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Research Article

Norms of Composition Operators from Weighted Harmonic Bloch Spaces into Weighted Harmonic Zygmund Spaces

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This article examines the norms of composition operators from the weighted harmonic Bloch space $\mathscr{B}_H^\lambda, (0 < \lambda < \infty)$ to the weighted harmonic Zygmund space $\mathscr{Z}_H^\beta, (0 < \beta < \infty)$. The critical norm is on the open unit disk. We first give necessary and sufficient conditions where the composition operator between $\mathscr{B}_H^\lambda$ and $\mathscr{Z}_H^\beta$ is bounded. Secondly, we will study the compactness case of the composition operator between $\mathscr{B}_H^\lambda$ and $\mathscr{Z}_H^\beta$. Finally, we will estimate the essential norms of the composition operator between $\mathscr{B}_H^\lambda$ and $\mathscr{Z}_H^\beta$.

1. Introduction

Operator theory for spaces of analytic functions has been described in various settings, and there is a rich volume of studies in the academic literature that focus on the operator theory of spaces related to analytic functions on the unit disk. These studies delve into diverse environments, and the references will be highlighted below.

In [1], the second author discusses the essential norms of Stević-Sharma operators from general Banach spaces into Zygmund-type spaces, and in [2], the authors characterize the bounded and compact Stević-Sharma operator from a general class X of Banach function spaces into Zygmund-type space. In [3], the authors show a new essential norm estimate of composition operators from weighted Bloch space into μ-Bloch spaces. Cowen and MacCluer in [4] investigated composition operators on spaces of analytic functions. In [5], the necessary and sufficient conditions for the compactness and boundedness of product operator from H^∞ to Zygmund spaces were characterized.

Yet, there is a noticeable lack of investigations that offer a comprehensive look into the harmonic setting. We would like to highlight several of these references below. Characterization composition operators on some Banach spaces of harmonic mappings were discussed in [6]. Colonna in [7] discussed the Bloch constant of bounded harmonic mappings. Lusky studied the weighted spaces of harmonic and holomorphic functions in [8] and then in [9] determined the isomorphism classes of weighted spaces of harmonic and analytic functions. Characterization of the harmonic Bloch space and the harmonic Besov spaces by an oscillation in [10]. Jordá and Zarco studied the weighted Banach spaces of harmonic functions and the isomorphisms on weighted Banach spaces of harmonic and holomorphic functions in [11, 12].

This paper is part of a series of works that address several different properties of composition operators between weighted Banach spaces of harmonic mappings. We discussed the boundedness, compactness, and the essential norm of composition operators from the space of bounded harmonic mappings $\mathscr{H}^\infty$ into the harmonic Zygmund space $\mathscr{Z}_H$ in [13]. Bakhit et al. in [14] discussed the boundedness, compactness, and the essential norm of composition operators from harmonic Lipschitz space into $\mathscr{Z}_H$.

A *harmonic mapping* is a complex-valued function f with simply connected domain Ω such that

$$\Delta f := 4\frac{\partial^2 f}{\partial\xi\partial\bar{\xi}} \equiv 0. \quad (1)$$

Here, let $Hol(\mathbb{U})$ be the space containing all analytic functions on the unit disk $\mathbb{U}$ and $\mathscr{H}(\mathbb{U})$ be the space of harmonic mappings, while $\mathbb{G}$ be a compact subset of the unit disk $\mathbb{U}$. Further, let $\mathscr{H}^{\infty}_{\mathscr{H}}(\mathbb{U})$ be the space of all bounded mappings $f \in \mathscr{H}(\mathbb{U})$ equipped with the norm

$$\|f\|_{\infty} = \sup_{\xi\in\mathbb{U}}|f(\xi)|. \quad (2)$$

The harmonic mapping f always can invariably be represented in the form $g+\bar{h}$, where both $g, h \in Hol(\mathbb{U})$. Up to an additive constant, this representation attains uniqueness. For the scope of our study, we will focus on harmonic mappings with the domain $\mathbb{U}$. Therefore,

$$f \in \mathscr{H}(\mathbb{U}) \Leftrightarrow f = g+\bar{h}, \forall g, h \in Hol(\mathbb{U}) \text{ where } h(0)=0. \quad (3)$$

See [15], as an excellent reference on the harmonic function theory.

The composition operator C_{ψ} induced by analytic self-maps $\psi : \mathbb{U} \longrightarrow \mathbb{U}$ (or conjugate analytic self-maps) can be expressed as

$$C_{\psi}f = f\circ\psi, \forall f \in \mathscr{H}(\mathbb{U}). \quad (4)$$

Surely, this operator preserves harmonicity (see [6]).

In this work, we begin with some preliminaries that we use to derive the main results. We continue our research in [13, 14] by focusing on the boundedness and the compactness of C_{ψ} from harmonic λ-Bloch space $\mathscr{B}^{\lambda}_{H}$ into the weighted harmonic Zygmund space $\mathscr{Z}^{\beta}_{H}$. We conclude by estimating the essential norm from $\mathscr{B}^{\lambda}_{H}$ into $\mathscr{Z}^{\beta}_{H}$.

Let Q and W be two normed linear spaces. Then, the linear operator $T : Q \longrightarrow W$ is bounded if there exists a positive constant C such that

$$\|Tf\|_{W} \le C\|f\|_{Q}, \forall f \in Q. \quad (5)$$

Further, the operator $T : Q \longrightarrow W$ is compact if every bounded set in Q whose closure is compact, while the essential norm $\|T\|_{e}$ of $T : Q \longrightarrow W$ is its distance from the compact operators in the operator norm. Then, the essential norm of $T : Q \longrightarrow W$ is given by $\|T\|_{e,Q\longrightarrow W} = \inf\{\|T-\mathscr{C}\|_{Q\longrightarrow W}\}$, where $\mathscr{C} : Q \longrightarrow W$ is a compact operator.

1.1. The Harmonic λ-Bloch Space $\mathscr{B}^{\lambda}_{H}$. For $\lambda \in (0,\infty)$, the harmonic λ-Bloch space $\mathscr{B}^{\lambda}_{H}$ contains all $f \in \mathscr{H}(\mathbb{U})$ which is defined such that

$$\beta^{\lambda}_{f} := \sup_{\xi\in\mathbb{U}}\left(1-|\xi|^2\right)^{\lambda}\left(\left|\frac{\partial f(\xi)}{\partial\xi}\right| + \left|\frac{\partial f(\xi)}{\partial\bar{\xi}}\right|\right) < \infty. \quad (6)$$

If $f \in \mathscr{B}^{\lambda}_{H}$ is represented as $f = g+\bar{h}$, with $g, h \in Hol(\mathbb{U})$, the harmonic λ-Bloch seminorm β^{λ}_{f} can be characterized as

$$\beta^{\lambda}_{f} = \sup_{\xi\in\mathbb{U}}\left(1-|\xi|^2\right)^{\lambda}\left(\left|g'(\xi)\right| + \left|h'(\xi)\right|\right) < \infty. \quad (7)$$

The quantity

$$\|f\|_{\mathscr{B}^{\lambda}_{H}} := |f(0)| + \beta^{\lambda}_{f} \quad (8)$$

gives a Banach space structure on $\mathscr{B}^{\lambda}_{H}$ (see [16]).

The little harmonic λ-Bloch space $\mathscr{B}^{\lambda}_{H,0}$ is considered as the subspace of $\mathscr{B}^{\lambda}_{H}$ consisting of $f \in \mathscr{H}(\mathbb{U})$ such that

$$\beta^{\lambda}_{f,0} = \lim_{|\xi|\longrightarrow 1}\left(1-|\xi|^2\right)^{\lambda}\left(\left|\frac{\partial f(\xi)}{\partial\xi}\right| + \left|\frac{\partial f(\xi)}{\partial\bar{\xi}}\right|\right) = 0. \quad (9)$$

1.2. The Weighted Harmonic Zygmund Space $\mathscr{Z}^{\beta}_{H}$. For $\beta \in (0,\infty)$, $\mathscr{Z}^{\beta}_{H}$ consists of all mappings $f \in \mathscr{H}(\mathbb{U})$ such that

$$\|f\|_{*,\beta} := \sup_{\xi\in\mathbb{U}}\left(1-|\xi|^2\right)^{\beta}\left(\left|\frac{\partial^2 f}{\partial\xi^2}(\xi)\right| + \left|\frac{\partial^2 f}{\partial\bar{\xi}^2}(\xi)\right|\right) < \infty. \quad (10)$$

Define

$$\|f\|_{\mathscr{Z}^{\beta}_{H}} := |f(0)| + \left|\frac{\partial f}{\partial\xi}(0)\right| + \left|\frac{\partial f}{\partial\bar{\xi}}(0)\right| + \|f\|_{*,\beta}. \quad (11)$$

Obviously, $\|.\|_{\mathscr{Z}^{\beta}_{H}}$ is a norm on $\mathscr{Z}^{\beta}_{H}$, and $\mathscr{Z}^{\beta}_{H}$ is a Banach space. For $\beta=1$, $\mathscr{Z}^{1}_{H}$ is with the harmonic Zygmund space $\mathscr{Z}_{H}$ (see [13]).

