varpi_{2n-1},\varpi_{2n})]}\hbar(\wp)d\wp\\ &= k_1\int_0^{\Lambda(\varpi_{2n-1},\varpi_{2n-1},\varpi_{2n})}\hbar(\wp)d\wp+k_2\int_0^{\Lambda(\varpi_{2n-1},\varpi_{2n-1},\varpi_{2n})}\hbar(\wp)d\wp+k_3\int_0^{\Lambda(\varpi_{2n},\varpi_{2n-1},\varpi_{2n+1})}\hbar(\wp)d\wp+k_4\\ &\cdot\int_0^{\Lambda(\varpi_{2n},\varpi_{2n-1},\varpi_{2n+1})[1+\Lambda(\varpi_{2n-1},\varpi_{2n-1},\varpi_{2n})]/[1+\Lambda(\varpi_{2n-1},\varpi_{2n-1},\varpi_{2n})]}\hbar(\wp)d\wp+k_5\\ &\cdot\int_0^{\Lambda(\varpi_{2n-1},\varpi_{2n-1},\varpi_{2n+1})[1+\Lambda(\varpi_{2n-1},\varpi_{2n-1},\varpi_{2n})]/[1+\Lambda(\varpi_{2n-1},\varpi_{2n-1},\varpi_{2n})]}\hbar(\wp)d\wp+k_6\\ &\cdot\int_0^{\Lambda(\varpi_{2n-1},\varpi_{2n-1},\varpi_{2n})\Lambda(\varpi_{2n},\varpi_{2n-1},\varpi_{2n+1})/[2\Lambda(\varpi_{2n-1},\varpi_{2n-1},\varpi_{2n+1})+\Lambda(\varpi_{2n},\varpi_{2n-1},\varpi_{2n})+\Lambda(\varpi_{2n-1},\varpi_{2n-1},\varpi_{2n})]}\hbar(\wp)d\wp\\ &\leq (k_1+k_2+2k_3+2k_4+2k_5+k_6)\int_0^{\Lambda(\varpi_{2n-1},\varpi_{2n-1},\varpi_{2n})}\hbar(\wp)d\wp+(2k_3+2k_4+k_5+k_6)\int_0^{\Lambda(\varpi_{2n},\varpi_{2n},\varpi_{2n+1})}\hbar(\wp)d\wp,\end{aligned} \tag{7}$$

which implies

$$\int_0^{\Lambda(\varpi_{2n},\varpi_{2n},\varpi_{2n+1})} \hbar(\wp)d\wp \le \left(\frac{k_1+k_2+2k_3+2k_4+2k_5+k_6}{1-2k_3-2k_4-k_5-k_6}\right), \tag{8}$$

$$\int_0^{\Lambda(\varpi_{2n-1},\varpi_{2n-1},\varpi_{2n})} \hbar(\wp)d\wp. \tag{9}$$

If we take $\mu = ((k_1+k_2+2k_3+2k_4+2k_5+k_6)/(1-2k_3-2k_4-k_5-k_6))$, then we find $\mu<1$ since $k_1+k_2+4k_3+4k_4+3k_5+2k_6<1$. Using the inequality (9) again, we obtain

$$\int_0^{\Lambda(\varpi_{2n},\varpi_{2n},\varpi_{2n+1})} \hbar(\wp)d\wp \le \mu^n \int_0^{\Lambda(\varpi_0,\varpi_0,\varpi_1)} \hbar(\wp)d\wp. \tag{10}$$

Thus, in general, for $n=0,1,2,\cdots$, we have

$$\int_0^{\Lambda(\varpi_n,\varpi_n,\varpi_{n+1})} \hbar(\wp)d\wp \le \mu^n \int_0^{\Lambda(\varpi_0,\varpi_0,\varpi_1)} \hbar(\wp)d\wp. \tag{11}$$

Passing limit $n\longrightarrow\infty$ (11),

$$\lim_{n\longrightarrow\infty}\int_0^{\Lambda(\varpi_n,\varpi_n,\varpi_{n+1})} \hbar(\wp)d\wp = 0, \tag{12}$$

since $0<\mu<1$. Condition (4) implies

$$\lim_{n\longrightarrow\infty}\Lambda(\varpi_n,\varpi_n,\varpi_{n+1}) = 0. \tag{13}$$

Next to show that the sequence $\{\varpi_n\}$ is a Cauchy sequence. Assume that $\{\varpi_n\}$ is not a Cauchy sequence. Then, there exists an $\varepsilon>0$ for which we can find subsequences $\{\varpi_{2m(j)}\}$ and $\{\varpi_{2n(j)}\}$ of $\{\varpi_{2n}\}$ and increasing sequences of integers $\{2m(j)\}$ and $\{2n(j)\}$ such that $n(j)$ is smallest index for which,

$$n(j) > m(j) > k, \tag{14}$$

$$\Lambda\left(\varpi_{2m(j)},\varpi_{2m(j)},\varpi_{2n(j)}\right) \ge \varepsilon. \tag{15}$$

Further corresponding to $m(j)$, we can choose $n(j)$ in such a way that it is the smallest integer with $n(j)>m(j)$ and satisfying (14). Then,

$$\Lambda\left(\varpi_{2m(j)},\varpi_{2m(j)},\varpi_{2n(j)-1}\right) < \varepsilon. \tag{16}$$

Now, using (16), (S2), and Lemma 6, we have the following equation (by (16)):

$$\begin{aligned}\varepsilon &\le \Lambda\left(\varpi_{2m(j)},\varpi_{2m(j)},\varpi_{2n(j)}\right) = \Lambda\left(\varpi_{2n(j)},\varpi_{2n(j)},\varpi_{2m(j)}\right)\\ &\le 2\Lambda\left(\varpi_{2n(j)},\varpi_{2n(j)},\varpi_{2n(j)-1}\right) + \Lambda\left(\varpi_{2m(j)},\varpi_{2m(j)},\varpi_{2n(j)-1}\right)\\ &\le \varepsilon + 2\Lambda\left(\varpi_{2n(j)},\varpi_{2n(j)},\varpi_{2n(j)-1}\right).\end{aligned} \tag{17}$$

Passing $k\longrightarrow\infty$ in (17) and using (13),

$$\lim_{k\longrightarrow\infty}\Lambda\left(\varpi_{2m(j)},\varpi_{2m(j)},\varpi_{2n(j)}\right) = \varepsilon. \tag{18}$$

Owing (S2) and Lemma 6,

$$\begin{aligned}&\Lambda\left(\varpi_{2m(j)},\varpi_{2m(j)},\varpi_{2n(j)}\right)\\ &\quad\le 2\Lambda\left(\varpi_{2m(j)},\varpi_{2m(j)},\varpi_{2m(j)-1}\right) + \Lambda\left(\varpi_{2n(j)},\varpi_{2n(j)},\varpi_{2m(j)-1}\right)\\ &\quad= 2\Lambda\left(\varpi_{2m(j)},\varpi_{2m(j)},\varpi_{2m(j)-1}\right) + \Lambda\left(\varpi_{2m(j)-1},\varpi_{2m(j)-1},\varpi_{2n(j)}\right),\\ &\Lambda\left(\varpi_{2m(j)-1},\varpi_{2m(j)-1},\varpi_{2n(j)-1}\right)\\ &\quad\le 2\Lambda\left(\varpi_{2m(j)-1},\varpi_{2m(j)-1},\varpi_{2m(j)}\right) + \Lambda\left(\varpi_{2n(j)-1},\varpi_{2n(j)-1},\varpi_{2m(j)}\right)\\ &\quad= 2\Lambda\left(\varpi_{2m(j)},\varpi_{2m(j)},\varpi_{2m(j)-1}\right) + \Lambda\left(\varpi_{2m(j)},\varpi_{2m(j)},\varpi_{2n(j)-1}\right),\end{aligned} \tag{19}$$

$$\lim_{k\longrightarrow\infty}\int_0^{\Lambda\left(\varpi_{2m(j)-1},\varpi_{2m(j)-1},\varpi_{2n(j)-1}\right)} \hbar(\wp)d\wp \le \int_0^{\varepsilon}\hbar(\wp)d\wp. \tag{20}$$

Using the inequality (6) for $\varpi=\varkappa=\varpi_{2m(j)-1}$ and $\vartheta=\varpi_{2n(j)-1}$ and equations (9), (15), and (20), then we obtain

$$\begin{aligned}\int_0^{\varepsilon}\hbar(\wp)d\wp &\le \int_0^{\Lambda\left(\varpi_{2m(j)},\varpi_{2m(j)},\varpi_{2n(j)}\right)}\hbar(\wp)d\wp\\ &= \int_0^{\Lambda\left(\mathfrak{I}_1\varpi_{2m(j)-1},\mathfrak{I}_1\varpi_{2m(j)-1},\mathfrak{I}_2\varpi_{2n(j)-1}\right)}\hbar(\wp)d\wp\\ &\le \mu\int_0^{\Lambda\left(\varpi_{2m(j)-1},\varpi_{2m(j)-1},\varpi_{2n(j)-1}\right)}\hbar(\wp)d\wp\\ &\le \mu\int_0^{\varepsilon}\hbar(\wp)d\wp,\end{aligned} \tag{21}$$

a contradiction to the assumption as $0<\mu<1$. Thus, the sequence $\{\varpi_n\}$ is a Cauchy sequence in Ξ. Thus, $\varpi^*\in\Xi$, i.e, $\lim_{n\longrightarrow\infty}\varpi_n=\varpi^*$, by completeness of Ξ.

