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Research Article

Solvability of a Hadamard Fractional Boundary Value Problem at Resonance on Infinite Domain

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This paper investigates the existence of solutions for Hadamard fractional differential equations with integral boundary conditions at resonance on infinite domain. By constructing two suitable Banach spaces, establishing an appropriate compactness criterion, and defining appropriate projectors, we study an existence theorem upon the coincidence degree theory of Mawhin. An example is given to illustrate our main result.

1. Introduction

In this paper, we study the following Hadamard fractional boundary value problem (BVP for short) on an infinite domain:

$$\begin{cases} {}^HD^{\alpha}u(t) = f\left(t, u(t), {}^HD^{\alpha-2}u(t), {}^HD^{\alpha-1}u(t)\right), t \in (1,+\infty), \\ u(1) = 0, {}^HD^{\alpha-2}u(1) = 0, {}^HD^{\alpha-1}u(+\infty) = \sum_{i=1}^{m} \lambda_i {}^HI^{\beta_i}u(\eta), \end{cases} \tag{1}$$

where $2 < \alpha \leq 3$, $\lambda_i \geq 0, \eta \in (1,\infty)$, ${}^HD^{\alpha}$ is the Hadamard fractional derivative of α, and ${}^HI^{\beta_i}$ denotes the Hadamard fractional integral of $\beta_i \geq 0$, $i = 1, 2, \cdots, m$.

In recent decades, due to the broad application of fractional calculus in many fields, such as chemical physics, electrical networks, signal and image processing, modeling for anomalous diffusion, and fluid flow, there has been significant development in both the theory and applications of fractional calculus [1–6]. There exist many types of fractional derivatives, such as Riemann-Liouville, Caputo, Hadamard, Grunwald-Letnikov, and Weyl. In contrast to Riemann-Liouville and Caputo derivatives, Hadamard involves logarithmic function with arbitrary exponent. Many researchers have studied the Hadamard and Caputo-Hadamard fractional differential equations; for more details, see [7–17].

In [15], the authors discussed a Hadamard fractional differential equation on infinite intervals:

$$\begin{cases} D^{\alpha}u(t) + a(t)f(u(t)) = 0, 1 < \alpha \leq 2, t \in (1,+\infty), \\ u(1) = 0, D^{\alpha-1}u(+\infty) = \sum_{i=1}^{m} \lambda_i I^{\beta_i}u(\eta), \end{cases} \tag{2}$$

where D^{α} is the Hadamard fractional derivative of α and I^{β_i} denotes the Hadamard fractional integral of β_i, and $\lambda_i \geq 0, \eta > 1$, $i = 1, 2, \cdots, m$.

In [14], by using monotone iterative technique, the authors obtained the existence of positive solutions for a Hadamard fractional differential equation on infinite intervals:

$$\begin{cases} {}^HD^{\alpha}u(t) + f\left(t, u(t), {}^HI^{r}u(t), {}^HD^{\alpha-1}u(t)\right) = 0, 1 < \alpha \leq 2, t \in (1,+\infty), \\ u(1) = 0, {}^HD^{\alpha-1}u(+\infty) = \sum_{i=1}^{m} \lambda_i {}^HI^{\beta_i}u(\eta), \end{cases} \tag{3}$$

where ${}^{H}D^{\alpha}$ denotes the Hadamard fractional derivative of α, $\eta \in (1,+\infty)$, and ${}^{H}I^{(\cdot)}$ is the Hadamard fractional integral. r, $\beta_i, \lambda_i \geq 0 (i = 1, 2, \cdots, m)$ are given constants.

In [11], Li and Zhai obtained the existence and uniqueness of positive solutions to Hadamard fractional differential equations on infinite intervals by making use of a fixed point theorem for generalized concave operators:

$$\begin{cases} {}^{H}D^{\alpha}x(t) + b(t)f(t, x(t)) + c(t) = 0, 1 < \alpha \leq 2, t \in (1,+\infty), \\ x(1) = 0, {}^{H}D^{\alpha-1}x(+\infty) = \sum_{i=1}^{m} \lambda_i {}^{H}I^{\beta_i}x(\eta), \end{cases} \tag{4}$$

where ${}^{H}D^{\alpha}$ is the Hadamard fractional derivative of α and ${}^{H}I^{\beta_i}$ denotes the Hadamard fractional integral of β_i, and $\gamma_i \geq 0, \eta > 1, i = 1, 2, \cdots, m$.

The above three articles studied the solutions of Hadamard fractional differential equation at nonresonance, i.e., $\Gamma(\alpha) - \sum_{i=1}^{m}(\lambda_i \Gamma(\alpha)/\Gamma(\alpha + \beta_i))(\log \eta)^{\alpha+\beta_i-1} > 0$. To the best of our knowledge, there are few papers that have investigated the boundary value problems of Hadamard fractional differential equations at resonance on infinite domain. Inspired by the excellent results in [18–21], we will discuss this problem by constructing two suitable Banach spaces, establishing an appropriate compactness criterion, defining appropriate operators and the coincidence degree theory due to Mawhin.

In this paper, we will assume that the following conditions hold.

(H_1) $\sum_{i=1}^{m}(\lambda_i/\Gamma(\alpha + \beta_i))(\log \eta)^{\alpha+\beta_i-1} = 1$.

(H_2) There exists $h(t) \in L^1[1,+\infty)$ such that $0 < \int_1^{\infty} h(s)(ds/s) < +\infty$, for $t \in [1,+\infty), h(t) \neq 0$ and

$$\int_1^{\infty} h(s)\frac{ds}{s} - \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\alpha + \beta_i)} \int_1^{\eta} (\log \eta - \log s)^{\alpha+\beta_i-1} h(s) \frac{ds}{s} \neq 0. \tag{5}$$

(H_3) $f : [1,+\infty) \times R^3 \longrightarrow R$ satisfies the following conditions:

(i) For each $u \in R^3, f(t, u)$, $f(t, u)/t$ is Lebesgue measurable

(ii) For a.e. $t \in [1,+\infty)$, $f(t, u)$ is continuous

(iii) For each $r > 0$, there exists $\varphi_r(t)/t \in L^1[1,+\infty)$ such that $|f(t, u, v, w)| \leq \varphi_r(t)$ for all $|u|/(1 + (\log t)^{\alpha-1})$, $|v|/(1 + \log t)$, $|w| \in [0, r]$, a.e. $t \in [1,+\infty)$

2. Preliminaries and Lemmas

For convenience, we introduce some definitions and fundamental results of fractional calculus theory [22].

Definition 1. The Hadamard fractional integral of order $q > 0$ for a function $g : [1,+\infty) \longrightarrow R$ is defined as

$${}^{H}I^{q}g(t) = \frac{1}{\Gamma(q)} \int_1^t (\log t - \log s)^{q-1} g(s) \frac{ds}{s}, \tag{6}$$

provided that the integral exists and $\log(.) = \log_e(.)$.

Definition 2. The Hadamard fractional derivative of order $q > 0$ for a function $g : [1,+\infty) \longrightarrow R$ is given by

$${}^{H}D^{q}g(t) = \frac{1}{\Gamma(n-q)} \left(t\frac{d}{dt}\right)^n \int_1^t (\log t - \log s)^{n-q-1} g(s) \frac{ds}{s}, \tag{7}$$

where $n = [q] + 1$ and $[q]$ denotes the integral part of number q.

