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Research Article

New Integral Operator for Analytic Functions

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Let $A_p(n)$ be the class of functions $f(z)$ given by $f(z) = z^p + a_{p+n}z^{p+n} + a_{p+n+1}z^{p+n+1} + \cdots$ which are analytic in the open unit disk $\mathbb{U}$. For $f(z) \in A_p(n)$, new integral operators $\mathcal{O}_{-j}f(z)$ and $\mathcal{O}_j f(z)(j = 0, 1, 2, \cdots)$ are considered. The operators $\mathcal{O}_{-j}f(z)$ and $\mathcal{O}_j f(z)$ satisfy $\mathcal{O}_j(\mathcal{O}_{-j}f(z)) = \mathcal{O}_{-j}(\mathcal{O}_j f(z)) = f(z)$ and $(\mathcal{O}_{-j} * \mathcal{O}_j)f(z) = (\mathcal{O}_j * \mathcal{O}_{-j})f(z) = (f * f)(z)$ for the convolution $*$ of $\mathcal{O}_{-j}f(z)$ and $\mathcal{O}_j f(z)$. In the present paper, the dominants for both operators $\mathcal{O}_{-j}f(z)$ and $\mathcal{O}_j f(z)$ and subordinations for $\mathcal{O}_{-j}f(z)$ and $\mathcal{O}_j f(z)$ are discussed. Also, new subclass $\mathcal{T}_p(\gamma_m, \delta, \rho; m, j)$ concerning with m different boundary points is defined and discussed. Moreover, some interesting problems of $\mathcal{T}_p(\gamma_m, \delta, \rho; m, j)$ associated with $\mathcal{O}_j f(z)$ are obtained. Furthermore, some interesting examples for our results are considered.

1. Introduction

The study of operators in geometric function theory started early. Many differential and integral operators began to be studied and introduced around twentieth century by many mathematicians like Alexander [1], Libera [2] (see also [3]), and Bernardi [4] (see also [5]), and they used integral operators early and are known in the GFT; moreover, Ruscheweyh [6] and Salagean [7] used differential operators which are also known in GFT. The Alexander integral operator [1] is an example of them, which was defined by Alexander in 1915. The importance of studying operators in geometric function theory highlights many geometric properties of families of analytic functions such as convexity, starlikeness, coefficient estimates, distortion properties, and subordination and superordination relations; also, [11–14] used the Srivastava-Attiya operator; in addition, [15–17] used integral operators, and furthermore, [18–20] used linear operators.

The new operators we introduce in this paper are generalizations of an extension of the Alexander and Libera integral operators [3].

Let $\mathcal{A}_p(n)$ be the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k \,;\, n \in \mathbb{N} = \{1, 2, 3, \cdots\}, \tag{1}$$

that are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For $f(z) \in \mathcal{A}_p(n)$, we define

$$\mathcal{O}_{-1}f(z) = \frac{p+1}{pz}\int_0^z t f'(t)dt = z^p + \sum_{k=p+n}^{\infty} \frac{(p+1)k}{p(k+1)} a_k z^k, \tag{2}$$

$$\mathcal{O}_{-2}f(z) = \frac{p+1}{pz}\int_0^z t(\mathcal{O}_{-1}f(t))'dt = z^p + \sum_{k=p+n}^{\infty} \left(\frac{(p+1)k}{p(k+1)}\right)^2 a_k z^k, \tag{3}$$

$$\mathcal{O}_{-j}f(z) = \frac{p+1}{pz}\int_0^z t\left(\mathcal{O}_{-j+1}f(t)\right)' dt = z^p + \sum_{k=p+n}^{\infty}\left(\frac{(p+1)k}{p(k+1)}\right)^j a_k z^k, \tag{4}$$

where $j \in \mathbb{N}$ and $\mathcal{O}_0 f(z) = f(z)$.

The operator $\mathcal{O}_{-1}f(z)$ for $p=1$ is defined by Güney and Owa [3]. Also, see [21] for related operator of generalized Libera operator in $\mathcal{A}_p(1)$.

For the above operator $\mathcal{O}_{-1}f(z)$ for $f(z) \in \mathcal{A}_p(n)$, we consider the following operators:

$$\mathcal{O}_1 f(z) = \frac{p}{p+1}\int_0^z \frac{(tf(t))'}{t} dt = z^p + \sum_{k=p+n}^{\infty}\frac{p(k+1)}{(p+1)k} a_k z^k, \tag{5}$$

$$\mathcal{O}_2 f(z) = \frac{p}{p+1}\int_0^z \frac{(t\mathcal{O}_1 f(t))'}{t} dt = z^p + \sum_{k=p+n}^{\infty}\left(\frac{p(k+1)}{(p+1)k}\right)^2 a_k z^k, \tag{6}$$

$$\mathcal{O}_j f(z) = \frac{p}{p+1}\int_0^z \frac{\left(t\mathcal{O}_{j-1} f(t)\right)'}{t} dt = z^p + \sum_{k=p+n}^{\infty}\left(\frac{p(k+1)}{(p+1)k}\right)^j a_k z^k, \tag{7}$$

for $j \in \mathbb{N}$.

In view of (4) and (7), we have

$$\mathcal{O}_j\left(\mathcal{O}_{-j}f(z)\right) = \mathcal{O}_{-j}\left(\mathcal{O}_j f(z)\right) = f(z), \tag{8}$$

for $j = \cdots, -2, -1, 0, 1, 2, \cdots$.

Let us consider $f(z) \in \mathcal{A}_p(n)$ given by (1) and $g(z) \in \mathcal{A}_p(n)$ which is given by

$$g(z) = z^p + \sum_{k=p+n}^{\infty} b_k z^k. \tag{9}$$

Then, the convolution (or Hadamard product) of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = (g * f)(z) = z^p + \sum_{k=p+n}^{\infty} a_k b_k z^k, \tag{10}$$

(see [22, 23]). This convolution shows that

$$\left(\mathcal{O}_{-j}f * \mathcal{O}_j f\right)(z) = \left(\mathcal{O}_j f * \mathcal{O}_{-j} f\right)(z) = (f * f)(z) = z^p + \sum_{k=p+n}^{\infty} a_k^2 z^k, \tag{11}$$

for $f(z) \in \mathcal{A}_p(n)$.

The function $f(z) \in \mathcal{A}_p(n)$ is said to be p-valently starlike of order α in $\mathbb{U}$ if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U}), \tag{12}$$

for some real $\alpha (0 \le \alpha < p)$. We denote the class of such p-valently starlike of order α in $\mathbb{U}$ by $\mathcal{S}_p^*(\alpha)$.

If $f(z) \in \mathcal{A}_p(n)$ satisfies

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathbb{U}), \tag{13}$$

for some real $\alpha (0 \le \alpha < p)$, then $f(z)$ is said to be p-valently convex of order α in $\mathbb{U}$. We also denote by $\mathcal{K}_p(\alpha)$ the class of $f(z)$ which are p-valently convex of order α in $\mathbb{U}$.

With the above definitions for $\mathcal{S}_p^*(\alpha)$ and $\mathcal{K}_p(\alpha)$, we see that

$$f(z) \in \mathcal{S}_p^*(\alpha) \Leftrightarrow p\int_0^z \frac{f(t)}{t} dt \in \mathcal{K}_p(\alpha), \tag{14}$$

$$f(z) \in \mathcal{K}_p(\alpha) \Leftrightarrow \frac{1}{p} zf'(z) \in \mathcal{S}_p^*(\alpha). \tag{15}$$

Furthermore, if $f(z) \in \mathcal{A}_p(n)$ satisfies

$$\operatorname{Re}\left(\frac{f'(z)}{z^{p-1}}\right) > \alpha \quad (z \in \mathbb{U}), \tag{16}$$

for some real $\alpha (0 \le \alpha < p)$, then $f(z)$ is said to be p-valently close-to-convex of order α in $\mathbb{U}$. Also, we write that $f(z) \in \mathcal{C}_p(\alpha)$ for such functions.

For the above classes, Owa [24] has shown the following lemmas.

Lemma 1. *If $f(z) \in \mathcal{A}_p(n)$ satisfies*

$$\sum_{k=p+n}^{\infty} (k-\alpha)|a_k| \le p - \alpha, \tag{17}$$

for some $\alpha (0 \le \alpha < p)$, then $f(z) \in \mathcal{S}_p^*(\alpha)$.