Remark 1. Let $f \in Hol(\mathbb{U})$; then, $\partial f/\partial\xi$ is simplified to f' and $\partial f/\partial\bar{\xi} = \partial^2 f/\partial\bar{\xi}^2 = 0$. Thus, for all $0<\beta<\infty$, the collection of all $f \in Hol(\mathbb{U})$ in $\mathscr{Z}^{\beta}_{H}$ is the classical β-Zygmund space $\mathscr{Z}^{\beta}$, and both norms are identical.

For $0<\lambda<\infty$, let $f \in \mathscr{H}(\mathbb{U})$ be represented as $f = g+\bar{h}$, with $g, h \in Hol(\mathbb{U})$. For given $n \in \mathbb{N}$, let us define

$$B^{\lambda,n}_{H}(f) = \left(1-|\xi|^2\right)^{\lambda+n-1}\left(\left|g^{(n)}(\xi)\right| + \left|h^{(n)}(\xi)\right|\right). \quad (12)$$

The following lemma will help to characterize the boundedness of $C_{\psi} : \mathscr{B}^{\lambda}_{H} \longrightarrow \mathscr{Z}^{\beta}_{H}$. Its proof is the proof of Theorem 19 in [16].

Lemma 2. *Given $n \ge 2$ and $0<\lambda<\infty$, let $f \in \mathscr{H}(\mathbb{U})$ be represented as $f = g+\bar{h}$, with $g, h \in Hol(\mathbb{U})$. Then,*

(1) $f \in \mathscr{B}^{\lambda}_{H}$ if and only if $\sup_{\xi\in\mathbb{U}} B^{\lambda,n}_{H}(f) < \infty$

(2) $f \in \mathscr{B}^{\lambda}_{H,0}$ if and only if $\lim_{|\xi|\longrightarrow 1} B^{\lambda,n}_{H}(f) = 0$

Let $x \in \mathbb{U}$ be a fixed point, and let $\alpha \in \{1,2,3\}$. For any $\xi \in \mathbb{U}$, we consider the test functions $F^{\lambda}_{x,\alpha}$ defined as

$$F^{\lambda}_{x,\alpha}(\xi) = \frac{\left(1-|x|^2\right)^{\alpha}}{(1-\bar{x}\xi)^{\alpha+\lambda-1}} + \frac{\left(1-|x|^2\right)^{\alpha}}{\left(1-x\bar{\xi}\right)^{\alpha+\lambda-1}}. \tag{13}$$

Moreover, it is evident that $\lim_{|x|\longrightarrow 1} F^{\lambda}_{x,\alpha} = 0$ uniformly on $\mathbb{G}$. Recall that the power series representation of $F^{\lambda}_{x,\alpha}$ is

$$\begin{aligned} F^{\lambda}_{x,\alpha}(\xi) &= \left(1-|x|^2\right)^{\alpha} \sum_{i=0}^{\infty} \frac{\Gamma(i+\alpha+\lambda-1)}{i!\Gamma(\alpha+\lambda-1)} \left\{(\bar{x}\xi)^j + \left(x\bar{\xi}\right)^j\right\} \\ &\approx \left(1-|x|^2\right)^{\alpha} \sum_{i=0}^{\infty} i^{\alpha+\lambda-1} \left\{(\bar{x}\xi)^j + \left(x\bar{\xi}\right)^j\right\}. \end{aligned} \tag{14}$$

For all $n \in \mathbb{N}$ and $\alpha \in \{1,2,3\}$, by direct calculations, we know that

$$\frac{\partial^n F^{\lambda}_{x,\alpha}(\xi)}{\partial \xi^n} = \frac{(\alpha+\lambda+n-2)!}{(\alpha+\lambda-2)!} \left[\frac{\bar{x}^n\left(1-|x|^2\right)^{\alpha}}{(1-\bar{x}\xi)^{\alpha+\lambda+n-1}}\right], \tag{15}$$

$$\frac{\partial^n F^{\lambda}_{x,\alpha}(\xi)}{\partial \bar{\xi}^n} = \frac{(\alpha+\lambda+n-2)!}{(\alpha+\lambda-2)!} \left[\frac{x^n\left(1-|x|^2\right)^{\alpha}}{\left(1-x\bar{\xi}\right)^{\alpha+\lambda+n-1}}\right]. \tag{16}$$

Then, we have

$$\begin{aligned} \frac{\partial^n F^{\lambda}_{x,\alpha}(x)}{\partial \xi^n} &= \frac{(\alpha+\lambda+n-2)!}{(\alpha+\lambda-2)!} \frac{\bar{x}^n}{\left(1-|x|^2\right)^{\lambda+n-1}}, \\ \frac{\partial^n F^{\lambda}_{x,\alpha}(x)}{\partial \bar{\xi}^n} &= \frac{(\alpha+\lambda+n-2)!}{(\alpha+\lambda-2)!} \frac{x^n}{\left(1-|x|^2\right)^{\lambda+n-1}}. \end{aligned} \tag{17}$$

As before, for all $\xi \in \mathbb{U}$,

$$\begin{aligned} \left|\frac{\partial}{\partial \xi} F^{\lambda}_{x,\alpha}(\xi)\right| &= (\alpha+\lambda-1)\left|\frac{\bar{x}\left(1-|x|^2\right)^{\alpha}}{(1-\bar{x}\xi)^{\alpha+\lambda}}\right| \leq \frac{2(\alpha+\lambda)2^{\lambda}}{(1-|\xi|)^{\lambda}}, \\ \left|\frac{\partial}{\partial \bar{\xi}} F^{\lambda}_{x,\alpha}(\xi)\right| &= (\alpha+\lambda-1)\left|\frac{b\left(1-|x|^2\right)^{\alpha}}{(1-\bar{x}\xi)^{\alpha+\lambda}}\right| \leq \frac{2(\alpha+\lambda)2^{\lambda}}{(1-|\xi|)^{\lambda}}. \end{aligned} \tag{18}$$

Then, we have

$$\left|\frac{\partial}{\partial \xi} F^{\lambda}_{x,\alpha}(\xi)\right| + \left|\frac{\partial}{\partial \bar{\xi}} F^{\lambda}_{x,\alpha}(\xi)\right| \leq \frac{(\alpha+\lambda)2^{\lambda+2}}{\left(1-|\xi|^2\right)^{\lambda}}. \tag{19}$$

Thus, it can be demonstrated that $F^{\lambda}_{x,\alpha} \in \mathscr{B}^{\lambda}_H$ and $\sup_{x\in\mathbb{U}} \|F^{\lambda}_{x,\alpha}\|_{\mathscr{B}^{\lambda}_H} \preceq 1$, for every $\alpha \in \mathbb{N}$.

Throughout this article, the notation $X \preceq Y$ means that $X \leq CY$, where $C > 0$ is a constant. Therefore, the notation $X \approx Y$ means that X and Y are equivalent, when $Y \preceq X \preceq Y$.

2. Boundedness

In this section, we work on the boundedness of the operator $C_{\psi} : \mathscr{B}^{\lambda}_H \longrightarrow \mathscr{Z}^{\beta}_H$.

Theorem 3. *Let $\psi : \mathbb{U} \longrightarrow \mathbb{U}$ and let $0 < \lambda, \beta < \infty$. Then, $C_{\psi} : \mathscr{B}^{\lambda}_H \longrightarrow \mathscr{Z}^{\beta}_H$ is bounded if and only if*

$$\sup_{i\in\mathbb{N}} \left\| i^{\lambda-1}\left(\psi^i + \bar{\psi}^i\right)\right\|_{\mathscr{Z}^{\beta}_H} < \infty. \tag{20}$$

Proof. Let the sequence $p_i(z) = i^{\lambda-1}(z^i + \bar{z}^i)$, for $z \in \mathbb{U}$ and $i \in \mathbb{N}_0$. The sequence $\{p_i\}$ is bounded in $\mathscr{B}^{\lambda}_H$ with supremum norms $\|p_i\|_{\mathscr{B}^{\lambda}_H} \preceq 1$ (the authors in Theorem 2.9 of [17] have demonstrated that $\|\psi^i + \bar{\psi}^i\|_{\mathscr{B}^{\lambda}_H} \preceq i^{1-\lambda}$). If $C_{\psi} : \mathscr{B}^{\lambda}_H \longrightarrow \mathscr{Z}^{\beta}_H$ is bounded, then for each $i \geq 0$ and $0 < \beta < \infty$, we have

$$\left\| i^{\lambda-1}\left(\psi^i + \bar{\psi}^i\right)\right\|_{\mathscr{Z}^{\beta}_H} = \left\|C_{\psi} p_i\right\|_{\mathscr{Z}^{\beta}_H} \preceq \left\|C_{\psi}\right\|_{*,\beta}. \tag{21}$$

Therefore,

$$\sup_{i\in\mathbb{N}} \left\| i^{\lambda-1}\left(\psi^i + \bar{\psi}^i\right)\right\|_{\mathscr{Z}^{\beta}_H} < \infty. \tag{22}$$

Conversely, suppose that (20) holds and set

$$L = \sup_{i\in\mathbb{N}} \left\| i^{\lambda-1}\left(\psi^i + \bar{\psi}^i\right)\right\|_{\mathscr{Z}^{\beta}_H} < \infty. \tag{23}$$

Since $C_{\psi} p_0 = 0 \in \mathscr{Z}^{\beta}_H$, we see that

$$\|0\|_{\mathscr{Z}^{\beta}_H} = \left\|C_{\psi} p_0\right\|_{\mathscr{Z}^{\beta}_H} \leq L. \tag{24}$$