Lemma 3 (see [22], Property 2.24). *If $a, \alpha, \beta > 0$, then*

$$\left({}^{H}D_a^{\alpha}\left(\log \frac{t}{a}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} \left(\log \frac{x}{a}\right)^{\beta-\alpha-1}. \tag{8}$$

Lemma 4. *Let $q > 0$ and $u \in C[1,+\infty) \bigcap L^1[1,+\infty)$, and then, the Hadamard fractional differential equation ${}^{H}D^{q}u(t) = 0$ has the solution*

$$u(t) = \sum_{i=1}^{n} c_i (\log t)^{q-i}, \tag{9}$$

and the following formula holds:

$${}^{H}I^{q}{}^{H}D^{q}u(t) = u(t) + \sum_{i=1}^{n} c_i (\log t)^{q-i}, \tag{10}$$

where $c_i \in R, i = 1, 2, \cdots, n$ and $n - 1 < q < n$.

The following notations and a theorem can be found in [23].

Let X and Z be real Banach spaces, $L : \mathrm{dom}\,(L) \subset X \longrightarrow Z$ be a Fredholm operator with index zero, and $P : X \longrightarrow X, Q : Z \longrightarrow Z$ be projectors such that

$$\begin{aligned} \mathrm{Im}\, P &= \mathrm{Ker}\, L, \\ \mathrm{Ker}\, Q &= \mathrm{Im}\, L, \\ X &= \mathrm{Ker}\, L \oplus \mathrm{Ker}\, P, \\ Z &= \mathrm{Im}\, L \oplus \mathrm{Im}\, Q, \\ L|_{\mathrm{dom}\, L \cap \mathrm{Ker}\, P} &\longrightarrow \mathrm{Im}\, L \end{aligned} \tag{11}$$

is invertible, denoting its inverse by K_p.

Assume Ω is an open bounded subset of X, $\mathrm{dom}\, L \cap \Omega \neq \emptyset$. The map $N : X \longrightarrow Z$ is said to be L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_p(I - Q)N : \bar{\Omega} \longrightarrow X$ is compact.

Theorem 5 (see [23]). *Let $\Omega \subset X$ be open and bounded, let $L : \text{dom}\,(L) \subset X \longrightarrow Z$ be a Fredholm operator of index zero, and let $N : X \longrightarrow Z$ be L-compact on $\bar{\Omega}$. Assume that the following conditions are satisfied:*

(1) $Lu \neq \lambda Nu$ for every $(u,\lambda) \in [(\text{dom}\, L \setminus \text{Ker}\, L) \cap \partial\Omega] \times (0,1)$

(2) $Nu \notin \text{Im}\, L$ for every $u \in \text{Ker}\, L \cap \partial\Omega$

(3) $\deg\,(QN|_{\text{Ker}\, L}, \text{Ker}\, L \cap \Omega, 0) \neq 0$, where $Q : Z \longrightarrow Z$ is a projection such that $\text{Ker}\, Q = \text{Im}\, L$

Then, the equation $Lu = Nu$ has at least one solution in $\text{dom}\, L \cap \bar{\Omega}$.

Define the spaces

$$X = \left\{ u(t): u(t), {}^H D^{\alpha-2}u(t), {}^H D^{\alpha-1}u(t) \in C[1,+\infty), \left\| \frac{u}{1+(\log t)^{\alpha-1}} \right\|_\infty < +\infty, \left\| \frac{{}^H D^{\alpha-2}u}{1+\log t} \right\|_\infty < +\infty, \left\| {}^H D^{\alpha-1}u \right\|_\infty < +\infty \right\}, \tag{12}$$

with the norm

$$\|u\|_X = \max \left\{ \left\| \frac{u}{1+(\log t)^{\alpha-1}} \right\|_\infty, \left\| \frac{{}^H D^{\alpha-2}u}{1+\log t} \right\|_\infty, \left\| {}^H D^{\alpha-1}u \right\|_\infty \right\}, \tag{13}$$

where $\|u\|_\infty = \sup_{t\in[1,+\infty)} |u(t)|$.

Similar to the proof of Lemma 2.4 in [20], we can prove that $(X, \|\cdot\|_\infty)$ and $(X, \|\cdot\|_X)$ are Banach spaces.

Let $Z = \{y(t): y(t), (y(t)/t) \in L^1[1,+\infty)\}$ with the norm

$$\|y\|_Z = \|y(t)\|_1 + \left\| \frac{y(t)}{t} \right\|_1 = \int_1^\infty |y(t)|dt + \int_1^\infty \frac{|y(t)|}{t} dt. \tag{14}$$

Then, it is easy for us to prove that $(Z, \|\bullet\|_Z)$ is a Banach space.

We define the operator $L : \text{dom}\, L \subset X \longrightarrow Z$ by

$$Lu = {}^H D^\alpha u, u \in \text{dom}\, L, \tag{15}$$

where

$$\text{dom}\, L = \left\{ u \in X : u(1) = 0, {}^H D^{\alpha-2}u(1) = 0, {}^H D^{\alpha-1}u(+\infty) = \sum_{i=1}^m \lambda_i {}^H I^{\beta_i} u(\eta) \right\}, \tag{16}$$

and we define $N : X \longrightarrow Z$ by

$$Nu(t) = f\left(t, u(t), {}^H D^{\alpha-2}u(t), {}^H D^{\alpha-1}u(t)\right). \tag{17}$$

Then, BVP (1) can be written as $Lu = Nu$.

Next, similar to the compactness criterion in [15, 20], we establish the following criterion, and it can be proved in a similar way.

Lemma 6. *Assume V is bounded in X, and then, V is relatively compact in X if the following conditions hold:*

(i) *For any $u(t) \in V$, $u(t)/(1+(\log t)^{\alpha-1})$, ${}^H D^{\alpha-2}u(t)/(1+(\log t))$, and ${}^H D^{\alpha-1}u(t)$ are equicontinuous on any compact interval of $[1,+\infty)$*

(ii) *For any $\varepsilon > 0$, there exists a constant $T = T(\varepsilon) > 1$ such that*

$$\begin{aligned} &\left| \frac{u(t_1)}{1+(\log t_1)^{\alpha-1}} - \frac{u(t_2)}{1+(\log t_2)^{\alpha-1}} \right| < \varepsilon, \\ &\left| \frac{{}^H D^{\alpha-2}u(t_1)}{1+\log t_1} - \frac{{}^H D^{\alpha-2}u(t_2)}{1+\log t_2} \right| < \varepsilon, \\ &\left| {}^H D^{\alpha-1}u(t_1) - {}^H D^{\alpha-1}u(t_2) \right| < \varepsilon. \end{aligned} \tag{18}$$

for any $u(t) \in V$ and $t_1, t_2 \geq T$.

3. The Main Results

For convenience, for $y \in Z$, we define the operator A by

$$Ay(t) = \int_1^\infty y(s) \frac{ds}{s} - \sum_{i=1}^m \frac{\lambda_i}{\Gamma(\alpha+\beta_i)} \int_1^\eta (\log \eta - \log s)^{\alpha+\beta_i-1} y(s) \frac{ds}{s}. \tag{19}$$

Lemma 7. *If (H_1) holds, then*

(1) $\text{Ker}\, L = \{u \in \text{dom}\, L : u(t) = c(\log t)^{\alpha-1}, c \in R\}$

(2) $\text{Im}\, L = \{y \in Z : Ay(t) = 0\}$

Proof. We can easily obtain that

$$\text{Ker}\, L = \left\{ u \in \text{dom}\, L : u(t) = c(\log t)^{\alpha-1}, c \in \mathrm{R} \right\}. \tag{20}$$

Now, we show that $\text{Im}\, L = \{y \in Z : Ay(t) = 0\}$.