Now, from the given inequality (6) for $\varpi = \varkappa = \varpi^*$ and $\vartheta = \varpi_{2n}$, we find

$$\begin{aligned}
&\int_0^{\Lambda(\mathfrak{T}_1\varpi^*,\mathfrak{T}_1\varpi^*,\varpi_{2n+1})} \hbar(\wp)d\wp = \int_0^{\Lambda(\mathfrak{T}_1\varpi^*,\mathfrak{T}_1\varpi^*,\mathfrak{T}_2\varpi_{2n})} \hbar(\wp)d\wp \\
&\le k_1 \int_0^{\Lambda(\varpi^*,\varpi^*,\varpi_{2n})} \hbar(\wp)d\wp + k_2 \int_0^{\Lambda(\varpi^*,\varpi^*,\mathfrak{T}_1\varpi^*)} \hbar(\wp)d\wp + k_3 \\
&\quad \cdot \int_0^{\Lambda(\varpi_{2n},\varpi^*,\mathfrak{T}_2\varpi_{2n})} \hbar(\wp)d\wp + k_4 \\
&\quad \cdot \int_0^{\Lambda(\varpi_{2n},\varpi^*,\mathfrak{T}_2\varpi_{2n})[1+\Lambda(\varpi^*,\varpi^*,\mathfrak{T}_1\varpi^*)]/[1+\Lambda(\varpi^*,\varpi^*,\varpi_{2n})]} \hbar(\wp)d\wp + k_5 \\
&\quad \cdot \int_0^{\Lambda(\varpi^*,\varpi^*,\mathfrak{T}_2\varpi_{2n})[1+\Lambda(\varpi^*,\varpi^*,\mathfrak{T}_1\varpi^*)]/[1+\Lambda(\varpi^*,\varpi^*,\varpi_{2n})]} \hbar(\wp)d\wp + k_6 \\
&\quad \cdot \int_0^{\Lambda(\varpi^*,\varpi^*,\mathfrak{T}_1\varpi^*)\Lambda(\varpi_{2n},\varpi^*,\mathfrak{T}_2\varpi_{2n})/[2\Lambda(\varpi^*,\varpi^*,\mathfrak{T}_2\varpi_{2n})+\Lambda(\varpi_{2n},\varpi^*,\mathfrak{T}_1\varpi^*)+\Lambda(\varpi^*,\varpi^*,\varpi_{2n})]} \hbar(\wp)d\wp \\
&= k_1 \int_0^{\Lambda(\varpi^*,\varpi^*,\mathfrak{T}_1\varpi^*)} \hbar(\wp)d\wp + k_3 \int_0^{\Lambda(\varpi_{2n},u,\varpi_{2n+1})} \hbar(\wp)d\wp + k_4 \\
&\quad \cdot \int_0^{\Lambda(\varpi_{2n},\varpi^*,\varpi_{2n+1})[1+\Lambda(\varpi^*,\varpi^*,\mathfrak{T}_1\varpi^*)]/[1+\Lambda(\varpi^*,\varpi^*,\varpi_{2n})]} \hbar(\wp)d\wp + k_5 \\
&\quad \cdot \int_0^{\Lambda(\varpi^*,\varpi^*,\varpi_{2n+1})[1+\Lambda(\varpi^*,\varpi^*,\mathfrak{T}_1\varpi^*)]/[1+\Lambda(\varpi^*,\varpi^*,\varpi_{2n})]} \hbar(\wp)d\wp + k_6 \\
&\quad \cdot \int_0^{\Lambda(\varpi^*,\varpi^*,\mathfrak{T}_1\varpi^*)\Lambda(\varpi_{2n},\varpi^*,\varpi_{2n+1})/[2\Lambda(\varpi^*,\varpi^*,\varpi_{2n+1})+\Lambda(\varpi_{2n},\varpi^*,\mathfrak{T}_1\varpi^*)+\Lambda(\varpi^*,\varpi^*,\varpi_{2n})]} \hbar(\wp)d\wp.
\end{aligned} \tag{22}$$

Passing $n \longrightarrow \infty$ in (23) and owing (S1) and Lemma 6,

$$\int_0^{\Lambda(\mathfrak{T}_1\varpi^*,\mathfrak{T}_1\varpi^*,\varpi^*)} \hbar(\wp)d\wp \le k_2 \int_0^{\Lambda(\mathfrak{T}_1\varpi^*,\mathfrak{T}_1\varpi^*,\varpi^*)} \hbar(\wp)d\wp, \tag{23}$$

which implies $\Lambda(\mathfrak{T}_1\varpi^*, \mathfrak{T}_1\varpi^*, \varpi^*) = 0$, that is, $\mathfrak{T}_1\varpi^* = \varpi^*$ since $k_2 < 1$. This implies that $\varpi^* \in \mathrm{Fix}(\mathfrak{T}_1)$. Similarly, $\varpi^* \in \mathrm{Fix}(\mathfrak{T}_2)$; hence, $\varpi^* \in CFix(\mathfrak{T}_1, \mathfrak{T}_2)$.

Let $\varpi^* \neq \varpi_1 \in \mathrm{CFix}(\mathfrak{T}_1, \mathfrak{T}_2)$ be the another CFP. Using the inequality (6) for $\varpi = \varkappa = \varpi^*$, $\vartheta = \varpi_1$, and Lemma 6, we obtain

$$\begin{aligned}
&\int_0^{\Lambda(\varpi^*,\varpi^*,\varpi_1)} \hbar(\wp)d\wp = \int_0^{\Lambda(\mathfrak{T}_1\varpi^*,\mathfrak{T}_1\varpi^*,\mathfrak{T}_2\varpi_1)} \hbar(\wp)d\wp \\
&\le k_1 \int_0^{\Lambda(\varpi^*,\varpi^*,\varpi_1)} \hbar(\wp)d\wp + k_2 \int_0^{\Lambda(\varpi^*,\varpi^*,\mathfrak{T}_1\varpi^*)} \hbar(\wp)d\wp + k_3 \int_0^{\Lambda(\varpi_1,\varpi^*,\mathfrak{T}_2\varpi_1)} \hbar(\wp)d\wp + k_4 \\
&\quad \cdot \int_0^{\Lambda(\varpi_1,\varpi^*,\mathfrak{T}_2\varpi_1)[1+\Lambda(\varpi^*,\varpi^*,\mathfrak{T}_1\varpi^*)]/[1+\Lambda(\varpi^*,\varpi^*,\varpi_1)]} \hbar(\wp)d\wp + k_5 \\
&\quad \cdot \int_0^{\Lambda(\varpi^*,\varpi^*,\mathfrak{T}_2\varpi_1)[1+\Lambda(\varpi^*,\varpi^*,\mathfrak{T}_1\varpi^*)]/[1+\Lambda(\varpi^*,\varpi^*,\varpi_1)]} \hbar(\wp)d\wp + k_6 \\
&\quad \cdot \int_0^{\Lambda(\varpi^*,\varpi^*,\mathfrak{T}_1\varpi^*)\Lambda(\varpi_1,\varpi^*,\mathfrak{T}_2\varpi_1)/[2\Lambda(\varpi^*,\varpi^*,\mathfrak{T}_2\varpi_1)+\Lambda(\varpi_1,\varpi^*,\mathfrak{T}_1\varpi^*)+\Lambda(\varpi^*,\varpi^*,\varpi_1)]} \hbar(\wp)d\wp, \\
&= k_1 \int_0^{\Lambda(\varpi^*,\varpi^*,\varpi_1)} \hbar(\wp)d\wp + k_2 \int_0^{\Lambda(\varpi^*,\varpi^*,\varpi^*)} \hbar(\wp)d\wp + k_3 \\
&\quad \cdot \int_0^{\Lambda(\varpi_1,\varpi^*,\varpi_1)} \hbar(\wp)d\wp + k_4 \int_0^{\Lambda(\varpi_1,\varpi^*,\varpi_1)[1+\Lambda(\varpi^*,\varpi^*,\varpi^*)]/[1+\Lambda(\varpi^*,\varpi^*,\varpi_1)]} \hbar(\wp)d\wp + k_5 \\
&\quad \cdot \int_0^{\Lambda(u,u,\varpi_1)[1+\Lambda(\varpi^*,\varpi^*,\varpi^*)]/[1+\Lambda(\varpi^*,\varpi^*,\varpi_1)]} \hbar(\wp)d\wp + k_6 \\
&\quad \cdot \int_0^{\Lambda(\varpi^*,\varpi^*,\varpi^*)\Lambda(\varpi_1,\varpi^*,\varpi_1)/[2\Lambda(\varpi^*,\varpi^*,\varpi_1)+\Lambda(\varpi_1,\varpi^*,\varpi^*)+\Lambda(\varpi^*,\varpi^*,\varpi_1)]} \hbar(\wp)d\wp \\
&\le (k_1 + 2k_3 + 2k_4 + k_5) \int_0^{\Lambda(\varpi^*,\varpi^*\varpi_1)} \hbar(\wp)d\wp,
\end{aligned} \tag{24}$$

which implies $\Lambda(\varpi^*, \varpi^*, \varpi_1) = 0$, that is, $\varpi^* = \varpi_1$ since $k_1 + 2k_3 + 2k_4 + k_5 < 1$. Thus, $\mathrm{CFix}(\mathfrak{T}_1, \mathfrak{T}_2)$ is a singleton set.

