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Research Article

Existence Results of Random Impulsive Integrodifferential Inclusions with Time-Varying Delays

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This study examines the existence of mild solutions for nonlinear random impulsive integrodifferential inclusions with time-varying delays under sufficient conditions. Our study is based on the Martelli fixed point theorem, Pachpatte's inequality, and the fixed point theorem due to Covitz and Nadler. Besides, we generalize, extend, and develop some well-known results in the existing literature.

1. Introduction

Differential equations (DEs) with impulses are applied to simulate processes that experience rapid changes at discrete moments that is why the dynamics of impulsive DEs have drawn the interest of many academicians in recent decades (see [1–3]). Additionally, generally speaking, impulsive effects emerge as prevalent occurrences in the realm of natural phenomena, induced by sudden disturbances that transpire at precise instances, like in the case of different threshold-based biological models, bursting explosive biological medicine models, and the optimal control model in economics (for more details, see [4, 5]).

Real-world systems and natural events are virtually always affected by stochastic disruptions. Mathematical representations cannot disregard stochastic aspects due to a mix of uncertainty and complexity. To encompass a variety of abrupt incidents, incorporating impulsive occurrences, partial DEs driven by stochastic processes or equations with random impulses offer a natural and efficient approach. Many researchers have thought about the study of integro-DEs with random impulses, including those in [6–10]. Dassios explored the singular system within a category of DEs with multiple delays in his work documented in [11]. Keten et al., as referenced in [12], examined the conditions for the existence and singularity of solutions for nonlinear DEs incorporating the Caputo-Fabrizio operator in the Banach spaces, leveraging the exponential decay principle. Sakar, in [13], enhanced the homotopy analysis method by incorporating an optimally determined auxiliary parameter based on the residual error function, applied to solve neutral functional DEs with proportional delays. Shafqat et al. [14] delved into the investigation of both local and global existence as well as the singularity of mild solutions for the Navier-Stokes equations involving the time fractional differential operator. Recently, many new results related to random impulsive DEs have been obtained. For example, Yin et al. in [15] investigated existence and multiplicity of mild solutions for first-order Hamilton random impulsive DEs with the Dirichlet boundary conditions, employing the generalized saddle point theorem. Another notable contribution comes from Guo et al., as documented in [16], where they delved into the optimal control problem associated with random impulsive DEs, utilizing fundamental analytical techniques and theories related to stochastic processes. The scholarly discourse also extends to the study by Liu et al. in [17], which focuses on the existence of positive solutions within a class of impulsive fractional DEs involving ϕ-Hilfer fractional derivatives. Moreover, Li et al., featured in [18], undertook an investigation into the existence of solutions for the Sturm-Liouville DEs featuring random impulses and boundary value problems, employing Dhage's fixed point theorem as a pivotal tool in their analysis. The existence, singularity, and stability of the delay integro-DEs with random impulse were established by [6, 7].

The existence of semilinear functional differential inclusions accompanied by random impulses of a particular nature was examined by Vinodkumar in [19] of the type

$$\begin{cases} z'(\kappa) \in \mathscr{A}(\kappa)z(\kappa) + \mathscr{H}(\kappa, z_\kappa)\, \mathscr{T} \geq \kappa \geq \mu, \kappa \neq \zeta_\wp, \\ z(\zeta_\wp) = a_\wp\left(\mu_\wp\right) z\left(\zeta_\wp^-\right), \wp = 1, 2, \cdots, \\ z_{\kappa_0} = \upsilon. \end{cases} \tag{1}$$

The existence of integrodifferential inclusions with impulsive was examined by Kadam et al. in [9] of the type

$$\begin{cases} z'(\kappa) \in \mathscr{A}(\kappa)z(\kappa) + \mathscr{H}\left(\kappa, z(\kappa), \int_{\kappa_0}^{\kappa} g(\kappa, \nu)h(\nu, z(\kappa))d\kappa\right) \mathscr{T} \geq \kappa \geq 0, \\ z(\kappa) = g\left(z_{\kappa_1}, \cdots, z_{\kappa_i}\right) = \upsilon(\kappa), i = 1, 2, \cdots, \\ \Delta z(\mathscr{T}_1) = I_1 z(\mathscr{T}_1), I_1 = 1(1)m. \end{cases} \tag{2}$$

Motivated by the above-mentioned works and to our knowledge, no notable work yet that conducts that research has been published on the nonlinear random impulsive integrodifferential inclusions (NRIIDIns) with time-varying delays under sufficient conditions; here, we consider NRIIDIns with time-varying delays of the type

$$\begin{cases} z'(\kappa) \in \mathscr{A}(\kappa)z(\kappa) + \mathscr{H}\left(\kappa, z(\sigma_1(\kappa)), \cdots, z(\sigma_n(\kappa)), \int_{\kappa_0}^{\kappa} g(\kappa, \nu, z(\sigma_{n+1}(\nu)))d\nu\right), \kappa \geq \mu, \kappa \neq \zeta_\wp, \\ z(\zeta_\wp) = a_\wp\left(\mu_\wp\right) z\left(\zeta_\wp^-\right), \wp = 1, 2, \cdots, \\ z_{\kappa_0} = \upsilon, \end{cases} \tag{3}$$

where $\mathscr{A}$ is the infinitesimal generator of strongly continuous semigroup of linear operators $\{\mathfrak{U}(\kappa), \kappa \leq 0\}$ with $\mathfrak{D}(\mathscr{A}) \subset \mathbb{X}$, the function $\mathscr{H} : \mathbb{R}_\mu \times \mathscr{C} \longrightarrow \mathscr{P}(\mathbb{X}), g : \Delta \times \mathscr{C} \longrightarrow \mathscr{P}(\mathbb{X})$,$\mathscr{P}(\mathbb{X})$ is the family of all nonempty measurable subsets of $\mathbb{X}$, and $\sigma_i : [\kappa_0, \mathscr{T}] \longrightarrow [\kappa_0, \mathscr{T}], i = 1, 2, \cdots, n+1$ are continuous functions with $\sigma_i(\kappa) \leq \kappa, i = 1, 2, \cdots, n+1$. The set of piecewise continuous functions is $\mathscr{C} = \mathscr{C}([-\rho, 0], \mathbb{X})$ which maps $[-\rho, 0]$ into $\mathbb{X}$ with some given $\rho > 0$, $\zeta_0 = \kappa_0$, and $\zeta_\wp = \zeta_{\wp-1} + \tau_\wp$ for $\wp = 1, 2, \cdots$. Here, $\kappa_0 \in \mathbb{R}_\mu$ is an arbitrary real number. Obviously, $\kappa_0 = \zeta_0 < \zeta_1 < \zeta_2 < \cdots < \lim_{k \longrightarrow \infty} \zeta_\wp = \infty$, $a_\wp : \mathscr{D}_\wp \longrightarrow \mathbb{R}$ is a matrix-valued function for all $\wp = 1, 2, \cdots$, $z(\zeta_\wp^-) = \lim_{\kappa \uparrow \zeta_q} z(\kappa)$ according to their paths with the norm $\|z\|_\kappa = \sup_{\kappa-\rho \leq \nu \leq \kappa} |z(\nu)|$ for all κ fulfilling $\mu \leq \kappa \leq \mathscr{T}$, $\mu, \mathscr{T} \in \mathbb{R}^+$ are given numbers, $\|.\|$ is any given norm in $\mathbb{X}$, and φ is a function defined from $[-\rho, 0] \longrightarrow \mathbb{X}$; here, the set $\{(\kappa, \nu)\text{: } 0 \leq \nu \leq \kappa < \infty\}$ is denoted by Δ. Denote by $\{\mathbb{G}_\kappa, \kappa \geq 0\}$ the simple counting process generated by ζ_n, that is, $\{\mathbb{G}_\kappa \geq n\} = \{\zeta_n \leq \kappa\}$, and denote $\mathscr{H}_\kappa$ the σ-algebra generated by $\{\mathbb{G}_\kappa, \kappa \geq 0\}$. Then, $(\Omega, \mathscr{P}, \{\mathscr{H}_\kappa\})$ is a probability space.

The basic idea behind this work is to extend the work of Vinodkumar [19] and Kadam et al. [9]. At the same time, our approach is smooth and depends on the Martelli fixed point theorem, Pachpatte's inequality, and the fixed point theorem due to Covitz and Nadler. Moreover, we are achieving the same results with fewer hypotheses just by using Pachpatte's inequality to establish boundedness condition.

The structure of this article is as follows: Section 2 introduces fundamental concepts and preliminary information. In Section 3, we employ Martelli's fixed point theorem for condensing maps to explore the existence of NRIIDIns with time-varying delays, specifically focusing on the convex case of the multivalued function (MF). Our investigation delves into the problems of existence while incorporating the Wintner growth condition. In Section 4, we investigate the existence of RIFDIns in the non-convex case of the MF, employing the fixed point theorem attributed to Covitz and Nadler. Section 5 gives an illustrative example that effectively applies the insights garnered from Section 3. Finally, Section 6 provides the concluding remarks, along with acknowledgments of the study.

2. Preliminaries

Consider a real separable Hilbert space $\mathbb{X}$ and a nonempty set Ω. The random variable μ_q is defined from Ω to $\mathscr{D}_q \stackrel{\text{def}}{=} (0, d_q)$ for $q = 1, 2, \cdots$, where $0 < d_q < \infty$. Moreover, suppose that μ_q follow the Erlang distribution, where $q = 1, 2, \cdots$, and let μ_i and μ_j be independent of each other as $i \neq j$ for $i, j = 1, 2 \cdots$. For simplification, we denote $\mathbb{R}_\tau = [\mu, \infty)$. Let $£_p = £_p(\Omega, \mathscr{H}_\kappa, \mathbb{X})$ denote the Banach space of each $\mathscr{H}_\kappa$-measurable square integrable random variables in $\mathbb{X}$. Suppose that $\mathscr{T} > \kappa_0$ is any fixed time and G denotes the Banach space $G([\kappa_0 - \rho, \mathscr{T}], £_2)$, which is the family of all $\mathscr{H}_\kappa$-measurable, $\mathscr{C}$-valued random variables with the norm

$$\|\phi\|_G = \left(\sup_{\kappa_0 \le \kappa \le \mathcal{T}} \mathbb{E}\|\phi\|_\kappa^2\right)^{1/2}. \quad (4)$$

The family of all $\mathcal{H}_0$-measurable, G-valued random variable υ is denoted by $\pounds_p^0(\Omega, G)$.

We use the following notations: $\mathcal{P}_{cl}(\mathbb{X}) = \{\mathbb{W} \in \mathcal{P}(\mathbb{X}), \mathbb{W} \text{ closed}\}$, $\mathcal{P}_{bd}(\mathbb{X}) = \{\mathbb{W} \in \mathcal{P}(\mathbb{X}), \mathbb{W} \text{ bounded}\}$, $\mathcal{P}_{cv}(\mathbb{X}) = \{\mathbb{W} \in \mathcal{P}(\mathbb{X}), \mathbb{W} \text{ convex}\}$, and $\mathcal{P}_{cp}(\mathbb{X}) = \{\mathbb{W} \in \mathcal{P}(\mathbb{X}), \mathbb{W} \text{ compact}\}$. In a Hilbert space $\mathbb{X}$, a multivalued map $F : \mathbb{X} \longrightarrow \mathcal{P}(\mathbb{X})$ is a convex (closed) valued, if $F(z)$ is convex (closed) $\forall z \in \mathbb{X}$. F is bounded on bounded sets if $F(\mathbb{I}) = \cup_{z\in\mathbb{I}} F(z)$ is bounded in $\mathbb{X}$, $\forall \mathbb{I} \in \mathcal{P}_{bd}(\mathbb{X})$ that is

$$\sup_{x\in\mathbb{I}}\{\sup\{|w|: w \in F(z)\}\} < \infty. \quad (5)$$

F is called upper semicontinuous (u.s.c.) on $\mathbb{X}$, if $\forall z_0 \in \mathbb{X}$. The set $F(z_0)$ is nonempty, closed subset of $\mathbb{X}$, and if $\forall$ open set $\mathbb{I}$ of $\mathbb{X}$ containing $F(z_0)\exists$ an open neighbourhood $\mathbb{N}$ of z_0 s.t $F(\mathbb{M}) \subset \mathbb{I}$. F is called completely continuous if $F(\mathbb{I})$ is relatively compact, $\forall \mathbb{I} \in \mathcal{P}_{bd}(\mathbb{X})$.