Let

$${}^H D^\alpha u(t) = y(t) \tag{21}$$

have a solution $u(t)$ that satisfies the conditions in BVP (1). From Lemma 4, we have

$$\begin{aligned} u(t) &= {}^H I^\alpha y(t) + c_1(\log t)^{\alpha-1} + c_2(\log t)^{\alpha-2} + c_3(\log t)^{\alpha-3} \\ &= \frac{1}{\Gamma(\alpha)} \int_1^t (\log t - \log s)^{\alpha-1} y(s) \frac{ds}{s} + c_1(\log t)^{\alpha-1} \\ &\quad + c_2(\log t)^{\alpha-2} + c_3(\log t)^{\alpha-3}, \end{aligned} \tag{22}$$

where c_1, c_2, and $c_3 \in R$. By $u(1)=0$, we get $c_3=0$. By Lemma 3, we get

$$^{H}D^{\alpha-2}u(t) = {}^{H}I^{2}y(t) + c_1\Gamma(\alpha)\log t + c_2\Gamma(\alpha-1). \quad (23)$$

From $^{H}D^{\alpha-2}u(1)=0$, we obtain $c_2=0$. Hence,

$$\begin{aligned} u(t) &= {}^{H}I^{\alpha}y(t) + c_1(\log t)^{\alpha-1}, \\ {}^{H}D^{\alpha-1}u(t) &= {}^{H}Iy(t) + c_1\Gamma(\alpha) = \int_1^t y(s)\frac{ds}{s} + c_1\Gamma(\alpha). \end{aligned} \quad (24)$$

By $^{H}D^{\alpha-1}u(+\infty) = \sum_{i=1}^{m}\lambda_i {}^{H}I^{\beta_i}u(\eta)$ and (H_1), we conclude

$$\begin{aligned} &\int_1^{\infty} y(s)\frac{ds}{s} + c_1\Gamma(\alpha) \\ &= \sum_{i=1}^{m}\lambda_i\left({}^{H}I^{\alpha+\beta_i}y(\eta) + {}^{H}I^{\beta_i}c_1(\log\eta)^{\alpha-1}\right) \\ &= \sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha+\beta_i)}\int_1^{\eta}(\log\eta-\log s)^{\alpha+\beta_i-1}y(s)\frac{ds}{s} + \sum_{i=1}^{m}\frac{\lambda_i c_1}{\Gamma(\beta_i)} \\ &\quad\cdot\int_1^{\eta}(\log\eta-\log s)^{\beta_i-1}(\log s)^{\alpha-1}\frac{ds}{s} \\ &= \sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha+\beta_i)}\int_1^{\eta}(\log\eta-\log s)^{\alpha+\beta_i-1}y(s)\frac{ds}{s} \\ &\quad + \sum_{i=1}^{m}\frac{\lambda_i c_1(\log\eta)^{\beta_i-1}}{\Gamma(\beta_i)}\int_1^{\eta}\left(1-\frac{\log s}{\log\eta}\right)^{\beta_i-1}(\log s)^{\alpha-1}\frac{ds}{s} \\ &= \sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha+\beta_i)}\int_1^{\eta}(\log\eta-\log s)^{\alpha+\beta_i-1}y(s)\frac{ds}{s} \\ &\quad + \sum_{i=1}^{m}\frac{\lambda_i c_1(\log\eta)^{\alpha+\beta_i-1}}{\Gamma(\beta_i)}\int_0^{1}(1-w)^{\beta_i-1}w^{\alpha-1}dw \\ &= \sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha+\beta_i)}\int_1^{\eta}(\log\eta-\log s)^{\alpha+\beta_i-1}y(s)\frac{ds}{s} \\ &\quad + \sum_{i=1}^{m}\frac{\lambda_i c_1(\log\eta)^{\alpha+\beta_i-1}}{\Gamma(\beta_i)}B(\beta_i,\alpha) \\ &= \sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha+\beta_i)}\int_1^{\eta}(\log\eta-\log s)^{\alpha+\beta_i-1}y(s)\frac{ds}{s} \\ &\quad + \sum_{i=1}^{m}\frac{\lambda_i c_1\Gamma(\alpha)(\log\eta)^{\alpha+\beta_i-1}}{\Gamma(\alpha+\beta_i)} \\ &= \sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha+\beta_i)}\int_1^{\eta}(\log\eta-\log s)^{\alpha+\beta_i-1}y(s)\frac{ds}{s} + c_1\Gamma(\alpha), \end{aligned} \quad (25)$$

where $B(\beta_i,\alpha)$ is the beta-function. Thus,

$$\int_1^{\infty} y(s)\frac{ds}{s} - \sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha+\beta_i)}\int_1^{\eta}(\log\eta-\log s)^{\alpha+\beta_i-1}y(s)\frac{ds}{s} = 0. \quad (26)$$

On the other hand, if (26) holds, setting

$$u(t) = b(\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_1^t(\log t-\log s)^{\alpha-1}y(s)\frac{ds}{s}, \quad (27)$$

where b is an arbitrary constant, then $u(t)$ is solution of (21). □

Lemma 8. *If (H_1) and (H_2) hold, then $L:\mathrm{dom}\,L\subset X\longrightarrow Z$ is a Fredholm operator of index zero, and the linear continuous projectors $P:X\longrightarrow X$ and $Q:Y\longrightarrow Y$ can be defined as*

$$\begin{aligned} Pu(t) &= \frac{^{H}D^{\alpha-1}u(1)}{\Gamma(\alpha)}(\log t)^{\alpha-1}, \\ Qy(t) &= \frac{Ay}{Ah}h(t), \end{aligned} \quad (28)$$

respectively, and the linear operator $K_p:\mathrm{Im}\,L\longrightarrow\mathrm{dom}\,L\cap\mathrm{Ker}\,P$ can be given by

$$K_p y(t) = {}^{H}I^{\alpha}y(t). \quad (29)$$

Proof. We can easily get $P^2u=Pu$, $u\in X$, and $Q^2y=Qy$, $y\in Z$, which imply that P,Q are projectors. Clearly, $\mathrm{Im}\,P=\mathrm{Ker}\,L$, and $\mathrm{Im}\,L=\mathrm{Ker}\,Q$. For $y\in Z$, we have $y=(y-Qy)+Qy$. Thus, $Z=\mathrm{Ker}\,Q+\mathrm{Im}\,Q=\mathrm{Im}\,L+\mathrm{Im}\,Q$.

Let $y\in\mathrm{Im}\,L\cap\mathrm{Im}\,Q$. Since $y\in\mathrm{Im}\,Q$, then there exists $b\in R$ such that $y(t)=bh(t)$, $t\in[1,+\infty)$. Since $y\in\mathrm{Im}\,L$, we get

$$Ay(t) = b\left(\int_1^{\infty}h(s)\frac{ds}{s} - \sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha+\beta_i)}\int_1^{\eta}(\log\eta-\log s)^{\alpha+\beta_i-1}h(s)\frac{ds}{s}\right) = 0. \quad (30)$$

From (H_2), we obtain $b=0$ and $y(t)=0$. So $\mathrm{Im}\,L\cap\mathrm{Im}\,Q=\{0\}$ and $Z=\mathrm{Im}\,L\oplus\mathrm{Im}\,Q$. Obviously, $\dim\mathrm{Ker}L=\dim\mathrm{Im}\,Q=1$; this means that L is a Fredholm operator of index zero.

Similarly, we can get $X=\mathrm{Ker}\,L\oplus\mathrm{Ker}\,P$.