If we take $\mathfrak{T}_1 = \mathfrak{T}_2 = \mathfrak{T}$ in Theorem 9; then, we obtain the following result. □

Corollary 10. *Let (Ξ, Λ) be a CSMS and $\mathfrak{T} : \Xi \longrightarrow \Xi$ such that*

$$\begin{aligned}
&\int_0^{\Lambda(\mathfrak{T}\varpi,\mathfrak{T}\varkappa,\mathfrak{T}\vartheta)} \hbar(\wp)d\wp \le k_1 \int_0^{\Lambda(\varpi,\varkappa,\vartheta)} \hbar(\wp)d\wp + k_2 \\
&\quad \cdot \int_0^{\Lambda(\varpi,\varkappa,\mathfrak{T}\varpi)} \hbar(\wp)d\wp + k_3 \int_0^{\Lambda(\vartheta,\varkappa,\mathfrak{T}\vartheta)} \hbar(\wp)d\wp + k_4 \\
&\quad \cdot \int_0^{\Lambda(\vartheta,\varkappa,\mathfrak{T}\vartheta)[1+\Lambda(\varpi,\varkappa,\mathfrak{T}\varpi)]/[1+\Lambda(\varpi,\varkappa,\vartheta)]} \hbar(\wp)d\wp + k_5 \\
&\quad \cdot \int_0^{\Lambda(\varpi,\varkappa,\mathfrak{T}\vartheta)[1+\Lambda(\varpi,\varkappa,\mathfrak{T}\varpi)]/[1+\Lambda(\varpi,\varkappa,\vartheta)]} \hbar(\wp)d\wp + k_6 \\
&\quad \cdot \int_0^{\Lambda(\varpi,\varkappa,\mathfrak{T}\varpi)\Lambda(\vartheta,\varkappa,\mathfrak{T}\vartheta)/[2\Lambda(\varpi,\varkappa,\mathfrak{T}\vartheta)+\Lambda(\vartheta,\varkappa,\mathfrak{T}\varpi)+\Lambda(\varpi,\varkappa,\vartheta)]} \hbar(\wp)d\wp,
\end{aligned} \tag{25}$$

for all $\varpi, \varkappa, \vartheta \in \Xi$, where k_1, k_2, k_3, k_4, k_5, and k_6 are nonnegative reals such that $k_1 + k_2 + 4k_3 + 4k_4 + 3k_5 + 2k_6 < 1$ and $\hbar$ is defined in (4). Then, $\varpi^* \in Fix(\mathfrak{T})$ is a singleton set.

If we take $k_1 = h$ and $k_2 = k_3 = k_4 = k_5 = k_6 = 0$ in Corollary 10, then we obtain the following result.

Corollary 11 (see [16]). *Let (Ξ, Λ) be a CSMS and let $\mathfrak{T} : \Xi \longrightarrow \Xi$ such that*

$$\int_0^{\Lambda(\mathfrak{T}\varpi,\mathfrak{T}\varkappa,\mathfrak{T}\vartheta)} \hbar(\wp)d\wp \le h \int_0^{\Lambda(\varpi,\varkappa,\vartheta)} \hbar(\wp)d\wp, \tag{26}$$

for all $\varpi, \varkappa, \vartheta \in \Xi$, where $h \in (0, 1)$ is a constant and $\hbar$ is defined in (4). Then, $\varpi^* \in Fix(\mathfrak{T})$ is a singleton set and $\mathfrak{T}^n \rho = \varpi^*$ for each $\rho \in \Xi$.

Remark 12.

(1) Corollary 11 is a generalization of Branciari [11] fixed point result from complete metric space to the setup of CSMS

(2) In Corollary 11, if we set $\hbar(\wp) = 1$ for all $\wp \in [0, \infty)$, we get Theorem 3.1 in [2], Theorem 3.1 in [22], and Corollary 2.5 in [8] for $n = 1$

(3) Corollary 11 is also a generalization of Sedghi et al.'s result [2] to the case of integral type contraction condition

(4) Theorem 9 and Corollary 10 are generalization of Theorem 2.4 of Özgür and Taş [16]. Indeed, if we take $\mathfrak{T}_1 = \mathfrak{T}_2 = \mathfrak{T}$, $k_1 = h$, and $k_2 = k_3 = k_4 = k_5 = k_6 = 0$ in Theorem 9; then, we get Theorem 2.4 in [16]

Theorem 13. *Let (Ξ, Λ) be a CSMS, and let $\mathfrak{I}_1, \mathfrak{I}_2, \mathfrak{I}_3, \mathfrak{I}_4 : \Xi \longrightarrow \Xi$ be four mappings satisfying the following conditions:*

$$\begin{aligned}\int_0^{\Lambda(\mathfrak{I}_1\varpi,\mathfrak{I}_1\varkappa,\mathfrak{I}_2\vartheta)}\hbar(\wp)d\wp &\le k_1\int_0^{\Lambda(\mathfrak{I}_3\varpi,\mathfrak{I}_3\varkappa,\mathfrak{I}_4\vartheta)}\hbar(\wp)d\wp+k_2\\ &\cdot\int_0^{\Lambda(\mathfrak{I}_3\varpi,\mathfrak{I}_3\varkappa,\mathfrak{I}_1\varpi)}\hbar(\wp)d\wp+k_3\int_0^{\Lambda(\mathfrak{I}_4\vartheta,\mathfrak{I}_4\varkappa,\mathfrak{I}_2\vartheta)}\hbar(\wp)d\wp+k_4\\ &\cdot\int_0^{\Lambda(\mathfrak{I}_4\vartheta,\mathfrak{I}_4\varkappa,\mathfrak{I}_1\varpi)}\hbar(\wp)d\wp+k_5\int_0^{\Lambda(\mathfrak{I}_3\varpi,\mathfrak{I}_3\varkappa,\mathfrak{I}_2\vartheta)}\hbar(\wp)d\wp+k_6\\ &\cdot\int_0^{\max\{\Lambda(\mathfrak{I}_3\varpi,\mathfrak{I}_3\varkappa,\mathfrak{I}_2\vartheta),\Lambda(\mathfrak{I}_4\vartheta,\mathfrak{I}_4\varkappa,\mathfrak{I}_1\varpi)\}}\hbar(\wp)d\wp,\end{aligned} \tag{27}$$

(i) *$\mathfrak{I}_1(\Xi) \subseteq \mathfrak{I}_3(\Xi)$ and $\mathfrak{I}_2(\Xi) \subseteq \mathfrak{I}_4(\Xi)$*

(ii) *The pairs $(\mathfrak{I}_2, \mathfrak{I}_3)$ and $(\mathfrak{I}_1, \mathfrak{I}_4)$ are weakly compatible*

for all $\varpi, \varkappa, \vartheta \in \Xi$, where k_1, k_2, k_3, k_4, k_5, and k_6 are nonnegative reals such that $k_1 + k_2 + k_3 + 2k_4 + 2k_6 < 1$ and $\hbar$ is defined in (4). Then, $CFix(\mathfrak{I}_1, \mathfrak{I}_2, \mathfrak{I}_3, \mathfrak{I}_4)$ is a singleton set.