Lemma 2. *If $f(z) \in \mathcal{A}_p(n)$ satisfies*

$$\sum_{k=p+n}^{\infty} k(k-\alpha)|a_k| \le p(p-\alpha), \tag{18}$$

for some $\alpha (0 \le \alpha < p)$, then $f(z) \in \mathcal{K}_p(\alpha)$.

Lemma 3. *If $f(z) \in \mathcal{A}_p(n)$ satisfies*

$$\sum_{k=p+n}^{\infty} k|a_k| \le p - \alpha, \tag{19}$$

for some $\alpha (0 \le \alpha < p)$, then $f(z) \in \mathcal{C}_p(\alpha)$.

With the above lemmas, we have the following remark.

Remark 4. Let us consider a function $f(z) \in \mathscr{A}_p(n)$ given by

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k, \tag{20}$$

with $a_k = |a_k| e^{i(\pi+p-k)}$ $(k = p+n, p+n+1, \cdots)$. For such function $f(z)$, we have

$$\begin{aligned} \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) &= \operatorname{Re}\left(\frac{p + \sum_{k=p+n}^{\infty} k a_k z^{k-p}}{1 + \sum_{k=p+n}^{\infty} a_k z^{k-p}}\right) \\ &= \operatorname{Re}\left(\frac{p + \sum_{k=p+n}^{\infty} k|a_k||z|^{k-p} e^{i\pi}}{1 + \sum_{k=p+n}^{\infty} |a_k||z|^{k-p} e^{i\pi}}\right) \\ &= \frac{p - \sum_{k=p+n}^{\infty} k|a_k||z|^{k-p}}{1 - \sum_{k=p+n}^{\infty} |a_k||z|^{k-p}} \ (z \in \mathbb{U}). \end{aligned} \tag{21}$$

If $f(z) \in \mathcal{S}_p^*(\alpha)$, then

$$\frac{p - \sum_{k=p+n}^{\infty} k|a_k||z|^{k-p}}{1 - \sum_{k=p+n}^{\infty} |a_k||z|^{k-p}} > \alpha \quad (z \in \mathbb{U}). \tag{22}$$

That is,

$$\sum_{k=p+n}^{\infty} (k-\alpha)|a_k||z|^{k-p} \le p - \alpha \quad (z \in \mathbb{U}). \tag{23}$$

Letting $|z| \longrightarrow 1^-$, we see

$$\sum_{k=p+n}^{\infty} (k-\alpha)|a_k| \le p - \alpha. \tag{24}$$

Conversely, if $f(z)$ satisfies the coefficient inequality (24), then $f(z)$ satisfies

$$\begin{aligned} \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) &= \frac{p - \sum_{k=p+n}^{\infty} k|a_k||z|^{k-p}}{1 - \sum_{k=p+n}^{\infty} |a_k||z|^{k-p}} \\ &= 1 - \frac{1 - p + \sum_{k=p+n}^{\infty} (k-1)|a_k||z|^{k-p}}{1 - \sum_{k=p+n}^{\infty} |a_k||z|^{k-p}} \\ &> 1 - \frac{(1-\alpha) - (1-\alpha)\sum_{k=p+n}^{\infty} |a_k|}{1 - \sum_{k=p+n}^{\infty} |a_k|} = \alpha, \end{aligned} \tag{25}$$

which shows that $f(z) \in \mathcal{S}_p^*(\alpha)$. Therefore, the coefficient inequality (17) in Lemma 1 is necessary and sufficient condition for $f(z) \in \mathcal{S}_p^*(\alpha)$ of $f(z)$ given by (20).

Using the same manner, we have that the coefficient inequality (18) in Lemma 2 is a necessary and sufficient condition for $f(z) \in \mathscr{K}_p(\alpha)$ of $f(z)$ given by (20). Also, we see that the coefficient inequality (19) in Lemma 3 is a necessary and sufficient condition for $f(z) \in \mathscr{C}_p(\alpha)$ of $f(z)$ given by (20).

In the next section, we have some interested results associated with the dominants.

2. Dominants

Let us consider a function $g(z) \in \mathscr{A}_p(n)$ given by

$$g(z) = z^p + \sum_{k=p+n}^{\infty} b_k z^k, \tag{26}$$

with $b_k \ge 0$. If $f(z)$ given by (1) and $g(z)$ given by (26) satisfy

$$|a_k| \le b_k \quad (k = p+n, p+n+1, \cdots), \tag{27}$$

then $f(z)$ is said to be dominated by $g(z)$ (or $g(z)$ dominants $f(z)$). We write this dominant by

$$f(z) \ll g(z) \quad (z \in \mathbb{U}). \tag{28}$$

It follows from the definition for the dominant that

$$f(z) \ll \mathcal{O}_{-1} f(z) \ll \cdots \ll \mathcal{O}_{-j} f(z) \ll \cdots, \tag{29}$$

$$\cdots \ll \mathcal{O}_j f(z) \ll \cdots \ll \mathcal{O}_1 f(z) \ll f(z), \tag{30}$$

for $j \in \mathbb{N}$.

We note that the function $f(z) \in \mathscr{A}_p(1)$ given by

$$f(z) = \frac{z^p}{(1-z)^{2(p-\alpha)}} \quad (0 \le \alpha < p) \tag{31}$$

is the function in the class $\mathcal{S}_p^*(\alpha)$, which can be written as

$$f(z) = z^p + \sum_{k=p+1}^{\infty} \frac{\prod_{\ell=0}^{k-p-1} (2(p-\alpha) + \ell)}{(k-p)!} z^k. \tag{32}$$

For the function $f(z)$ in (31), we know that

$$\begin{aligned} F(z) &= \int_0^z f(t)\,dt = p\int_0^z \frac{t^{p-1}}{(1-t)^{2(p-\alpha)}}\,dt \\ &= z^p + \sum_{k=p+1}^{\infty} \left(\frac{p}{k}\right) \frac{\prod_{\ell=0}^{k-p-1} (2(p-\alpha) + \ell)}{(k-p)!} z^k \end{aligned} \tag{33}$$

belongs to the class $\mathscr{K}_p(\alpha)$.

Further, we consider a function $f(z) \in \mathscr{A}_p(1)$ defined by

$$f(z) = \int_0^z t^{p-1}\left(2\alpha - p + \frac{2(p-\alpha)}{1-t}\right) dt \quad (z \in \mathbb{U}), \tag{34}$$

with $0 \le \alpha < p$. This function satisfies

$$\frac{f'(z)}{z^{p-1}} = 2\alpha - p + \frac{2(p-\alpha)}{1-z}, \tag{35}$$

$$\operatorname{Re}\left(\frac{f'(z)}{z^{p-1}}\right) = 2\alpha - p + 2(p-\alpha)\operatorname{Re}\left(\frac{1}{1-z}\right) > \alpha. \tag{36}$$

Therefore, $f(z) \in \mathscr{C}_p(\alpha)$ and

$$f(z) = z^p + \sum_{k=p+1}^{\infty} \frac{2(p-\alpha)}{k} z^k. \tag{37}$$

Now, we derive the following theorems concerning with the coefficient inequalities for $\mathcal{O}_j f$.