For $y\in\mathrm{Im}\,L$, we have

$$(LK_p)y(t) = {}^{H}D^{\alpha H}I^{\alpha}y(t) = y(t). \quad (31)$$

And for $u\in\mathrm{dom}\,L\cap\mathrm{Ker}\,P$, by Lemma 4 and $K_pLu\in\mathrm{dom}\,L$, we get

$$(K_pL)u(t) = {}^{H}I^{\alpha H}D^{\alpha}u(t) = u(t) + c(\log t)^{\alpha-1}. \quad (32)$$

It follows from $u\in\mathrm{Ker}\,P$ that $^{H}D^{\alpha-1}u(1)=0$. This, together with $K_pLu\in\mathrm{Ker}\,P$, means that $c=0$. So, $(K_pL)u(t)=u(t)$. This shows that $K_p=(L|_{\mathrm{dom}\,L\cap\mathrm{Ker}\,P})^{-1}$. □

Lemma 9. *Suppose that (H_1)–(H_3) hold, then N is L-compact.*

Proof. Let $\Omega \subseteq X$ be bounded; i.e., there exists $r > 0$ such that $\|u\|_X \leq r$, $u \in \bar{\Omega}$. By (H$_3$), we get

$$\begin{aligned}
|Nu(t)| &= \left|f\left(t, u(t), {}^H D^{\alpha-2}u(t), {}^H D^{\alpha-1}u(t)\right)\right| \leq \varphi_r(t), t \in [1,+\infty), u \in \bar{\Omega},\\
|ANu| &= \left|\int_1^\infty Nu(s)\frac{ds}{s} - \sum_{i=1}^m \frac{\lambda_i}{\Gamma(\alpha+\beta_i)}\int_1^\eta (\log\eta - \log s)^{\alpha+\beta_i-1} Nu(s)\frac{ds}{s}\right|\\
&\leq \int_1^\infty |\varphi_r(s)|\frac{ds}{s} + \left|\sum_{i=1}^m \frac{\lambda_i(\log\eta)^{\alpha+\beta_i-1}}{\Gamma(\alpha+\beta_i)}\int_1^\eta \left(1-\frac{\log s}{\log\eta}\right)^{\alpha+\beta_i-1} Nu(s)\frac{ds}{s}\right|\\
&\leq \int_1^\infty \left|\frac{\varphi_r(s)}{s}\right| ds + \sum_{i=1}^m \frac{\lambda_i(\log\eta)^{\alpha+\beta_i-1}}{\Gamma(\alpha+\beta_i)}\int_1^\eta |Nu(s)|\frac{ds}{s}\\
&\leq \int_1^\infty |\varphi_r(s)|\frac{ds}{s} + \int_1^\infty |\varphi_r(s)|\frac{ds}{s} = 2\left\|\frac{\varphi_r}{s}\right\|_1 < +\infty,\\
\|QNu\|_Z &= \left|\frac{ANu}{Ah}\right|\|h(t)\|_Z \leq 2\left\|\frac{\varphi_r}{s}\right\|_1 \frac{\|h\|_Z}{|Ah|} < +\infty.
\end{aligned} \tag{33}$$

Hence, $QN(\bar{\Omega})$ is bounded. Next, we will prove that $K_p(I-Q)N : \bar{\Omega} \longrightarrow X$ is compact.

Firstly, for $u \in \bar{\Omega}$,

$$\begin{aligned}
\left|\frac{K_p(I-Q)Nu(t)}{1+(\log t)^{\alpha-1}}\right| &= \left|\frac{1}{\Gamma(\alpha)}\int_1^t \frac{(\log t - \log s)^{\alpha-1}}{1+(\log t)^{\alpha-1}}(Nu(s) - QNu(s))\frac{ds}{s}\right|\\
&\leq \frac{1}{\Gamma(\alpha)}\left(\left\|\frac{\varphi_r}{s}\right\|_1 + 2\left\|\frac{\varphi_r}{s}\right\|_1 \frac{1}{|Ah|}\left\|\frac{h}{s}\right\|_1\right),\\
\left|\frac{{}^H D^{\alpha-2}K_p(I-Q)Nu(t)}{1+\log t}\right| &= \left|\int_1^t \frac{\log t - \log s}{1+\log t}(Nu(s) - QNu(s))\frac{ds}{s}\right|\\
&\leq \left\|\frac{\varphi_r}{s}\right\|_1 + 2\left\|\frac{\varphi_r}{s}\right\|_1 \frac{1}{|Ah|}\left\|\frac{h}{s}\right\|_1,\\
\left|{}^H D^{\alpha-1}K_p(I-Q)Nu(t)\right| &= \left|\int_1^t (Nu(s) - QNu(s))\frac{ds}{s}\right|\\
&\leq \left\|\frac{\varphi_r}{s}\right\|_1 + 2\left\|\frac{\varphi_r}{s}\right\|_1 \frac{1}{|Ah|}\left\|\frac{h}{s}\right\|_1.
\end{aligned} \tag{34}$$

Thus, $K_p(I-Q)N(\bar{\Omega})$ is bounded.

Secondly, for $u \in \bar{\Omega}$ and any $1 < T_0 < +\infty, 1 \leq t_1 < t_2 \leq T_0$,

$$\begin{aligned}
&\left|\frac{K_p(I-Q)Nu(t_2)}{1+(\log t_2)^{\alpha-1}} - \frac{K_p(I-Q)Nu(t_1)}{1+(\log t_1)^{\alpha-1}}\right|\\
&= \frac{1}{\Gamma(\alpha)}\left|\int_1^{t_2} \frac{(\log t_2 - \log s)^{\alpha-1}}{1+(\log t_2)^{\alpha-1}}(Nu(s) - QNu(s))\frac{ds}{s}\right.\\
&\quad \left. - \int_1^{t_1} \frac{(\log t_1 - \log s)^{\alpha-1}}{1+(\log t_1)^{\alpha-1}}(Nu(s) - QNu(s))\frac{ds}{s}\right|\\
&\leq \frac{1}{\Gamma(\alpha)}\int_1^{t_1}\left|\frac{(\log t_2 - \log s)^{\alpha-1}}{1+(\log t_2)^{\alpha-1}} - \frac{(\log t_1 - \log s)^{\alpha-1}}{1+(\log t_1)^{\alpha-1}}\right|\\
&\quad \cdot |Nu(s) - QNu(s)|\frac{ds}{s} + \frac{1}{\Gamma(\alpha)}\\
&\quad \cdot \left|\int_{t_1}^{t_2} \frac{(\log t_2 - \log s)^{\alpha-1}}{1+(\log t_2)^{\alpha-1}}(Nu(s) - QNu(s))\frac{ds}{s}\right|\\
&\leq \frac{1}{\Gamma(\alpha)}\int_1^{t_1}\left|\frac{(\log t_2 - \log s)^{\alpha-1}}{1+(\log t_2)^{\alpha-1}} - \frac{(\log t_1 - \log s)^{\alpha-1}}{1+(\log t_1)^{\alpha-1}}\right|\\
&\quad \cdot \left(\varphi_r(s) + 2\left\|\frac{\varphi_r}{s}\right\|_1 \frac{1}{|Ah|}|h(s)|\right)\frac{ds}{s} + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\\
&\quad \cdot \left(\varphi_r(s) + 2\left\|\frac{\varphi_r}{s}\right\|_1 \frac{1}{|Ah|}|h(s)|\right)\frac{ds}{s},
\end{aligned}$$