Assuming the multivalued mapping denoted as F exhibits complete continuity and has a nonempty compact value, it can be deduced that F is u.s.c. $\Leftrightarrow F$ if its graph is closed. A mapping $F : \mathbb{X} \longrightarrow \mathbb{X}$ is classified as condensing, being u.s.c., when applied to any bounded subset $\mathbb{I} \subset \mathbb{X}$ s.t the measure of noncompactness, denoted by β, satisfying $\beta(F(\mathbb{I})) < \beta(\mathbb{I})$, where β is the Kuratowski measure of noncompactness as defined in [20].

Remark 1 (see [21]). An utterly continuous multivalued function stands as the simplest instance of a condensing map. A fixed point for a function F emerges when an element z exists within $\mathbb{X}$ such that z resides in the range of $F(\mathbb{X})$. The collection of fixed points for the multivalued operator F shall be referred to as the set fix F.

Define the function $\mathbb{T} : \mathcal{P}_{bd,cl}(\mathbb{X}) \times \mathcal{P}_{bd,cl}(\mathbb{X}) \longrightarrow \mathbb{R}^+$ defined by

$$\mathbb{T}(\mathbb{E}, \mathbb{G}) = \max\left\{\sup_{e\in\mathbb{E}} d(e, \mathbb{G}), \sup_{g\in\mathbb{G}} d(\mathbb{E}, g)\right\}, \quad (6)$$

where $d(\mathbb{E}, b) = \inf\{\|e - g\|^p, e \in \mathbb{E}\}$ and $d(e, \mathbb{G}) = \inf\{\|e - g\|^p, g \in \mathbb{G}\}$. The function $\mathbb{T}$ is called a Hausdorff metric on $\mathcal{P}_{bd,cl}(\mathbb{X})$.

The multivalued map $\mathbb{Z} : [\mu, \mathcal{T}] \longrightarrow \mathcal{P}_{bd,cl}(\mathbb{X})$ is called measurable if for all $z \in \mathbb{X}$ the function $\mathbb{W} : [\mu, \mathcal{T}] \longrightarrow \mathbb{R}^+$ defined by

$$\mathbb{W}(\kappa) = d(z, \mathbb{Z}(\kappa)) = \inf\left\{\|z - y\|^p, y \in \mathbb{Z}(\kappa)\right\} \text{ is measurable.} \quad (7)$$

Definition 2. A multivalued operator $\mathbb{Z} : [\mu, \mathcal{T}] \longrightarrow \mathcal{P}_{cl}(\mathbb{X})$ is said to be

(a) contraction $\Leftrightarrow \exists \eta > 0$ such that

$$\mathbb{H}(\mathbb{Z}(z), \mathbb{Z}(w)) < \eta\|z - w\|^p, \text{for each } z, w \in \mathbb{X} \text{ with } \eta < 1. \quad (8)$$

(b) $\mathbb{Z}$ has a fixed point if $\exists z \in \mathbb{X}$ s.t $z \in \mathbb{Z}(z)$

For additional information about multivalued maps, consult references [21–23]. Our results regarding existence are rooted in the fixed point theorems of Martelli [24] and the work of Covitz and Nadler [25].

Theorem 3 (see [24]). *Let $\mathbb{X}$ be a Hilbert space and $\mathbb{Z} : \mathbb{X} \longrightarrow \mathcal{P}_{bd,cl,cv}(\mathbb{X})$ a u.s.c. and condensing map. If the set*

$$\mathbb{S} = \{s \in \mathbb{X} | \lambda s \in \mathbb{Z}(\kappa), \textit{for some } \lambda > 1\} \textit{ is bounded}, \quad (9)$$

then $\mathbb{Z}$ has a fixed point.

Theorem 4 (see [25]). *Let $\mathbb{X}$ be a Banach space. If $\mathbb{Z} : \mathbb{X} \longrightarrow \mathcal{P}_{cl}(\mathbb{X})$ is a contraction, then fix $\mathbb{Z} \neq \phi$.*

Definition 5. A map $\mathcal{H} : \mathbb{R}_\tau \times \mathcal{C} \longrightarrow \mathcal{P}(\mathbb{X})$ and $g : \Delta \times \mathcal{C} \longrightarrow \mathcal{P}(\mathbb{X})$ are given functions. Moreover, $\sigma_i : [\kappa_0, \mathcal{T}] \longrightarrow [\kappa_0, \mathcal{T}], i = 1, 2, \cdots, n + 1$, are continuous functions with $\sigma_i(\kappa) \le \kappa, i = 1, 2, \cdots, n + 1$. We assume the following.

(1) $\kappa \longrightarrow \mathcal{H}(\kappa, z_1, \cdots, z_{n+1})$ is measurable for each $z_1, \cdots, z_n \in \mathcal{C}$

(2) Each $z_1, \cdots, z_{n+1} \longrightarrow \mathcal{H}(\kappa, z_1, \cdots, z_{n+1})$ is continuous for almost all $\kappa \in [\mu, \mathcal{T}]$

(3) $\forall m > 0$ (m is positive integer), $\exists \alpha_m \in \pounds_p([\mu, \mathcal{T}], \mathbb{R}^+)$ s.t

$$\|\mathcal{H}(t, z_1, \cdots, z_{n+1})\|^p \coloneqq \sup_{\|z\|^p \le m}\{\mathbb{E}\|h\|^p : h \in \mathcal{H}(\kappa, z_1, \cdots, z_{n+1})\} \le \alpha_m(\kappa), \quad (10)$$

for $\kappa \in [\mu, T]$. $\forall z \in \pounds_p(\mathbb{X})$define the set of selections of $\mathcal{H}$ by

$$S_{\mathcal{H},z} = \left\{h \in \pounds_p(\mathbb{X}): h(\kappa) \in \mathcal{H}\left(\kappa, z_{\kappa_1}, \cdots, z_{\kappa_{n+1}}\right) \text{for } \kappa \in [\mu, \mathcal{T}]\right\}. \quad (11)$$

Lemma 6 (see [26]). *Consider a compact interval denoted as J and let $\mathbb{X}$ represent a Hilbert space. Let $\mathcal{H}$ be an Lp-Carathéodory multivalued map with $\mathfrak{S}_{\mathcal{H},x} = \phi$ and β be a linear continuous mapping from $\beta \circ \mathfrak{S}_{\mathcal{H}} : C(J, \mathbb{X}) \longrightarrow \mathcal{P}_{bd,cl,co}(C(J, \mathbb{X})), x \mapsto \beta \circ \mathfrak{S}_{\mathcal{H}}(z) = \beta(\mathfrak{S}_{\mathcal{H},z})$, which is a closed graph operator in $C(J, \mathbb{X}) \times C(J, \mathbb{X})$.*

Definition 7 (see [8]). A semigroup $\{\mathfrak{U}(\kappa), \kappa \geq 0\}$ is said to be uniformly bounded if $\exists K \geq 1$ (K is constant) s.t

$$\|\mathfrak{U}(\kappa)\| \leq K, \text{for } \kappa > 0. \tag{12}$$

The following Pachpatte's inequalities play the crucial role in our analysis.

Lemma 8 (see [27], p. 33). *Let z, ϑ, and ψ be nonnegative continuous functions defined on $\mathbb{R}_+$, for which the inequality*

$$z(t) \leq z_0 + \int_0^t \vartheta(s)z(s)ds + \int_0^t \vartheta(s)\left(\int_0^s \psi(r)z(r)dr\right)ds, t \in \mathbb{R}_+, \tag{13}$$

holds, where z_0 is nonnegative constant. Then,

$$z(t) \leq z_0\left[1 + \int_0^t \vartheta(s) \exp\left(\int_0^s \vartheta(s) + \psi(r)dr\right)ds\right], t \in \mathbb{R}_+. \tag{14}$$

Definition 9. For a given $\mathcal{T} \in (\kappa_0, +\infty)$, a stochastic process $\{z(\kappa) \in G, \kappa_0 - \rho \leq \kappa \leq \mathcal{T}\}$ is said to be a solution to equation (3) in $(\Omega, \mathcal{P}, \{\mathcal{H}_\kappa\})$, if

(1) $z(\kappa) \in \mathbb{R}^n$ is $\mathcal{H}_\kappa$-adapted for $\kappa \geq \kappa_0$

(2) $z(\kappa_0 + \nu) = \upsilon(\nu) \in \pounds_2^0(\Omega, \mathcal{H})$ when $\nu \in [-\rho, 0]$

$$\begin{aligned} z(\kappa) = \sum_{\wp=0}^{\infty} \Bigg[& \prod_{i=1}^{\wp} a_i(\mu_i)\mathfrak{U}(\kappa - \kappa_0)\upsilon(0) + \sum_{i=1}^{\wp}\prod_{j=i}^{\wp} a_j\left(\mu_j\right) \\ & \cdot \int_{\zeta_{i-1}}^{\zeta_i} \mathfrak{U}(\kappa - \nu)h(\nu)d\nu + \int_{\zeta_\wp}^{\kappa} \mathfrak{U}(\kappa - \nu)h(\nu)d\nu \Bigg] \\ & \cdot I_{[\zeta_\wp, \zeta_{\wp+1})}(\kappa), h \in \mathfrak{U}_{\mathcal{H},z}, \kappa \in [\kappa_0, \mathcal{T}], \end{aligned} \tag{15}$$

where $\prod_{j=i}^{\wp} a_j(\mu_j) = a_\wp(\mu_\wp)a_{\wp-1}(\mu_{\wp-1}), \cdots a_i(\mu_i)$ and $I_{\mathcal{A}}(.)$ is the index function

3. Existence Result: Convex Case

Here, we list the following assumptions for our convenience.

A1. There exists a continuous function $\rho_1 : [\kappa_0, \mathcal{T}] \longrightarrow (0,+\infty)$ such that

$$\begin{aligned} \|\mathcal{H}(\kappa, z_1, \cdots, z_{n+1})\|^p &= \sup_{\|z\|^p \leq m} \left\{\mathbb{E}\|h\|^p : h \in \mathcal{H}(\kappa, z_1, \cdots, z_{n+1})\right\} \\ &\leq \rho_1(\kappa) W\left(\|z_1\|_\nu^p + \cdots + \|z_{n+1}\|_\nu^p\right), \end{aligned} \tag{16}$$

$z_i \in \mathcal{C}, i = 1, 2, \cdots, n+1$, where $W : \mathbb{R}^+ \longrightarrow (0,+\infty)$ is a continuous increasing function satisfying $W(\eta(\kappa)z) \leq \eta(\kappa) W(z)$.

A2. There exists a continuous function $\rho_2 : [\kappa_0, \mathcal{T}] \longrightarrow (0,+\infty)$ such that

$$\mathbb{E}\left\|\int_{\kappa_0}^t g(\kappa, \nu, z)ds\right\|^p \leq \rho_2(\kappa)\mathbb{E}\|z\|_\sigma^p, \kappa, \sigma \geq 0, z \in \mathbb{X}. \tag{17}$$

A3. $\nu_i : [\kappa_0, \mathcal{T}] \longrightarrow [\kappa_0, \mathcal{T}], i = 1, 2, \cdots, n+1$, are continuous functions with $\nu_i(\kappa) \leq \kappa, i = 1, 2, \cdots, n+1$.

A4. $\mathbb{E}\{\max_{i,q}\prod_{j=i}^q\|a_\iota(\tau_j)\|\}$ is uniformly bounded if there is a constant $C > 0$ such that

$$\mathbb{E}\left\{\max_{i,q}\prod_{j=i}^{k}\left\|a_j\left(\mu_j\right)\right\|\right\} \leq C \text{ for each } \mu_j \in \mathcal{D}_j, j = 1, 2, \cdots \tag{18}$$

A5. $\mathcal{A} : D(\mathcal{A}) \subset \mathbb{X} \longrightarrow \mathbb{X}$ is the infinitesimal generator of strongly continuous $\mathfrak{U}(\kappa)$ in $\mathbb{X}$.

A6. $\mathcal{H} : \mathbb{R}_\mu \times \mathcal{C} \longrightarrow \mathcal{P}(\mathbb{X})$ is a compact convex Lp-Carathéodory multivalued function.