For any $\xi \in \mathbb{U}$ and $f \in \mathscr{H}(\mathbb{U})$ represented as $f = g + \bar{h}$, with $g, h \in Hol(\mathbb{U})$, note that $|(C_{\psi} f)(0)| = |f(\psi(0))| \leq \|f\|_{\mathscr{B}^{\lambda}_H}$. Therefore, because $|\psi(0)| < 1$, we see that

$$\begin{aligned} \left|\frac{\partial (C_{\psi} f)}{\partial \xi}(0)\right| + \left|\frac{\partial (C_{\psi} f)}{\partial \bar{\xi}}(0)\right| &= \left|\frac{\partial f(\psi(0))}{\partial \xi}\psi'(0)\right| \\ &+ \left|\frac{\partial f(\psi(0))}{\partial \bar{\xi}}\bar{\psi}'(0)\right| = \left|h'(\psi(0))\psi'(0)\right| \\ &+ \left|g'(\psi(0))\bar{\psi}'(0)\right| \preceq \frac{|\psi'(0)|}{\left(1-|\psi(0)|^2\right)^{\lambda}} \|f\|_{\mathscr{B}^{\lambda}_H} < \infty. \end{aligned} \tag{25}$$

For any $\xi \in \mathbb{U}$ and $f \in \mathscr{H}(\mathbb{U})$, we note that

$$\left|\frac{\partial^2 (C_\psi f)}{\partial \xi^2}(\xi)\right| = \left|\frac{\partial^2 f(\psi(\xi))}{\partial \xi^2}\left[\psi'(\xi)\right]^2 + \frac{\partial f(\psi(\xi))}{\partial \xi}\psi''(\xi)\right| \leq \left|\psi'(\xi)\right|^2\left|\frac{\partial^2 f(\psi(\xi))}{\partial \xi^2}\right| + \left|\psi''(\xi)\right|\left|\frac{\partial f(\psi(\xi))}{\partial \xi}\right|, \tag{26}$$

$$\left|\frac{\partial^2 (C_\psi f)}{\partial \bar{\xi}^2}(\xi)\right| = \left|\frac{\partial^2 f(\psi(\xi))}{\partial \bar{\xi}^2}\left[\bar{\psi}'(\xi)\right]^2 + \frac{\partial f(\psi(\xi))}{\partial \bar{\xi}}\bar{\psi''}(\xi)\right| \leq \left|\psi'(\xi)\right|^2\left|\frac{\partial^2 f(\psi(\xi))}{\partial \bar{\xi}^2}\right| + \left|\psi''(\xi)\right|\left|\frac{\partial f(\psi(\xi))}{\partial \bar{\xi}}\right|. \tag{27}$$

Now, multiplying the above expressions (26) and (27) by $(1-|\xi|^2)^\beta$, we have

$$\begin{aligned}&\left(1-|\xi|^2\right)^\beta\left(\left|\frac{\partial^2 (C_\psi f)}{\partial \xi^2}(\xi)\right| + \left|\frac{\partial^2 (C_\psi f)}{\partial \bar{\xi}^2}(\xi)\right|\right)\\ &\leq \left(1-|\xi|^2\right)^\beta\left|\psi'(\xi)\right|^2\left(\left|\frac{\partial^2 f(\psi(\xi))}{\partial \xi^2}\right| + \left|\frac{\partial^2 f(\psi(\xi))}{\partial \bar{\xi}^2}\right|\right)\\ &\quad + \left(1-|\xi|^2\right)^\beta\left|\psi''(\xi)\right|\left(\left|\frac{\partial f(\psi(\xi))}{\partial \xi}\right| + \left|\frac{\partial f(\psi(\xi))}{\partial \bar{\xi}}\right|\right).\end{aligned} \tag{28}$$

By Lemma 2, we know that

$$\begin{aligned}B_H^{\lambda,2}(f) &= \left(1-|\psi(\xi)|^2\right)^{\lambda+1}\left(\left|h''(\psi(\xi))\right| + \left|g''(\psi(\xi))\right|\right) \preceq \|f\|_{\mathscr{B}_H^\lambda},\\ B_H^{\lambda,1}(f) &= \left(1-|\psi(\xi)|^2\right)^{\lambda}\left(\left|h'(\psi(\xi))\right| + \left|g'(\psi(\xi))\right|\right) \preceq \|f\|_{\mathscr{B}_H^\lambda}.\end{aligned} \tag{29}$$

Since $f \in \mathscr{H}(\mathbb{U})$ can be expressed as $f = g + \bar{h}$, with $g, h \in Hol(\mathbb{U})$, we obtain

$$\begin{aligned}&\left(1-|\xi|^2\right)^\beta\left(\left|\frac{\partial^2 (C_\psi f)}{\partial \xi^2}(\xi)\right| + \left|\frac{\partial^2 (C_\psi f)}{\partial \bar{\xi}^2}(\xi)\right|\right)\\ &\leq \left(1-|\xi|^2\right)^\beta\left|\psi'(\xi)\right|^2\left(\left|g''(\psi(\xi))\right| + \left|h''(\psi(\xi))\right|\right)\\ &\quad + \left(1-|\xi|^2\right)^\beta\left|\psi''(\xi)\right|\left(\left|g'(\psi(\xi)) + \right|h'(\psi(\xi))\right|\Big)\\ &\preceq \frac{\left(1-|\xi|^2\right)^\beta\left|\psi'(\xi)\right|^2}{\left(1-|\psi(\xi)|^2\right)^{\lambda+1}}B_H^{\lambda,2}(f) + \frac{\left(1-|\xi|^2\right)^\beta\left|\psi''(\xi)\right|}{\left(1-|\psi(\xi)|^2\right)^{\lambda}}B_H^{\lambda,1}(f)\\ &\preceq (L_1 + L_2)\|f\|_{\mathscr{B}_H^\lambda},\end{aligned} \tag{30}$$

where $L_1 = (1-|\xi|^2)^\beta|\psi'(\xi)|^2/(1-|\psi(\xi)|^2)^{\lambda+1}$ and $L_2 = (1-|\xi|^2)^\beta|\psi''(\xi)|/(1-|\psi(\xi)|^2)^\lambda$. To prove that $C_\psi : \mathscr{B}_H^\lambda \longrightarrow \mathscr{Z}_H^\beta$ is a bounded operator, it suffices to show that both quantities L_1 and L_2 are finite. For $\xi \in \mathbb{U}$ since $C_\psi p_1 = \psi + \bar{\psi}$, we have

$$\frac{\partial^2\left[C_\psi p_1(\xi)\right]}{\partial \xi^2} + \frac{\partial^2\left[C_\psi p_1(\xi)\right]}{\partial \bar{\xi}^2} = \psi''(\xi) + \bar{\psi''}(\xi). \tag{31}$$

Then,

$$\sup_{\xi\in\mathbb{U}}\left(1-|\xi|^2\right)^\beta\left|\psi''(\xi)\right| \leq \frac{1}{2}\left\|C_\psi p_1\right\|_{\mathscr{Z}_H^\beta} \leq \frac{L}{2}. \tag{32}$$

Moreover, we know that $C_\psi p_2 = \psi^2 + \bar{\psi}^2$,

$$\begin{aligned}\frac{\partial^2\left[C_\psi p_2(\xi)\right]}{\partial \xi^2} &= 2\left(\psi'(\xi)\right)^2 + 2\psi(\xi)\psi''(\xi),\\ \frac{\partial^2\left[C_\psi p_2(\xi)\right]}{\partial \bar{\xi}^2} &= 2\left(\bar{\psi}'(\xi)\right)^2 + 2\bar{\psi(\xi)}\bar{\psi''}(\xi).\end{aligned} \tag{33}$$

For $\xi \in \mathbb{U}$ since $|\psi(\xi)| \leq 1$, so we have

$$\left|\psi'(\xi)\right|^2 \leq \frac{1}{4}\left\{\left|\frac{\partial^2\left[C_\psi p_2(\xi)\right]}{\partial \xi^2}\right| + \left|\frac{\partial^2\left[C_\psi p_2(\xi)\right]}{\partial \bar{\xi}^2}\right|\right\} + \left|\psi''(\xi)\right|. \tag{34}$$

Thus,

$$\begin{aligned}&\sup_{\xi\in\mathbb{U}}\left(1-|\xi|^2\right)^\beta\left|\psi'(\xi)\right|^2 \leq \frac{1}{4}\sup_{\xi\in\mathbb{U}}\left(1-|\xi|^2\right)^\beta\\ &\quad\cdot\left(\left|\frac{\partial^2\left[C_\psi p_2(\xi)\right]}{\partial \xi^2}\right| + \left|\frac{\partial^2\left[C_\psi p_2(\xi)\right]}{\partial \bar{\xi}^2}\right|\right)\\ &\quad + \sup_{\xi\in\mathbb{U}}\left(1-|\xi|^2\right)^\beta\left|\psi''(\xi)\right| \leq \frac{1}{4}\left\|C_\psi p_2\right\|_{\mathscr{Z}_H^\beta}\\ &\quad + \frac{1}{2}\left\|C_\psi p_1\right\|_{\mathscr{Z}_H^\beta} \leq \frac{L}{2}.\end{aligned} \tag{35}$$