$$\begin{aligned}
&\left|\frac{{}^H D^{\alpha-2}K_p(I-Q)Nu(t_2)}{1+\log t_2} - \frac{{}^H D^{\alpha-2}K_p(I-Q)Nu(t_1)}{1+\log t_1}\right|\\
&= \left|\int_1^{t_2} \frac{\log t_2 - \log s}{1+\log t_2}(Nu(s) - QNu(s))\frac{ds}{s}\right.\\
&\quad \left. - \int_1^{t_1} \frac{\log t_1 - \log s}{1+\log t_1}(Nu(s) - QNu(s))\frac{ds}{s}\right|\\
&\leq \int_1^{t_1}\left|\frac{\log t_2 - \log s}{1+\log t_2} - \frac{\log t_1 - \log s}{1+\log t_1}\right| |Nu(s) - QNu(s)|\frac{ds}{s}\\
&\quad + \left|\int_{t_1}^{t_2} \frac{\log t_2 - \log s}{1+\log t_2}(Nu(s) - QNu(s))\frac{ds}{s}\right|\\
&\leq \int_1^{t_1}\left|\frac{\log t_2 - \log s}{1+\log t_2} - \frac{\log t_1 - \log s}{1+\log t_1}\right|\\
&\quad \cdot \left(\varphi_r(s) + 2\left\|\frac{\varphi_r}{s}\right\|_1 \frac{1}{|Ah|}|h(s)|\right)\frac{ds}{s}\\
&\quad + \int_{t_1}^{t_2}\left(\varphi_r(s) + 2\left\|\frac{\varphi_r}{s}\right\|_1 \frac{1}{|Ah|}|h(s)|\right)\frac{ds}{s},
\end{aligned}$$

$$\begin{aligned}
&\left|{}^H D^{\alpha-1}K_p(I-Q)Nu(t_2) - {}^H D^{\alpha-1}K_p(I-Q)Nu(t_1)\right|\\
&= \left|\int_1^{t_2}(Nu(s) - QNu(s))\frac{ds}{s} - \int_1^{t_1}(Nu(s) - QNu(s))\frac{ds}{s}\right|\\
&\leq \left|\int_{t_1}^{t_2}(Nu(s) - QNu(s))\frac{ds}{s}\right|\\
&\leq \int_{t_1}^{t_2}\left(\varphi_r(s) + 2\left\|\frac{\varphi_r}{s}\right\|_1 \frac{1}{|Ah|}|h(s)|\right)\frac{ds}{s}.
\end{aligned} \tag{35}$$

This shows that $(K_p(I-Q)Nu(t))/(1+(\log t)^{\alpha-1})$, $({}^H D^{\alpha-2}K_p(I-Q)Nu(t))/(1+\log t)$, and ${}^H D^{\alpha-1}K_p(I-Q)Nu(t)$ are equicontinuous on $[1, T_0]$.

Next, we prove that for any $u(t) \in \bar{\Omega}$, $(K_p(I-Q)Nu(t))/(1+(\log t)^{\alpha-1})$, $({}^H D^{\alpha-2}K_p(I-Q)Nu(t))/(1+\log t)$, and ${}^H D^{\alpha-1}K_p(I-Q)Nu(t)$ satisfy the condition (ii) of Lemma 6.

Since $\varphi_r(s) + 2\|\varphi_r/s\|_1 1/|Ah||h(s)|$ is Lebesgue integrable on $[1, +\infty)$, for any $\varepsilon > 0$, there exists a constant $M > 1$ such that for any $t_1, t_2 \geq M$,

$$\int_M^{+\infty}\left(\varphi_r(s)+2\left\|\frac{\varphi_r}{s}\right\|_1\frac{1}{|Ah|}|h(s)|\right)\frac{ds}{s} < \frac{\varepsilon}{4}, \int_{t_1}^{t_2}\left(\varphi_r(s)+2\left\|\frac{\varphi_r}{s}\right\|_1\frac{1}{|Ah|}|h(s)|\right)\frac{ds}{s} < \varepsilon. \tag{36}$$

In addition, due to $\lim_{t\longrightarrow+\infty}(((\log t-\log s)^{\alpha-1})/(1+(\log t)^{\alpha-1}))=1$ and $\lim_{t\longrightarrow+\infty}((\log t-\log s)/(1+\log t))=1$, there exists a constant $T>M>1$ so that for any $t_1,t_2\ge T$,

$$\begin{aligned}&\left|\frac{(\log t_2-\log s)^{\alpha-1}}{1+(\log t_2)^{\alpha-1}}-\frac{(\log t_1-\log s)^{\alpha-1}}{1+(\log t_1)^{\alpha-1}}\right|\\&\le\left|1-\frac{(\log t_2-\log s)^{\alpha-1}}{1+(\log t_2)^{\alpha-1}}\right|+\left|1-\frac{(\log t_1-\log s)^{\alpha-1}}{1+(\log t_1)^{\alpha-1}}\right|\\&<\frac{\Gamma(\alpha)\varepsilon}{2\int_1^M(\varphi_r(s)+2\|\varphi_r/s\|_1 1/|Ah||h(s)|)(ds/s)},\end{aligned}\tag{37}$$

$$\begin{aligned}&\left|\frac{\log t_2-\log s}{1+\log t_2}-\frac{\log t_1-\log s}{1+\log t_1}\right|\\&\le\left|1-\frac{\log t_2-\log s}{1+\log t_2}\right|+\left|1-\frac{\log t_1-\log s}{1+\log t_1}\right|\\&<\frac{\varepsilon}{2\int_1^M(\varphi_r(s)+2\|\varphi_r/s\|_1 1/|Ah||h(s)|)(ds/s)}.\end{aligned}\tag{38}$$

Then, for any $t_1,t_2\ge T$, by (36)–(38), we can get

$$\begin{aligned}&\left|\frac{K_p(I-Q)Nu(t_2)}{1+(\log t_2)^{\alpha-1}}-\frac{K_p(I-Q)Nu(t_1)}{1+(\log t_1)^{\alpha-1}}\right|\\&=\frac{1}{\Gamma(\alpha)}\left|\int_1^{t_2}\frac{(\log t_2-\log s)^{\alpha-1}}{1+(\log t_2)^{\alpha-1}}(Nu(s)-QNu(s))\frac{ds}{s}\right.\\&\quad\left.-\int_1^{t_1}\frac{(\log t_1-\log s)^{\alpha-1}}{1+(\log t_1)^{\alpha-1}}(Nu(s)-QNu(s))\frac{ds}{s}\right|\\&\le\frac{1}{\Gamma(\alpha)}\int_1^M\left|\frac{(\log t_2-\log s)^{\alpha-1}}{1+(\log t_2)^{\alpha-1}}-\frac{(\log t_1-\log s)^{\alpha-1}}{1+(\log t_1)^{\alpha-1}}\right|\\&\quad\cdot|Nu(s)-QNu(s)|\frac{ds}{s}+\frac{1}{\Gamma(\alpha)}\\&\quad\cdot\left|\int_M^{t_2}\frac{(\log t_2-\log s)^{\alpha-1}}{1+(\log t_2)^{\alpha-1}}(Nu(s)-QNu(s))\frac{ds}{s}\right|+\frac{1}{\Gamma(\alpha)}\\&\quad\cdot\left|\int_M^{t_1}\frac{(\log t_1-\log s)^{\alpha-1}}{1+(\log t_1)^{\alpha-1}}(Nu(s)-QNu(s))\frac{ds}{s}\right|\\&\le\frac{1}{\Gamma(\alpha)}\int_1^M\left|\frac{(\log t_2-\log s)^{\alpha-1}}{1+(\log t_2)^{\alpha-1}}-\frac{(\log t_1-\log s)^{\alpha-1}}{1+(\log t_1)^{\alpha-1}}\right|\\&\quad\cdot\left(\varphi_r(s)+2\left\|\frac{\varphi_r}{s}\right\|_1\frac{1}{|Ah|}|h(s)|\right)\frac{ds}{s}+\frac{2}{\Gamma(\alpha)}\int_M^{+\infty}\\&\quad\cdot\left(\varphi_r(s)+2\left\|\frac{\varphi_r}{s}\right\|_1\frac{1}{|Ah|}|h(s)|\right)\frac{ds}{s}<\varepsilon,\end{aligned}$$