Theorem 5. *If $f(z) \in \mathscr{A}_p(1)$ satisfies*

$$|a_k| \leq \left(\frac{(p+1)k}{p(k+1)}\right)^j \frac{\prod_{\ell=0}^{k-p-1}(2(p-\alpha)+\ell)}{(k-p)!} \quad (k = p+1, p+2, \cdots), \tag{38}$$

for some $j = \cdots, -1, 0, 1, 2, \cdots$, then

$$\mathcal{O}_j f(z) \ll \frac{z^p}{(1-z)^{2(p-\alpha)}} \quad (z \in \mathbb{U}). \tag{39}$$

Proof. Since

$$\mathcal{O}_j f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{p(k+1)}{(p+1)k}\right)^j a_k z^k, \tag{40}$$

$$\frac{z^p}{(1-z)^{2(p-\alpha)}} = z^p + \sum_{k=p+1}^{\infty} \frac{\prod_{\ell=0}^{k-p-1}(2(p-\alpha)+\ell)}{(k-p)!} z^k. \tag{41}$$

Therefore,

$$\left(\frac{p(k+1)}{(p+1)k}\right)^j |a_k| \leq \frac{\prod_{\ell=0}^{k-p-1}(2(p-\alpha)+\ell)}{(k-p)!} \quad (k = p+1, p+2, \cdots). \tag{42}$$

We have the dominant (39). This implies the proof of theorem. □

Theorem 6. *If $f(z) \in \mathscr{A}_p(1)$ satisfies*

$$|a_k| \leq \left(\frac{(p+1)k}{p(k+1)}\right)^j \left(\frac{p}{k}\right) \frac{\prod_{l=0}^{k-p-1}(2(p-\alpha)+\ell)}{(k-p)!} \quad (k = p+1, p+2, \cdots), \tag{43}$$

for some $j = \cdots, -1, 0, 1, 2, \cdots$, then

$$\mathcal{O}_j f(z) \ll p \int_0^z \frac{t^{p-1}}{(1-t)^{2(p-\alpha)}} dt \quad (z \in \mathbb{U}). \tag{44}$$

Proof. Using (33), we obtain the dominant (44). □

Theorem 7. *If $f(z) \in \mathscr{A}_p(1)$ satisfies*

$$|a_k| \leq \left(\frac{(p+1)k}{p(k+1)}\right)^j \frac{2(p-\alpha)}{k} \quad (k = p+1, p+2, \cdots), \tag{45}$$

for some $j = \cdots, -1, 0, 1, 2, \cdots$, then

$$\mathcal{O}_j f(z) \ll \int_0^z t^{p-1}\left(2\alpha - p + \frac{2(p-\alpha)}{1-t}\right) dt \quad (z \in \mathbb{U}). \tag{46}$$

Proof. It is clear using (34). □

Theorem 8. *For $f(z) \in \mathscr{A}_p(1)$,*

$$\mathcal{O}_j f(z) \ll \frac{z^p}{(1-z)^{2(p-\alpha)}} \quad (z \in \mathbb{U}), \tag{47}$$

if and only if

$$\mathcal{O}_{j-1} f(z) \ll \frac{p+1}{pz} \int_0^z \frac{t^p(p+(p-2\alpha)t)}{(1-t)^{2p-2\alpha+1}} dt \quad (z \in \mathbb{U}), \tag{48}$$

where $j = \cdots, -1, 0, 1, 2, \cdots$ and $0 \leq \alpha < p$.

Proof. If $f(z)$ satisfies the dominant (47), then

$$\begin{aligned} \mathcal{O}_{j-1} f(z) &\ll \frac{p+1}{pz} \int_0^z t\left(\frac{t^p}{(1-t)^{2(p-\alpha)}}\right)' dt \\ &= \frac{p+1}{pz} \int_0^z \frac{t^p(p+(p-2\alpha)t)}{(1-t)^{2p-2\alpha+1}} dt \quad (z \in \mathbb{U}). \end{aligned} \tag{49}$$

Conversely, if $f(z)$ satisfies the dominant (48), then we have

$$\frac{pz}{p+1} \mathcal{O}_{j-1} f(z) \ll \int_0^z \frac{t^p(p+(p-2\alpha)t)}{(1-t)^{2p-2\alpha+1}} dt \quad (z \in \mathbb{U}). \tag{50}$$

This gives

$$z\left(\mathcal{O}_{j-1} f(z)\right)' \ll \frac{z^p(p+(p-2\alpha)z)}{(1-z)^{2p-2\alpha+1}} \quad (z \in \mathbb{U}). \tag{51}$$

That is, that

$$\mathcal{O}_j f(z) \ll \int_0^z \frac{t^{p-1}(p+(p-2\alpha)t)}{(1-t)^{2p-2\alpha+1}} dt = \frac{z^p}{(1-z)^{2(p-\alpha)}} \quad (z \in \mathbb{U}). \tag{52}$$

This completes the proof of the theorem. □

Making $j = 0$ in Theorem 8, we have the following.

Corollary 9. *For* $f(z) \in \mathscr{A}_p(1)$,

$$f(z) \ll \frac{z^p}{(1-z)^{2(p-\alpha)}} \quad (z \in \mathbb{U}), \tag{53}$$

if and only if

$$\mathcal{O}_{-1}f(z) \ll \frac{p+1}{pz}\int_0^z \frac{t^p(p+(p-2\alpha)t)}{(1-t)^{2p-2\alpha+1}}dt \quad (z \in \mathbb{U}), \tag{54}$$

where $0 \le \alpha < p$.

Further taking $p = 1$ in Corollary 9, we have the following.

Corollary 10. *For* $f(z) \in \mathscr{A}_1(1)$,

$$f(z) \ll \frac{z}{(1-z)^{2(1-\alpha)}} \quad (z \in \mathbb{U}), \tag{55}$$

if and only if

$$\mathcal{O}_{-1}f(z) \ll h(z) \quad (z \in \mathbb{U}), \tag{56}$$

where $0 < \alpha < 1$, $\alpha \neq 1/2$, *and*

$$h(z) = \frac{2(1-z)^{2\alpha}}{z}\left\{\frac{2\alpha-1}{2\alpha} - \frac{4\alpha-3}{(2\alpha-1)(1-z)} + \frac{1}{(1-z)^2}\right\} - \frac{1}{\alpha(2\alpha-1)z} \quad (z \in \mathbb{U}). \tag{57}$$

Using the same manner of Theorem 8, we obtain the following.

Theorem 11. *For* $f(z) \in \mathscr{A}_p(1)$,

$$\mathcal{O}_j f(z) \ll \frac{z^p}{(1-z)^{2(p-\alpha)}} \quad (z \in \mathbb{U}), \tag{58}$$

if and only if

$$\mathcal{O}_{j+1}f(z) \ll \frac{p}{p+1}\int_0^z \frac{t^{p-1}(p+1+(p-2\alpha-1)t)}{(1-t)^{2p-2\alpha+1}}dt \quad (z \in \mathbb{U}), \tag{59}$$

where $j = \cdots, -1, 0, 1, 2, \cdots$ *and* $0 \le \alpha < p$.

Letting $j = 0$ in Theorem 11, we have the following theorem.

Corollary 12. *For* $f(z) \in \mathscr{A}_p(1)$,

$$f(z) \ll \frac{z^p}{(1-z)^{2(p-\alpha)}} \quad (z \in \mathbb{U}), \tag{60}$$

if and only if

$$\mathcal{O}_1 f(z) \ll \frac{p}{p+1}\int_0^z \frac{t^{p-1}(p+1+(p-2\alpha-1)t)}{(1-t)^{2p-2\alpha+1}}dt \quad (z \in \mathbb{U}), \tag{61}$$

where $0 \le \alpha < p$.

Taking $p = 1$ in Corollary 12, we get the following corollary.

Corollary 13. *For* $f(z) \in \mathscr{A}_1(1)$,

$$f(z) \ll \frac{z}{(1-z)^{2(1-\alpha)}} \quad (z \in \mathbb{U}), \tag{62}$$

if and only if

$$\mathcal{O}_1 f(z) \ll \frac{1}{2(2\alpha-1)}\left(1 - \frac{4\alpha-1-2\alpha z}{(1-z)^{2(1-\alpha)}}\right) \quad (z \in \mathbb{U}), \tag{63}$$

where $0 \le \alpha < 1$ *and* $\alpha \neq 1/2$.

Proof. We only need to check that

$$\frac{1}{2}\int_0^z \frac{2-2\alpha t}{(1-t)^{3-2\alpha}}dt = \int_0^z \left(\frac{\alpha}{(1-t)^{2-2\alpha}} + \frac{1-\alpha}{(1-t)^{3-2\alpha}}\right) \cdot dt = \int_{1-z}^1 \left(\frac{\alpha}{u^{2-2\alpha}} + \frac{1-\alpha}{u^{3-2\alpha}}\right) \cdot du = \frac{1}{2(2\alpha-1)}\left(1 - \frac{4\alpha-1-2\alpha z}{(1-z)^{2(1-\alpha)}}\right). \tag{64}$$

□

Applying the function $f(z)$ given by (34), for operator $\mathcal{O}_j f$, we have the following.