By the linearity of the test functions $F_{\psi(\xi),\alpha}^\lambda$ in (14), for $\alpha = 1, 2, 3$ and $\xi \in \mathbb{U}$, we have

$$\left\|C_\psi F_{\psi(\xi),\alpha}^\lambda\right\|_{\mathscr{Z}_H^\beta} \preceq \left(1-|\psi(\xi)|^2\right)^\alpha\sum_{i=0}^{\infty} i^\alpha|\psi(\xi)|^i\left\|C_\psi p_i\right\|_{\mathscr{Z}_H^\beta} \preceq L. \tag{36}$$

From (16), for $\alpha = 1, 2, 3$ and $\xi \in \mathbb{U}$, we obtain

$$\begin{aligned}\frac{\partial^2\left[C_\psi F^\lambda_{\psi(\xi),\alpha}(\xi)\right]}{\partial \xi^2} &= \frac{(\lambda+\alpha-1)(\lambda+\alpha)\left[\overline{\psi(\xi)}\psi'(\xi)\right]^2}{\left(1-|\psi(\xi)|^2\right)^{\lambda+1}} + \frac{(\lambda+\alpha-1)\left[\overline{\psi(\xi)}\psi''(\xi)\right]}{\left(1-|\psi(\xi)|^2\right)^{\lambda}},\\ \frac{\partial^2\left[C_\psi F^\lambda_{\psi(\xi),\alpha}(\xi)\right]}{\partial \bar{\xi}^2} &= \frac{(\lambda+\alpha-1)(\lambda+\alpha)\left[\psi(\xi)\overline{\psi'(\xi)}\right]^2}{\left(1-|\psi(\xi)|^2\right)^{\lambda+1}} + \frac{(\lambda+\alpha-1)\left[\psi(\xi)\overline{\psi''(\xi)}\right]}{\left(1-|\psi(\xi)|^2\right)^{\lambda}}.\end{aligned} \tag{37}$$

Next, for $\alpha = 1, 2, 3$, we let

$$\begin{aligned}Q_{\psi(\xi),\alpha} &= \frac{\partial^2\left[C_\psi F^\lambda_{\psi(\xi),\alpha}(\xi)\right]}{\partial \xi^2} + \frac{\partial^2\left[C_\psi F^\lambda_{\psi(\xi),\alpha}(\xi)\right]}{\partial \bar{\xi}^2}\\ &= \frac{(\lambda+\alpha-1)(\lambda+\alpha)\left[\overline{\psi(\xi)}\psi'(\xi)\right]^2}{\left(1-|\psi(\xi)|^2\right)^{\lambda+1}} + \frac{(\lambda+\alpha-1)\left[\overline{\psi(\xi)}\psi''(\xi)\right]}{\left(1-|\psi(\xi)|^2\right)^{\lambda}}\\ &\quad + \frac{(\lambda+\alpha-1)(\lambda+\alpha)\left[\psi(\xi)\overline{\psi'(\xi)}\right]^2}{\left(1-|\psi(\xi)|^2\right)^{\lambda+1}} + \frac{(\lambda+\alpha-1)\left[\psi(\xi)\overline{\psi''(\xi)}\right]}{\left(1-|\psi(\xi)|^2\right)^{\lambda}}.\end{aligned} \tag{38}$$

By equation (38), for $\alpha = 1, 2, 3$, we obtain

$$\frac{2\left[\overline{\psi(\xi)}\psi'(\xi)\right]^2}{\left(1-|\psi(\xi)|^2\right)^{\lambda+1}} + \frac{2\left[\psi(\xi)\overline{\psi'(\xi)}\right]^2}{\left(1-|\psi(\xi)|^2\right)^{\lambda+1}} = Q_{\psi(\xi),1} - 2Q_{\psi(\xi),2} + Q_{\psi(\xi),3}. \tag{39}$$

Moreover,

$$\frac{\left[\overline{\psi(\xi)}\psi''(\xi)\right]}{\left(1-|\psi(\xi)|^2\right)^{\lambda}} + \frac{\left[\psi(\xi)\overline{\psi''(\xi)}\right]}{\left(1-|\psi(\xi)|^2\right)^{\lambda}} = -(\lambda+2)Q_{\psi(\xi),1} + (2\lambda+3)Q_{\psi(\xi),2} - (\lambda+1)Q_{\psi(\xi),3}. \tag{40}$$

Thus, from (39), we obtain

$$\begin{aligned}\frac{\left(1-|\xi|^2\right)^\beta |\psi(\xi)|^2|\psi'(\xi)|^2}{\left(1-|\psi(\xi)|^2\right)^{\lambda+1}} &\le \frac{1}{4}\sup_{\xi\in\mathbb{U}}\left(1-|\xi|^2\right)^\beta \cdot \left(\left|Q_{\psi(\xi),1}\right| + 2\left|Q_{\psi(\xi),2}\right| + \left|Q_{\psi(\xi),3}\right|\right)\\ &\le \frac{1}{4}\left(\left\|C_\psi F^\lambda_{\psi(\xi),1}\right\|_{\mathscr{Z}^\beta_H} + 2\left\|C_\psi F^\lambda_{\psi(\xi),2}\right\|_{\mathscr{Z}^\beta_H} + \left\|C_\psi F^\lambda_{\psi(\xi),3}\right\|_{\mathscr{Z}^\beta_H}\right) \le L.\end{aligned} \tag{41}$$

Moreover, from (40), we obtain

$$\begin{aligned}\frac{\left(1-|\xi|^2\right)^\beta |\psi(\xi)||\psi''(\xi)|}{\left(1-|\psi(\xi)|^2\right)^{\lambda}} &\le \frac{1}{4}\sup_{\xi\in\mathbb{U}}\left(1-|\xi|^2\right)^\beta \left((\lambda+2)\left|Q_{\psi(\xi),1}\right| + (2\lambda+3)\left|Q_{\psi(\xi),2}\right| + (\lambda+1)\left|Q_{\psi(\xi),3}\right|\right)\\ &\le \frac{(\lambda+2)}{4}\left\|C_\psi F^\lambda_{\psi(\xi),1}\right\|_{\mathscr{Z}^\beta_H} + \frac{(2\lambda+3)}{4}\left\|C_\psi F^\lambda_{\psi(\xi),2}\right\|_{\mathscr{Z}^\beta_H} + \frac{(\lambda+1)}{4}\left\|C_\psi F^\lambda_{\psi(\xi),3}\right\|_{\mathscr{Z}^\beta_H}\\ &\le \left(\lambda + \frac{3}{2}\right)L.\end{aligned} \tag{42}$$

Now we let $0 < s < 1$; then, if $|\psi(\xi)| > s$ in (41), we have

$$L_1 = \frac{\left(1-|\xi|^2\right)^\beta |\psi'(\xi)|^2}{\left(1-|\psi(\xi)|^2\right)^{\lambda+1}} \le \frac{L}{s^2}. \tag{43}$$

Conversely, if we let $|\psi(\xi)| \le s$ in (35), we have

$$L_1 = \frac{\left(1-|\xi|^2\right)^\beta |\psi'(\xi)|^2}{\left(1-|\psi(\xi)|^2\right)^{\lambda+1}} \le \frac{3L}{8(1-s^2)}. \tag{44}$$

From (43) and (44), it follows that the quantity L_1 is finite.

Similarly, for L_2, we let $0 < s < 1$. Then, if $|\psi(\xi)| > s$ in (42), we have

$$L_2 = \frac{\left(1-|\xi|^2\right)^\beta |\psi''(\xi)|}{\left(1-|\psi(\xi)|^2\right)^{\lambda}} \le \left(\frac{\lambda}{s} + \frac{3}{2s}\right)L. \tag{45}$$

If we let $|\psi(\xi)| \le s$ in (32), we have

$$L_2 = \frac{\left(1-|\xi|^2\right)^{\beta}|\psi''(\xi)|}{\left(1-|\psi(\xi)|^2\right)^{\lambda}} \le \frac{L}{4(1-s^2)}. \quad (46)$$

Therefore, the quantity L_2 is finite, and the proof is complete. □

3. Compactness

In this section, we focus on discussing the compactness of the operator $C_\psi : \mathscr{B}_H^\lambda \longrightarrow \mathscr{Z}_H^\beta$. The proof of the following lemma is a slight modification of the proof of Proposition 3.11 in [4] (the case of Banach spaces of analytic functions).

Lemma 4. *Let $T : \mathscr{B}_H^\lambda \longrightarrow \mathscr{Z}_H^\beta$ be bounded operator; then, $T : \mathscr{B}_H^\lambda \longrightarrow \mathscr{Z}_H^\beta$ is compact if and only if $\|Tf_k\|_{\mathscr{Z}_H^\beta} \longrightarrow 0$ as $k \longrightarrow \infty$, for any bounded sequence $\{f_k\}_{k\in\mathbb{N}}$ in $\mathscr{B}_H^\lambda$ converges to zero uniformly on $\mathbb{G}$.*

The following theorem shows that the compactness of $C_\psi : \mathscr{B}_H^\lambda \longrightarrow \mathscr{Z}_H^\beta$ can be characterized in terms of the sequence $\|C_\psi p_i\|_{\mathscr{Z}_H^\beta}$, where $p_i(z) = z^i + \bar{z}^i$, for $z \in \mathbb{U}$ and when $i \in \mathbb{N}_0$.