Proof. Let $\varpi_0 \in \Xi$ and the sequence $\{\varkappa_n\}$ be defined as

$$\begin{aligned}\varkappa_n &= \mathfrak{I}_2\varpi_n = \mathfrak{I}_4\varpi_{n+1},\\ \varkappa_{n+1} &= \mathfrak{I}_1\varpi_{n+1} = \mathfrak{I}_3\varpi_{n+2},\end{aligned} \tag{28}$$

for $n = 1, 2, \cdots$. Then, from (27) for $\varpi = \varkappa = \varpi_{n+1}$ and $\vartheta = \varpi_n$ and owing (28), (S1), (S2), and Lemma 6,

$$\begin{aligned}\int_0^{\Lambda(\varkappa_{n+1},\varkappa_{n+1},\varkappa_n)}\hbar(\wp)d\wp &= \int_0^{\Lambda(\mathfrak{I}_1\varpi_{n+1},\mathfrak{I}_1\varpi_{n+1},\mathfrak{I}_2\varpi_n)}\hbar(\wp)d\wp\\ &\le k_1\int_0^{\Lambda(\mathfrak{I}_3\varpi_{n+1},\mathfrak{I}_3\varpi_{n+1},\mathfrak{I}_4\varpi_n)}\hbar(\wp)d\wp+k_2\\ &\cdot\int_0^{\Lambda(\mathfrak{I}_3\varpi_{n+1},\mathfrak{I}_3\varpi_{n+1},\mathfrak{I}_1\varpi_{n+1})}\hbar(\wp)d\wp+k_3\\ &\cdot\int_0^{\Lambda(\mathfrak{I}_4\varpi_n,\mathfrak{I}_4\varpi_{n+1},\mathfrak{I}_2\varpi_n)}\hbar(\wp)d\wp+k_4\\ &\cdot\int_0^{\Lambda(\mathfrak{I}_4\varpi_n,\mathfrak{I}_4\varpi_{n+1},\mathfrak{I}_1\varpi_{n+1})}\hbar(\wp)d\wp+k_5\\ &\cdot\int_0^{\Lambda(\mathfrak{I}_3\varpi_{n+1},\mathfrak{I}_3\varpi_{n+1},\mathfrak{I}_2\varpi_n)}\hbar(\wp)d\wp+k_6\\ &\cdot\int_0^{\max\{\Lambda(\mathfrak{I}_3\varpi_{n+1},\mathfrak{I}_3\varpi_{n+1},\mathfrak{I}_2\varpi_n),\Lambda(\mathfrak{I}_4\varpi_n,\mathfrak{I}_4\varpi_{n+1},\mathfrak{I}_1\varpi_{n+1})\}}\hbar(\wp)d\wp,\\ &= k_1\int_0^{\Lambda(\varkappa_n,\varkappa_n,\varkappa_{n-1})}\hbar(\wp)d\wp+k_2\int_0^{\Lambda(\varkappa_n,\varkappa_n,\varkappa_{n+1})}\hbar(\wp)d\wp+k_3\\ &\cdot\int_0^{\Lambda(\varkappa_{n-1},\varkappa_n,\varkappa_n)}\hbar(\wp)d\wp+k_4\int_0^{\Lambda(\varkappa_{n-1},\varkappa_n,\varkappa_{n+1})}\hbar(\wp)d\wp+k_5\\ &\cdot\int_0^{\Lambda(\varkappa_n,\varkappa_n,\varkappa_n)}\hbar(\wp)d\wp+k_6\int_0^{\max\{\Lambda(\varkappa_n,\varkappa_n,\varkappa_n),\Lambda(\varkappa_{n-1},\varkappa_n,\varkappa_{n+1})\}}\hbar(\wp)d\wp\\ &\le (k_1+k_3+k_4+k_6)\int_0^{\Lambda(\varkappa_n,\varkappa_n,\varkappa_{n-1})}\hbar(\wp)d\wp+(k_2+k_4+k_6)\\ &\cdot\int_0^{\Lambda(\varkappa_{n+1},\varkappa_{n+1},\varkappa_n)}\hbar(\wp)d\wp,\end{aligned} \tag{29}$$

which implies

$$\int_0^{\Lambda(\varkappa_{n+1},\varkappa_{n+1},\varkappa_n)}\hbar(\wp)d\wp \le \left(\frac{k_1+k_3+k_4+k_6}{1-k_2-k_4-k_6}\right)\int_0^{\Lambda(\varkappa_n,\varkappa_n,\varkappa_{n-1})}\hbar(\wp)d\wp. \tag{30}$$

If we take $\alpha = (k_1 + k_3 + k_4 + k_6)/(1 - k_2 - k_4 - k_6)$, then we find $\alpha < 1$ since $k_1 + k_2 + k_3 + 2k_4 + 2k_6 < 1$. Using the inequality (30) again, we obtain

$$\int_0^{\Lambda(\varkappa_{n+1},\varkappa_{n+1},\varkappa_n)}\hbar(\wp)d\wp \le \alpha^n\int_0^{\Lambda(\varkappa_0,\varkappa_0,\varkappa_1)}\hbar(\wp)d\wp. \tag{31}$$

Passing $n \longrightarrow \infty$ in (35),

$$\lim_{n\longrightarrow\infty}\int_0^{\Lambda(\varkappa_{n+1},\varkappa_{n+1},\varkappa_n)}\hbar(\wp)d\wp = 0, \tag{32}$$

since $0 < \alpha < 1$. Condition (4) implies

$$\lim_{n\longrightarrow\infty}\Lambda(\varkappa_{n+1},\varkappa_{n+1},\varkappa_n) = 0. \tag{33}$$

Next to prove that the sequence $\{\varkappa_n\}$ is a Cauchy sequence. Let $m > n$, where $m, n \in \mathbb{N}$, from equations (27) and (28), we find

$$\begin{aligned}\int_0^{\Lambda(\varkappa_n,\varkappa_n,\varkappa_m)}\hbar(\wp)d\wp &= \int_0^{\Lambda(\mathfrak{I}_1\varpi_n,\mathfrak{I}_1\varpi_n,\mathfrak{I}_2\varpi_m)}\hbar(\wp)d\wp\\ &\le k_1\int_0^{\Lambda(\mathfrak{I}_3\varpi_n,\mathfrak{I}_3\varpi_n,\mathfrak{I}_4\varpi_m)}\hbar(\wp)d\wp+k_2\int_0^{\Lambda(\mathfrak{I}_3\varpi_n,\mathfrak{I}_3\varpi_n,\mathfrak{I}_1\varpi_n)}\hbar(\wp)d\wp+k_3\\ &\cdot\int_0^{\Lambda(\mathfrak{I}_4\varpi_m,\mathfrak{I}_4\varpi_n,\mathfrak{I}_2\varpi_m)}\hbar(\wp)d\wp+k_4\int_0^{\Lambda(\mathfrak{I}_4\varpi_m,\mathfrak{I}_4\varpi_n,\mathfrak{I}_1\varpi_n)}\hbar(\wp)d\wp+k_5\\ &\cdot\int_0^{\Lambda(\mathfrak{I}_3\varpi_n,\mathfrak{I}_3\varpi_n,\mathfrak{I}_2\varpi_m)}\hbar(\wp)d\wp+k_6\\ &\cdot\int_0^{\max\{\Lambda(\mathfrak{I}_3\varpi_n,\mathfrak{I}_3\varpi_n,\mathfrak{I}_2\varpi_m),\Lambda(\mathfrak{I}_4\varpi_m,\mathfrak{I}_4\varpi_n,\mathfrak{I}_1\varpi_n)\}}\hbar(\wp)d\wp,\end{aligned} \tag{34}$$

$$\begin{aligned}&= k_1\int_0^{\Lambda(\varkappa_{n-1},\varkappa_{n-1},\varkappa_{m-1})}\hbar(\wp)d\wp+k_2\int_0^{\Lambda(\varkappa_{n-1},\varkappa_{n-1},\varkappa_n)}\hbar(\wp)d\wp+k_3\\ &\cdot\int_0^{\Lambda(\varkappa_{m-1},\varkappa_{n-1},\varkappa_m)}\hbar(\wp)d\wp+k_4\int_0^{\Lambda(\varkappa_{m-1},\varkappa_{n-1},\varkappa_n)}\hbar(\wp)d\wp+k_5\\ &\cdot\int_0^{\Lambda(\varkappa_{n-1},\varkappa_{n-1},\varkappa_m)}\hbar(\wp)d\wp+k_6\\ &\cdot\int_0^{\max\{\Lambda(\varkappa_{n-1},\varkappa_{n-1},\varkappa_m),\Lambda(\varkappa_{m-1},\varkappa_{n-1},\varkappa_n)\}}\hbar(\wp)d\wp.\end{aligned} \tag{35}$$

□

By the similar manner, for $m > n$, we have following cases:

Case 1. If $\Lambda(\varkappa_{n-1}, \varkappa_{n-1}, \varkappa_m) > \Lambda(\varkappa_{m-1}, \varkappa_{n-1}, \varkappa_n)$,

$$\int_0^{\Lambda(\varkappa_n,\varkappa_n,\varkappa_m)} \hbar(\wp)d\wp \leq \left(\frac{2k_1 + k_2 + 2k_3 + 2k_4 + 2k_5 + 2k_6}{1 - k_1 - k_3 - 2k_4 - k_5 - k_6}\right) \cdot \int_0^{\Lambda(\varkappa_{n-1},\varkappa_{n-1},\varkappa_n)} \hbar(\wp)d\wp + \left(\frac{2k_1 + k_3 + k_4 + k_6}{1 - k_1 - k_3 - 2k_4 - k_5 - k_6}\right) \cdot \int_0^{\Lambda(\varkappa_{m-1},\varkappa_{m-1},\varkappa_m)} \hbar(\wp)d\wp. \tag{36}$$