Theorem 10. *Under the assumptions A1-A6, problem (3) has at least one mild solution defined on $[\kappa_0, \mathcal{T}]$ provided that the following estimate is fulfilled:*

$$\int_{\kappa_0}^{\mathcal{T}} M_*(\nu)d\nu < \int_{c_0}^{\infty} \frac{d\nu}{W(\nu)}, \tag{19}$$

where $M_*(\nu) = 2^{p-1}K^p \max\{1, C\}(\mathcal{T} - \kappa_0)\varrho_1(\kappa)(n + \varrho_2(\kappa))$, $c_0 = 2^{p-1}K^pC^p\mathbb{E}\|\upsilon\|^p$, and $K^pC^p \geq 1/2^{p-1}$.

Proof. Let $\mathcal{T}$ be an arbitrary number $\kappa_0 < \mathcal{T} < \infty$ fulfilling (19). We transform the system (3) into a fixed point problem. We consider the operator $\mathbb{Z} : G \longrightarrow G$ defined as

$$\mathbb{Z}(z) = f \in G : f(\kappa) = \begin{cases} \upsilon(\kappa - \kappa_0), \kappa \in [\kappa_0 - \rho, \kappa_0], \\ \displaystyle\sum_{\wp=0}^{\infty}\left[\prod_{i=1}^{\wp} a_i(\mu_i)\mathfrak{U}(\kappa - \kappa_0)\upsilon(0) + \sum_{i=1}^{\wp}\prod_{j=i}^{\wp} a_j\left(\mu_j\right)\int_{\zeta_{i-1}}^{\zeta_i}\mathfrak{U}(\kappa - \nu)h(\nu)d\nu,\right. \\ \displaystyle\left. + \int_{\zeta_\wp}^{\kappa}\mathfrak{U}(\kappa - \nu)h(\nu)d\nu\right] I_{[\zeta_\wp, \zeta_{\wp+1})}(\kappa), h \in \mathfrak{U}_{\mathcal{H},z}, \kappa \in [\kappa_0, \mathcal{T}]. \end{cases} \tag{20}$$

We will demonstrate that the operator $\mathbb{Z}$ satisfies all the conditions outlined in Theorem 3. The proof is presented through the following sequence of steps.

Step 1. To prove $\mathbb{Z}$ is convex, $\forall y \in G$. Since $\mathcal{H}$ has convex values, it follows that $\mathfrak{U}_{\mathcal{H},z}$ is convex, so that $h_1, h_2 \in S_{\mathcal{H},z}$; then, $\mu h_1 + (1-\mu)h_2 \in S_{\mathcal{H},z}, 0 \leq \mu \leq 1$, which implies clearly that ϕ is convex.

Step 2. To prove $\mathbb{Z}$ is bounded on G.

Let $\mathbb{G}_m = \{z \in G | \|z\|_G^p \le m\}$, $m \ge 0$, be a bounded subset of G. We prove that $\mathbb{Z}(\mathbb{G}_m)$ is a bounded subset of G. $\forall z \in \mathbb{G}_m$, let $f \in \mathbb{Z}(z)$. Then, $\exists h \in \mathfrak{S}_{\mathscr{H},z}$ s.t $\forall \kappa \in [\kappa_0, \mathscr{T}]$, we get

$$\begin{aligned}
\|f(\kappa)\|^p \le & \left[\sum_{\wp=0}^{\infty}\left[\left\|\prod_{i=1}^{\wp} a_i(\mu_i)\right\| \|\mathfrak{U}(\kappa-\kappa_0)\| \|v(0)\|\right.\right. \\
& + \sum_{i=1}^{\wp}\left\|\prod_{j=i}^{\wp} a_j\left(\mu_j\right)\right\|\left\{\int_{\zeta_{i-1}}^{\zeta_i} \|\mathfrak{U}(\kappa-\nu)h(\nu)\| d\nu\right\} \\
& \left.\left. + \int_{\zeta_\wp}^{\kappa} \|\mathfrak{U}(\kappa-\nu)h(\nu)\| d\nu\right] I_{[\zeta_\wp,\zeta_{\wp+1})}(\kappa)\right]^p \\
\le & 2^{p-1} \max_{\wp}\left\{\prod_{i=1}^{\wp} \|a_i(\mu_i)\|^p\right\} \|\mathfrak{U}(\kappa-\kappa_0)\|^p \|v(0)\|^p \\
& + 2^{p-1}\left[\max_{i,\wp}\left\{1, \prod_{j=i}^{q}\left\|a_j\left(\mu_j\right)\right\|\right\}\right]^p \\
& \cdot \|\mathfrak{U}(\kappa-\nu)\|^p\left(\int_{\kappa_0}^{\kappa} \|h(\nu)\| d\nu\right)^p, \\
\|f\|_\kappa^p \le & 2^{p-1} \max_{\wp}\left\{\prod_{i=1}^{\wp} \|a_i(\mu_i)\|^p\right\} \|\mathfrak{U}(\kappa-\kappa_0)\|^p \|v(0)\|^p \\
& + 2^{p-1}\left[\max_{i,\wp}\left\{1, \prod_{j=i}^{\wp}\|a_j(\tau_j)\|\right\}\right]^p \\
& \cdot \|\mathfrak{U}(\kappa-\nu)\|^p (\mathscr{T}-\kappa_0)^{p-1}\left(\int_{\kappa_0}^{\kappa} \|h(\nu)\|^p d\nu\right).
\end{aligned} \tag{21}$$

Then,

$$\begin{aligned}
\mathbb{E}\|f\|_\kappa^p \le & 2^{p-1} C^p K^p \mathbb{E}\left[\|v\|^p\right] + 2^{p-1} \max\{1, C^p\}(\mathscr{T}-\kappa_0)^{p-1} \\
& \cdot \int_{\kappa_0}^{\kappa} \mathbb{E}\left[\|h(\nu)\|^p\right] d\nu \le 2^{p-1} C^p K^p \mathbb{E}\left[\|v\|^p\right] \\
& + 2^{p-1} K^p e^{p\nu(\mathscr{T}-\kappa_0)} \max\{1, C^p\}(\mathscr{T}-\kappa_0)^{p-1} \\
& \times \int_{\kappa_0}^{\kappa} \varrho_1(\nu) W\left[\mathbb{E}\|z(\sigma_1(\nu))\|^p + \cdots + \mathbb{E}\|z(\sigma_n(\nu))\|^p\right. \\
& \left. + \varrho_2(\nu)\mathbb{E}\|z(\sigma_{n+1}(r))\|^p dr\right] d\nu \le 2^{p-1} K^p C^p \mathbb{E}\left[\|v\|^p\right] \\
& + 2^{p-1} K^p \max\{1, C^p\}(\mathscr{T}-\kappa_0)^{p-1} \\
& \cdot \int_{\kappa_0}^{\kappa} \varrho_1(\nu)(n+\varrho_2(\nu)) \sup_{\kappa_0 \le \nu \le \kappa} W\left[\mathbb{E}\left(\|z\|_\nu^p\right)\right] d\nu.
\end{aligned} \tag{22}$$

Hence, $\forall f \in \phi(\mathbb{G}_m)$, we get

$$\begin{aligned}
\|f\|_G^p \le & 2^{p-1} K^p C^p \mathbb{E}\left[\|v\|^p\right] + 2^{p-1} K^p \max\{1, C^p\}(\mathscr{T}-\kappa_0)^{p-1} \\
& \cdot \int_{\kappa_0}^{\kappa} \varrho_1(\nu)(n+\varrho_2(\nu)) \sup_{\kappa_0 \le \nu \le \kappa} W\left[\mathbb{E}\left(\|z\|_\nu^p\right)\right] d\nu = \bar{\mathcal{E}}.
\end{aligned} \tag{23}$$

Then, $\forall f \in \mathbb{Z}(z)$, we get $\|f\|_G^p \le \bar{\mathcal{E}}$.

Step 3. $\mathbb{Z}$ sends bounded sets into equicontinuous sets of G.

Let $z \in \mathbb{G}_m$ be a bounded set in G and $\kappa_1, \kappa_2 \in [\kappa_0, \mathscr{T}]$. If $\kappa_0 < \kappa_1 < \kappa_2 < \mathscr{T}$, $f \in \mathbb{Z}(z)$,$\exists$ a function $h \in \mathfrak{S}_{\mathscr{H},z}$ s.t $\forall \kappa \in [\kappa_0, \mathscr{T}]$, we get

$$\begin{aligned}
f(\kappa_1) - f(\kappa_2) = & \sum_{\wp=0}^{\infty}\left[\prod_{i=1}^{\wp} a_i(\mu_i)\mathfrak{U}(\kappa_1-\kappa_0)v(0)\right. \\
& + \sum_{i=1}^{\wp}\prod_{j=i}^{\wp} a_j\left(\mu_j\right)\int_{\zeta_{i-1}}^{\zeta_i} \mathfrak{U}(\kappa_1-\nu)h(\nu)d\nu \\
& \left. + \int_{\zeta_\wp}^{\kappa_1} \mathfrak{U}(\kappa_1-\nu)h(\nu)d\nu\right] I_{[\zeta_\wp,\zeta_{\wp+1})}(\kappa_1) \\
& - \sum_{\wp=0}^{\infty}\left[\prod_{i=1}^{\wp} a_i(\mu_i)\mathfrak{U}(\kappa_2-t_0)v(0)\right. \\
& + \sum_{j=1}^{\wp}\prod_{j=i}^{\wp} a_j\left(\mu_j\right)\int_{\zeta_{i-1}}^{\zeta_i} \mathfrak{U}(\kappa_2-\nu)h(\nu)d\nu \\
& \left. + \int_{\zeta_\wp}^{\kappa_2} \mathfrak{U}(\kappa_2-\nu)h(\nu)d\nu\right] I_{[\zeta_\wp,\zeta_{\wp+1})}(t_2).
\end{aligned} \tag{24}$$

Thus,

$$\begin{aligned}
f(\kappa_1) - f(\kappa_2) = & \sum_{\wp=0}^{\infty}\left[\prod_{i=1}^{\wp} a_i(\mu_i)\mathfrak{U}(\kappa_1-\kappa_0)v(0)\right. \\
& + \sum_{i=1}^{\wp}\prod_{j=i}^{\wp} a_j\left(\mu_j\right)\int_{\zeta_{i-1}}^{\zeta_i} \mathfrak{U}(\kappa_1-\nu)h(\nu)d\nu \\
& \left. + \int_{\zeta_\wp}^{\kappa_1} \mathfrak{U}(\kappa_1-\nu)h(\nu)d\nu\right] \\
& \cdot \left(I_{[\zeta_\wp,\zeta_{\wp+1})}(\kappa_1) - I_{[\zeta_\wp,\zeta_{\wp+1})}(t_2)\right) \\
& + \sum_{\wp=0}^{\infty}\left[\prod_{i=1}^{\wp} a_i(\mu_i)\mathfrak{U}(\kappa-\kappa_0) - \mathfrak{U}(\kappa_2-\kappa_0)v(0)\right. \\
& + \sum_{i=1}^{\wp}\prod_{j=i}^{\wp} a_j\left(\mu_j\right)\int_{\zeta_{i-1}}^{\zeta_i} (\mathfrak{U}(\kappa-\kappa_0) \\
& - \mathfrak{U}(\kappa_2-\kappa_0))h(\nu)d\nu + \int_{\zeta_i}^{\kappa_1} (\mathfrak{U}(\kappa-\kappa_0) \\
& - \mathfrak{U}(\kappa_2-\kappa_0))h(\nu)d\nu \\
& \left. + \int_{\kappa_1}^{\kappa_2} \mathfrak{U}(\kappa_2-\kappa_0)h(\nu)d\nu\right] I_{[\zeta_\wp,\zeta_{\wp+1})}(\kappa_2).
\end{aligned} \tag{25}$$