Theorem 5. *Let $C_\psi : \mathscr{B}_H^\lambda \longrightarrow \mathscr{Z}_H^\beta$ be bounded operator, where $\psi : \mathbb{U} \longrightarrow \mathbb{U}$. Then, $C_\psi : \mathscr{B}_H^\lambda \longrightarrow \mathscr{Z}_H^\beta$ is compact if and only if*

$$\lim_{j\longrightarrow\infty} \left\| i^{\lambda-1}\left(\psi^i + \bar{\psi}^i\right) \right\|_{\mathscr{Z}_H^\beta} = 0. \quad (47)$$

Proof. First, we consider the sequence $p_i(z) = i^{\lambda-1}(z^i + \bar{z}^i)$, for $z \in \mathbb{U}$ and $i \in \mathbb{N}_0$. Since the sequence $\{p_i\}$ is bounded in $\mathscr{B}_H^\lambda$ and converges to zero uniformly on $\mathbb{G}$, if $C_\psi : \mathscr{B}_H^\lambda \longrightarrow \mathscr{Z}_H^\beta$ is compact, then $C_\psi : \mathscr{B}_H^\lambda \longrightarrow \mathscr{Z}_H^\beta$ is a bounded operator, and (47) holds by Lemma 4.

On the other hand, assume that $C_\psi : \mathscr{B}_H^\lambda \longrightarrow \mathscr{Z}_H^\beta$ is a bounded operator and

$$\lim_{i\longrightarrow\infty} \left\| i^{\lambda-1}\left(\psi^i + \bar{\psi}^i\right) \right\|_{\mathscr{Z}_H^\beta} = 0. \quad (48)$$

Now, we define a sequence $\{h_i\}$ in $\mathscr{B}_H^\lambda$ with $M = \sup_{j\in\mathbb{N}} \|h_j\|_{\mathscr{B}_H^\lambda} < \infty$, and $h_i \longrightarrow 0$ uniformly on $\mathbb{G}$, as $i \longrightarrow \infty$.

By Lemma 4, to prove that $C_\psi : \mathscr{B}_H^\lambda \longrightarrow \mathscr{Z}_H^\beta$ is compact, it is sufficient to show that

$$\lim_{i\longrightarrow\infty} \|C_\psi h_i\|_{\mathscr{Z}_H^\beta} = 0. \quad (49)$$

Next we suppose $\|i^{\lambda-1}(\psi^i + \bar{\psi}^i)\|_{\mathscr{Z}_H^\beta} \le L$ (L is an upper bound for $\|i^{\lambda-1}(\psi^i + \bar{\psi}^i)\|_{\mathscr{Z}_H^\beta}$). For $\varepsilon > 0$, then there is $N \in \mathbb{N}$ such that

$$\left\| i^{\lambda-1}\left(\psi^i + \bar{\psi}^i\right) \right\|_{\mathscr{Z}_H^\beta} = \|C_\psi p_i\|_{\mathscr{Z}_H^\beta} < \varepsilon, \quad \forall i \ge N. \quad (50)$$

For $\alpha = 1, 2, 3$ and $\xi \in \mathbb{U}$, let us use (14) the power series representation of the test function $F_{x,\alpha}^\lambda$; then, we have

$$\begin{aligned} \left\|C_\psi F_{\psi(\xi),1}^\lambda\right\|_{\mathscr{Z}_H^\beta} &\preceq \left(1-|\psi(\xi)|^2\right) \sum_{i=0}^{N-1} |\psi(\xi)|^i \|C_\psi p_i\|_{\mathscr{Z}_H^\beta} \Bigg] \\ &\quad + \left(1-|\psi(\xi)|^2\right) \sum_{i=N}^{\infty} |\psi(\xi)|^i \|C_\psi p_i\|_{\mathscr{Z}_H^\beta} \\ &< \left(1-|\psi(\xi)|^2\right) NL + \varepsilon. \end{aligned} \quad (51)$$

Moreover,

$$\begin{aligned} \left\|C_\psi F_{\psi(\xi),2}^\lambda\right\|_{\mathscr{Z}_H^\beta} &\preceq \left(1-|\psi(\xi)|^2\right)^2 \left[\left\{\sum_{i=1}^{N} + \sum_{i=N+1}^{\infty}\right\} i|\psi(\xi)|^{i-1} \left\|C_\psi p_{j-1}\right\|_{\mathscr{Z}_H^\beta}\right] \\ &< \left(1-|\psi(\xi)|^2\right)^2 \frac{N(N+1)}{2} L + \varepsilon, \\ \left\|C_\psi F_{\psi(\xi),3}^\lambda\right\|_{\mathscr{Z}_H^\beta} &\preceq \left(1-|\psi(\xi)|^2\right)^3 \left[\left\{\sum_{i=2}^{N+1} + \sum_{i=(N+2)}^{\infty}\right\} i(i-1)|\psi(\xi)|^{i-2} \left\|C_\psi p_{i-2}\right\|_{\mathscr{Z}_H^\beta}\right] \\ &< \left(1-|\psi(\xi)|^2\right)^3 \frac{N(N+1)(N+2)}{6} L + \varepsilon. \end{aligned} \quad (52)$$

Next, for any $\xi \in \mathbb{U}$, let $0 < s < 1$ be sufficiently close to 1 such that $|\psi(\xi)| > s$; thus,

$$\left\|C_\psi F_{\psi(\xi),\alpha}^\lambda\right\|_{\mathscr{Z}_H^\beta} < 2\varepsilon, \text{for } \alpha = 1, 2, 3. \quad (53)$$

For $\alpha = 1, 2, 3$, since ε is arbitrary, so it follows that

$$\lim_{|\psi(\xi)|\longrightarrow 1} \left\|C_\psi F_{\psi(\xi),\alpha}^\lambda\right\|_{\mathscr{Z}_H^\beta} = 0. \quad (54)$$

From (41), we know

$$\frac{\left(1-|\xi|^2\right)^{\beta}|\psi'(\xi)|^2}{\left(1-|\psi(\xi)|^2\right)^{\lambda+1}} \le \frac{1}{|\psi(\xi)|^2} \max_{1\le\alpha\le 3} \left\|C_\psi F_{\psi(\xi),\alpha}^\lambda\right\|_{\mathscr{Z}_H^\beta}. \quad (55)$$

Further, from (42), we know

$$\frac{\left(1-|\xi|^2\right)^{\beta}|\psi''(\xi)|}{\left(1-|\psi(\xi)|^2\right)^{\lambda}} \le \frac{(2\lambda+3)}{2|\psi(\xi)|} \max_{1\le\alpha\le 3} \left\|C_\psi F_{\psi(\xi),\alpha}^\lambda\right\|_{\mathscr{Z}_H^\beta}. \quad (56)$$

Using (55), we obtain

$$\lim_{|\psi(\xi)|\longrightarrow 1}\frac{\left(1-|\xi|^2\right)^{\beta}\left|\psi'(\xi)\right|^2}{\left(1-|\psi(\xi)|^2\right)^{\lambda+1}}=0. \tag{57}$$

Moreover, by (56), we obtain

$$\lim_{|\psi(\xi)|\longrightarrow 1}\frac{\left(1-|\xi|^2\right)^{\beta}\left|\psi''(\xi)\right|}{\left(1-|\psi(\xi)|^2\right)^{\lambda}}=0. \tag{58}$$

Thus, sufficiently close to 1 if $|\psi(\xi)|>s$, for any $0<s<1$. Then,

$$\begin{aligned}&\frac{\left(1-|\xi|^2\right)^{\beta}\left|\psi'(\xi)\right|^2}{\left(1-|\psi(\xi)|^2\right)^{\lambda+1}}<\varepsilon,\\&\frac{\left(1-|\xi|^2\right)^{\beta}\left|\psi''(\xi)\right|}{\left(1-|\psi(\xi)|^2\right)^{\lambda}}<\varepsilon.\end{aligned} \tag{59}$$

For any $m\in\mathbb{N}$ and $n\geq 2$, by using Lemma 2, if $f_k\in\mathscr{B}_H^{\lambda}$, then we have $B_H^{\lambda,n}(f_k)\preceq\|f_k\|_{\mathscr{B}_H^{\lambda}}$. From (32) and (35), we know

$$\begin{aligned}&\sup_{\xi\in\mathbb{U}}\left(1-|\xi|^2\right)^{\beta}\left(\left|\frac{\partial^2\left[C_\psi f_k(\xi)\right]}{\partial\xi^2}\right|+\left|\frac{\partial^2\left[C_\psi f_k(\xi)\right]}{\partial\bar{\xi}^2}\right|\right)\\&\quad\leq\sup_{\xi\in\mathbb{U}}\left(1-|\xi|^2\right)^{\beta}\left|\psi'(\xi)\right|^2\left(\left|\frac{\partial^2 f_k(\psi(\xi))}{\partial\xi^2}\right|+\left|\frac{\partial^2 f_k(\psi(\xi))}{\partial\bar{\xi}^2}\right|\right)\\&\qquad+\sup_{\xi\in\mathbb{U}}\left(1-|\xi|^2\right)^{\beta}\left|\psi''(\xi)\right|\left(\left|\frac{\partial f_k(\psi(\xi))}{\partial\xi}\right|+\left|\frac{\partial f_k(\psi(\xi))}{\partial\bar{\xi}}\right|\right)\\&\quad\preceq\|f_k\|_{\mathscr{B}_H^{\lambda}}\left(\frac{\left(1-|\xi|^2\right)^{\beta}\left|\psi'(\xi)\right|^2}{\left(1-|\psi(\xi)|^2\right)^{\lambda+1}}+\frac{\left(1-|\xi|^2\right)\left|\psi''(\xi)\right|}{\left(1-|\psi(\xi)|^2\right)^{\lambda}}\right)\\&\quad\preceq\varepsilon M.\end{aligned} \tag{60}$$