Case 2. If $\Lambda(\varkappa_{n-1}, \varkappa_{n-1}, \varkappa_m) < \Lambda(\varkappa_{m-1}, \varkappa_{n-1}, \varkappa_n)$,

$$\int_0^{\Lambda(\varkappa_n,\varkappa_n,\varkappa_m)} \hbar(\wp)d\wp \leq \left(\frac{2k_1 + k_2 + 2k_3 + 2k_4 + 2k_5 + 2k_6}{1 - k_1 - k_3 - k_4 - k_5 - 2k_6}\right) \cdot \int_0^{\Lambda(\varkappa_{n-1},\varkappa_{n-1},\varkappa_n)} \hbar(\wp)d\wp + \left(\frac{2k_1 + k_3 + k_4 + k_6}{1 - k_1 - k_3 - k_4 - k_5 - 2k_6}\right) \cdot \int_0^{\Lambda(\varkappa_{m-1},\varkappa_{m-1},\varkappa_m)} \hbar(\wp)d\wp. \tag{37}$$

Passing $m, n \longrightarrow \infty$ in both the cases,

$$\int_0^{\Lambda(\varkappa_n,\varkappa_n,\varkappa_m)} \hbar(\wp)d\wp \leq \left(\alpha^{n-1} + \alpha^{m-1}\right) \int_0^{\Lambda(\varkappa_0,\varkappa_0,\varkappa_1)} \hbar(\wp)d\wp \longrightarrow 0 \quad \text{as} \quad n, m \longrightarrow \infty, \text{since} \quad 0 < \alpha < 1. \tag{38}$$

Hence, $\{\varkappa_n\}$ is a Cauchy sequence in a CSMS Ξ, so it is convergent to the point, say $\varpi^* \in \Xi$, that is, $\lim_{n\longrightarrow\infty}\varkappa_n = \varpi^*$ and

$$\varpi^* = \lim_{n\longrightarrow\infty} \mathfrak{J}_2\varpi_n = \lim_{n\longrightarrow\infty} \mathfrak{J}_4\varpi_{n+1} = \lim_{n\longrightarrow\infty} \mathfrak{J}_1\varpi_{n+1} = \lim_{n\longrightarrow\infty} \mathfrak{J}_3\varpi_{n+2}. \tag{39}$$

Since $\mathfrak{J}_1(\Xi) \subseteq \mathfrak{J}_3(\Xi)$, there exists a point $\rho \in \Xi$ such that $\varpi^* = \mathfrak{J}_3\rho$. If $\varpi^* \neq \mathfrak{J}_2\rho$, then from equation (30), we get

$$\begin{aligned}\int_0^{\Lambda(\varkappa_{n+1},\varkappa_{n+1},\mathfrak{J}_2\rho)} \hbar(\wp)d\wp &= \int_0^{\Lambda(\mathfrak{J}_1\varpi_{n+1},\mathfrak{J}_1\varpi_{n+1},\mathfrak{J}_2 w)} \hbar(\wp)d\wp \\ &\leq k_1 \int_0^{\Lambda(\mathfrak{J}_3\varpi_{n+1},\mathfrak{J}_3\varpi_{n+1},\mathfrak{J}_4\rho)} \hbar(\wp)d\wp + k_2 \cdot \int_0^{\Lambda(\mathfrak{J}_3\varpi_{n+1},\mathfrak{J}_3\varpi_{n+1},\mathfrak{J}_1\varpi_{n+1})} \hbar(\wp)d\wp + k_3 \\ &\cdot \int_0^{\Lambda(\mathfrak{J}_4\rho,\mathfrak{J}_4\varpi_{n+1},\mathfrak{J}_2\rho)} \hbar(\wp)d\wp + k_4 \cdot \int_0^{\Lambda(\mathfrak{J}_4\rho,\mathfrak{J}_4\varpi_{n+1},\mathfrak{J}_1\varpi_{n+1})} \hbar(\wp)d\wp + k_5 \\ &\cdot \int_0^{\Lambda(\mathfrak{J}_3\varpi_{n+1},\mathfrak{J}_3\varpi_{n+1},\mathfrak{J}_2\rho)} \hbar(\wp)d\wp + k_6 \cdot \int_0^{\max\left\{\Lambda(\mathfrak{J}_3\varpi_{n+1},\mathfrak{J}_3\varpi_{n+1},\mathfrak{J}_2\rho),\Lambda(\mathfrak{J}_4\rho,\mathfrak{J}_4\varpi_{n+1},\mathfrak{J}_1\varpi_{n+1})\right\}} \hbar(\wp)d\wp,\end{aligned} \tag{40}$$

$$\begin{aligned}&= k_1 \int_0^{\Lambda(\mathfrak{J}_3\varpi_{n+1},\mathfrak{J}_3\varpi_{n+1},\mathfrak{J}_4\rho)} \hbar(\wp)d\wp + k_2 \cdot \int_0^{\Lambda(\mathfrak{J}_3\varpi_{n+1},\mathfrak{J}_3\varpi_{n+1},\mathfrak{J}_1\varpi_{n+1})} \hbar(\wp)d\wp + k_3 \\ &\cdot \int_0^{\Lambda(\mathfrak{J}_4\rho,\mathfrak{J}_4\varpi_{n+1},\mathfrak{J}_2\rho)} \hbar(\wp)d\wp + k_4 \int_0^{\Lambda(\mathfrak{J}_4\rho,\mathfrak{J}_4\varpi_{n+1},\mathfrak{J}_1\varpi_{n+1})} \hbar(\wp)d\wp + k_5 \\ &\cdot \int_0^{\Lambda(\mathfrak{J}_3\varpi_{n+1},\mathfrak{J}_3\varpi_{n+1},\mathfrak{J}_2\rho)} \hbar(\wp)d\wp + k_6 \cdot \int_0^{\max\left\{\Lambda(\mathfrak{J}_3\varpi_{n+1},\mathfrak{J}_3\varpi_{n+1},\mathfrak{J}_2\rho),\Lambda(\mathfrak{J}_4\rho,\mathfrak{J}_4\varpi_{n+1},\mathfrak{J}_1\varpi_{n+1})\right\}} \hbar(\wp)d\wp.\end{aligned} \tag{41}$$

Passing $n \longrightarrow \infty$ in (41) and owing (S1), (S1), (39), and Lemma 6,

$$\int_0^{\Lambda(\varpi^*,\varpi^*,\mathfrak{J}_2\rho)} \hbar(\wp)d\wp \leq (k_1 + k_3 + k_4 + k_6) \cdot \int_0^{\Lambda(\varpi^*,\varpi^*,\mathfrak{J}_4\rho)} \hbar(\wp)d\wp + (k_3 + k_5 + k_6) \int_0^{\Lambda(\varpi^*,\varpi^*,\mathfrak{J}_2\rho)} \hbar(\wp)d\wp, \tag{42}$$

or

$$\int_0^{\Lambda(\varpi^*,\varpi^*,\mathfrak{J}_2\rho)} \hbar(\wp)d\wp \leq \left(\frac{k_1 + k_3 + k_4 + k_6}{1 - k_3 - k_5 - k_6}\right) \int_0^{\Lambda(\varpi^*,\varpi^*,\mathfrak{J}_4\rho)} \hbar(\wp)d\wp = \alpha \int_0^{\Lambda(\varpi^*,\varpi^*,\mathfrak{J}_4\rho)} \hbar(\wp)d\wp, \tag{43}$$

where $\alpha = ((k_1 + k_3 + k_4 + k_6)/(1 - k_3 - k_5 - k_6)) < 1$, since $k_1 + 2k_3 + k_4 + k_5 + 2k_6 < 1$. This implies a contradiction as $0 < \alpha < 1$; therefore, $\Lambda(\varpi^*, \varpi^*, \mathfrak{J}_2\rho) = 0$, that is, $\varpi^* = \mathfrak{J}_2\rho$. Thus, $\varpi^* = \mathfrak{J}_3\rho = \mathfrak{J}_2\rho$.

Hence, $\rho \in CP(\mathfrak{J}_2, \mathfrak{J}_3)$. Since the pair $(\mathfrak{J}_2, \mathfrak{J}_3)$ is weakly compatible, then

$$\mathfrak{J}_3\mathfrak{J}_2\rho = \mathfrak{J}_2\mathfrak{J}_3\rho \Rightarrow \mathfrak{J}_3\varpi^* = \mathfrak{J}_2\varpi^*. \tag{44}$$

Similarly, $\mathfrak{J}_2(\Xi) \subseteq \mathfrak{J}_4(\Xi)$, and there exists a point $\upsilon \in \Xi$ such that $\varpi^* = \mathfrak{J}_4\upsilon$. Then, from equation (27) and using the same method as above, we can find that $\mathfrak{J}_1\upsilon = \varpi^*$, so $\varpi^* = \mathfrak{J}_1\upsilon = \mathfrak{J}_4\upsilon$.