Then,

$$\mathbb{E}\|\phi z(\kappa_1) - \phi z(\kappa_2)\|^p \le 2^{p-1}\mathbb{E}\|I_1\|^p + 2^{p-1}\mathbb{E}\|I_2\|^p, \tag{26}$$

where

$$I_1 = \sum_{\wp=0}^{\infty}\left[\prod_{i=1}^{\wp} a_i(\mu_i)\mathfrak{U}(\kappa-\kappa_0)v(0) + \sum_{i=1}^{\wp}\prod_{j=i}^{\wp} a_j\left(\mu_j\right)\int_{\zeta_{i-1}}^{\zeta_i}\mathfrak{U}(\kappa-\nu)h(\nu)d\nu + \int_{\zeta_\wp}^{\kappa_1}\mathfrak{U}(\kappa-\nu)h(\nu)d\nu\right]\left(I_{[\zeta_\wp,\zeta_{\wp+1})}(\kappa_1) - I_{[\zeta_\wp,\zeta_{\wp+1})}(\kappa_2)\right), \tag{27}$$

$$I_2 = \sum_{\wp=0}^{\infty}\left[\prod_{i=1}^{\wp} a_i(\mu_i)\mathfrak{U}(\kappa-\kappa_0) - \mathfrak{U}(\kappa_2-\kappa_0)v(0) + \sum_{i=1}^{\wp}\prod_{j=i}^{\wp} a_j\left(\mu_j\right)\int_{\zeta_{i-1}}^{\zeta_i}(\mathfrak{U}(\kappa-\kappa_0) - \mathfrak{U}(\kappa_2-\kappa_0))h(\sigma)d\sigma + \int_{\zeta_i}^{\kappa_1}(\mathfrak{U}(\kappa-\kappa_0) - \mathfrak{U}(\kappa_2-\kappa_0))h(\nu)d\nu + \int_{\kappa_1}^{\kappa_2}\mathfrak{U}(\kappa_2-\kappa_0)h(\nu)d\nu\right]I_{[\zeta_q,\zeta_{\wp+1})}(\kappa_2), \tag{28}$$

$$\begin{aligned}
\mathbb{E}\|I_1\|^p &\le 2^{p-1}C^pK^p\mathbb{E}\|v(0)\|^p\mathbb{E}\left(I_{[\zeta_\wp,\zeta_{\wp+1})}(\kappa_1) - I_{[\zeta_\wp,\zeta_{\wp+1})}(\kappa_2)\right) \\
&\quad + 2^{p-1}K^p \max\{1, C^p\}(\kappa_1-\kappa_0)\mathbb{E}\int_{\kappa_0}^{\kappa_1} \\
&\quad \cdot \|h(\nu)\|^p d\nu\mathbb{E}\left(I_{[\zeta_\wp,\zeta_{q+1})}(\kappa_1) - I_{[\zeta_\wp,\zeta_{\wp+1})}(\kappa_2)\right) \\
&\le 2^{p-1}K^pC^p\mathbb{E}\|v(0)\|^p\mathbb{E}\left(I_{[\zeta_q,\zeta_{\wp+1})}(\kappa_1) - I_{[\zeta_\wp,\zeta_{q+1})}(\kappa_2)\right) \\
&\quad + 2^{p-1}K^p \max\{1, C^p\}(\kappa_1-\kappa_0) \\
&\quad \times \int_{\kappa_0}^{\kappa_1}\varrho_1(\sigma)W\left(n\mathbb{E}\|z\|_\nu^p + \int_{\kappa_0}^{\nu}\varrho_2(\nu)\mathbb{E}\|z\|_\nu^p\right) \\
&\quad \cdot d\nu\mathbb{E}\left(I_{[\zeta_q,\zeta_{q+1})}(\kappa_1) - I_{[\zeta_\wp,\zeta_{q+1})}(\kappa_2)\right) \\
&\le 2^{p-1}K^pC^p\mathbb{E}\|v(0)\|^p\mathbb{E}\left(I_{[\zeta_q,\zeta_{q+1})}(\kappa_1) - I_{[\zeta_\wp,\zeta_{q+1})}(\kappa_2)\right) \\
&\quad + 2^{p-1}K^p \max\{1, C^p\}(\kappa_1-\kappa_0) \\
&\quad \times \int_{\kappa_0}^{\kappa_1} M_1W\left(\mathbb{E}\|z\|_\nu^p\right)d\nu\mathbb{E}\left(I_{[\zeta_\wp,\zeta_{\wp+1})}(\kappa_1) - I_{[\zeta_\wp,\zeta_{\wp+1})}(\kappa_2)\right) \\
&\longrightarrow 0 \, as \, \kappa_2 \longrightarrow \kappa_1,
\end{aligned} \tag{29}$$

where $M_1 = \sup\{\rho_1(\kappa)(n+\rho_2(\kappa))\}$,

$$\begin{aligned}
\mathbb{E}\|I_2\|^p &\le K^pC^p(\kappa_2-\kappa_1)\mathbb{E}\int_{\kappa_1}^{\kappa_2}\|h(\nu)\|^p d\nu \\
&\le K^pC^p(\kappa_2-\kappa_1)\int_{\kappa_1}^{\kappa_2} M_1\rho_1(\nu)W\left(n\mathbb{E}\|z\|_\nu^p + \int_{\kappa_0}^{\nu}\rho_2(\nu)\mathbb{E}\|z\|_\nu^p big\right)d\nu \\
&\longrightarrow 0 \text{ as } \kappa_2 \longrightarrow \kappa_1.
\end{aligned} \tag{30}$$

Equations (29) and (30) are independent of $z \in \mathbb{G}_m$. This demonstrates that as κ_2 approaches κ_1, the right-hand side of equation (26) approaches zero. The compact nature of $\mathfrak{U}(\kappa-\kappa_0)$ for $\kappa_0 < \kappa$ implies continuity within the uniform operator topology.

Step 4. $\mathbb{Z}$ maps bounded sets into relatively compact sets in G.

Let $\kappa_0 < \kappa \le \mathcal{T}$ be fixed and ε a real number fulfilling $\varepsilon \in (0, \kappa-\kappa_0)$. For $z \in \mathbb{G}_m$. We define a function f_ε by

$$f_\varepsilon(\kappa) = \sum_{\wp=0}^{\infty}\left[\prod_{i=1}^{\wp} a_i(\mu_i)\mathfrak{U}(\kappa-\kappa_0)v(0) + \sum_{i=1}^{\wp}\prod_{j=i}^{\wp} a_j(\tau_j)\int_{\zeta_{i-1}}^{\zeta_i}\mathfrak{U}(\kappa-\nu)h(\nu)d\nu + \int_{\zeta_\wp}^{\kappa-\varepsilon}\mathfrak{U}(\kappa-\nu)h(\nu)d\nu\right]I_{[\zeta_\wp,\zeta_{\wp+1})}(\kappa),$$
$$\kappa \in (\kappa_0, \kappa-\varepsilon), h \in \mathfrak{U}_{\mathcal{H},z}, h \in \mathfrak{S}_{\mathcal{H},z}. \tag{31}$$

Since $\mathfrak{U}(\kappa-\kappa_0)$ is a compact operator, the set $W_\varepsilon(\kappa) = \{f_\varepsilon(\kappa): f_\varepsilon \in \phi(z)\}$ is relatively compact in G, $\forall \varepsilon \in (0, \kappa_0-\kappa)$. Additionally, $\forall f \in \mathbb{Z}(z)$, we get

$$\begin{aligned}
f(\kappa) - f(\kappa_\varepsilon) &= \sum_{\wp=0}^{\infty}\left[\prod_{i=1}^{\wp} a_i(\mu_i)\mathfrak{U}(\kappa-\kappa_0)v(0) + \sum_{i=1}^{\wp}\prod_{j=i}^{\wp} a_j\left(\mu_j\right)\int_{\zeta_{i-1}}^{\zeta_i}\mathfrak{U}(\kappa-\nu)h(\nu)d\nu + \int_{\zeta_\wp}^{\kappa}\mathfrak{U}(\kappa-\nu)h(\nu)d\nu\right]I_{[\zeta_\wp,\zeta_{q+1})}(\kappa) \\
&\quad - \sum_{\wp=0}^{\infty}\left[\prod_{i=1}^{\wp} a_i(\mu_i)\mathfrak{U}(\kappa-\kappa_0)v(0) + \sum_{i=1}^{\wp}\prod_{j=i}^{\wp} a_j\left(\mu_j\right)\int_{\zeta_{i-1}}^{\zeta_i}\mathfrak{U}(\kappa-\nu)h(\nu)d\nu + \int_{\zeta_\wp}^{\kappa-\varepsilon}\mathfrak{U}(\kappa_1-\nu)h(\nu)d\nu\right]I_{[\zeta_\wp,\zeta_{\wp+1})}(\kappa).
\end{aligned} \tag{32}$$

Using A1–A6, we get

$$\mathbb{E}\|f - f_\varepsilon z\|_\kappa^p \le K^p \max\{1, C^p\}(\mathcal{T}-\kappa_0)^{p-1}\int_{\kappa-\varepsilon}^{t} M_1W(m)ds. \tag{33}$$

Hence, there exist precompact sets that can be brought arbitrarily close to the collection $\{\mathbb{Z}z(\kappa): z \in \mathbb{G}_m\}$. The collection $\{\mathbb{Z}(z): z \in \mathbb{G}_m\}$ is itself precompact within the space G. By combining the outcomes of steps 1 to 4 and invoking the Ascoli-Arzela theorem, it follows that $\mathbb{Z}$ qualifies as a compact multivalued map, consequently establishing its character as a condensing map.

Step 5. $\mathbb{Z}$ has a closed graph.

Let $z^{(n)} \longrightarrow z^*$ and $f^n \in \mathbb{Z}(z^{(n)})$ with $f^{(n)} \longrightarrow f^*$. We shall show that $f^* \in \mathbb{Z}(z^*)$. There exists $h^{(n)} \in \mathfrak{S}_{\mathscr{H},z^{(n)}}$, such that

$$\begin{aligned} f^{(n)}(\kappa) = \sum_{\wp=0}^{\infty}\Bigg[\prod_{i=1}^{\wp} a_i(\mu_i)\mathfrak{U}(\kappa-\kappa_0) \\ + \sum_{i=1}^{\wp}\prod_{j=i}^{\wp} a_j\left(\mu_j\right)\int_{\zeta_{i-1}}^{\zeta_i}\mathfrak{U}(\kappa-\nu)h^{(n)}(\nu)d\nu \\ + \int_{\zeta_\wp}^{\kappa}\mathfrak{U}(\kappa-\nu)h^{(n)}(\nu)d\nu\Bigg]I_{[\zeta_\wp,\zeta_{\wp+1})}(\kappa). \end{aligned} \tag{34}$$

Now, we prove that $\exists h^* \in \mathfrak{S}_{\mathscr{H},z^*}$, s.t

$$\begin{aligned} f^*(\kappa) = \sum_{q=0}^{\infty}\Bigg[\prod_{i=1}^{\wp} a_i(\tau_i)\mathfrak{U}(\kappa-\kappa_0)\upsilon(0) \\ + \sum_{i=1}^{\wp}\prod_{j=i}^{\wp} a_j(\tau_j)\int_{\zeta_{i-1}}^{\zeta_i}\mathfrak{U}(\kappa-\nu)h^*(\sigma)d\sigma \\ + \int_{\zeta_\wp}^{\kappa}\mathfrak{U}(\kappa-\nu)h^*(\nu)d\sigma\Bigg]I_{[\zeta_\wp,\zeta_{\wp+1})}(\kappa), \end{aligned} \tag{35}$$

where $h^*(\sigma) = \mathscr{H}(\nu, z^*(\sigma_1(\nu), \cdots, z^*(\sigma_n(\nu)), \int_{\kappa_0}^{\nu} g(\nu, r, z^*(\sigma_{n+1}(r)))dr)$.