From (32) and (35) in the proof of Theorem 2, we know

$$\begin{aligned}&\sup_{\xi\in\mathbb{U}}\left(1-|\xi|^2\right)^{\beta}\left|\psi''(\xi)\right|\leq\frac{L}{2},\\&\sup_{\xi\in\mathbb{U}}\left(1-|\xi|^2\right)^{\beta}\left|\psi'(\xi)\right|^2\leq\frac{L}{2}.\end{aligned} \tag{61}$$

We know that from Cauchy's estimates, all the sequences $\{\partial f_k/\partial\xi\}$, $\{\partial f_k/\partial\bar{\xi}\}$, $\{\partial^2 f_k/\partial\xi^2\}$, and $\{\partial^2 f_k/\partial\bar{\xi}^2\}$ are convergent to zero on compact subsets $\mathbb{G}$ of the unit disk $\mathbb{U}$. Thus, using (61), for any $0<s<1$ if $|\psi(\xi)|\leq s$, we obtain

$$\begin{aligned}&\left(1-|\xi|^2\right)^{\beta}\left(\left|\frac{\partial^2\left[C_\psi f_k(\xi)\right]}{\partial\xi^2}\right|+\left|\frac{\partial^2\left[C_\psi f_k(\xi)\right]}{\partial\bar{\xi}^2}\right|\right)\\&\quad\leq\left(1-|\xi|^2\right)^{\beta}\left|\psi'(\xi)\right|^2\left(\left|\frac{\partial^2 f_k(\psi(\xi))}{\partial\xi^2}\right|+\left|\frac{\partial^2 f_k(\psi(\xi))}{\partial\bar{\xi}^2}\right|\right)\\&\qquad+\left(1-|\xi|^2\right)^{\beta}\left|\psi''(\xi)\right|\left(\left|\frac{\partial f_k(\psi(\xi))}{\partial\xi}\right|+\left|\frac{\partial f_k(\psi(\xi))}{\partial\bar{\xi}}\right|\right)\\&\quad\leq\frac{L}{2}\left(\left|\frac{\partial^2 f_k(\psi(\xi))}{\partial\xi^2}\right|+\left|\frac{\partial^2 f_k(\psi(\xi))}{\partial\bar{\xi}^2}\right|+\left|\frac{\partial f_k(\psi(\xi))}{\partial\xi}\right|+\left|\frac{\partial f_k(\psi(\xi))}{\partial\bar{\xi}}\right|\right),\end{aligned} \tag{62}$$

which implies that

$$\begin{aligned}&\lim_{k\longrightarrow\infty}\left(1-|\xi|^2\right)^{\beta}\left(\left|\frac{\partial^2\left[C_\psi f_k(\xi)\right]}{\partial\xi^2}\right|+\left|\frac{\partial^2\left[C_\psi f_k(\xi)\right]}{\partial\bar{\xi}^2}\right|\right)\\&\quad\leq\lim_{k\longrightarrow\infty}\left|\frac{\partial^2 f_k(\psi(\xi))}{\partial\xi^2}\right|+\lim_{k\longrightarrow\infty}\left|\frac{\partial^2 f_k(\psi(\xi))}{\partial\bar{\xi}^2}\right|\\&\qquad+\lim_{k\longrightarrow\infty}\left|\frac{\partial f_k(\psi(\xi))}{\partial\xi}\right|+\lim_{k\longrightarrow\infty}\left|\frac{\partial f_k(\psi(\xi))}{\partial\bar{\xi}}\right|=0.\end{aligned} \tag{63}$$

Therefore, $\lim_{k\longrightarrow\infty}|C_\psi f_k(0)|=0$ and $\lim_{k\longrightarrow\infty}|\partial[C_\psi f_k](0)/\partial\bar{\xi}|=0$. Thus, we obtain

$$\lim_{k\longrightarrow\infty}\left\|C_\psi f_k\right\|_{\mathscr{Z}_H^{\beta}}=0. \tag{64}$$

From Lemma 4, we verify that $C_\psi:\mathscr{B}_H^{\lambda}\longrightarrow\mathscr{Z}_H^{\beta}$ is compact. □

4. Essential Norm

In this section, our emphasis shifts to a comprehensive discussion regarding the essential norms of the operator $C_\psi:\mathscr{B}_H^{\lambda}\longrightarrow\mathscr{Z}_H^{\beta}$. First, we define

$$\begin{aligned}B_1&=\limsup_{|\psi(\xi)|\longrightarrow 1}\frac{\left(1-|\xi|^2\right)^{\beta}\left|\psi'(\xi)\right|^2}{\left(1-|\psi(\xi)|^2\right)^{\lambda+1}},\\B_2&=\limsup_{|\psi(\xi)|\longrightarrow 1}\frac{\left(1-|\xi|^2\right)^{\beta}\left|\psi''(\xi)\right|}{\left(1-|\psi(\xi)|^2\right)^{\lambda}}.\end{aligned} \tag{65}$$

Theorem 6. *For $\psi:\mathbb{U}\longrightarrow\mathbb{U}$, let $C_\psi:\mathscr{B}_H^{\lambda}\longrightarrow\mathscr{Z}_H^{\beta}$ be bounded operator. Then,*

$$\left\|C_\psi\right\|_{e,\mathscr{B}_H^{\lambda}\longrightarrow\mathscr{Z}_H^{\beta}}\approx\max_{1\leq\alpha\leq 3}\left\{\limsup_{|x|\longrightarrow 1}\left\|C_\psi F_{x,\alpha}^{\lambda}(\xi)\right\|_{\mathscr{Z}_H^{\beta}}\right\}\approx\max\{B_1,B_2\}. \tag{66}$$

Proof. First, for $\alpha = 1, 2, 3$ and $\xi \in \mathbb{U}$, by the test function (13), we will prove that

$$\max_{1\le\alpha\le3}\left\{\limsup_{|x|\longrightarrow1}\left\|C_\psi F^\lambda_{x,\alpha}\right\|_{\mathscr{Z}^\beta_H}\right\}\le\left\|C_\psi\right\|_{e,\mathscr{B}^\lambda_H\longrightarrow\mathscr{Z}^\beta_H}. \quad (67)$$

Fix $x \in \mathbb{U}$, since for all $1 \le \alpha \le 3$, $F^\lambda_{x,\alpha} \in \mathscr{B}^\lambda_H$ and $F^\lambda_{x,\alpha}$ converges uniformly to 0 on $\mathbb{G}$. Then, for a compact operator $\mathscr{T} : \mathscr{B}^\lambda_H \longrightarrow \mathscr{Z}^\beta_H$, we have

$$\lim_{|x|\longrightarrow1}\left\|\mathscr{T}F^\lambda_{x,\alpha}\right\|_{\mathscr{Z}^\beta_H}=0,\quad \forall\alpha=1,2,3. \quad (68)$$

Thus,

$$\begin{aligned}\left\|C_\psi-\mathscr{T}\right\|_{\mathscr{B}^\lambda_H\longrightarrow\mathscr{Z}^\beta_H}&\ge\limsup_{|x|\longrightarrow1}\left\|\left(C_\psi-\mathscr{T}\right)F^\lambda_{x,\alpha}\right\|_{\mathscr{Z}^\beta_H}\\&\ge\limsup_{|x|\longrightarrow1}\left\|C_\psi F^\lambda_{x,\alpha}\right\|_{\mathscr{Z}^\beta_H}-\limsup_{|x|\longrightarrow1}\left\|\mathscr{T}F^\lambda_{x,\alpha}\right\|_{\mathscr{Z}^\beta_H}.\end{aligned} \quad (69)$$

Hence, we obtain

$$\begin{aligned}\left\|C_\psi\right\|_{e,\mathscr{B}^\lambda_H\longrightarrow\mathscr{Z}^\beta_H}&=\inf_{\mathscr{T}}\left\|C_\psi-\mathscr{T}\right\|\\&\ge\max_{1\le\alpha\le3}\left\{\limsup_{|x|\longrightarrow1}\left\|C_\psi F^\lambda_{x,\alpha}\right\|_{\mathscr{Z}^\beta_H}\right\}.\end{aligned} \quad (70)$$

Next, to prove that $\|C_\psi\|_{e,\mathscr{B}^\lambda_H\longrightarrow\mathscr{Z}^\beta_H} \ge \max\{B_1, B_2\}$, we define the sequence $\{w_i\}$ such that $\lim_{i\longrightarrow\infty}|\psi(w_i)| = 1$, for $w \in \mathbb{U}$. We also define

$$\begin{aligned}G_i(\xi)&=F^\lambda_{\psi(w_i),1}(\xi)-\frac{2\lambda+3}{\lambda+2}F^\lambda_{\psi(w_i),2}(\xi)+\frac{\lambda+1}{\lambda+2}F^\lambda_{\psi(w_i),3}(\xi),\\K_i(\xi)&=F^\lambda_{\psi(w_i),1}(\xi)-2F^\lambda_{\psi(w_i),2}(\xi)+F^\lambda_{\psi(w_i),3}(\xi).\end{aligned} \quad (71)$$