Theorem 14. *For* $f(z) \in \mathscr{A}_1(1)$,

$$\mathcal{O}_j f(z) \ll (2\alpha-1)z - 2(1-\alpha)\log(1-z) \quad (z \in \mathbb{U}), \tag{65}$$

if and only if

$$\mathcal{O}_{j-1}f(z) \ll 2\alpha - 3 + (2\alpha-1)(1+z) - 4(1-\alpha)\frac{\log(1-z)}{z} \quad (z \in \mathbb{U}), \tag{66}$$

where $j = \cdots, -1, 0, 1, 2, \cdots$ *and* $0 \le \alpha < 1$.

Proof. We note that

$$\int_0^z \left(2\alpha - 1 + \frac{2(1-\alpha)}{1-t}\right)dt = (2\alpha-1)z - 2(1-\alpha)\log(1-z). \tag{67}$$

Therefore, if $f(z)$ satisfies the dominant (65), then

$$\mathcal{O}_{j-1}f(z) \ll \frac{2}{z}\int_0^z t\left(2\alpha - 1 + \frac{2(1-\alpha)}{1-t}\right) \cdot dt = 2\alpha - 3 + (2\alpha-1)(1+z) - 4(1-\alpha)\frac{\log(1-z)}{z} \quad (z \in \mathbb{U}). \tag{68}$$

Conversely, if $f(z)$ satisfies the dominant (66), then

$$\frac{z}{2}\mathcal{O}_{j-1}f(z) \ll \int_0^z t\left(2\alpha - 1 + \frac{2(1-\alpha)}{1-t}\right)dt \quad (z \in \mathbb{U}). \tag{69}$$

This implies that

$$\left(\mathcal{O}_j f(z)\right)' \ll 2\alpha - 1 + \frac{2(1-\alpha)}{1-z} \quad (z \in \mathbb{U}). \tag{70}$$

That is, that

$$\mathcal{O}_j f(z) \ll \int_0^z \left(2\alpha - 1 + \frac{2(1-\alpha)}{1-t}\right)dt = (2\alpha-1)z - 2(1-\alpha) \cdot \log(1-z) \quad (z \in \mathbb{U}). \tag{71}$$

This completes the proof of the theorem. □

Letting $j = 0$ in Theorem 14, we see the following.

Corollary 15. *For $f(z) \in \mathcal{A}_1(1)$,*

$$f(z) \ll (2\alpha-1)z - 2(1-\alpha)\log(1-z) \quad (z \in \mathbb{U}), \tag{72}$$

if and only if

$$\mathcal{O}_{-1}f(z) \ll 2\alpha - 3 + (2\alpha-1)(1+z) - 4(1-\alpha)\frac{\log(1-z)}{z} \quad (z \in \mathbb{U}), \tag{73}$$

where $0 \le \alpha < 1$.

Similarly, we have the following.

Theorem 16. *For $f(z) \in \mathcal{A}_1(1)$,*

$$\mathcal{O}_j f(z) \ll (2\alpha-1)z - 2(1-\alpha)\log(1-z) \quad (z \in \mathbb{U}), \tag{74}$$

if and only if

$$\mathcal{O}_{j+1}f(z) \ll (2\alpha-1)z - (1-\alpha)\log(1-z) - (1-\alpha)\int_0^z \frac{\log(1-t)}{t}dt \quad (z \in \mathbb{U}), \tag{75}$$

where $j = \cdots, -1, 0, 1, 2, \cdots$ and $0 \le \alpha < 1$.

Proof. If $f(z)$ satisfies the dominant (74), then

$$\mathcal{O}_{j+1}f(z) \ll \frac{1}{2}\int_0^z \frac{1}{t}\left(2(2\alpha-1)t - 2(1-\alpha)\log(1-t) + 2(1-\alpha)\frac{t}{1-t}\right) \cdot dt = (2\alpha-1)z - (1-\alpha)\log(1-z) - (1-\alpha) \cdot \int_0^z \frac{\log(1-t)}{t}dt \quad (z \in \mathbb{U}). \tag{76}$$

Further, if $f(z)$ satisfies the dominant (75), then we have

$$\frac{\left(z\mathcal{O}_j f(z)\right)'}{z} \ll 2\left((2\alpha-1) + \frac{1-\alpha}{1-z} - (1-\alpha)\frac{\log(1-z)}{z}\right) \quad (z \in \mathbb{U}). \tag{77}$$

That is,

$$z\mathcal{O}_j f(z) \ll (2\alpha-1)z^2 - 2(1-\alpha)z\log(1-z) \quad (z \in \mathbb{U}), \tag{78}$$

which is the same as the dominant (74). □

Making $j = 0$ in Theorem 16, we have the following.

Corollary 17. *For $f(z) \in \mathcal{A}_1(1)$,*

$$f(z) \ll (2\alpha-1)z - 2(1-\alpha)\log(1-z) \quad (z \in \mathbb{U}), \tag{79}$$

if and only if

$$\mathcal{O}_1 f(z) \ll (2\alpha-1)z - (1-\alpha)\log(1-z) - (1-\alpha)\int_0^z \frac{\log(1-t)}{t}dt \quad (z \in \mathbb{U}), \tag{80}$$

where $0 \le \alpha < 1$.

Next, we have the following.

Theorem 18. *For $f(z) \in \mathcal{A}_1(1)$,*

$$\mathcal{O}_j f(z) \ll \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) \quad (z \in \mathbb{U}), \tag{81}$$

if and only if

$$\mathcal{O}_{j-1}f(z) \ll \frac{1}{z}\log\left(\frac{1}{1-z^2}\right) \quad (z \in \mathbb{U}), \tag{82}$$

where $j = \cdots, -1, 0, 1, 2, \cdots$.

Proof. Note that

$$z\left(\mathcal{O}_j f(z)\right)' \ll \frac{1}{2}\left(\frac{1}{1+z} + \frac{1}{1-z}\right) \quad (z \in \mathbb{U}). \tag{83}$$

Therefore,

$$\mathcal{O}_{j-1}f(z) = \frac{2}{z}\int_0^z t\left(\mathcal{O}_j f(t)\right)' dt \ll \frac{1}{z}\int_0^z \left(\frac{1}{1+t} + \frac{1}{1-t}\right) \cdot dt = \frac{1}{z}\log\left(\frac{1}{1-z^2}\right) \quad (z \in \mathbb{U}). \tag{84}$$

Conversely, if $f(z)$ satisfies the dominant (82), then

$$\frac{\left(z\mathcal{O}_{j-1}f(z)\right)'}{z} \ll \frac{2}{1-z^2} \quad (z \in \mathbb{U}). \tag{85}$$

That is, that

$$\mathcal{O}_j f(z) = \frac{1}{2}\int_0^z \frac{\left(t\mathcal{O}_{j-1}f(t)\right)'}{t} dt \ll \int_0^z \frac{1}{1-t^2} dt = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right). \tag{86}$$

□

Making $j = 0$ in Theorem 18, we see the following.

Corollary 19 (see [3]). *For $f(z) \in \mathcal{A}_1(1)$,*

$$f(z) \ll \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) \quad (z \in \mathbb{U}), \tag{87}$$

if and only if

$$\mathcal{O}_{-1}f(z) \ll \frac{1}{z}\log\left(\frac{1}{1-z^2}\right) \quad (z \in \mathbb{U}). \tag{88}$$

Remark 20. The function $f(z)$ given by

$$f(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) = z + \sum_{k=2}^{\infty} \frac{1}{2k-1} z^{2k-1} \quad (z \in \mathbb{U}) \tag{89}$$

is convex in $\mathbb{U}$. Also, this function maps $\mathbb{U}$ onto the strip region with $-\pi/4 < Imf(z) < \pi/4$.

In the following section, we have some subordination relations for functions associated with the operator $\mathcal{O}_j f$.

3. Subordinations

Let $f(z)$ and $F(z)$ belong to the class $\mathcal{A}_p(n)$. Then, $f(z)$ is said to be subordinate to $F(z)$ if there exists a function $w(z)$ which is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$ such that $f(z) = F(w(z))$. We denote this subordination by

$$f(z) \prec F(z) \quad (z \in \mathbb{U}). \tag{90}$$

If $F(z) \in \mathcal{A}_1(1)$ is univalent in $\mathbb{U}$, then the subordination (90) for $f(z) \in \mathcal{A}_1(1)$ is equivalent that $f(0) = F(0)$ and $f(\mathbb{U}) \subset F(\mathbb{U})$ (see Miller and Mocanu [25]).