Take into account the linear continuous operator $\Gamma : \pounds_p(\mathbb{X}) \longrightarrow G$ defined by

$$\begin{aligned} \Gamma z(\kappa) = \sum_{\wp=0}^{\infty}\Bigg[\prod_{i=1}^{\wp} a_i(\mu_i)\mathfrak{U}(\kappa-\kappa_0)\upsilon(0) \\ + \sum_{i=1}^{\wp}\prod_{j=i}^{\wp} a_j\left(\mu_j\right)\int_{\zeta_{i-1}}^{\zeta_i}\mathfrak{U}(\kappa-\nu)h(\nu)d\nu \\ + \int_{\zeta_\wp}^{\kappa}\mathfrak{U}(\kappa-\nu)h(\nu)d\nu\Bigg]I_{[\zeta_\wp,\zeta_{\wp+1})}(\kappa). \end{aligned} \tag{36}$$

Then, we get

$$\begin{aligned} \Bigg\|\left(f^{(n)}(\kappa) - \sum_{\wp=0}^{\infty}\left[\prod_{i=1}^{\wp} a_i(\tau_i)\mathfrak{U}(\kappa-\kappa_0)\upsilon(0)\right]I_{[\zeta_\wp,\zeta_{\wp+1})}(\kappa)\right) \\ - \left(f^*(\kappa) - \sum_{\wp=0}^{\infty}\left[\prod_{i=1}^{\wp} a_i(\mu_i)\mathfrak{U}(\kappa-\kappa_0)\upsilon(0)\right]I_{[\zeta_\wp,\zeta_{\wp+1})}(\kappa)\right)\Bigg\|^p \\ \longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned} \tag{37}$$

From Lemma 6, it becomes evident that the composite operator $\Gamma \circ \mathfrak{S}_{\mathscr{H}}$ possesses a closed graph. This observation aligns with the definition of Γ, which yields the following relationship:

$$f^{(n)}(\kappa) - \sum_{\wp=0}^{\infty}\left[\prod_{i=1}^{\wp} a_i(\mu_i)\mathfrak{U}(\kappa-\kappa_0)\upsilon(0)\right]I_{[\zeta_\wp,\zeta_{\wp+1})}(\kappa) \in \Gamma \circ \mathfrak{U}_{\mathscr{H},z^{(n)}}. \tag{38}$$

As $z^{(n)} \longrightarrow z^*$ and $f^{(n)} \longrightarrow f^*$, $\exists h^* \in \mathfrak{U}_{\mathscr{H},z^*}$, s.t

$$\begin{aligned} f^*(\kappa) - \sum_{\wp=0}^{\infty}\left[\prod_{i=1}^{\wp} a_i(\tau_i)\mathfrak{U}(\kappa-\kappa_0)\upsilon(0)\right]I_{[\zeta_\wp,\zeta_{\wp+1})}(\kappa) \\ = \sum_{i=1}^{\wp}\prod_{j=i}^{\wp} a_j\left(\mu_j\right)\int_{\zeta_{i-1}}^{\zeta_i}\mathfrak{U}(t-\sigma)h^*(\nu)d\nu \\ + \int_{\zeta_\wp}^{\kappa}\mathfrak{U}(\kappa-\nu)h^*(\nu)d\nu\Bigg]I_{[\zeta_\wp,\zeta_{\wp+1})}(\kappa). \end{aligned} \tag{39}$$

$\forall \kappa \in [\kappa_0, \mathscr{T}]$. Hence, $f^* \in \mathbb{Z}(z^*)$, which follows that the graph $\mathbb{Z}$ is closed.

Step 6. A priori bounds.

Here, it remains to show that the set

$$U(\mathbb{Z}) = \{z \in G : \eta z \in \mathbb{Z}(z) \text{ for some } 0 < \eta < 1\} \text{ is bounded.} \tag{40}$$

Let $z \in U$; then for $\eta > 1$, $z \in \mathbb{Z}(z)$ and there exists $h \in \mathfrak{S}_{\mathscr{H},z}$ s.t

$$\begin{aligned} z(\kappa) = \eta\sum_{\wp=0}^{\infty}\Bigg[\prod_{i=1}^{\wp} a_i(\mu_i)\mathfrak{U}(\kappa-\kappa_0)\upsilon(0) \\ + \sum_{i=1}^{\wp}\prod_{j=i}^{\wp} a_j\left(\mu_j\right)\int_{\zeta_{i-1}}^{\zeta_i}\mathfrak{U}(\kappa-\nu)h(\nu)d\nu \\ + \int_{\zeta_\wp}^{\kappa_1}\mathfrak{U}(\kappa-\nu)h(\nu)d\nu\Bigg]I_{[\zeta_\wp,\zeta_{\wp+1})}(\kappa_1), \kappa \in [\kappa_0, \mathscr{T}], \end{aligned} \tag{41}$$

where

$$h(\nu) = \mathscr{H}\left(\nu, z\left(\sigma_1(\nu), \cdots, z(\sigma_n(\nu)), \int_{\kappa_0}^{\nu} g(\nu, r, z(\sigma_{n+1}(r)))dr\right)\right). \tag{42}$$

Then, by A1-A6, we have

$$\|z(\kappa)\|^p \le \eta^p \left[\sum_{\wp=0}^{\infty}\left[\left\|\prod_{i=1}^{\wp} a_i(\mu_i)\right\| \|\mathfrak{U}(\kappa-\kappa_0)\|\|v(0)\| + \sum_{i=1}^{\wp}\left\|\prod_{j=i}^{\wp} a_j\left(\mu_j\right)\right\|\left\{\int_{\zeta_{i-1}}^{\zeta_i}\|\mathfrak{U}(\kappa,\nu)h(\nu)\|d\nu\right\} + \int_{\zeta_q}^{\kappa}\|\mathfrak{U}(\kappa,\nu)h(\nu)\|d\nu\right] I_{[\zeta_\wp,\zeta_{\wp+1})}(\kappa)\right]^p. \tag{43}$$

Observing that the right-hand side's final term in the inequality above increases in κ and selecting $K^p C^p \ge 1/2^{p-1}$, we arrive at the conclusion that

$$\begin{aligned}\|z\|_t^p \le{}& 2^{p-1}\max_{\wp}\left\{\prod_{i=1}^{\wp}\|a_i(\mu_i)\|^p\right\}\|\mathfrak{U}(\kappa-\kappa_0)\|^p\|v(0)\|^p \\ &+ 2^{p-1}\left[\max_{i,\wp}\left\{1, \prod_{j=i}^{\wp}\left\|a_j\left(\mu_j\right)\right\|\right\}\right]^p \\ &\times \|\mathfrak{U}(\kappa,\nu)\|^p(\mathcal{T}-\kappa_0) \\ &\cdot \left(\int_{\kappa_0}^{\kappa}\left\|\mathscr{H}(\nu, z(\sigma_1(\nu)), \cdots, z(\sigma_n(\nu)),\right.\right. \\ &\left.\left.\cdot\int_{\kappa_0}^{\nu} g(\nu, r, z(\sigma_{n+1}(r)))dr\right\|^p d\nu\right).\end{aligned} \tag{44}$$

Then,

$$\begin{aligned}\mathbb{E}\|z\|_\kappa^p \le{}& 2^{p-1}C^pK^p\mathbb{E}\left[\|v\|^p\right] + 2^{p-1}\max\{1, C^p\} \\ &\times(\mathcal{T}-\kappa_0)\int_{\kappa_0}^{\kappa}\mathbb{E}\left[\|\mathscr{H}(\sigma, z(\sigma_1(\sigma)), \cdots, z(\sigma_n(\nu)),\right. \\ &\left.\int_{\kappa_0}^{\nu} g(\nu, r, z(\sigma_{n+1}(r)))dr\right\|^p\Bigg] d\sigma \\ \le{}& 2^{p-1}C^pK^p\mathbb{E}\left[\|v\|^p\right] + 2^{p-1}K^p\max\{1, C^p\} \\ &\times(\mathcal{T}-\kappa_0)\int_{\kappa_0}^{\kappa}\varrho_1(\nu)W\left[\mathbb{E}\|z(\sigma_1(\sigma))\|^p\right. \\ &+\cdots+\mathbb{E}\|z(\sigma_n(\nu))\|^p + \varrho_2(\nu)\mathbb{E}\|z(\sigma_{n+1}(r))\|^p dr]d\nu \\ \le{}& 2^{p-1}C^pK^p\mathbb{E}\left[\|v\|^p\right] + 2^{p-1}K^p\max\{1, C^p\}(\mathcal{T}-\kappa_0)^{p-1} \\ &\cdot\int_{\kappa_0}^{\kappa}\varrho_1(\nu)(n+\varrho_2(\nu))W\left[\mathbb{E}\left(\|z\|_\nu^p\right)\right]d\nu.\end{aligned} \tag{45}$$

From the above inequality, the last term of the right side increases in κ, we would obtain

$$\begin{aligned}\sup_{\kappa_0\le\nu\le\kappa}\mathbb{E}\|z\|_\nu^p \le{}& 2^{p-1}K^pC^p\mathbb{E}\left[\|v\|^p\right] + 2^{p-1}K^p\max\{1, C^p\} \\ &\cdot(\mathcal{T}-\kappa_0)^{p-1}\int_{\kappa_0}^{\kappa}\varrho_1(\nu)(n+\varrho_2(\nu))W\left[\mathbb{E}\left(\|z\|_\nu^p\right)\right]d\nu \\ \le{}& 2^{p-1}K^pC^p\mathbb{E}\left[\|v\|^p\right] + 2^{p-1}K^p\max\{1, C^p\} \\ &\cdot(\mathcal{T}-\kappa_0)^{p-1}\int_{\kappa_0}^{\kappa}\varrho_1(\nu)(n+\varrho_2(\sigma)) \\ &\cdot\sup_{\kappa_0\le\nu\le\kappa}W\left[\mathbb{E}\left(\|z\|_\nu^p\right)\right]d\nu.\end{aligned} \tag{46}$$

Let us define the function $\pounds(t)$ as follows:

$$\pounds(\kappa) = \sup_{\kappa_0\le\nu\le\kappa}\mathbb{E}\left[\|z\|_\nu^p\right], \kappa\in[\kappa_0, \mathcal{T}]. \tag{47}$$

From the above inequality (47), the right side is denoted by $\rho(\kappa)$, we get

$$\pounds(t)\le\rho(\kappa), \kappa\in[\kappa_0, \mathcal{T}], \rho(\kappa_0) = 2^{p-1}K^pC^p\mathbb{E}\|v\|^p = c_0,$$

$$\begin{aligned}\rho'(\kappa) &= 2^{p-1}K^p\max\{1, C^p\}(\mathcal{T}-\kappa_0)\varrho_1(\nu)(n+\varrho_2(\sigma))W(\pounds(\kappa)) \\ &\le 2^{p-1}K^p\max\{1, C^p\}(\mathcal{T}-\kappa_0)\varrho_1(\nu)(n+\varrho_2(\nu))W(\rho(\nu)),\end{aligned}$$

$$\frac{\rho'(\kappa)}{W(\rho(\kappa))}\le\rho^*(\kappa), \kappa\in[\kappa_0, \mathcal{T}]. \tag{48}$$

This implies

$$\int_{\rho(\kappa_0)}^{\rho(\kappa)}\frac{d\nu}{W(\nu)}\le\int_{\kappa_0}^{\mathcal{T}}M_*(nu)d\nu<\int_{c_0}^{\infty}\frac{d\nu}{W(\nu)}, \kappa\in[\kappa_0, \mathcal{T}], \tag{49}$$

where the last inequality is obtained by (19). From (57) and applying the mean value theorem, it can be deduced that there exists a constant η_1 such that $\rho(\kappa)\le\eta_1$ and here $\pounds(\kappa)\le\eta_1$. Since $\pounds(\kappa) = \sup_{\kappa_0\le\nu\le\kappa}\mathbb{E}[\|z\|_\nu^p]$ holds for every $\kappa\in[\kappa_0, \mathcal{T}]$, we have $\eta_1\ge\sup_{\kappa_0\le\nu\le\mathcal{T}}\mathbb{E}[\|z\|_\nu^p]$, where η_1 depends only on $\mathcal{T}$ and the functions ρ and W.

$$\eta_1\ge\sup_{\kappa_0\le\nu\le\mathcal{T}}\mathbb{E}\|z\|_\nu^p = \mathbb{E}\|z\|_G^p. \tag{50}$$

U is bounded, as evidenced by this. Theorem 3 leads us to conclude that $\mathbb{Z}$ has a fixed point z defined on $[-\rho; \mathcal{T}]$, which is a solution of (3). □

Theorem 11. *Assume that there exists continuous functions $\varrho_1, \varrho_2 : [\kappa_0, \mathcal{T}] \longrightarrow (0, +\infty)$ such that*

$$\mathbb{E}\|\mathcal{H}(\kappa, z_1, \cdots, z_{n+1})\|^p \leq \varrho_1(\kappa)\mathbb{E}\left(\|z_1\|_s^p + \cdots + \|z_{n+1}\|_s^p\right), \quad z_i \in \mathbb{R}^n, i = 1, 2, \cdots, n+1,$$

$$\mathbb{E}\|g(\kappa, s, z)ds\|^p \leq \varrho_2(\kappa)\mathbb{E}\|z\|_s^p, \kappa, s \geq 0, z \in \mathbb{R}^n. \tag{51}$$

Under the assumptions A3-A6, system (3) has a solution $z(\kappa)$ defined on $[\kappa_0, \mathcal{T}]$.