Hence, $\rho \in CP(\mathfrak{J}_1, \mathfrak{J}_4)$. Also, the pair $(\mathfrak{J}_1, \mathfrak{J}_4)$ is weakly compatible; then,

$$\mathfrak{J}_1\mathfrak{J}_4\upsilon = \mathfrak{J}_4\mathfrak{J}_1\upsilon \Rightarrow \mathfrak{J}_1\varpi^* = \mathfrak{J}_4\varpi^*. \tag{45}$$

Next to prove is $\varpi^* \in \mathrm{Fix}(\mathfrak{J}_2)$. Owing (27), (S1), (S2), and Lemma 6,

$$\int_0^{\Lambda(\mathfrak{J}_1\varpi_{n+1},\mathfrak{J}_1\varpi_{n+1},\mathfrak{J}_2\varpi^*)} \hbar(\wp)d\wp, \tag{46}$$

$$\leq k_1 \int_0^{\Lambda(\mathfrak{T}_3\varpi_{n+1},\mathfrak{T}_3\varpi_{n+1},\mathfrak{T}_4\varpi^*)} \hbar(\wp)d\wp + k_2 \cdot \int_0^{\Lambda(\mathfrak{T}_3\varpi_{n+1},\mathfrak{T}_3\varpi_{n+1},\mathfrak{T}_1\varpi_{n+1})} \hbar(\wp)d\wp + k_3 \cdot \int_0^{\Lambda(\mathfrak{T}_4\varpi^*,\mathfrak{T}_4\varpi_{n+1},\mathfrak{T}_2\varpi^*)} \hbar(\wp)d\wp + k_4 \cdot \int_0^{\Lambda(\mathfrak{T}_3\varpi^*,\mathfrak{T}_3\varpi_{n+1},\mathfrak{T}_1\varpi_{n+1})} \hbar(\wp)d\wp + k_5 \cdot \int_0^{\Lambda(\mathfrak{T}_3\varpi_{n+1},\mathfrak{T}_3\varpi_{n+1},\mathfrak{T}_2\varpi^*)} \hbar(\wp)d\wp + k_6 \cdot \int_0^{\max\{\Lambda(\mathfrak{T}_3\varpi_{n+1},\mathfrak{T}_3\varpi_{n+1},\mathfrak{T}_2\varpi^*),\Lambda(\mathfrak{T}_4\varpi^*,\mathfrak{T}_4\varpi_{n+1},\mathfrak{T}_1\varpi_{n+1})\}} \hbar(\wp)d\wp. \quad (47)$$

Passing $n \longrightarrow \infty$ in (47),

$$\int_0^{\Lambda(\varpi^*,\varpi^*,\mathfrak{T}_2\varpi^*)} \hbar(\wp)d\wp \leq (k_1 + k_3 + k_4 + k_6) \cdot \int_0^{\Lambda(\varpi^*,\varpi^*,\mathfrak{T}_4\varpi^*)} \hbar(\wp)d\wp + (k_3 + k_5 + k_6) \cdot \int_0^{\Lambda(\varpi^*,\varpi^*,\mathfrak{T}_2\varpi^*)} \hbar(\wp)d\wp. \quad (48)$$

This implies

$$\int_0^{\Lambda(\varpi^*,\varpi^*,g\varpi^*)} \hbar(\wp)d\wp \leq \left(\frac{k_1 + k_3 + k_4 + k_6}{1 - k_3 - k_5 - k_6}\right) \int_0^{\Lambda(\varpi^*,\varpi^*,\mathfrak{T}_4\varpi^*)} \hbar(\wp)d\wp = \alpha \int_0^{\Lambda(\varpi^*,\varpi^*,\mathfrak{T}_4\varpi^*)} \hbar(\wp)d\wp, \quad (49)$$

where $\alpha = ((k_1 + k_3 + k_4 + k_6)/(1 - k_3 - k_5 - k_6)) < 1$, since $k_1 + 2k_3 + k_4 + k_5 + 2k_6 < 1$, a contradiction as $0 < \alpha < 1$; therefore, $\Lambda(\varpi^*, \varpi^*, \mathfrak{T}_2\varpi^*) = 0$, that is, $\varpi^* = \mathfrak{T}_2\varpi^*$; also, from (44), we get

$$\mathfrak{T}_3\varpi^* = \mathfrak{T}_2\varpi^* = \varpi^*. \quad (50)$$

By the similar method, we can show that ϖ^* is a fixed point of $\mathfrak{T}_1$, that is, $\varpi^* = \mathfrak{T}_1\varpi^*$, so from (45), we have

$$\mathfrak{T}_1\varpi^* = \mathfrak{T}_3\varpi^* = \varpi^*. \quad (51)$$

From equations (50) and (51), we get that $\mathrm{Fix}(\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \mathfrak{T}_4) = \{\varpi^*\}$. Finally, we prove uniqueness of common fixed point. To do this, $\varpi_1 \neq \varpi^* = \mathrm{Fix}(\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \mathfrak{T}_4)$ be an another CFP. Using (27), (28), (S1), (S2), and Lemma 6, we have

$$\int_0^{\Lambda(\varpi^*,\varpi^*,\varpi_1)} \hbar(\wp)d\wp = \int_0^{\Lambda(\mathfrak{T}_1\varpi^*,\mathfrak{T}_1\varpi^*,\mathfrak{T}_2\varpi_1)} \hbar(\wp)d\wp \leq k_1 \int_0^{\Lambda(\mathfrak{T}_3\varpi^*,\mathfrak{T}_3\varpi^*,\mathfrak{T}_4\varpi_1)} \hbar(\wp)d\wp + k_2 \cdot \int_0^{\Lambda(\mathfrak{T}_3\varpi^*,\mathfrak{T}_3\varpi^*,\mathfrak{T}_1\varpi^*)} \hbar(\wp)d\wp + k_3 \cdot \int_0^{\Lambda(\mathfrak{T}_4\varpi_1,\mathfrak{T}_4\varpi^*,\mathfrak{T}_2\varpi_1)} \hbar(\wp)d\wp + k_4 \cdot \int_0^{\Lambda(\mathfrak{T}_4\varpi_1,\mathfrak{T}_4\varpi^*,\mathfrak{T}_1\varpi^*)} \hbar(\wp)d\wp + k_5 \cdot \int_0^{\Lambda(\mathfrak{T}_3\varpi^*,\mathfrak{T}_3\varpi^*,\mathfrak{T}_2\varpi_1)} \hbar(\wp)d\wp + k_6 \cdot \int_0^{\max\{\Lambda(\mathfrak{T}_3\varpi^*,\mathfrak{T}_3\varpi^*,\mathfrak{T}_2\varpi_1),\Lambda(\mathfrak{T}_4\varpi_1,\mathfrak{T}_4\varpi^*,\mathfrak{T}_1\varpi^*)\}} \hbar(\wp)d\wp = k_1 \int_0^{\Lambda(\varpi^*,\varpi^*,\varpi_1)} \hbar(\wp)d\wp + k_2 \int_0^{\Lambda(\varpi^*,\varpi^*,\varpi^*)} \hbar(\wp)d\wp + k_3 \cdot \int_0^{\Lambda(\varpi_1,\varpi^*,\varpi_1)} \hbar(\wp)d\wp + k_4 \int_0^{\Lambda(\varpi_1,\varpi^*,\varpi^*)} \hbar(\wp)d\wp + k_5 \cdot \int_0^{\Lambda(\varpi^*,\varpi^*,\varpi_1)} \hbar(\wp)d\wp + k_6 \cdot \int_0^{\max\{\Lambda(\varpi^*,\varpi^*,\varpi_1),\Lambda(\varpi_1,\varpi^*,\varpi^*)\}} \hbar(\wp)d\wp \leq (k_1 + 2k_3 + k_4 + k_5 + k_6) \int_0^{\Lambda(\varpi^*,\varpi^*,\varpi_1)} \hbar(\wp)d\wp, \quad (52)$$

a contradiction since $k_1 + 2k_3 + k_4 + k_5 + k_6 < 1$. Thus, we conclude that $\Lambda(\varpi^*, \varpi^*, \varpi_1) = 0$; that is, $\varpi^* = \varpi_1$.

Remark 14. Theorem 13 generalizes Theorem 2.4 in [16] and Banach fixed point theorem [1].