For subordinations, Miller and Mocanu [26] have given the following.

Lemma 21. *Let n be a positive integer, $\lambda > 0$, and let $\beta_0 = \beta_0(\lambda, n)$ be the root of the equation*

$$\beta\pi = \frac{3}{2}\pi - \tan^{-1}(n\lambda\beta). \tag{91}$$

Also, let

$$\alpha = \alpha(\beta, \lambda, n) = \beta + \frac{2}{\pi}\tan^{-1}(n\lambda\beta) \quad (0 < \beta < \beta_0). \tag{92}$$

If $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$, then

$$p(z) + \lambda z p'(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha} \quad (z \in \mathbb{U}) \tag{93}$$

satisfies

$$p(z) \prec \left(\frac{1+z}{1-z}\right)^{\beta} \quad (z \in \mathbb{U}). \tag{94}$$

Using the above lemma, we derive the following.

Theorem 22. *If $f(z) \in \mathcal{A}_p(n)$ satisfies*

$$\frac{1}{z^p}\left\{\lambda z\left(\mathcal{O}_j f(z)\right)' - (\lambda p - 1)\mathcal{O}_j f(z)\right\} \prec \left(\frac{1+z}{1-z}\right)^{\alpha} \quad (z \in \mathbb{U}), \tag{95}$$

then

$$\frac{\mathcal{O}_j f(z)}{z^p} \prec \left(\frac{1+z}{1-z}\right)^{\beta} \quad (z \in \mathbb{U}), \tag{96}$$

where $j = \cdots, -1, 0, 1, 2, \cdots$ and α and β are defined in Lemma 21.

Proof. We define a function $g(z)$ by

$$g(z) = \frac{\mathcal{O}_j f(z)}{z^p} \quad (j = \cdots, -1, 0, 1, 2, \cdots, z \in \mathbb{U}). \tag{97}$$

Then, $g(z)$ is analytic in $\mathbb{U}$ and $g(0) = 1$. It follows that

$$g(z) + \lambda z g'(z) = \frac{1}{z^p}\left\{\lambda z\left(\mathcal{O}_j f(z)\right)' - (\lambda p - 1)\mathcal{O}_j f(z)\right\} \prec \left(\frac{1+z}{1-z}\right)^{\alpha} \quad (z \in \mathbb{U}). \tag{98}$$

Therefore, using Lemma 21, we have

$$g(z)=\frac{\mathcal{O}_j f(z)}{z^p}\prec\left(\frac{1+z}{1-z}\right)^{\beta}\quad(z\in\mathbb{U}).\tag{99}$$

□

Letting $j=0$ in Theorem 22, we see the following.

Corollary 23. *If $f(z)\in\mathcal{A}_p(n)$ satisfies*

$$\frac{1}{z^p}\left\{\lambda z f'(z)-(\lambda p-1)f(z)\right\}\prec\left(\frac{1+z}{1-z}\right)^{\alpha}\quad(z\in\mathbb{U}),\tag{100}$$

then

$$\frac{f(z)}{z^p}\prec\left(\frac{1+z}{1-z}\right)^{\beta}\quad(z\in\mathbb{U}),\tag{101}$$

where α and β are defined in Lemma 21.

If we take $n=\lambda=\beta=1$ in Lemma 21, then $\alpha=\alpha(1,1,1)=3/2$. For such n,λ,β, we know the following.

Corollary 24. *If $f(z)\in\mathcal{A}_p(n)$ satisfies*

$$\frac{1}{z^p}\left\{z\left(\mathcal{O}_j f(z)\right)'-(p-1)\mathcal{O}_j f(z)\right\}\prec\left(\frac{1+z}{1-z}\right)^{3/2}(z\in\mathbb{U}),\tag{102}$$

then

$$\frac{\mathcal{O}_j f(z)}{z^p}\prec\frac{1+z}{1-z}\quad(z\in\mathbb{U}),\tag{103}$$

where $j=\cdots,-1,0,1,2,\cdots$.

If we consider $n=\lambda=\alpha=1$ in Lemma 21, then we see $\beta=0.638\cdots$. For such n,λ,α, we have the following.

Corollary 25. *If $f(z)\in\mathcal{A}_p(n)$ satisfies*

$$\frac{1}{z^p}\left\{z\left(\mathcal{O}_j f(z)\right)'-(p-1)\mathcal{O}_j f(z)\right\}\prec\frac{1+z}{1-z}\quad(z\in\mathbb{U}),\tag{104}$$

then

$$\frac{\mathcal{O}_j f(z)}{z^p}\prec\left(\frac{1+z}{1-z}\right)^{\beta}\quad(z\in\mathbb{U}),\tag{105}$$

where $\beta=0.638\cdots$ and $j=\cdots,-1,0,1,2,\cdots$.

Taking $p=1$ in Corollary 24, we know the following.

Corollary 26. *If $f(z)\in\mathcal{A}_1(n)$ satisfies*

$$\left(\mathcal{O}_j f(z)\right)'\prec\left(\frac{1+z}{1-z}\right)^{3/2}\quad(z\in\mathbb{U}),\tag{106}$$

then

$$\frac{\mathcal{O}_j f(z)}{z}\prec\frac{1+z}{1-z}\quad(z\in\mathbb{U}),\tag{107}$$

where $j=\cdots,-1,0,1,2,\cdots$.

Remark 27. The function $w=(1+z)/(1-z)$ maps $\mathbb{U}$ onto the domain such that $\operatorname{Re}w>0$. Therefore, a function $w=((1+z)/(1-z))^{3/2}$ maps $\mathbb{U}$ onto the domain such that $-(3/4)\pi<\arg w<(3/4)\pi$.

Letting $\lambda=1/(p+1)$ in Theorem 22, we see the following.

Corollary 28. *If $f(z)\in\mathcal{A}_p(n)$ satisfies*

$$\frac{1}{p+1}\frac{\left(z\mathcal{O}_j f(z)\right)'}{z^p}\prec\left(\frac{1+z}{1-z}\right)^{\alpha}\quad(z\in\mathbb{U}),\tag{108}$$

then

$$\frac{\mathcal{O}_j f(z)}{z^p}\prec\left(\frac{1+z}{1-z}\right)^{\beta}\quad(z\in\mathbb{U}),\tag{109}$$

where $j=\cdots,-1,0,1,2,\cdots$ and α and β are defined in Lemma 21.

In the next section, we get some results for subclasses of analytic functions concerning with m different boundary points.

4. A Subclass concerning with m Different Boundary Points

Now, we consider m different boundary points z_ℓ ($\ell=1,2,3,\cdots,m$) with $|z_\ell|=1$. For such boundary points z_ℓ, we define

$$\gamma_m=\frac{1}{m}\sum_{\ell=1}^{m}\frac{\mathcal{O}_j f(z_\ell)}{z_\ell^p}\quad(j=\cdots,-1,0,1,2,\cdots),\tag{110}$$

with $\gamma_m\in e^{i\delta}\mathcal{O}_j f(\mathbb{U}),\gamma_m\neq 1$ and $-\pi/2\leq\delta\leq\pi/2$.

With the above γ_m, if $f(z)\in\mathcal{A}_p(n)$ satisfies

$$\left|\frac{e^{i\delta}\left(\left(\mathcal{O}_j f(z)\right)/z^p\right)-\gamma_m}{e^{i\delta}-\gamma_m}-1\right|<\rho\quad(z\in\mathbb{U}),\tag{111}$$

for some $\rho>0$, we say that $f(z)\in\mathcal{T}_p(\gamma_m,\delta,\rho;m,j)$.