Proof. Similar to steps 1-5, in Theorem 10, we can prove steps 1-5. In the following, we prove step 6.

Step 6. A priori bounds.

Here, it remains to show that the set

$$U(\mathbb{Z}) = \{z \in G : \eta z \in \mathbb{Z}(z) \text{ for some } 0 < \eta < 1\} \text{ is bounded,}$$

$$\begin{aligned}
\mathbb{E}\|z\|_\kappa^p &\leq 2^{p-1}C^pK^p\mathbb{E}\left[\|v\|^p\right] + 2^{p-1}\max\{1, C^p\}(\mathcal{T} - \kappa_0)\int_{\kappa_0}^{\kappa} \\
&\quad \cdot \mathbb{E}\left[\left\|\mathcal{H}(s, z(\sigma_1(s)), \cdots, z(\sigma_n(s)), \int_{\kappa_0}^{s} g(s, r, z(\sigma_{n+1}(r)))dr\right\|^p\right] ds \\
&\leq 2^{p-1}C^pK^p\mathbb{E}\left[\|v\|^p\right] + 2^{p-1}K^p\max\{1, C^p\} \\
&\quad \times (\mathcal{T} - \kappa_0)\int_{t_0}^{\kappa} \varrho_1(s)\left[\mathbb{E}\|z(\sigma_1(s))\|^p + \cdots + \mathbb{E}\|z(\sigma_n(s))\|^p\right. \\
&\quad \left. + \int_{\kappa_0}^{s} \varrho_2(r)\mathbb{E}\|z(\sigma_{n+1}(r))\|^p dr\right]ds \leq 2^{p-1}C^pK^p\mathbb{E}\left[\|v\|^p\right] \\
&\quad + 2^{p-1}K^p\max\{1, C^p\} \times (\mathcal{T} - \kappa_0)\int_{\kappa_0}^{\kappa} \varrho_1(s)\left[\mathbb{E}\|z(s)\|^p\right. \\
&\quad \left. + \cdots + \mathbb{E}\|z(s)\|^p + \int_{\kappa_0}^{s} \varrho_2(r)\mathbb{E}\|z(r)\|^p dr\right]ds \\
&\leq 2^{p-1}C^pK^p\mathbb{E}\left[\|v\|^p\right] + 2^{p-1}K^p\max\{1, C^p\}(\mathcal{T} - \kappa_0) \\
&\quad \cdot \int_{\kappa_0}^{\kappa} \varrho_1(s)\left(n\mathbb{E}\|z\|_s^p + \int_{\kappa_0}^{s} \varrho_2(r)dr\mathbb{E}\|z\|_s^p\right)ds.
\end{aligned} \tag{52}$$

We note that from the above inequality, the last term of the right side increases in κ, we would obtain

$$\begin{aligned}
\sup_{t_0 \leq v \leq \kappa} \mathbb{E}\|z\|_v^p &\leq 2^{p-1}K^pC^p\mathbb{E}\left[\|v\|^p\right] + 2^{p-1}K^p\max\{1, C^p\}(\mathcal{T} - \kappa_0) \\
&\quad \cdot \int_{\kappa_0}^{\kappa} \varrho_1(s)\left(n\mathbb{E}\|z\|_s^p + \int_{\kappa_0}^{s} \varrho_2(r)\mathbb{E}\|z\|_s^p dr\right)ds \\
&\leq 2^{p-1}K^pC^p\mathbb{E}\left[\|v\|^p\right] + 2^{p-1}K^p\max\{1, C^p\}(\mathcal{T} - \kappa_0) \\
&\quad \cdot \int_{\kappa_0}^{\kappa} \varrho_1(s)\left(n\sup_{\kappa_0 \leq v \leq t}\left[\mathbb{E}\|z\|_v^p\right] + \int_{\kappa_0}^{s} \varrho_2(r)dr\sup_{t_0 \leq v \leq \kappa}\left[\mathbb{E}\|z\|_v^p\right]\right)ds.
\end{aligned} \tag{53}$$

Let us define the function $\pounds(\kappa)$ as follows:

$$\pounds(\kappa) = \sup_{\kappa_0 \leq v \leq \kappa} \mathbb{E}\left[\|z\|_v^p\right], \kappa \in [\kappa_0, \mathcal{T}]. \tag{54}$$

Then, for any $\kappa \in [\kappa_0, \mathcal{T}]$, it follows that

$$\pounds(\kappa) \leq 2^{p-1}K^pC^p\mathbb{E}\left[\|v\|^p\right] + 2^{p-1}K^p\max\{1, C^p\}(\mathcal{T} - \kappa_0)\int_{t_0}^{\kappa} \cdot \varrho_1(s)\left(n\pounds(s) + \int_{\kappa_0}^{s} \varrho_2(r)\pounds(r)dr\right)ds, \tag{55}$$

$$\pounds(\kappa) \leq 2^{p-1}K^pC^p\mathbb{E}\left[\|v\|^p\right] + 2^{p-1}K^p\max\{1, C^p\}(\mathcal{T} - \kappa_0) \cdot \left(\int_{\kappa_0}^{\kappa} n\varrho_1(s)\pounds(s)ds + \int_{\kappa_0}^{\kappa} \varrho_1(s)\int_{\kappa_0}^{s} \varrho_2(r)\pounds(r)drds\right), \tag{56}$$

$$\pounds(\kappa) \leq c_1 + \int_{\kappa_0}^{\kappa} c_2 n\rho_1(s)\pounds(s)ds + \int_{\kappa_0}^{\kappa} c_2\rho_1(s)\int_{\kappa_0}^{s} \rho_2(r)\pounds(r)drds, \tag{57}$$

where $c_0 = 2^{p-1}K^pC^p\mathbb{E}[\|v\|^p]$. Let $N(\kappa) = \sup\{M_*\}$ and $N_*(t) = \sup\{N(\kappa): \kappa \in [_\kappa 0, \mathcal{T}]\}$; then from eqn (57), we have

$$\pounds(\kappa) \leq c_0 + \int_{\kappa_0}^{\kappa} N(s)\pounds(s)ds + \int_{\kappa_0}^{\kappa} N(s)\int_{\kappa_0}^{s} N(r)\pounds(r)drds. \tag{58}$$

Applying Lemma 8 to inequality (58), we get

$$\begin{aligned}
\pounds(\kappa) &\leq c_0\left[1 + \int_{\kappa_0}^{\kappa} \vartheta(s)\exp\left(\int_{\kappa_0}^{s} N(r) + N(r)dr\right)ds\right] \\
&\leq c_0\left[1 + N_*\exp\left(2N_*(\mathcal{T} - \kappa_0)\right)(\mathcal{T} - \kappa_0)\right] = \mathfrak{A}.
\end{aligned} \tag{59}$$

□

We now provide another existence result for problem (3). In the following theorem, the Wintner-type growth condition is used to relax the multivalued $\mathcal{H}$.

Theorem 12. *Suppose that A4-A6 and the following condition hold:*

A7. $\exists$ some function $\ell_1 \in \pounds_p([\kappa_0, \mathcal{T}], \mathbb{R}^+)$ s.t

$$\begin{aligned}
&\mathcal{F}(\mathcal{H}(\kappa, z_1, \cdots, z_{n+1}), \mathcal{H}(\kappa, w_1, \cdots, w_{n+1})) \\
&\quad \leq \ell_1(\kappa)\left(\|z_1 - w_1\|_v^p + \cdots + \|z_{n+1} - w_{n+1}\|_v^p\right), \forall \kappa \in [\kappa_0, \mathcal{T}], \\
&\quad z_i, w_i \in \mathcal{C}, i = 1, 2, \cdots, n+1, \mathcal{F}((0, \mathcal{H}(\kappa, 0, \cdots, 0)) \\
&\quad \leq \ell_1(\kappa), \text{for a.e } \kappa \in [\kappa_0, \mathcal{T}].
\end{aligned} \tag{60}$$

A8. $\exists$ some function ℓ_2 s.t $\mathcal{F}(\int_{\kappa_0}^{\kappa} g(\kappa, \sigma, z)d\sigma) \leq \ell_2(\kappa)\|z\|_v^p, z \in \mathcal{C}$,

$$\mathcal{F}((0, g(\kappa, v, 0)) \leq \ell_2(\kappa), \text{for a.e } \kappa \in [\kappa_0, \mathcal{T}], \tag{61}$$

where $\int_{\kappa_0}^{\mathcal{T}} M_*(\kappa)d\nu < \infty$,

$$M_{1*}(\kappa) = 2^{p-1}C^pK^p\mathbb{E}\left[\|\upsilon\|^p\right] + 2^{p-1}K^p \max\{1, C^p\} \cdot (\mathcal{T}-\kappa_0)^{p-1}\int_{\kappa_0}^{\kappa} \ell_1(\nu)(n+\ell_2(\nu))d\nu,$$

$$M_{2*}(\kappa) = 2^{p-1}C^pK^p\mathbb{E}\left[\|\upsilon\|^p\right] + 2^{p-1}K^p \max\{1, C^p\}(\mathcal{T}-\kappa_0)^{p-1}. \tag{62}$$

Then, problem (3) has at least one mild solution on $[-\rho, \mathcal{T}]$.

Proof. Let $\mathbb{Z}$ be defined as in Theorem 10. It is possible to demonstrate that $\mathbb{Z}$ is upper semicontinuous and completely continuous, as in the proof of Theorem 10. Here, we prove that

$$\mathbb{U} = \{u \in \mathbb{X} | \lambda u \in \mathbb{Z}(\kappa), \text{for some } \lambda > 1\} \text{ is bounded.} \tag{63}$$

Let $z \in \mathbb{U}$; then, $\exists h \in \mathfrak{U}_{\mathcal{H},z}$, $\kappa \in [\kappa_0, \mathcal{T}]$,

$$z(\kappa) = \eta \sum_{q=0}^{\infty}\left[\prod_{i=1}^{q} a_i(\mu_i)\mathfrak{U}(\kappa-\kappa_0)\upsilon(0) + \sum_{i=1}^{q}\prod_{\iota=i}^{q} a_\iota(\mu_\iota)\int_{\zeta_{i-1}}^{\zeta_i}\mathfrak{U}(\kappa-\nu)h(\nu)d\nu + \int_{\zeta_q}^{\kappa}\mathfrak{U}(\kappa-\nu)h(\nu)d\nu\right] I_{[\zeta_q,\zeta_{q+1})}(\kappa), \kappa \in [\kappa_0, \mathcal{T}]. \tag{64}$$

Thus, by A4-A8, we have

$$\mathbb{E}\|z\|_\kappa^p \le 2^{p-1}C^pK^p\mathbb{E}\left[\|\upsilon\|^p\right] + 2^{p-1}\max\{1, C^p\} \times (\mathcal{T}-\kappa_0)^{p-1}\int_{\kappa_0}^{\kappa}\mathbb{E}\left[\left\|\mathcal{H}(\nu, z(\sigma_1(\nu)), \cdots, z(\sigma_n(\nu)), \int_{\kappa_0}^{\nu} g(\nu, r, z(\sigma_{n+1}(r)))dr\right\|^p\right]d\nu. \tag{65}$$

Observing that the right-hand side's final term in the inequality above increases in κ and selecting $K^pC^p \ge 1/2^{p-1}$, we arrive at the conclusion that

$$\begin{aligned}\mathbb{E}\|z\|_\kappa^p \le{}& 2^{p-1}C^pK^p\mathbb{E}\left[\|\upsilon\|^p\right] + 2^{p-1}K^p\max\{1, C^p\} \cdot (\mathcal{T}-\kappa_0)^{p-1}\int_{\kappa_0}^{\kappa}\ell_1(\nu)(n+\ell_2(\nu))d\nu \\ &+ 2^{p-1}K^p\max\{1, C^p\}(\mathcal{T}-\kappa_0)^{p-1} \cdot \int_{\kappa_0}^{\kappa}\ell_1(\nu)(n+\ell_2(\nu))\mathbb{E}\|z\|_\nu^p d\nu.\end{aligned} \tag{66}$$

By (47), we get

$$\pounds(\kappa) \le M_{1*}(\kappa) + M_{2*}(\kappa)\int_{\kappa_0}^{\kappa}\ell_1(\nu)(n+\ell_2(\nu))\pounds(\kappa)d\nu. \tag{67}$$

Using the Gronwall inequality, we obtain

$$\pounds(t) \le M_{1*}(\kappa)\exp\left(M_{2*}(\kappa)\int_{\kappa_0}^{\kappa}\ell_1(\nu)(n+\ell_2(\nu))d\nu\right), \forall\kappa \in [\kappa_0, \mathcal{T}]. \tag{68}$$

Therefore, $\exists\beta_2 > 0$ s.t $\pounds(\kappa) \le \beta_2$, $\forall\kappa \in [\kappa_0, \mathcal{T}]$; this gives

$$\|z\|_G^p \le \beta_2. \tag{69}$$

This demonstrates the bounded nature of the set $\mathbb{U}$. By virtue of Theorem 3, it can be inferred that $\mathbb{U}$ possesses a fixed point, which corresponds to a mild solution of equation (3). □

4. Existence Results: Nonconvex Case

Our study in this section is based on Covitz and Nadler's fixed point theorem for contraction multivalued operators. We use additional assumption:

A9. $(\mathcal{F}_{\mathcal{H}\text{cp}})$: $\mathcal{H} : [\mu, \mathcal{T}] \times \mathcal{C} \longrightarrow \mathcal{P}_{\text{cp}}(\mathbb{X})$ has the property that $\mathcal{H}(., z_1, \cdots, z_{n+1})$: $[\mu, \mathcal{T}] \longrightarrow \mathcal{P}_{\text{cp}}(\mathbb{X})$ is measurable $\forall z \in \mathcal{C}$.