Example 3. Let $\Xi = \mathbb{R}$ be the CSMS with the Λ-metric defined as $\Lambda(\varpi, \varkappa, \vartheta) = |y - z| + |y + z - 2x|$ for all $\varpi, \varkappa, \vartheta \in \Xi$. Define mapping $\mathfrak{T} : \Xi \longrightarrow \Xi$ as

$$\mathfrak{T}(\varpi) = \begin{cases} \varpi + 2, & \text{if } \varpi \in \{0, 2\}, \\ 3, & \text{if otherwise,} \end{cases} \quad (53)$$

for all $\varpi \in \Xi$ and $\hbar : [0,\infty) \longrightarrow [0,\infty)$ as $\hbar(\wp) = 2\wp$ for all $\wp \in [0, \infty)$. Then,

$$\int_0^{\varepsilon} \hbar(\wp)d\wp = \int_0^{\varepsilon} 2\wp d\wp = \varepsilon^2 > 0, \quad (54)$$

for each $\varepsilon > 0$. Therefore, $\mathfrak{T}$ satisfies (25) of Corollary 10 for (by taking $\varpi = \varkappa = 0$ and $\vartheta = 1$)

(i) $k_1 = 0$, $k_2 = 1/4$, and $k_3 = k_4 = k_5 = k_6 = 0$

(ii) $k_1 = k_2 = k_3 = 0$, $k_4 = 9/100$, and $k_5 = k_6 = 0$

(iii) $k_1 = k_2 = k_3 = k_4 = 0$, $k_5 = 1/25$, and $k_6 = 0$

It is noted that $\mathrm{Fix}(\mathfrak{J}) = \{3\}$. But $\mathfrak{J}$ does not satisfy (26) of Corollary 11 for the same values as

$$\int_0^2 2\wp d\wp = 4 \leq h \int_0^2 2\wp d\wp = 4h, \tag{55}$$

since $h \in (0, 1)$.

Example 4. Let $\Xi = [0, 1)$ and define metric on Ξ as

$$\Lambda(\varpi, \varkappa, \vartheta) = |\varpi - \vartheta| + |\varkappa - \vartheta|, \tag{56}$$

for all $\varpi, \varkappa, \vartheta \in [0, 1)$. Define the mappings $\mathfrak{J}_1$, $\mathfrak{J}_2$, $\mathfrak{J}_3$, and $\mathfrak{J}_4$ on Ξ by

$$\begin{gathered}
\mathfrak{J}_1(\varpi) = \begin{cases} 0, & \text{if } \varpi \in \left[0, \dfrac{1}{2}\right), \\ \dfrac{1}{8} & \text{if } \varpi \in \left[\dfrac{1}{2}, 1\right), \end{cases} \\
\mathfrak{J}_2(\varpi) = \begin{cases} 0, & \text{if } \varpi \in \left[0, \dfrac{1}{2}\right), \\ \dfrac{1}{5}, & \text{if } \varpi \in \left[\dfrac{1}{2}, 1\right), \end{cases} \\
\mathfrak{J}_3(\varpi) = \begin{cases} 0, & \text{if } \varpi \in \left[0, \dfrac{1}{2}\right), \\ \dfrac{1}{4}, & \text{if } \varpi \in \left[\dfrac{1}{2}, 1\right), \end{cases} \\
\mathfrak{J}_4(\varpi) = \begin{cases} 0, & \text{if } \varpi \in \left[0, \dfrac{1}{2}\right), \\ \dfrac{2}{5}, & \text{if } \varpi \in \left[\dfrac{1}{2}, 1\right). \end{cases}
\end{gathered} \tag{57}$$

Let the function $\hbar : [0,\infty) \longrightarrow [0,\infty)$ be defined as $\hbar(\wp) = 2\wp$ for all $\wp \in [0, \infty)$; then, for each $\varepsilon > 0$,

$$\int_0^\varepsilon \hbar(\wp) d\wp = \int_0^\varepsilon 2\wp d\wp = \varepsilon^2 > 0. \tag{58}$$

It is clear that $\mathfrak{J}_1(\Xi) \subseteq \mathfrak{J}_3(\Xi)$ and $\mathfrak{J}_2(\Xi) \subseteq \mathfrak{J}_4(\Xi)$, so at the points $\varpi = \varkappa = 1/2$ and $\vartheta = 0$, the pairs $(\mathfrak{J}_2, \mathfrak{J}_3)$ and $(\mathfrak{J}_1, \mathfrak{J}_4)$ are weakly compatible. Now for the points $\varpi = \varkappa = 1/2$ and $\vartheta = 0$, we calculate the following:

$$\begin{aligned}
L.H.S. &= \int_0^{\Lambda(\mathfrak{J}_1\varpi, \mathfrak{J}_1\varkappa, \mathfrak{J}_2\vartheta)} \hbar(\wp) d\wp = \int_0^{\Lambda(\mathfrak{J}_1\varpi, \mathfrak{J}_1\varkappa, \mathfrak{J}_2\vartheta)} 2\wp d\wp \\
&= \int_0^{|\mathfrak{J}_1\varpi - \mathfrak{J}_2\vartheta| + |\mathfrak{J}_1\varkappa - \mathfrak{J}_2\vartheta|} 2\wp d\wp \\
&= [|\mathfrak{J}_1\varpi - \mathfrak{J}_2\vartheta| + |\mathfrak{J}_1\varkappa - \mathfrak{J}_2\vartheta|]^2 = \left|\frac{1}{8} + \frac{1}{8}\right|^2 = \frac{1}{16}.
\end{aligned} \tag{59}$$

Also,

$$\begin{aligned}
k_1 \int_0^{\Lambda(\mathfrak{J}_3\varpi, \mathfrak{J}_3\varkappa, \mathfrak{J}_4\vartheta)} \hbar(\wp) d\wp &= \frac{k_1}{4}, \\
k_2 \int_0^{\Lambda(\mathfrak{J}_3\varpi, \mathfrak{J}_3\varkappa, \mathfrak{J}_1\varpi)} \hbar(\wp) d\wp &= \frac{k_2}{16}, \\
k_3 \int_0^{\Lambda(\mathfrak{J}_4\vartheta, \mathfrak{J}_4\varkappa, \mathfrak{J}_2\vartheta)} \hbar(\wp) d\wp &= \frac{4k_3}{25}, \\
k_4 \int_0^{\Lambda(\mathfrak{J}_4\vartheta, \mathfrak{J}_4\varkappa, \mathfrak{J}_1\varpi)} \hbar(\wp) d\wp &= \frac{2k_4}{5}, \\
k_5 \int_0^{\Lambda(\mathfrak{J}_3\varpi, \mathfrak{J}_3\varkappa, \mathfrak{J}_2\vartheta)} \hbar(\wp) d\wp &= \frac{k_5}{4}, \\
k_6 \int_0^{\max\{\Lambda(\mathfrak{J}_3\varpi, \mathfrak{J}_3\varkappa, \mathfrak{J}_2\vartheta), \Lambda(\mathfrak{J}_4\vartheta, \mathfrak{J}_4\varkappa, \mathfrak{J}_1\varpi)\}} \hbar(\wp) d\wp &= \frac{2k_6}{5}.
\end{aligned} \tag{60}$$

Thus,

$$R.H.S. = \frac{k_1}{4} + \frac{k_2}{16} + \frac{4k_3}{25} + \frac{2k_4}{5} + \frac{k_5}{4} + \frac{2k_6}{5} \geq \frac{1}{16} = L.H.S. \tag{61}$$

The inequality (61) is satisfied if $k_i \in [0, 1)$ for $i = 1, 2, \cdots, 6$. Hence, the inequality (27) is verified, and all the conditions of Theorem 13 are hold and $\mathrm{CFix}(\mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3, \mathfrak{J}_4) = \{0\}$.

4. Application on Fractional Integral Equation

The Riemann-Liouville integral is defined by

$$L^\vartheta \upsilon(\chi) = \frac{1}{\Gamma(\vartheta)} \int_a^\chi \upsilon(\varpi)(\chi - \varpi)^{\vartheta - 1} d\varpi, \tag{62}$$

where the gamma function is represented as Γ. If $\upsilon : [a, b] \longrightarrow \mathbb{R}$, $a, b \in (-\infty, \infty)$ is a locally integral function and ϑ is a complex integer in the half-plane $\mathfrak{R}(\vartheta) > 0$, the integral is well defined. Proceeding via (62), the fractional Fredholm integral equation is formulated by

$$\upsilon(\chi) = \ell(\chi) + \frac{\lambda}{\Gamma(\vartheta)} \int_a^\chi \upsilon(\varpi)(\chi - \varpi)^{\vartheta - 1} \kappa(\chi, \varpi) d\varpi, \tag{63}$$

where $\kappa(\chi, \varpi)$ is a continuous function in a square region $[a, b]^2$, such that $|\kappa(\chi, x)| \leq \mu(\mu \in (0, \infty))$ and ℓ is a continuous function on $[a, b]$. Define the space of continuous function

$$\Xi[a, b] \coloneqq \{\upsilon | \upsilon : [a, b] \longrightarrow \mathbb{R}\}. \tag{64}$$