We see that the condition (111) for the class $\mathcal{T}_p(\gamma_m, \delta, \rho\,; m, j)$ is equivalent to

$$\left|\frac{\mathcal{O}_j f(z)}{z^p} - 1\right| < \rho\left|e^{i\delta} - \gamma_m\right| \quad (z \in \mathbb{U}). \tag{112}$$

Noting that

$$\mathcal{O}_j f(z) = z^p + \sum_{k=p+n}^{\infty}\left(\frac{p(k+1)}{(p+1)k}\right)^j a_k z^k, \tag{113}$$

for $j = \cdots, -1, 0, 1, 2, \cdots$, if we consider a function $f(z) \in \mathcal{A}_p(n)$ such that

$$f(z) = z^p + \left(\frac{(p+n)(p+1)}{p(p+n+1)}\right)^j \rho\left(e^{i\delta} - \gamma_m\right) z^{p+n}, \tag{114}$$

then we have

$$\left|\frac{\mathcal{O}_j f(z)}{z^p} - 1\right| = \rho\left|e^{i\delta} - \gamma_m\right||z|^n < \rho\left|e^{i\delta} - \gamma_m\right| \quad (z \in \mathbb{U}). \tag{115}$$

Now we introduce the following lemma by Miller and Mocanu [25, 27] (also, due to Jack [28]).

Lemma 29. *Let the function $w(z)$ given by*

$$w(z) = a_n z^n + a_{n+1} z^{n+1} + \cdots \quad (n \in \mathbb{N} = \{1, 2, 3, \cdots\}) \tag{116}$$

be analytic in $\mathbb{U}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r$ at a point $z_0 \in \mathbb{U}$, then there exists a real number $k \geq n$ such that

$$\frac{z_0 w'(z_0)}{w(z_0)} = k, \tag{117}$$

$$\operatorname{Re}\left(1 + \frac{z_0 w''(z_0)}{w'(z_0)}\right) \geq k. \tag{118}$$

Using the above lemma, we derive the following.

Theorem 30. *If $f(z) \in \mathcal{A}_p(n)$ satisfies*

$$\left|\alpha\frac{\mathcal{O}_j f(z)}{z^p} + \beta\frac{\left(\mathcal{O}_j f(z)\right)'}{z^{p-1}} - (\alpha + \beta p)\right| < \rho(\alpha + \beta(p+n))\left|e^{i\delta} - \gamma_m\right| \quad (z \in \mathbb{U}), \tag{119}$$

for some γ_m given by (110) with $\gamma_m \neq 1$ such that $z_q \in \partial\mathbb{U}$ $(q = 1, 2, 3, \cdots, m)$ and for some real $\rho > 1$, then

$$\left|\frac{\mathcal{O}_j f(z)}{z^p} - 1\right| < \rho\left|e^{i\delta} - \gamma_m\right| \quad (z \in \mathbb{U}), \tag{120}$$

where $j = \cdots, -1, 0, 1, 2, \cdots$ and $\alpha \geq 0$ and $\beta \geq 0$. This implies that $f(z) \in \mathcal{T}_p(\gamma_m, \delta, \rho\,; m, j)$.

Proof. Define a function $w(z)$ by

$$\begin{aligned} w(z) &= \frac{e^{i\delta}\left(\left(\mathcal{O}_j f(z)\right)/z^p\right) - \gamma_m}{e^{i\delta} - \gamma_m} - 1 \\ &= \frac{e^{i\delta}}{e^{i\delta} - \gamma_m}\left(\sum_{k=p+n}^{\infty}\left(\frac{p(k+1)}{(p+1)k}\right)^j a_k z^{k-p}\right). \end{aligned} \tag{121}$$

Then, $w(z)$ is analytic in $\mathbb{U}$ and $w(0) = 0$. Since

$$\frac{\mathcal{O}_j f(z)}{z^p} = 1 + \left(1 - \gamma_m e^{-i\delta}\right) w(z), \tag{122}$$

$$\frac{\left(\mathcal{O}_j f(z)\right)'}{z^{p-1}} = p + \left(1 - \gamma_m e^{-i\delta}\right) w(z)\left(p + \frac{z w'(z)}{w(z)}\right), \tag{123}$$

we know that

$$\begin{aligned} &\left|\alpha\frac{\mathcal{O}_j f(z)}{z^p} + \beta\frac{\left(\mathcal{O}_j f(z)\right)'}{z^{p-1}} - (\alpha + \beta p)\right| \\ &= \left|\left(1 - \gamma_m e^{-i\delta}\right) w(z)\left(\alpha + \beta p + \beta\frac{z w'(z)}{w(z)}\right)\right| \\ &< \rho(\alpha + \beta(p+n))\left|e^{i\delta} - \gamma_m\right| \quad (z \in \mathbb{U}), \end{aligned} \tag{124}$$

by (119). Suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$\max\left\{w(z)\,; |z| \leq |z_0|\right\} = |w(z_0)| = \rho > 1. \tag{125}$$

Then, applying Lemma 29, we write that $w(z_0) = \rho e^{i\theta}$ $(0 \leq \theta < 2\pi)$ and $z_0 w'(z_0) = k w(z_0)$ $(k \geq n)$. This shows that

$$\begin{aligned} &\left|\alpha\frac{\mathcal{O}_j f(z_0)}{z_0^p} + \beta\frac{\left(\mathcal{O}_j f(z_0)\right)'}{z_0^{p-1}} - (\alpha + \beta p)\right| \\ &= \left|e^{i\delta} - \gamma_m\right|\rho(\alpha + \beta(p+k)) \geq \rho(\alpha + \beta(p+n))\left|e^{i\delta} - \gamma_m\right|. \end{aligned} \tag{126}$$

Since this contradicts our condition (119), we say that there is no $z_0 (0 < |z_0| < 1)$ such that $|w(z_0)| = \rho > 1$. Letting $|w(z)| < \rho$ for all $z \in \mathbb{U}$, we obtain

$$|w(z)| = \left|\frac{\left(\left(\mathcal{O}_j f(z)\right)/z^p\right) - 1}{e^{i\delta} - \gamma_m}\right| < \rho \quad (z \in \mathbb{U}). \tag{127}$$

This implies that $f(z) \in \mathcal{T}_p(\gamma_m, \delta, \rho\,; m, j)$. □

Taking $\alpha = 0$ in Theorem 30, we have the following.

Corollary 31. *If $f(z) \in \mathscr{A}_p(n)$ satisfies*

$$\left| \frac{\left(\mathcal{O}_j f(z)\right)'}{z^{p-1}} - p \right| < \rho(p+n)\left|e^{i\delta} - \gamma_m\right| \quad (z \in \mathbb{U}), \tag{128}$$

for some γ_m given by (110) with $\gamma_m \neq 1$ such that $z_q \in \partial\mathbb{U}$ $(q = 1, 2, 3, \cdots, m)$ and for some real $\rho > 1$, then

$$\left| \frac{\mathcal{O}_j f(z)}{z^p} - 1 \right| < \rho\left|e^{i\delta} - \gamma_m\right| \quad (z \in \mathbb{U}), \tag{129}$$

where $j = \cdots, -1, 0, 1, 2, \cdots$, that is that $f(z) \in \mathcal{T}_p(\gamma_m, \delta, \rho; m, j)$.

Example 1. Consider a function

$$f(z) = z^p + a_{p+n} z^{p+n}. \tag{130}$$

It follows that

$$\left| \frac{\left(\mathcal{O}_j f(z)\right)'}{z^{p-1}} - p \right| = \left| \frac{p^j (p+n+1)^j}{(p+1)^j (p+n)^{j-1}} a_{p+n} z^n \right| < \frac{p^j (p+n+1)^j}{(p+1)^j (p+n)^{j-1}} \left|a_{p+n}\right| \quad (z \in \mathbb{U}). \tag{131}$$

We consider five boundary points such that

$$z_1 = e^{-i\left(\arg\left(a_{p+n}\right)/n\right)}, \tag{132}$$

$$z_2 = e^{i\left(\pi - 6\arg\left(a_{p+n}\right)\right)/6n}, \tag{133}$$

$$z_3 = e^{i\left(\pi - 4\arg\left(a_{p+n}\right)\right)/4n}, \tag{134}$$

$$z_4 = e^{i\left(\pi - 3\arg\left(a_{p+n}\right)\right)/3n}, \tag{135}$$

$$z_5 = e^{i\left(\pi - 2\arg\left(a_{p+n}\right)\right)/2n}. \tag{136}$$

For these points $z_\ell (\ell = 1, 2, 3, 4, 5)$, we know that

$$\begin{aligned}
\frac{\mathcal{O}_j f(z_1)}{z_1^p} &= 1 + \left(\frac{p(p+n+1)}{(p+1)(p+n)}\right)^j \left|a_{p+n}\right|, \\
\frac{\mathcal{O}_j f(z_2)}{z_2^p} &= 1 + \left(\frac{p(p+n+1)}{(p+1)(p+n)}\right)^j \frac{\sqrt{3}+i}{2} \left|a_{p+n}\right|, \\
\frac{\mathcal{O}_j f(z_3)}{z_3^p} &= 1 + \left(\frac{p(p+n+1)}{(p+1)(p+n)}\right)^j \frac{\sqrt{2}(1+i)}{2} \left|a_{p+n}\right|, \\
\frac{\mathcal{O}_j f(z_4)}{z_4^p} &= 1 + \left(\frac{p(p+n+1)}{(p+1)(p+n)}\right)^j \frac{1+\sqrt{3}i}{2} \left|a_{p+n}\right|, \\
\frac{\mathcal{O}_j f(z_5)}{z_5^p} &= 1 + \left(\frac{p(p+n+1)}{(p+1)(p+n)}\right)^j i\left|a_{p+n}\right|.
\end{aligned} \tag{137}$$