For all $\xi \in \mathbb{U}$, it can be seen that $G_i, K_i \in \mathscr{B}^\lambda_H$ and

$$\lim_{|\psi(w_i)|\longrightarrow1}G_i=\lim_{|\psi(w_i)|\longrightarrow1}K_i=0, \quad (72)$$

uniformly on $\mathbb{G}$. Moreover, by simple calculation, we have

$$\begin{aligned}&G_i(\psi(w_i))=K_i(\psi(w_i))=0,\\&\left|\frac{\partial G_i(\psi(w_i))}{\partial\xi}\right|=\left|\frac{\partial G_i(\psi(w_i))}{\partial\bar\xi}\right|=\frac{1}{\lambda+2}\frac{|\psi(w_i)|}{\left(1-|\psi(w_i)|^2\right)^\lambda},\\&\frac{\partial^2G_i(\psi(w_i))}{\partial\xi^2}=\frac{\partial^2G_i(\psi(w_i))}{\partial\bar\xi^2}=0,\\&\frac{\partial K_i(\psi(w_i))}{\partial\xi}=\frac{\partial K_i(\psi(w_i))}{\partial\bar\xi}=0,\\&\left|\frac{\partial^2K_i(\psi(w_i))}{\partial\xi^2}\right|=\left|\frac{\partial^2K_i(\psi(w_i))}{\partial\bar\xi^2}\right|=\frac{2|\psi(w_i)|^2}{\left(1-|\psi(w_i)|^2\right)^{\lambda+1}}.\end{aligned} \quad (73)$$

Since $\mathscr{T} : \mathscr{B}^\lambda_H \longrightarrow \mathscr{Z}^\beta_H$ is a compact operator, by Lemma 4, we have

$$\begin{aligned}\left\|C_\psi-\mathscr{T}\right\|_{\mathscr{B}^\lambda_H\longrightarrow\mathscr{Z}^\beta_H}&\ge\limsup_{i\longrightarrow\infty}\left\|C_\psi G_i\right\|_{\mathscr{Z}^\beta_H}-\limsup_{i\longrightarrow\infty}\left\|\mathscr{T}G_i\right\|_{\mathscr{Z}^\beta_H}\\&\ge\limsup_{i\longrightarrow\infty}\left(1-|w_i|^2\right)^\beta\\&\quad\cdot\left\{\left|\frac{\partial^2(G_i(\psi(w_i)))}{\partial\xi^2}\right|+\left|\frac{\partial^2(G_i(\psi(w_i)))}{\partial\bar\xi^2}\right|\right\}\\&=\limsup_{i\longrightarrow\infty}\left(1-|w_i|^2\right)^\beta\left|\psi''(w_i)\right|\\&\quad\cdot\left\{\left|\frac{\partial(G_i)}{\partial\xi}(\psi(w_i))\right|+\left|\frac{\partial(G_i)}{\partial\bar\xi}(\psi(w_i))\right|\right\}\\&\ge\limsup_{i\longrightarrow\infty}\left(1-|w_i|^2\right)^\beta\frac{|\psi(w_i)|\left|\psi''(w_i)\right|}{\left(1-|\psi(w_i)|^2\right)^\lambda}.\end{aligned} \quad (74)$$

Similarly, we have

$$\begin{aligned}\left\|C_\psi-\mathscr{T}\right\|_{\mathscr{B}^\lambda_H\longrightarrow\mathscr{Z}^\beta_H}&\ge\limsup_{i\longrightarrow\infty}\left\|C_\psi K_i\right\|_{\mathscr{Z}^\beta_H}-\limsup_{i\longrightarrow\infty}\left\|\mathscr{T}K_i\right\|_{\mathscr{Z}^\beta_H}\\&\ge\limsup_{i\longrightarrow\infty}\left(1-|w_i|^2\right)^\beta\\&\quad\cdot\left\{\left|\frac{\partial^2(K_i(\psi(w_i)))}{\partial\xi^2}\right|+\left|\frac{\partial^2(K_i(\psi(w_i)))}{\partial\bar\xi^2}\right|\right\}\\&=\limsup_{i\longrightarrow\infty}\left(1-|w_i|^2\right)^\beta\left|\psi'(w_i)\right|^2\\&\quad\cdot\left\{\left|\frac{\partial^2(K_i)}{\partial\xi^2}(\psi(w_i))\right|+\left|\frac{\partial^2(K_i)}{\partial\bar\xi^2}(\psi(w_i))\right|\right\}\\&\ge\limsup_{i\longrightarrow\infty}\left(1-|w_i|^2\right)^\beta\frac{|\psi(w_i)|^2\left|\psi'(w_i)\right|^2}{\left(1-|\psi(w_i)|^2\right)^{\lambda+1}}.\end{aligned} \quad (75)$$

Thus,

$$\begin{aligned}\left\|C_\psi\right\|_{e,\mathscr{B}^\lambda_H\longrightarrow\mathscr{Z}^\beta_H}&=\inf_{\mathscr{T}}\left\|C_\psi-\mathscr{T}\right\|\ge\limsup_{i\longrightarrow\infty}\left(1-|w_i|^2\right)^\beta\frac{|\psi(w_i)|\left|\psi''(w_i)\right|}{\left(1-|\psi(w_i)|^2\right)^\lambda}\\&=\limsup_{|\psi(w)|\longrightarrow1}\left(1-|w|^2\right)^\beta\frac{|\psi(w)|\left|\psi''(w)\right|}{\left(1-|\psi(w)|^2\right)^\lambda}=B_2,\\\left\|C_\psi\right\|_{e,\mathscr{B}^\lambda_H\longrightarrow\mathscr{Z}^\beta_H}&=\inf_{\mathscr{T}}\left\|C_\psi-\mathscr{T}\right\|\ge\limsup_{i\longrightarrow\infty}\left(1-|w_i|^2\right)^\beta\frac{|\psi(w_i)|^2\left|\psi'(w_i)\right|^2}{\left(1-|\psi(w_i)|^2\right)^{\lambda+1}}\\&=\limsup_{|\psi(w)|\longrightarrow1}\left(1-|w|^2\right)^\beta\frac{|\psi(w)|^2\left|\psi'(w)\right|^2}{\left(1-|\psi(w)|^2\right)^{\lambda+1}}=B_1.\end{aligned} \quad (76)$$

Hence, we obtain

$$\left\|C_\psi\right\|_{e,\mathscr{B}^\lambda_H\longrightarrow\mathscr{Z}^\beta_H}=\inf_{\mathscr{T}}\left\|C_\psi-\mathscr{T}\right\|\ge\max\{B_1,B_2\}. \quad (77)$$

Secondly, we prove that

$$\|C_\psi\|_{e,\mathscr{B}_H^\lambda\longrightarrow\mathscr{Z}_H^\beta}\leq\max_{1\leq\alpha\leq3}\left\{\limsup_{|x|\longrightarrow1}\left\|C_\psi F_{x,\alpha}^\lambda\right\|_{\mathscr{Z}_H^\beta}\right\}. \tag{78}$$

Now, we consider the operator $\mathscr{T}_\gamma:\mathscr{H}(\mathbb{U})\longrightarrow\mathscr{H}(\mathbb{U})$, for any $0\leq\gamma<1$ such that

$$\left(\mathscr{T}_\gamma f\right)(w)=f_\gamma(w)=f(\gamma w),\quad f\in\mathscr{H}(\mathbb{U}). \tag{79}$$

Without a doubt, $f_\gamma\longrightarrow f$ uniformly on $\mathbb{G}$, as $\gamma\longrightarrow1$. Moreover, $\mathscr{T}_\gamma$ is compact on $\mathscr{B}_H^\lambda$ and $\|\mathscr{T}_\gamma\|_{\mathscr{B}_H^\lambda\longrightarrow\mathscr{B}_H^\lambda}\leq1$. For any sequence $\{\gamma_i\}\subset(0,1)$ such that $\gamma_i\longrightarrow1$ as $i\longrightarrow\infty$, we obtain

$$C_\psi\mathscr{T}_{\gamma_i}:\mathscr{B}_H^\lambda\longrightarrow\mathscr{Z}_H^\beta\text{ is compact},\forall i\in\mathbb{N}_0. \tag{80}$$

But the definition of the essential norm says

$$\|C_\psi\|_{e,\mathscr{B}_H^\lambda\longrightarrow\mathscr{Z}_H^\beta}\leq\lim_{i\longrightarrow\infty}\sup\left\|C_\psi-C_\psi\mathscr{T}_{\gamma_i}\right\|_{\mathscr{B}_H^\lambda\longrightarrow\mathscr{Z}_H^\beta}. \tag{81}$$

Thus, we only need to demonstrate that

$$\limsup_{i\longrightarrow\infty}\left\|C_\psi-C_\psi\mathscr{T}_{\gamma_i}\right\|_{\mathscr{B}_H^\lambda\longrightarrow\mathscr{Z}_H^\beta}\leq\max_{1\leq\alpha\leq3}\left\{\limsup_{|x|\longrightarrow1}\left\|C_\psi F_{x,\alpha}^\lambda\right\|_{\mathscr{Z}_H^\beta}\right\}. \tag{82}$$