$$\begin{aligned}&\left|\frac{{}^HD^{\alpha-2}K_p(I-Q)Nu(t_2)}{1+\log t_2}-\frac{{}^HD^{\alpha-2}K_p(I-Q)Nu(t_1)}{1+\log t_1}\right|\\&=\left|\int_1^{t_2}\frac{\log t_2-\log s}{1+\log t_2}(Nu(s)-QNu(s))\frac{ds}{s}\right.\\&\quad\left.-\int_1^{t_1}\frac{\log t_1-\log s}{1+\log t_1}(Nu(s)-QNu(s))\frac{ds}{s}\right|\\&\le\int_1^M\left|\frac{\log t_2-\log s}{1+\log t_2}-\frac{\log t_1-\log s}{1+\log t_1}\right||Nu(s)-QNu(s)|\frac{ds}{s}\\&\quad+\left|\int_M^{t_2}\frac{\log t_2-\log s}{1+\log t_2}(Nu(s)-QNu(s))\frac{ds}{s}\right|\\&\quad+\left|\int_M^{t_1}\frac{\log t_1-\log s}{1+\log t_1}(Nu(s)-QNu(s))\frac{ds}{s}\right|\\&\le\int_1^M\left|\frac{\log t_2-\log s}{1+\log t_2}-\frac{\log t_1-\log s}{1+\log t_1}\right|\\&\quad\cdot\left(\varphi_r(s)+2\left\|\frac{\varphi_r}{s}\right\|_1\frac{1}{|Ah|}|h(s)|\right)\frac{ds}{s}\\&\quad+2\int_M^{+\infty}\left(\varphi_r(s)+2\left\|\frac{\varphi_r}{s}\right\|_1\frac{1}{|Ah|}|h(s)|\right)\frac{ds}{s}<\varepsilon,\end{aligned}$$

$$\begin{aligned}&\left|{}^HD^{\alpha-1}K_p(I-Q)Nu(t_2)-{}^HD^{\alpha-1}K_p(I-Q)Nu(t_1)\right|\\&=\left|\int_1^{t_2}(Nu(s)-QNu(s))\frac{ds}{s}-\int_1^{t_1}(Nu(s)-QNu(s))\frac{ds}{s}\right|\\&\le\left|\int_{t_1}^{t_2}(Nu(s)-QNu(s))\frac{ds}{s}\right|\\&\le\int_{t_1}^{t_2}\left(\varphi_r(s)+2\left\|\frac{\varphi_r}{s}\right\|_1\frac{1}{|Ah|}|h(s)|\right)\frac{ds}{s}<\varepsilon.\end{aligned}\tag{39}$$

By Lemma 6, we obtain that $K_p(I-Q)N:\bar{\Omega}\longrightarrow X$ is compact. The proof is complete. □

To obtain the existence of the solution for BVP (1), we give the following conditions:

(H_4) There exists a constant $M_0>0$ such that $|{}^HD^{\alpha-1}u(t)|>M_0$, for $u\in\operatorname{dom}L\setminus\operatorname{Ker}L$, $t\in[1,+\infty)$, then $ANu\neq0$.

(H_5) There exist functions $a(t)/t$, $b(t)/t$, $c(t)/t$, and $d(t)/t\in L^1[1,+\infty)$ with $c_0+2b_0<1$ and $(2a_0/(1-c_0-2b_0))<\Gamma(\alpha)$ such that

$$|f(t,x,y,z)|\le a(t)\frac{|x(t)|}{1+(\log t)^{\alpha-1}}+b(t)\frac{|y(t)|}{1+\log t}+c(t)|z(t)|+d(t),\tag{40}$$

where $a_0=\|a(t)/t\|_1$, $b_0=\|b(t)/t\|_1$, $c_0=\|c(t)/t\|_1$, and $d_0=\|d(t)/t\|_1$.

(H_6) There exists a constant $k>0$ such that for any $u(t)=c(\log t)^{\alpha-1}$ with $|c|>k$, we have either $cAN(c(\log t)^{\alpha-1})>0$ or $cAN(c(\log t)^{\alpha-1})<0$.

Lemma 10. *Assume (H_1)-(H_5) hold, and then,*

$$\Omega_1 = \{u \in \operatorname{dom} L \setminus \operatorname{Ker} L : Lu = \lambda N u, \lambda \in (0,1)\} \tag{41}$$

is bounded.

Proof. Taking $u \in \Omega_1$, by $Lu = \lambda N u$, we have

$$u(t) = \frac{\lambda}{\Gamma(\alpha)} \int_1^t (\log t - \log s)^{\alpha-1} N u(s) \frac{ds}{s} + c(\log t)^{\alpha-1}. \tag{42}$$

So,

$${}^{H}D^{\alpha-2} u(t) = \lambda \int_1^t (\log t - \log s) N u(s) \frac{ds}{s} + c\Gamma(\alpha) \log t, \tag{43}$$

$${}^{H}D^{\alpha-1} u(t) = \lambda \int_1^t N u(s) \frac{ds}{s} + c\Gamma(\alpha). \tag{44}$$

From $Nu \in \operatorname{Im} L$ and Lemma 7, we get $ANu = 0$. Thus, by (H_4), there exists $t_0 \in [1, +\infty)$ such that

$$\left| {}^{H}D^{\alpha-1} u(t_0) \right| \le M_0. \tag{45}$$

By (44), we obtain

$${}^{H}D^{\alpha-1} u(t) = \lambda \int_{t_0}^t N u(s) \frac{ds}{s} + {}^{H}D^{\alpha-1} u(t_0). \tag{46}$$

It follows from (H_5) that

$$\begin{aligned} \left| {}^{H}D^{\alpha-1} u(t) \right| &\le \int_1^{+\infty} |Nu(s)| \frac{ds}{s} + M_0 \le a_0 \left\| \frac{u}{1+(\log t)^{\alpha-1}} \right\|_\infty \\ &\quad + b_0 \left\| \frac{{}^{H}D^{\alpha-2} u}{1+\log t} \right\|_\infty + c_0 \left\| {}^{H}D^{\alpha-1} u \right\|_\infty + d_0 + M_0. \end{aligned} \tag{47}$$

Hence,

$$\left\| {}^{H}D^{\alpha-1} u \right\|_\infty \le \frac{1}{1-c_0} \left(a_0 \left\| \frac{u}{1+(\log t)^{\alpha-1}} \right\|_\infty + b_0 \left\| \frac{{}^{H}D^{\alpha-2} u}{1+\log t} \right\|_\infty + d_0 + M_0 \right). \tag{48}$$