Furthermore, we define the function $\varsigma : \Xi[a, b] \times \Xi[a, b] \times \Xi[a, b] \longrightarrow [0, \infty)$ as follows:

$$\varsigma(\upsilon,\phi,\psi) \coloneqq \sup_{\chi\in[a,b]}|\upsilon(\chi)-\psi(\chi)| + \sup_{\chi\in[a,b]}|\upsilon(\chi)+\psi(\chi)-2\phi(\chi)|, \tag{65}$$

where $\upsilon, \phi, \psi \in \Xi[a, b]$. Thus, the function ς is an S-metric. We then go on to show that no other metric can provide the above metric. Assume not, that is, there exists a metric, say Σ such that

$$\varsigma(\upsilon,\phi,\psi) = \Sigma(\upsilon,\psi) + \Sigma(\phi,\psi). \tag{66}$$

Consequently, we obtain

$$\varsigma(\upsilon,\upsilon,\psi) = 2\Sigma(\upsilon,\psi) = 2\sup_{\chi\in[a,b]}|\upsilon(\chi)-\psi(\chi)|, \tag{67}$$

where

$$\Sigma(\upsilon,\psi) \coloneqq \sup_{\chi\in[a,b]}|\upsilon(\chi)-\psi(\chi)|. \tag{68}$$

Correspondingly, we have

$$\varsigma(\phi,\phi,\psi) = 2\Sigma(\phi,\psi) = 2\sup_{\chi\in[a,b]}|\phi(\chi)-\psi(\chi)|. \tag{69}$$

Combining (66)–(69), we have

$$\begin{aligned}&\sup_{\chi\in[a,b]}|\upsilon(\chi)-\psi(\chi)| + \sup_{\chi\in[a,b]}|\upsilon(\chi)+\psi(\chi)-2\phi(\chi)|\\ &\quad= \sup_{\chi\in[a,b]}|\upsilon(\chi)-\psi(\chi)| + \sup_{\chi\in[a,b]}|\phi(\chi)-\psi(\chi)|,\end{aligned} \tag{70}$$

which is not true. That is, the S-metric is not generated by any metric. Consequently, $(\Xi[a, b], \varsigma)$ is a CSMS.

Proposition 15. *Suppose that $(\Xi[a, b], \varsigma)$ is a CSMS achieving the metric (65). If*

$$|\lambda| < \frac{\Gamma(\vartheta+1)}{\mu(b-a)^{\vartheta}}, \vartheta > 0, \tag{71}$$

then the fractional integral equation (63) has a unique solution $\hbar : [a, b] \longrightarrow \mathbb{R}$.

Proof. Define the operator $\Omega : \Xi[a, b] \longrightarrow \Xi[a, b]$ by

$$(\Omega\upsilon)(\chi) = \ell(\chi) + \frac{\lambda}{\Gamma(\vartheta)}\int_a^{\chi}\upsilon(\varpi)(\chi-\varpi)^{\vartheta-1}\kappa(\chi,\varpi)d\varpi. \tag{72}$$

Next, we aim to prove that Ω achieves the contraction condition.

$$\begin{aligned}\varsigma((\Omega\upsilon_1),(\Omega\upsilon_1),(\Omega\upsilon_2)) &= 2\sup_{\chi\in[a,b]}|(\Omega\hbar_1)(\chi)-(\Omega\hbar_2)(\chi)|\\ &= 2\sup_{\chi\in[a,b]}\left|\left|\frac{\lambda}{\Gamma(\vartheta)}\int_a^{\chi}\left(\upsilon_1(x)-\upsilon_2(\varpi)\left((\chi-x)^{\vartheta-1}\kappa(\chi,\varpi)d\varpi\right)\right)\right|\right.\\ &\le 2\sup_{\chi\in[a,b]}\frac{|\lambda|\mu}{\Gamma(\vartheta+1)}|(\upsilon_1(\varpi)-\upsilon_2(\varpi))|\left|\left|\int_a^{\chi}(\chi-\varpi)^{\vartheta-1}dx\right|\right|\\ &= 2\sup_{\chi\in[a,b]}\frac{|\lambda|\mu}{\Gamma(\vartheta+1)}|(\upsilon_1(\varpi)-\upsilon_2(\varpi))|(b-a)^{\vartheta}\\ &\le \frac{|\lambda|\mu}{\Gamma(\vartheta+1)}(b-a)^{\vartheta}\varsigma(\upsilon_1,\upsilon_1,\upsilon_2) < \varsigma(\upsilon_1,\upsilon_1,\upsilon_2),\end{aligned} \tag{73}$$

which implies

$$\int_0^{\varsigma(\Omega\upsilon_1,\Omega\upsilon_1,\Omega\upsilon_2)}\hbar(\chi)d\chi < \int_0^{\varsigma(\upsilon_1,\upsilon_1,\upsilon_2)}\hbar(\chi)d\chi. \tag{74}$$

As a result, based on the assumptions, the contraction requirement is met at $F = G = \Omega$ and $k_1 = 1, h = 1, k_2 = k_3 = k_4 = k_5 = k_6 = 0$ in Corollary 11. Hence, the Fredholm integral equation (63) has just one solution. □

Example 5. Consider the fractional integral equation, which is written as

$$\upsilon(\chi) = \frac{\lambda}{\Gamma(\vartheta)}\int_0^{\chi}(\varpi)^{\alpha}(\chi-\varpi)^{\vartheta-1}d\varpi, \tag{75}$$

on the space $\Xi[0, 1]$, such that $\upsilon(0) = 0$. Then, the iteration solution becomes

$$\begin{aligned}\upsilon_1(\chi) &= \frac{\lambda}{\Gamma(\vartheta)}\int_0^{\chi}(\varpi)^{\alpha}(\chi-\varpi)^{\vartheta-1}d\varpi\\ &= \frac{\lambda\Gamma(\alpha+1)}{\Gamma(\alpha+1+\vartheta)}\chi^{\alpha+\vartheta},\\ \upsilon_2(\chi) &= \frac{\lambda\Gamma(\alpha+1)}{\Gamma(\alpha+1+\vartheta)}\frac{\lambda}{\Gamma(\vartheta)}\int_0^{\chi}(\varpi)^{\alpha+\vartheta}(\chi-\varpi)^{\vartheta-1}d\varpi\\ &= \lambda^2\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1+\vartheta)}\frac{\Gamma(\alpha+\vartheta+1)}{\Gamma(\alpha+1+2\vartheta)}\chi^{\alpha+2\vartheta},\\ \upsilon_3(\chi) &= \lambda^2\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1+\vartheta)}\frac{\Gamma(\alpha+\vartheta+1)}{\Gamma(\alpha+1+2\vartheta)}\frac{\lambda}{\Gamma(\vartheta)}\\ &\quad\cdot\int_0^{\chi}(\varpi)^{\alpha+2\vartheta}(\chi-\varpi)^{\vartheta-1}d\varpi\\ &= \lambda^3\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1+\vartheta)}\frac{\Gamma(\alpha+\vartheta+1)}{\Gamma(\alpha+1+2\vartheta)}\frac{\Gamma(\alpha+2\vartheta+1)}{\Gamma(\alpha+1+3\vartheta)}\chi^{\alpha+3\vartheta},\\ &\vdots\\ \upsilon_n(\chi) &= \lambda^n\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1+\vartheta)}\frac{\Gamma(\alpha+\vartheta+1)}{\Gamma(\alpha+1+2\vartheta)}\cdots\frac{\Gamma(\alpha+(n-1)\vartheta+1)}{\Gamma(\alpha+1+n\vartheta)}\chi^{\alpha+n\vartheta}\\ &= \lambda^n\left(\prod_{j=1}^{n}\frac{\Gamma(\alpha+(j-1)\vartheta+1)}{\Gamma(\alpha+1+j\vartheta)}\right)\chi^{\alpha+n\vartheta}.\end{aligned} \tag{76}$$

If

$$|\lambda| < \left(\frac{1}{\prod_{j=1}^{n}(\Gamma(\alpha+(j-1)\vartheta+1)/\Gamma(\alpha+1+j\vartheta))}\right)^{1/n}, \quad (77)$$

then according to Proposition 15, for $\mu = 1$ and $b - a = 1$, the integral equation (75) admits a unique solution. Note that when $n \longrightarrow \infty$, we get $|\lambda| < 1$. Moreover, inequality (77) can be realized by

$$|\lambda| < \left(\frac{\Gamma(\alpha+n\vartheta+1)}{\Gamma(\alpha+1)}\right)^{1/n}, \quad (78)$$

which fits the assumption on Proposition 15.

5. Conclusion

In this paper, we have proved some unique CFP theorems for contractive conditions of integral type involving rational terms in CSMS and gave some consequences as corollaries of the main results. Also, some illustrated examples are provided to validate the results. The results of findings in this work generalize and extend several results from the existing literature. We have employed the results to get a unique solution of a fractional integral equation.

Data Availability

No underlying data was collected.

Disclosure

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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