It follows from the above that

$$\gamma_5 = \frac{1}{5}\sum_{\ell=1}^{5} \frac{\mathcal{O}_j f(z_\ell)}{z_\ell^p} = 1 + \frac{\left(3+\sqrt{2}+\sqrt{3}\right)(1+i)}{10} \left(\frac{p(p+n+1)}{(p+1)(p+n)}\right)^j \left|a_{p+n}\right|. \tag{138}$$

Thus, we see that

$$\left|e^{i\delta} - \gamma_5\right| = \frac{\sqrt{2}\left(3+\sqrt{2}+\sqrt{3}\right)}{10} \left(\frac{p(p+n+1)}{(p+1)(p+n)}\right)^j \left|a_{p+n}\right|, \tag{139}$$

with $\delta = 0$. Using such γ_5 and $\delta = 0$, we consider $\rho > 1$ with

$$\frac{p^j (p+n+1)^j}{(p+1)^j (p+n)^{j-1}} \left|a_{p+n}\right| \le \rho(p+n)\left|e^{i\delta} - \gamma_5\right|. \tag{140}$$

That is, that

$$\rho \ge \left(\frac{p(p+n+1)}{(p+1)(p+n)}\right)^j \frac{\left|a_{p+n}\right|}{\left|e^{i\delta} - \gamma_5\right|} = \frac{10}{\sqrt{2}\left(3+\sqrt{2}+\sqrt{3}\right)} > 1. \tag{141}$$

With the above γ_5 and ρ, the function $f(z)$ satisfies

$$\left| \frac{\mathcal{O}_j f(z)}{z^p} - 1 \right| < \left(\frac{p(p+n+1)}{(p+1)(p+n)}\right)^j \left|a_{p+n}\right| \le \rho\left|e^{i\delta} - \gamma_5\right| \quad (z \in \mathbb{U}). \tag{142}$$

Taking $j = 0$ in Theorem 30, we have the following.

Corollary 32. *If $f(z) \in \mathscr{A}_p(n)$ satisfies*

$$\left| \alpha\frac{f(z)}{z^p} + \beta\frac{f'(z)}{z^{p-1}} - (\alpha + \beta p) \right| < \rho(\alpha + \beta(p+n))\left|e^{i\delta} - \gamma_m\right| \quad (z \in \mathbb{U}), \tag{143}$$

for some γ_m given by (110) with $\gamma_m \neq 1$ such that $z_q \in \partial\mathbb{U}$ $(q = 1, 2, 3, \cdots, m)$ and for some real $\rho > 1$, then

$$\left| \frac{f(z)}{z^p} - 1 \right| < \rho\left|e^{i\delta} - \gamma_m\right| \quad (z \in \mathbb{U}), \tag{144}$$

where $\alpha \ge 0$ and $\beta \ge 0$.

Taking $\alpha = \beta = 1$ in Theorem 30, we have the following.

Corollary 33. *If $f(z) \in \mathscr{A}_p(n)$ satisfies*

$$\left| \frac{\left(z\mathcal{O}_j f(z)\right)'}{z^p} - (p+1) \right| < \rho(p+n+1)\left|e^{i\delta} - \gamma_m\right| \quad (z \in \mathbb{U}), \tag{145}$$

for some γ_m *given by (110) with* $\gamma_m \neq 1$ *such that* $z_q \in \partial\mathbb{U}$ $(q = 1, 2, 3, \cdots, m)$ *and for some real* $\rho > 1$*, then*

$$\left|\frac{\mathcal{O}_j f(z)}{z^p} - 1\right| < \rho\left|e^{i\delta} - \gamma_m\right| \quad (z \in \mathbb{U}), \tag{146}$$

where $j = \cdots, -1, 0, 1, 2, \cdots$*, that is,* $f(z) \in \mathcal{T}_p(\gamma_m, \delta, \rho; m, j)$.

Example 2. We consider a function $f(z) \in \mathcal{A}_p(n)$ given by (130). Then, $f(z)$ satisfies

$$\begin{aligned}\left|\frac{\left(z\mathcal{O}_j f(z)\right)'}{z^p} - (p+1)\right| &= \left|\frac{p^j(p+n+1)^{j+1}}{(p+1)^j(p+n)^j} a_{p+n} z^n\right| \\ &< \frac{p^j(p+n+1)^{j+1}}{(p+1)^j(p+n)^j}\left|a_{p+n}\right| \quad (z \in \mathbb{U}).\end{aligned} \tag{147}$$

We also consider five boundary points $z_\ell (\ell = 1, 2, 3, 4, 5)$, given by (132)–(136). Since

$$|\gamma_5| = \frac{1}{5}\sum_{\ell=1}^{5} \frac{\mathcal{O}_j f(z_\ell)}{z_\ell^p} = 1 + \frac{\left(3 + \sqrt{2} + \sqrt{3}\right)(1+i)}{10}\left(\frac{p(p+n+1)}{(p+1)(p+n)}\right)^j \left|a_{p+n}\right|, \tag{148}$$

we have

$$\left|e^{i\delta} - \gamma_5\right| = \frac{\sqrt{2}\left(3 + \sqrt{2} + \sqrt{3}\right)}{10}\left(\frac{p(p+n+1)}{(p+1)(p+n)}\right)^j \left|a_{p+n}\right|, \tag{149}$$

with $\delta = 0$. With the above γ_5 and $\delta = 0$, we consider $\rho > 1$ such that

$$\frac{p^j(p+n+1)^{j+1}}{(p+1)^j(p+n)^j}\left|a_{p+n}\right| \le \rho(p+n+1)\left|e^{i\delta} - \gamma_5\right|. \tag{150}$$

Then, ρ satisfies

$$\rho \ge \left(\frac{p(p+n+1)}{(p+1)(p+n)}\right)^j \frac{\left|a_{p+n}\right|}{\left|e^{i\delta} - \gamma_5\right|} = \frac{10}{\sqrt{2}\left(3 + \sqrt{2} + \sqrt{3}\right)} > 1. \tag{151}$$

Using γ_5 and ρ, we have that

$$\left|\frac{\mathcal{O}_j f(z)}{z^p} - 1\right| < \left(\frac{p(p+n+1)}{(p+1)(p+n)}\right)^j \left|a_{p+n}\right| \le \rho\left|e^{i\delta} - \gamma_5\right| \quad (z \in \mathbb{U}). \tag{152}$$

Next, our theorem is the following.

Theorem 34. *If* $f(z) \in \mathcal{A}_p(n)$ *satisfies*

$$\left|\frac{z\left(\mathcal{O}_j f(z)\right)'}{z^p} - \frac{z\left(\mathcal{O}_j f(z)\right)'}{\mathcal{O}_j f(z)} - p\frac{\mathcal{O}_j f(z)}{z^p} - p\right| < \frac{\left|e^{i\delta} - \gamma_m\right|^2 n\rho^2}{1 + \left|e^{i\delta} - \gamma_m\right|\rho} \quad (z \in \mathbb{U}), \tag{153}$$

for some γ_m *given by (110) with* $\gamma_m \neq 1$ *such that* $z_q \in \partial\mathbb{U}$ $(q = 1, 2, 3, \cdots, m)$ *and for some real* $\rho > 1$*, then*

$$\left|\frac{\mathcal{O}_j f(z)}{z^p} - 1\right| < \rho\left|e^{i\delta} - \gamma_m\right| \quad (z \in \mathbb{U}). \tag{154}$$

This means that $f(z) \in \mathcal{T}_p(\gamma_m, \delta, \rho; m, j)$.