Theorem 13. *Suppose that hypotheses A4-A5 and A7-A9 are fulfilled, then the IVP (3) has at least one mild solution on* $[-\rho; \mathcal{T}]$, *provided*

$$\eta = K^p\max\{1, C^p\}(\mathcal{T}-\kappa_0)^{p-1}\int_{\kappa_0}^{\kappa}\ell_1(\nu)(n+\ell_2(\nu))d\nu < 1. \tag{70}$$

Proof. Transform problem (3) into a fixed point problem. Consider the multivalued operator $\mathbb{Z}$ as introduced in Theorem 10. Our aim is to show that $\mathbb{Z}$ fulfills the hypotheses of Theorem 11.

Step 1. To prove $\mathbb{Z}(z) \in \mathcal{P}_{cl} \in G$, $\forall z \in G$.

Indeed, let $(z^n)_{n\ge0} \in \mathbb{Z}(z)$ s.t $z^n \longrightarrow \bar{z}$ in G. Then, $\bar{z} \in G$ and $\exists h^n \in S_{\mathcal{H},z}$ s.t $\forall\kappa \in [\kappa_0, \mathcal{T}]$,

$$\begin{aligned}z^n(\kappa) ={}& \sum_{\wp=0}^{\infty}\left[\prod_{i=1}^{q} a_i(\mu_i)\mathfrak{U}(\kappa-\kappa_0)\upsilon(0) + \sum_{i=1}^{\wp}\prod_{j=i}^{q} a_j\left(\mu_j\right)\int_{\zeta_{i-1}}^{\zeta_i}\mathfrak{U}(\kappa, \nu) \cdot \mathcal{H}\left(\nu, z^n(\sigma_1(\nu)), \cdots, z^n(\sigma_m(\nu)), \int_{\kappa_0}^{\nu} g(\nu, r, z^n(\sigma_{m+1}(r)))dr\right)d\nu\right. \\ &\left. + \int_{\zeta_\wp}^{\kappa}\mathfrak{U}(\kappa, \nu)\mathcal{H} \cdot \left(\nu, z^n(\sigma_1(\nu)), \cdots, z^n(\sigma_m(\nu))\int_{\kappa_0}^{\nu} \cdot g(\nu, r, z^n(\sigma_{m+1}(r)))dr\right)d\nu\right] I_{[\zeta_\wp,\zeta_{\wp+1})}(\kappa).\end{aligned} \tag{71}$$

Using A7 and A8, $\mathscr{H}$ has compact values. If necessary, we can pass to a subsequence to ensure that h^n converges $f \in \pounds_p([\mu, \mathscr{T}], \mathbb{Z}]$, and hence, $h \in S_{\mathscr{H},z}$. Then, $\forall \kappa \in [\kappa_0, \mathscr{T}]$,

$$\begin{aligned} z^n(\kappa) \longrightarrow \bar{z}(\kappa) = \sum_{\wp=0}^{\infty} \Bigg[& \prod_{i=1}^{\wp} a_i(\mu_i) \mathfrak{U}(\kappa - \kappa_0) v(0) + \sum_{i=1}^{\wp} \prod_{j=i}^{\wp} a_j\left(\mu_j\right) \\ & \cdot \int_{\zeta_{i-1}}^{\zeta_i} \mathfrak{U}(\kappa, \nu) \mathscr{H}\Bigg(\nu, z^n(\sigma_1(\nu)), \cdots, z^n(\sigma_m(\nu)), \\ & \int_{\kappa_0}^{\nu} g(\nu, r, z^n(\sigma_{m+1}(r))) dr\Bigg) d\nu \\ & + \int_{\zeta_q}^{\kappa} \mathfrak{U}(\kappa, \nu) \mathscr{H}\Bigg(\nu, z^n(\sigma_1(\nu)), \cdots, z^n(\sigma_n(\nu)) \\ & \cdot \int_{\kappa_0}^{\nu} g(\nu, r, z^n(\sigma_{m+1}(r))) dr\Bigg) d\nu \Bigg] I_{[\zeta_\wp, \zeta_{\wp+1})}(\kappa). \end{aligned} \tag{72}$$

So $\bar{z} \in \mathbb{Z}(z)$.

Step 2. Contraction.

Let $z^1, z^2 \in G$ and $f^1 \in \mathbb{Z}(z^1)$; then, $\exists f^1(\kappa) \in \mathscr{H}(\kappa, z^1_\kappa)$ s.t $\kappa \in [\kappa_0, \mathscr{T}]$,

$$\begin{aligned} f^1(\kappa) \sum_{\wp=0}^{\infty} \Bigg[& \prod_{i=1}^{\wp} a_i(\tau_i) \mathfrak{U}(\kappa - \kappa_0) v(0) \\ & + \sum_{i=1}^{\wp} \prod_{j=i}^{\wp} a_j\left(\mu_j\right) \int_{\zeta_{i-1}}^{\zeta_i} \mathfrak{U}(\kappa, \nu) h^1(\nu) \\ & + \int_{\zeta_\wp}^{\kappa} \mathfrak{U}(\kappa, \nu) h^1(\nu) \Bigg] I_{[\zeta_\wp, \zeta_{\wp+1})}(\kappa). \end{aligned} \tag{73}$$

From A7 and A8, we get

$$\begin{aligned} & \mathscr{F}\left(\mathscr{H}\left(\kappa, z_i^1\right), \mathscr{H}\left(\kappa, z_i^2\right)\right) \le \ell_1(\kappa)\left(\left\|z_i^1 - z_i^2\right\|_\kappa^p\right), \\ & \qquad \kappa \in [\kappa_0, \mathscr{T}], i = 1, 2, \cdots, m+1, \\ & \mathscr{F}\left(\int_{\kappa_0}^{\kappa} g\left(\kappa, \nu, z^1\right) d\nu\right) \le \ell_2(\kappa)\left\|z^1\right\|_\kappa^p, \kappa \in [\kappa_0, \mathscr{T}]. \end{aligned} \tag{74}$$

Hence, there is $w \in \mathscr{H}(\kappa, z_i^2)$, $i = 1, 2, \cdots, m+1$, s.t

$$\left\|h^1(\kappa) - w\right\|_\kappa^p \le \ell_1(\kappa)(m + \ell_2(\kappa))\left(\left\|z^1 - w\right\|_\kappa^p\right), \kappa \in [\kappa_0, \mathscr{T}]. \tag{75}$$

Consider $\mathscr{N} : [\kappa_0, \mathscr{T}] \longrightarrow \mathscr{P}_{cp}(\mathbb{X})$, given by

$$\begin{aligned} \mathscr{N}(\kappa) = \Big\{ & w \in \mathbb{X} : \left\|h^1(\kappa) - w\right\|_\kappa^p \\ & \le \ell_1(\kappa)(m + \ell_2(\kappa))\left(\left\|z^1 - w\right\|_\kappa^p\right), \kappa \in [\kappa_0, \mathscr{T}]\Big\}. \end{aligned} \tag{76}$$

Since $\mathscr{N}^*(\kappa) = \mathscr{N}(\kappa) \cap \mathscr{H}(\kappa, z_i^\kappa)$, $i = 1, 2, \cdots, m+1$ is measurable, $\exists f^2(\kappa)$ a measurable selection for $\mathscr{N}^*$. So $h^2(\kappa) \in \mathscr{H}(\kappa, z_i^\kappa)$, $i = 1, 2, \cdots, m+1$, and

$$\left\|h^1(\kappa) - h^2(\kappa)\right\|_\kappa^p \le \ell_1(\kappa)(n + \ell_2(\kappa))\left(\left\|z^1 - w\right\|_\kappa^p\right), \forall \kappa \in [\kappa_0, \mathscr{T}]. \tag{77}$$

Let us define $\forall \kappa \in [\kappa_0, \mathscr{T}]$,

$$\begin{aligned} f^2(\kappa) = \sum_{\wp=0}^{\infty} \Bigg[& \prod_{i=1}^{\wp} a_i(\mu_i) \mathfrak{U}(\kappa - \kappa_0) v(0) \\ & + \sum_{i=1}^{\wp} \prod_{j=i}^{\wp} a_j(\tau_j) \int_{\zeta_{i-1}}^{\zeta_i} \mathfrak{U}(\kappa, \nu) h^2(\nu) \\ & + \int_{\zeta_\wp}^{\kappa} \mathfrak{U}(\kappa, \nu) h^2(\nu) \Bigg] I_{[\zeta_\wp, \zeta_{\wp+1})}(\kappa). \end{aligned} \tag{78}$$

Then, we get

$$\begin{aligned} f^1(\kappa) - f^2(\kappa) = & \sum_{\wp=0}^{\infty} \left[\sum_{i=1}^{\wp} \prod_{j=i}^{\wp} a_j\left(\mu_j\right) \int_{\zeta_{i-1}}^{\zeta_i} \mathfrak{U}(\kappa, \sigma)\left[h^1(\sigma) - h^2(\nu)\right] d\nu\right. \\ & + \int_{\zeta_q}^{\kappa} \mathfrak{U}(\kappa, \nu)\left[h^1(\nu) - h^2(\nu)\right] d\nu] I_{[\zeta_\wp, \zeta_{\wp+1})}(\kappa), \\ \left\|f^1(\kappa) - f^2(\kappa)\right\|^p = & \left(\sum_{\wp=0}^{\infty}\left[\sum_{i=1}^{\wp} \prod_{j=i}^{\wp} a_j\left(\mu_j\right) \int_{\zeta_{i-1}}^{\zeta_i}\right.\right. \\ & \cdot \|\mathfrak{U}(\kappa, \sigma)\|\left\|h^1(\nu) - h^2(\nu)\right\| d\nu \\ & \left.\left. + \int_{\zeta_\wp}^{\kappa} \|\mathfrak{U}(\kappa, \nu)\|\left\|h^1(\nu) - h^2(\nu)\right\| d\nu\right] I_{[\zeta_\wp, \zeta_{\wp+1})}(\kappa)\right)^p \\ & + \left[\max_{i, \wp}\left\{1, \prod_{j=i}^{\wp}\left\|a_j\left(\mu_j\right)\right\|\right\}\right]^p \|\mathfrak{U}(\kappa, \nu)\|^p \\ & \cdot (\mathscr{T} - \kappa_0)^{p-1}\left(\int_{\kappa_0}^{\kappa}\left\|h^1(\nu) - h^2(\nu)\right\|^p d\nu\right), \\ \mathbb{E}\left\|f^1 - f^2\right\|_\kappa^p \le & K^p \max\{1, C^p\}(\mathscr{T} - \kappa_0)^{p-1} \\ & \cdot \mathbb{E} \int_{\kappa_0}^{\kappa} \ell_1(\nu)(m + \ell_2(\nu))\left\|z^1 - z^2\right\|_\nu^p d\nu \\ \le & K^p \max\{1, C^p\}(\mathscr{T} - \kappa_0)^{p-1} \int_{\kappa_0}^{\kappa} \ell_1(\sigma)(m + \ell_2(\sigma)) \\ & \cdot \mathbb{E}\left\|z^1 - z^2\right\|_\nu^p d\nu \\ \le & K^p \max\{1, C^p\}(\mathscr{T} - \kappa_0)^{p-1} \\ & \cdot \int_{\kappa_0}^{\kappa} \ell_1(\nu)(m + \ell_2(\nu)) d\nu\, \mathbb{E}\left\|z^1 - z^2\right\|_\kappa^p. \end{aligned} \tag{79}$$