By (42)–(44), we have

$$\begin{aligned} u(t) &= \frac{\lambda}{\Gamma(\alpha)} \int_1^t (\log t - \log s)^{\alpha-1} N u(s) \frac{ds}{s} \\ &\quad + \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)} \left({}^{H}D^{\alpha-1} u(t_0) - \lambda \int_1^{t_0} N u(s) \frac{ds}{s} \right), \\ {}^{H}D^{\alpha-2} u(t) &= \lambda \int_1^t (\log t - \log s) N u(s) \frac{ds}{s} \\ &\quad + \log t \left({}^{H}D^{\alpha-1} u(t_0) - \lambda \int_1^{t_0} N u(s) \frac{ds}{s} \right). \end{aligned} \tag{49}$$

From (45) and (H_5), we get

$$\begin{aligned} \left| \frac{u(t)}{1+(\log t)^{\alpha-1}} \right| &\le \frac{1}{\Gamma(\alpha)} \left(2 \int_1^\infty |Nu(s)| \frac{ds}{s} + M_0 \right) \le \frac{2}{\Gamma(\alpha)} \\ &\quad \cdot \left(a_0 \left\| \frac{u}{1+(\log t)^{\alpha-1}} \right\|_\infty + b_0 \left\| \frac{{}^{H}D^{\alpha-2} u}{1+\log t} \right\|_\infty \right. \\ &\quad \left. + c_0 \left\| {}^{H}D^{\alpha-1} u \right\|_\infty + d_0 + \frac{M_0}{2} \right), \\ \left| \frac{{}^{H}D^{\alpha-2} u(t)}{1+\log t} \right| &\le 2 \int_1^\infty |Nu(s)| \frac{ds}{s} + M_0 \\ &\le 2 \left(a_0 \left\| \frac{u}{1+(\log t)^{\alpha-1}} \right\|_\infty + b_0 \left\| \frac{{}^{H}D^{\alpha-2} u}{1+\log t} \right\|_\infty \right. \\ &\quad \left. + c_0 \left\| {}^{H}D^{\alpha-1} u \right\|_\infty + d_0 + \frac{M_0}{2} \right). \end{aligned} \tag{50}$$

Thus,

$$\begin{aligned} \left\| \frac{u}{1+(\log t)^{\alpha-1}} \right\|_\infty &\le \frac{2}{\Gamma(\alpha) - 2a_0} \left(b_0 \left\| \frac{{}^{H}D^{\alpha-2} u}{1+\log t} \right\|_\infty \right. \\ &\quad \left. + c_0 \left\| {}^{H}D^{\alpha-1} u \right\|_\infty + d_0 + \frac{M_0}{2} \right), \end{aligned} \tag{51}$$

$$\begin{aligned} \left\| \frac{{}^{H}D^{\alpha-2} u}{1+\log t} \right\|_\infty &\le \frac{2}{1-2b_0} \left(a_0 \left\| \frac{u}{1+(\log t)^{\alpha-1}} \right\|_\infty \right. \\ &\quad \left. + c_0 \left\| {}^{H}D^{\alpha-1} u \right\|_\infty + d_0 + \frac{M_0}{2} \right). \end{aligned} \tag{52}$$

Substituting (48) into (51) and (52), we yield

$$\begin{aligned} \left\| \frac{u}{1+(\log t)^{\alpha-1}} \right\|_\infty &\le \frac{2}{\Gamma(\alpha)(1-c_0) - 2a_0} \\ &\quad \cdot \left(b_0 \left\| \frac{{}^{H}D^{\alpha-2} u}{1+\log t} \right\|_\infty + d_0 + \frac{M_0}{2}(1+c_0) \right), \\ \left\| \frac{{}^{H}D^{\alpha-2} u}{1+\log t} \right\|_\infty &\le \frac{2}{1-c_0-2b_0} \\ &\quad \cdot \left(a_0 \left\| \frac{u}{1+(\log t)^{\alpha-1}} \right\|_\infty + d_0 + \frac{M_0}{2}(1+c_0) \right). \end{aligned} \tag{53}$$

From (H_5), we have

$$\left\|\frac{u}{1+(\log t)^{\alpha-1}}\right\|_{\infty} \leq \frac{2d_0+M_0(1+c_0)}{\Gamma(\alpha)(1-c_0-2b_0)-2a_0}, \tag{54}$$

$$\left\|\frac{{}^{H}D^{\alpha-2}u}{1+\log t}\right\|_{\infty} \leq \frac{(2d_0+M_0(1+c_0))\Gamma(\alpha)}{\Gamma(\alpha)(1-c_0-2b_0)-2a_0}. \tag{55}$$

By (48), (54), and (55), we can deduce that $\|{}^{H}D^{\alpha-1}u\|_{\infty}$ is bounded. Therefore, Ω_1 is bounded. □

Lemma 11. *Assume (H_1)-(H_3) and (H_6) hold, and then,*

$$\Omega_2 = \{u \in \text{Ker } L : Nu \in \text{Im } L\} \tag{56}$$

is bounded.

Proof. For $u \in \Omega_2$, we have $u(t) = c(\log t)^{\alpha-1}$ and $ANu = 0$. From (H_6), we obtain that $|c| \leq k$. Thus,

$$\|u\|_X = \max\left\{\left\|\frac{c(\log t)^{\alpha-1}}{1+(\log t)^{\alpha-1}}\right\|_{\infty}, \left\|\frac{c\Gamma(\alpha)\log t}{1+\log t}\right\|_{\infty}, \|c\Gamma(\alpha)\|_{\infty}\right\} \tag{57}$$

is bounded, which means that Ω_2 is bounded. □

Lemma 12. *Assume (H_1)-(H_3) and (H_6) hold, and then,*

$$\Omega_3 = \{u \in \text{Ker } L : \lambda Ju + (1-\lambda)\theta QNu = 0, \lambda \in [0,1]\} \tag{58}$$

is bounded, where $J : \text{Ker } L \longrightarrow \text{Im } Q$ is a linear isomorphism given by

$$J\left(c(\log t)^{\alpha-1}\right) = \frac{c}{Ah}h(t), \forall c \in R, t \in [1,+\infty),$$
$$\theta = \begin{cases} 1, & \text{if } cAN\left(c(\log t)^{\alpha-1}\right) > 0, \\ -1, & \text{if } cAN\left(c(\log t)^{\alpha-1}\right) < 0. \end{cases} \tag{59}$$

Proof. For $u \in \Omega_3$, we have $u(t) = c(\log t)^{\alpha-1}$, and then, there exists $\lambda \in [0,1]$ such that

$$\lambda J\left(c(\log t)^{\alpha-1}\right) = -(1-\lambda)\theta QN\left(c(\log t)^{\alpha-1}\right), \tag{60}$$

that is,

$$\lambda c = -(1-\lambda)\theta AN\left(c(\log t)^{\alpha-1}\right). \tag{61}$$

If $\lambda = 0$, then $\text{AN}(c(\log t)^{\alpha-1}) = 0$, and by ($H_6$), we have $|c| \leq k$. If $\lambda = 1$, then $c = 0$. For $\lambda \in (0,1)$, if $|c| > k$, we have $\lambda c^2 = -(1-\lambda)\theta cAN(c(\log t)^{\alpha-1}) < 0$, which is a contradiction. Hence, Ω_3 is bounded. □

Theorem 13. *If (H_1)–(H_6) hold, then BVP (1) has at least one solution.*

Proof. Let Ω be a bounded open subset of X such that $\bigcup_{i=1}^{3}\bar{\Omega}_i \subset \Omega$. It follows from Lemma 9 that N is L-compact on $\bar{\Omega}$. From Lemmas 10 and 11, we obtain

(1) $Lu \neq \lambda Nu$ for every $(u,\lambda) \in [(\text{dom } L \setminus \text{Ker } L) \cap \partial\Omega] \times (0,1)$

(2) $Nu \notin \text{Im } L$ for every $u \in \text{Ker } L \cap \partial\Omega$

We need to verify the third condition.