Proof. Define a function $w(z)$ by

$$w(z) = \frac{e^{i\delta}\left(\left(\mathcal{O}_j f(z)\right)/z^p\right) - \gamma_m}{e^{i\delta} - \gamma_m} - 1 = \frac{e^{i\delta}\left(\left(\left(\mathcal{O}_j f(z)\right)/z^p\right) - 1\right)}{e^{i\delta} - \gamma_m} \quad (z \in \mathbb{U}). \tag{155}$$

Then, $w(z)$ is analytic in $\mathbb{U}, w(0) = 0$, and is given by (121). Noting that

$$\frac{\mathcal{O}_j f(z)}{z^p} = 1 + \left(1 - \gamma_m e^{-i\delta}\right) w(z), \tag{156}$$

$$\frac{z\left(\mathcal{O}_j f(z)\right)'}{z^p} = p + \left(1 - \gamma_m e^{-i\delta}\right)\left(p w(z) + z w'(z)\right). \tag{157}$$

We know that

$$\begin{aligned}&\left|\frac{z\left(\mathcal{O}_j f(z)\right)'}{z^p} - \frac{z\left(\mathcal{O}_j f(z)\right)'}{\mathcal{O}_j f(z)} - p\frac{\mathcal{O}_j f(z)}{z^p} - p\right| \\ &= \frac{\left|1 - e^{-i\delta}\gamma_m\right|^2 \left|w(z) z w'(z)\right|}{\left|1 + \left(1 - e^{-i\delta}\gamma_m\right) w(z)\right|} < \frac{\left|e^{i\delta} - \gamma_m\right|^2 n\rho^2}{1 + \left|e^{i\delta} - \gamma_m\right|\rho} \quad (z \in \mathbb{U}).\end{aligned} \tag{158}$$

Consider that there exists a point $z_0 \in \mathbb{U}$ such that

$$\max\left\{w(z); |z| \le |z_0|\right\} = |w(z_0)| = \rho > 1. \tag{159}$$

Then, Lemma 29 gives us that $w(z_0) = \rho e^{i\theta} (0 \le \theta \le 2\pi)$ and $z_0 w'(z_0) = k w(z_0) (k \ge n)$. It follows that

$$\begin{aligned}
&\left|\frac{z_0(\mathcal{O}_jf(z_0))'}{z_0^p}-\frac{z_0(\mathcal{O}_jf(z_0))'}{\mathcal{O}_jf(z_0)}-p\frac{\mathcal{O}_jf(z_0)}{z_0^p}-p\right| \\
&=\frac{\left|1-e^{-i\delta}\gamma_m\right|^2\rho^2k}{\left|1+(1-e^{-i\delta}\gamma_m)\rho e^{i\theta}\right|} \\
&=\frac{\left|1-e^{-i\delta}\gamma_m\right|^2\rho^2k}{\sqrt{\left(1+\left|1-e^{-i\delta}\gamma_m\right|\rho\cos\theta\right)^2+\left(\left|1-e^{-i\delta}\gamma_m\right|\rho\sin\theta\right)^2}} \\
&=\frac{\left|1-e^{-i\delta}\gamma_m\right|^2\rho^2k}{\sqrt{1+\left|1-e^{-i\delta}\gamma_m\right|^2\rho^2+2\left|1-e^{-i\delta}\gamma_m\right|\rho\cos\phi}} \\
&\geq\frac{\left|1-e^{-i\delta}\gamma_m\right|^2\rho^2k}{1+\left|1-e^{-i\delta}\gamma_m\right|\rho}\geq\frac{\left|e^{i\delta}-\gamma_m\right|^2\rho^2n}{1+\left|e^{i\delta}-\gamma_m\right|\rho},
\end{aligned} \tag{160}$$

where $\phi=\theta+\arg(1-e^{-i\delta}\gamma_m)$. Since this contradicts our condition (153), there is no $z_0\in\mathbb{U}$ such that $|w(z_0)|=\rho>1$. This means that $|w(z)|<\rho$ for all $z\in\mathbb{U}$. Thus, we have

$$|w(z)|=\left|\frac{e^{i\delta}\left(\left(\mathcal{O}_jf(z)\right)/z^p\right)-\gamma_m}{e^{i\delta}-\gamma_m}-1\right|<\rho \quad (z\in\mathbb{U}). \tag{161}$$

This completes the proof of the theorem. □

Finally, we consider the coefficient problem for the class $\mathcal{T}_p(\gamma_m,\delta,\rho;m,j)$.

Theorem 35. *Let $f(z)\in\mathcal{A}_p(n)$ satisfies*

$$|a_k|\leq\left(\frac{(p+1)k}{p(k+1)}\right)^j\rho\left|e^{i\delta}-\gamma_m\right|\left(\frac{1}{2\sum_{\ell=p+n-1}^{k-1}(k-\ell)}\right), \tag{162}$$

for $k=p+n,p+n+1,p+n+2,\cdots$; then,

$$\left|\frac{\mathcal{O}_jf(z)}{z^p}-1\right|<\rho\left|e^{i\delta}-\gamma_m\right| \quad (z\in\mathbb{U}), \tag{163}$$

where γ_m is given by (110) with $\gamma_m\neq1$ and $\rho>1$.

Proof. With a_k defined by (162), we know that

$$\begin{aligned}
\left|\frac{\mathcal{O}_jf(z)}{z^p}-1\right|&=\left|\sum_{k=p+n}^{\infty}\left(\frac{p(k+1)}{(p+1)k}\right)^ja_kz^k\right|<\sum_{k=p+n}^{\infty}\left(\frac{p(k+1)}{(p+1)k}\right)^j|a_k| \\
&\leq\sum_{k=p+n}^{\infty}\rho\left|e^{i\delta}-\gamma_m\right|\left(\frac{1}{2\sum_{\ell=p+n-1}^{k-1}(k-\ell)}\right) \\
&=\rho\left|e^{i\delta}-\gamma_m\right|\sum_{k=p+n}^{\infty}\left(\frac{1}{2\sum_{\ell=p+n-1}^{k-1}(k-\ell)}\right) \\
&=\rho\left|e^{i\delta}-\gamma_m\right|\left(\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\cdots\right) \\
&=\rho\left|e^{i\delta}-\gamma_m\right|\left\{\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots\right\} \\
&=\rho\left|e^{i\delta}-\gamma_m\right|.
\end{aligned} \tag{164}$$

□

Remark 36. From Theorem 35, we know that the function

$$f(z)=z^p+\sum_{k=p+n}^{\infty}a_kz^k, \tag{165}$$

with

$$a_k=\left(\frac{(p+1)k}{p(k+1)}\right)^j\rho\left|e^{i\delta}-\gamma_m\right|\left(\frac{e^{i\theta}}{2\sum_{\ell=p+n-1}^{k-1}(k-\ell)}\right), \tag{166}$$

for $k=p+n,p+n+1,p+n+2,\cdots$ is a member of the class $\mathcal{T}_p(\gamma_m,\delta,\rho;m,j)$.

5. Conclusion

In the present paper, we defined new integral operators $\mathcal{O}_{-j}f(z)$ and $\mathcal{O}_jf(z)$ of analytic functions $f(z)\in A_p(n)$. The dominants for both operators $\mathcal{O}_{-j}f(z)$ and $\mathcal{O}_jf(z)$ and subordinations for $\mathcal{O}_{-j}f(z)$ and $\mathcal{O}_jf(z)$ are discussed. Also, new subclass $\mathcal{T}_p(\gamma_m,\delta,\rho;m,j)$ concerning with m different boundary points is defined and discussed. Moreover, some interesting problems of $\mathcal{T}_p(\gamma_m,\delta,\rho;m,j)$ associated with $\mathcal{O}_jf(z)$ are obtained. Furthermore, some interesting examples for our results are considered.

Data Availability

Data used to support the findings of this study are included within the article.

Disclosure

A little portion of the results in this article were presented in GFTA 2021, 15-18 October 2021, Sibiu (http://gfta2021.uab.ro/upls/Abstract.pdf).

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

The authors contributed equally to the writing of this paper. All authors approved the final version of the manuscript.

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