Taking supremum over κ, we obtain

$$\begin{aligned} \left\|f^1 - f^2\right\|_G^p \le & K^p \max\{1, C^p\}(\mathscr{T} - \kappa_0)^{p-1} \\ & \cdot \int_{\kappa_0}^{\mathscr{T}} \ell_1(\nu)(m + \ell_2(\nu)) d\nu\left\|z^1 - z^2\right\|_G^p. \end{aligned} \tag{80}$$

From the equivalent relation, which is produced by switching the role ofz^1andz^2, we get

$$\mathcal{F}\left(M\left(z^1\right)\right), M\left(z^2\right)\right) \leq \eta\left\|z^1 - z^2\right\|_G^p. \tag{81}$$

Using (70), we get $0 < \eta < 1$ and hence $\mathbb{Z}$ is a contraction, and by Theorem 3, we can say that $\mathbb{Z}$ has a fixed point z, which is a mild solution of (3). □

5. An Application

The considered partial integrodifferential inclusion with time-varying delay is in the following form:

$$\begin{cases} \dfrac{\partial v(z,\kappa)}{\partial \kappa} \in \dfrac{\partial^2 v(z,\kappa)}{\partial^2 \kappa} + \displaystyle\int_{-r}^{\kappa_0}\int_{\kappa_0}^{\kappa} \mathcal{H}_1(\kappa, z, \eta)\mathcal{H}_2(v(\kappa+\eta, z))d\kappa d\eta, \\ 0 < z < \pi, \kappa_0 \leq \kappa \leq \mathcal{T}, \kappa \neq \zeta_{\wp}, \\ v(z, \zeta_{\wp}) = b_{\wp}\left(\mu_{\wp}\right) v\left(z, \zeta_{\wp}^-\right), \kappa = \zeta_{\wp}, \\ v(0,\kappa) = v(\pi,\kappa) = 0, \\ v(z,\kappa) = \upsilon(z,\kappa), -\rho \leq \kappa \leq 0, 0 \leq z \leq \pi. \end{cases} \tag{82}$$

Let $\mathbb{X} = \pounds_p[0,\pi]$ and $\mathcal{A} = \partial^2/\partial^2\kappa$ with the domain

$$\mathfrak{D}(\mathcal{A}) = \left\{ v \in \mathbb{X} \text{ and } \frac{\partial v}{\partial z} \text{ are absolutely continuous}, \frac{\partial^2 v}{\partial^2 z} \in \mathbb{X}, v(0) = v(\pi) = 0 \right\}. \tag{83}$$

Then,

$$\mathcal{A}v = \sum_{m=1}^{\infty} m(v, v_m), v \in \mathfrak{D}(\mathcal{A}), \tag{84}$$

where $v_v(z) = \sqrt{2/v}\sin(vz), v = 1, 2, \cdots n$ is the orthogonal set of eigenvectors in $\mathcal{A}$. It is well known that $\mathcal{A}$ generates a strongly continuous semigroup $\mathfrak{U}(\kappa)$ which is compact, analytic, and self-adjoint and

$$\|\mathfrak{U}(\kappa)\| \leq K, \text{for } \kappa \geq 0. \tag{85}$$

Thus, $\mathfrak{U}(\kappa)$ is uniformly bounded.

(1) The function $\mathcal{H} \geq 0$ is continuous in $[\kappa_0, \mathcal{T}] \times [0,\pi] \times [-\rho, 0]$, with

$$\int_{-\alpha}^{0} \mathcal{H}_1(\kappa, z, \eta)d\eta = u_1(\kappa, z), u_1(\kappa) = \left(\int_{-\alpha}^{0} u_1(\kappa, z)dz\right)^{1/p} < \infty. \tag{86}$$

For $z(\kappa+\eta) \in \mathfrak{C}$, we consider the validity of the following conditions:

(2) $\max_{i,\wp}\{\prod_{j=i}^{\wp}\|b_i(\delta_i)\|\} \leq \mathcal{C}^* < \infty$

(3) The multifunction $\mathcal{H}_2(.)$ is an Lp-Carathéodory multivalued function with compact and convex values and

$$0 \leq \left\| \int_{\kappa_0}^{\kappa} \mathcal{H}_2(v(\eta, z))d\eta \right\| \leq \varrho\|v(\eta,.)\|, (\eta, z) \in [\kappa_0, \mathcal{T}] \times [0,\pi], \tag{87}$$

where $\varrho : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is continuous and nondecreasing

As conditions (1)–(3) satisfy, the problem presented by equation (82) can be structured as abstract integrodifferential inclusions featuring a random impulse, as illustrated in equation (3), with

$$\int_{\kappa_0}^{\kappa} \mathcal{H}(\kappa, z(\kappa)) = \int_{-r}^{\kappa_0}\int_{\kappa_0}^{\kappa} \mathcal{H}_1(\kappa, z, \eta)\mathcal{H}_2(v(\kappa+\eta, z))d\kappa d\eta, \tag{88}$$

and $a_{\wp}(\mu_{\wp}) = a(\wp)\mu_{\wp}$.

6. Conclusion

In this manuscript, we have established the existence of mild solutions for the NRIIDIns with time-varying delays. This accomplishment was achieved through the utilization of the Martelli fixed point theorem, Pachpatte's inequality, and the fixed point theorem originally formulated by Covitz and Nadler. A delightful extension of our findings would involve exploring controllability aspects for a specific category of NRIIDIns with time-varying delays in the future. The investigation into the fractional order of NRIIDIns with time-varying delays holds significant intrigue and will be a primary area of emphasis in our upcoming research endeavors.

Data Availability

The data used to support the findings of this study are included within the article [9, 19].

Conflicts of Interest

All authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest or nonfinancial interest in the subject matter or materials discussed in this manuscript.

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References

[1] M. A. Almalahi and S. K. Panchal, "Some properties of implicit impulsive coupled system via φ-Hilfer fractional operator," *Boundary Value Problems*, vol. 2021, no. 1, 2021.

[2] M. A. Almalahi, S. K. Panchal, and F. Jarad, "Results on implicit fractional pantograph equations with Mittag-Leffler kernel and nonlocal condition," *Journal of Mathematics*, vol. 2022, Article ID 9693005, 19 pages, 2022.

[3] R. S. Jain, B. S. Reddy, and S. D. Kadam, "On existence of mild and classical solutions of second order impulsive implicit differential equations," *Fasciculi Mathematici*, vol. 61, pp. 71–81, 2018.

[4] V. Lakshmikantham, D. D. Bainov, and P. S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.

[5] X. Yang, X. Li, Q. Xi, and P. Duan, "Review of stability and stabilization for impulsive delayed systems," *Mathematical Biosciences and Engineering*, vol. 15, no. 6, pp. 1495–1515, 2018.

[6] A. Vinodkumar, M. Gowrisankar, and P. Mohankumar, "Existence and stability results on nonlinear delay integro-differential equations with random impulses," *Kyungpook National University*, vol. 56, no. 2, pp. 431–450, 2016.

[7] R. Kasinathan, B. Kandasamy, and V. Sandrasekaran, "Existence and stability results for integrodifferential evolution system with random impulse," *Applied Mathematics E-Notes*, vol. 21, pp. 550–562, 2021.

[8] S. A. Jose, W. Yukunthorn, J. E. N. Valdes, and L. Hugo, "Some existence, uniqueness and stability results of nonlocal random impulsive integro-differential equations," *Applied Mathematics E-Notes*, vol. 20, pp. 481–492, 2020.

[9] S. D. Kadam, S. Reddy, R. Menon, and R. S. Jain, "Existence and controllability of mild solution of impulsive integro-differential inclusions," *Nonlinear Functional Analysis and Applications*, vol. 25, no. 4, pp. 657–670, 2020.

[10] S. M. A. Maqbol, R. S. Jain, and B. S. Reddy, "On stability of nonlocal neutral stochastic integro differential equations with random impulses and Poisson jumps," *CUBO*, vol. 4, pp. 211–229, 2023.

[11] I. Dassios, "On the relations between a singular system of differential equations and a system with delays," *Mathematical Modelling and Numerical Simulation with Applications*, vol. 2, no. 4, pp. 221–227, 2022.

[12] A. Keten, M. Yavuz, and D. Baleanu, "Nonlocal Cauchy problem via a fractional operator involving power kernel in Banach spaces," *Fractal and Fractional*, vol. 3, no. 2, p. 27, 2019.

[13] M. G. Sakar, "Numerical solution of neutral functional-differential equations with proportional delays," *An International Journal of Optimization and Control: Theories & Applications (IJOCTA)*, vol. 7, no. 2, pp. 186–194, 2017.

[14] R. Shafqat, A. U. K. Niazi, M. Yavuz, M. B. Jeelani, and K. Saleem, "Mild solution for the time-fractional Navier–Stokes equation incorporating MHD effects," *Fractal and Fractional*, vol. 6, no. 10, p. 580, 2022.

[15] Q. B. Yin, Y. Guo, D. Wu, and X. B. Shu, "Existence and multiplicity of mild solutions for first-order Hamilton random impulsive differential equations with Dirichlet boundary conditions," *Qualitative Theory of Dynamical Systems*, vol. 22, no. 2, p. 47, 2023.

[16] Y. Guo, X. B. Shu, F. Xu, and C. Yang, "HJB equation for optimal control system with random impulses," *Optimization*, vol. 8, pp. 1–25, 2022.

[17] M. Liu, L. Chen, and X. B. Shu, "The existence of positive solutions for ϕ-Hilfer fractional differential equation with random impulses and boundary value conditions," *Waves in Random and Complex Media*, pp. 1–9, 2023.

[18] Z. Li, X. B. Shu, and T. Miao, "The existence of solutions for Sturm–Liouville differential equation with random impulses and boundary value problems," *Boundary Value Problems*, vol. 2021, 2021.

[19] A. Vinodkumar, "Existence results on random impulsive semilinear functional differential inclusions with delays," *Annals of Functional Analysis*, vol. 3, no. 2, pp. 89–106, 2012.

[20] J. Banas and K. Goebl, *Measures of Noncompactness in Banach Spaces*, Dekker, New York, 1980.

[21] K. Deimling, *Multivalued Differential Equations*, de Gruyter, New York, 1992.

[22] J. P. Aubin and A. Cellina, *Differential Inclusions*, Springer-Verlag, Berlin, 1984.

[23] S. C. Hu and N. S. Papageorgiou, "On the existence of periodic solutions for non-convex valued differential inclusions in $\mathbf{R}^N$," *Proceedings of the American Mathematical Society*, vol. 123, pp. 3043–3050, 1995.

[24] M. Martelli, "A Rothe's type theorem for non-compact acyclic-valued map," *Bollettino dell'Unione Matematica Italiana*, vol. 4, no. 11, pp. 70–76, 1975.

[25] H. Covitz and S. B. Nadler Jr., "Multi-valued contraction mappings in generalized metric spaces," *Israel Journal of Mathematics*, vol. 8, no. 1, pp. 5–11, 1970.

[26] A. Lasota and Z. Opial, "An application of the Kakutani-Ky-Fan theorem in the theory of ordinary differential equations," *Bulletin L'Académie Polonaise des Science, Série des Sciences Mathématiques, Astronomiques et Physiques*, vol. 13, pp. 781–786, 1965.

[27] B. G. Pachpatte, *Inequalities for Differential and Integral Equations*, Academic press, 1998.