(3) $\deg\left(QN|_{\text{Ker } L}, \text{Ker } L \cap \Omega, 0\right) \neq 0$

Define

$$H(u,\lambda) = \lambda Ju + (1-\lambda)\theta QNu. \tag{62}$$

According to Lemma 12, we know $H(u,\lambda) \neq 0$, for $u \in \partial\Omega \cap \text{Ker } L$. By the homotopy of degree, we have

$$\begin{aligned}\deg\left(QN|_{\text{Ker } L}, \Omega \cap \text{Ker } L, 0\right) &= \deg\left(\theta H(\cdot,0), \Omega \cap \text{Ker } L, 0\right) \\ &= \deg\left(\theta H(\cdot,1), \Omega \cap \text{Ker } L, 0\right) \\ &= \deg\left(\theta J, \Omega \cap \text{Ker } L, 0\right) \neq 0.\end{aligned} \tag{63}$$

We conclude from Theorem 5 that $Lu = Nu$ has has at least one solution in $\text{dom } L \cap \bar{\Omega}$; i.e., BVP (1) has at least one solution in X. The proof is complete. □

4. Example

Now, we provide an example to illustrate the usefulness of our result.

Example 1. Consider the following BVP:

$$\begin{cases} {}^{H}D^{5/2}u(t) = \dfrac{|u(t)|}{(\log t+6)^2\left(1+\log^{3/2}t\right)} + \dfrac{\left|{}^{H}D^{1/2}u(t)\right|}{(\log t+4)^2(1+\log t)} + \dfrac{\left|{}^{H}D^{3/2}u(t)\right|}{(\log t+5)^2} + \dfrac{|\sin t|}{(\log t+3)^2}, \\ u(1) = 0, {}^{H}D^{1/2}u(1) = 0, {}^{H}D^{3/2}u(+\infty) = {}^{H}I^{1/2}u(e) + \dfrac{3}{2}\,{}^{H}I^{3/2}u(e) + 6\,{}^{H}I^{5/2}u(e), \end{cases} \tag{64}$$

where $\alpha = 5/2$, $\lambda_1 = 1$, $\lambda_2 = 3/2$, $\lambda_3 = 6$, $\beta_1 = 1/2$, $\beta_2 = 3/2$, $\beta_3 = 5/2$, $\eta = e$, and

$$\begin{aligned} f\left(t, u(t), {}^{H}D^{1/2}u(t), {}^{H}D^{3/2}u(t)\right) &= \frac{|u(t)|}{(\log t+6)^2\left(1+\log^{3/2}t\right)} + \frac{\left|{}^{H}D^{1/2}u(t)\right|}{(\log t+4)^2(1+\log t)} \\ &\quad + \frac{\left|{}^{H}D^{3/2}u(t)\right|}{(\log t+5)^2} + \frac{|\sin t|}{(\log t+3)^2}. \end{aligned} \tag{65}$$

Then, $\sum_{i=1}^{m}(\lambda_i/(\Gamma(\alpha+\beta_i)))(\log \eta)^{\alpha+\beta_i-1} = 1$. So, the condition ($H_1$) holds.

Take $h(t)=1/t^2$, and then,

$$\begin{aligned}
&\int_1^\infty h(s)\frac{ds}{s}-\sum_{i=1}^m \frac{\lambda_i}{\Gamma(\alpha+\beta_i)}\int_1^\eta (\log\eta-\log s)^{\alpha+\beta_i-1}h(s)\frac{ds}{s}\\
&=\int_1^\infty \frac{1}{s^3}ds-\left(\frac{1}{\Gamma(3)}\int_1^e(1-\log s)^2\frac{1}{s^2}\frac{ds}{s}+\frac{3/2}{\Gamma(4)}\right.\\
&\left.\cdot\int_1^e(1-\log s)^3\frac{1}{s^2}\frac{ds}{s}+\frac{6}{\Gamma(5)}\int_1^e(1-\log s)^4\frac{1}{s^2}\frac{ds}{s}\right)\\
&=\frac{1}{2}-\left(\frac{1}{2}\int_0^1(1-v)^2e^{-2v}dv+\frac{1}{4}\int_0^1(1-v)^3e^{-2v}dv+\frac{1}{4}\right.\\
&\left.\cdot\int_0^1(1-v)^4e^{-2v}dv\right)=\frac{9+7e^{-2}}{32}\neq 0.
\end{aligned}\tag{66}$$

Thus, the condition (H_2) is satisfied.

Obviously, the condition (H_3) holds. The condition (H_4) can be estimated as follows:

$$\begin{aligned}
ANu&=\int_1^\infty Nu(s)\frac{ds}{s}-\sum_{i=1}^m \frac{\lambda_i}{\Gamma(\alpha+\beta_i)}\int_1^\eta(\log\eta-\log s)^{\alpha+\beta_i-1}\\
&\cdot Nu(s)\frac{ds}{s}=\int_1^\infty Nu(s)\frac{ds}{s}-\left(\frac{1}{\Gamma(3)}\int_1^e(1-\log s)^2\right.\\
&\cdot Nu(s)\frac{ds}{s}+\frac{3/2}{\Gamma(4)}\int_1^e(1-\log s)^3Nu(s)\frac{ds}{s}+\frac{6}{\Gamma(5)}\\
&\left.\cdot\int_1^e(1-\log s)^4Nu(s)\frac{ds}{s}\right)=\int_e^\infty Nu(s)\frac{ds}{s}\\
&+\int_1^e Nu(s)\frac{ds}{s}-\left(\frac{1}{2}\int_1^e(1-\log s)^2Nu(s)\frac{ds}{s}++\frac{1}{4}\right.\\
&\left.\cdot\int_1^e(1-\log s)^3Nu(s)\frac{ds}{s}+\frac{1}{4}\int_1^e(1-\log s)^4Nu(s)\frac{ds}{s}\right)\\
&\geq\int_e^\infty Nu(s)\frac{ds}{s}+\int_1^e\left(1-\frac{1}{2}(1-\log s)^2-\frac{1}{4}(1-\log s)^2\right.\\
&\left.-\frac{1}{4}(1-\log s)^2\right)Nu(s)\frac{ds}{s}=\int_e^\infty Nu(s)\frac{ds}{s}\\
&+\int_1^e\left(1-(1-\log s)^2\right)Nu(s)\frac{ds}{s}\neq 0.
\end{aligned}\tag{67}$$

Let $a(t)=1/((\log t+6)^2)$, $b(t)=1/((\log t+4)^2)$, $c(t)=1/((\log t+5)^2)$, and $d(t)=1/((\log t+3)^2)$, and then, $a(t)/t$, $b(t)/t$, $c(t)/t$, $d(t)/t\in L^1[1,+\infty)$:

$$|f(t,x,y,z)|\leq a(t)\frac{|x(t)|}{1+\log^{3/2}t}+b(t)\frac{|y(t)|}{1+\log t}+c(t)|z(t)|+d(t).\tag{68}$$

$a_0=1/6$, $b_0=1/4$, $c_0=1/5$, and $d_0=1/3$, so $c_0+2b_0<1$, $2a_0/(1-c_0-2b_0)=10/9<\Gamma(5/2)=3/4\sqrt{\pi}\approx 1.3293$. Thus, the condition (H_5) holds.

Using the same method as calculating in (H_4), we can get $ANu>0$. So the condition (H_6) holds. Hence, by Theorem 13, BVP (64) has at least one solution.

Data Availability

No data was used in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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