Let $f\in\mathscr{B}_H^\lambda$ such that $\|f\|_{\mathscr{B}_H^\lambda}\leq1$; then,

$$\begin{aligned}\left\|\left(C_\psi-C_\psi\mathscr{T}_{\gamma_i}\right)f\right\|_{\mathscr{Z}_H^\beta}&=|f(\psi(0))-f(\gamma_i\psi(0))|+|\psi'(0)|\\&\cdot\left\{\left|\frac{\partial\left(f-f_{\gamma_i}\right)}{\partial\xi}(\psi(0))\right|+\left|\frac{\partial\left(f-f_{\gamma_i}\right)}{\partial\bar{\xi}}(\psi(0))\right|\right\}\\&+\sup_{\xi\in\mathbb{U}}\left(1-|\xi|^2\right)^\beta\\&\cdot\left\{\left|\frac{\partial^2\left[\left(f-f_{\gamma_i}\right)\circ\psi(\xi)\right]}{\partial\xi^2}\right|+\left|\frac{\partial^2\left[\left(f-f_{\gamma_i}\right)\circ\psi(\xi)\right]}{\partial\bar{\xi}^2}\right|\right\}.\end{aligned} \tag{83}$$

It is clear that

$$\begin{aligned}\lim_{i\longrightarrow\infty}|f(\psi(0))-f(\gamma_i\psi(0))|&=\lim_{i\longrightarrow\infty}\left|\frac{\partial\left(f-f_{\gamma_i}\right)}{\partial\xi}(\psi(0))\right||\psi'(0)|\\&=\lim_{i\longrightarrow\infty}\left|\frac{\partial\left(f-f_{\gamma_i}\right)}{\partial\bar{\xi}}(\psi(0))\right||\psi'(0)|=0.\end{aligned} \tag{84}$$

On the other hand, we consider

$$\begin{aligned}&\limsup_{i\longrightarrow\infty}\left(1-|\xi|^2\right)^\beta\left\{\left|\frac{\partial^2\left[\left(f-f_{\gamma_i}\right)\circ\psi(\xi)\right]}{\partial\xi^2}\right|+\left|\frac{\partial^2\left[\left(f-f_{\gamma_i}\right)\circ\psi(\xi)\right]}{\partial\bar{\xi}^2}\right|\right\}\\&\leq\limsup_{i\longrightarrow\infty}\sup_{|\psi(\xi)|\leq\gamma_N}\left(1-|\xi|^2\right)^\beta\\&\quad\times\left\{\left|\frac{\partial^2\left[\left(f-f_{\gamma_i}\right)\circ\psi(\xi)\right]}{\partial\xi^2}\right|+\left|\frac{\partial^2\left[\left(f-f_{\gamma_i}\right)\circ\psi(\xi)\right]}{\partial\bar{\xi}^2}\right|\right\}\\&\quad+\limsup_{i\longrightarrow\infty}\sup_{|\psi(\xi)|>\gamma_N}\left(1-|\xi|^2\right)^\beta\\&\quad\times\left\{\left|\frac{\partial^2\left[\left(f-f_{\gamma_i}\right)\circ\psi(\xi)\right]}{\partial\xi^2}\right|+\left|\frac{\partial^2\left[\left(f-f_{\gamma_i}\right)\circ\psi(\xi)\right]}{\partial\bar{\xi}^2}\right|\right\}\\&=I_{\psi,i}+J_{\psi,i}.\end{aligned} \tag{85}$$

Now, we let $N\in\mathbb{N}$ be large enough and $\gamma_i\geq1/2$, for all $i\geq N$. Then,

$$\begin{aligned}I_{\psi,i}&\leq\limsup_{i\longrightarrow\infty}\sup_{|\psi(\xi)|\leq\gamma_N}\left(1-|\xi|^2\right)^\beta|\psi''(\xi)|\\&\quad\cdot\left\{\left|\frac{\partial\left[\left(f-f_{\gamma_i}\right)(\psi(\xi))\right]}{\partial\xi}\right|+\left|\frac{\partial\left[\left(f-f_{\gamma_i}\right)(\psi(\xi))\right]}{\partial\bar{\xi}}\right|\right\}\\&\quad+\limsup_{i\longrightarrow\infty}\sup_{|\psi(\xi)|\leq\gamma_N}\left(1-|\xi|^2\right)^\beta|\psi'(\xi)|^2\\&\quad\cdot\left\{\left|\frac{\partial^2\left[\left(f-f_{\gamma_i}\right)(\psi(\xi))\right]}{\partial\xi^2}\right|+\left|\frac{\partial^2\left[\left(f-f_{\gamma_i}\right)(\psi(\xi))\right]}{\partial\bar{\xi}^2}\right|\right\}.\end{aligned} \tag{86}$$

Since $C_\psi:\mathscr{B}_H^\lambda\longrightarrow\mathscr{Z}_H^\beta$ is bounded, from Theorem 2, we see that

$$\begin{aligned}&\sup_{\xi\in\mathbb{U}}\left(1-|\xi|^2\right)^\beta|\psi''(\xi)|<\infty,\\&\sup_{\xi\in\mathbb{U}}\left(1-|\xi|^2\right)^\beta|\psi'(\xi)|^2<\infty.\end{aligned} \tag{87}$$

Moreover, all the limits,

$$\begin{aligned}\lim_{i\longrightarrow\infty}\gamma_i\frac{\partial f_{\gamma_i}}{\partial\xi}&=\frac{\partial f}{\partial\xi},\\\lim_{i\longrightarrow\infty}(\gamma_i)^2\frac{\partial^2 f_{\gamma_i}}{\partial\xi^2}&=\frac{\partial^2 f}{\partial\xi^2},\\\lim_{i\longrightarrow\infty}\gamma_i\frac{\partial f_{\gamma_i}}{\partial\bar{\xi}}&=\frac{\partial f}{\partial\bar{\xi}},\\\lim_{i\longrightarrow\infty}(\gamma_i)^2\frac{\partial^2 f_{\gamma_i}}{\partial\bar{\xi}^2}&=\frac{\partial^2 f}{\partial\bar{\xi}^2},\end{aligned} \tag{88}$$

are uniformly on $\mathbb{G}$. Then, we have

$$\limsup_{i\longrightarrow\infty}\sup_{|w|\le\gamma_N}\left\{\left|\frac{\partial f(w)}{\partial\xi}-\frac{\partial f_{\gamma_i}(w)}{\partial\xi}\right|+\left|\frac{\partial f(w)}{\partial\bar{\xi}}-\frac{\partial f_{\gamma_i}(w)}{\partial\bar{\xi}}\right|\right\}=0,$$
$$\limsup_{i\longrightarrow\infty}\sup_{|w|\le\gamma_N}\left\{\left|\frac{\partial^2 f(w)}{\partial\xi^2}-\frac{\partial^2 f_{\gamma_i}(w)}{\partial\xi^2}\right|+\left|\frac{\partial^2 f(w)}{\partial\bar{\xi}^2}-\frac{\partial^2 f_{\gamma_i}(w)}{\partial\bar{\xi}^2}\right|\right\}=0. \tag{89}$$

Hence, by the above equations, we have

$$I_{\psi,i}=0. \tag{90}$$

Next, assume $|\psi(\xi)|>\gamma_N$, and we have

$$\begin{aligned}J_{\psi,i}\le{}&\limsup_{i\longrightarrow\infty}\sup_{|\psi(\xi)|>\gamma_N}\left(1-|\xi|^2\right)^\beta|\psi''(\xi)|\\&\times\left\{\left|\frac{\partial\left[\left(f-f_{\gamma_i}\right)(\psi(\xi))\right]}{\partial\xi}\right|+\left|\frac{\partial\left[\left(f-f_{\gamma_i}\right)(\psi(\xi))\right]}{\partial\bar{\xi}}\right|\right\}\\&+\limsup_{i\longrightarrow\infty}\sup_{|\psi(\xi)|>\gamma_N}\left(1-|\xi|^2\right)^\beta|\psi'(\xi)|^2\\&\times\left\{\left|\frac{\partial^2\left[\left(f-f_{\gamma_i}\right)(\psi(\xi))\right]}{\partial\xi^2}\right|+\left|\frac{\partial^2\left[\left(f-f_{\gamma_i}\right)(\psi(\xi))\right]}{\partial\bar{\xi}^2}\right|\right\}\\\le{}&\limsup_{i\longrightarrow\infty}\sup_{|\psi(\xi)|>\gamma_N}\left(1-|\xi|^2\right)^\beta|\psi''(\xi)|\left\{\left|\frac{\partial f(\psi(\xi))}{\partial\xi}\right|+\left|\frac{\partial f(\psi(\xi))}{\partial\bar{\xi}}\right|\right\}\\&+\limsup_{i\longrightarrow\infty}\sup_{|\psi(\xi)|>\gamma_N}\left(1-|\xi|^2\right)^\beta|\psi''(\xi)|\gamma_i\\&\times\left\{\left|\frac{\partial f(\gamma_i\psi(\xi))}{\partial\xi}\right|+\left|\frac{\partial f(\gamma_i\psi(\xi))}{\partial\bar{\xi}}\right|\right\}\\&+\limsup_{i\longrightarrow\infty}\sup_{|\psi(\xi)|>\gamma_N}\left(1-|\xi|^2\right)^\beta|\psi'(\xi)|^2\\&\times\left\{\left|\frac{\partial^2 f(\psi(\xi))}{\partial\xi^2}\right|+\left|\frac{\partial^2 f(\psi(\xi))}{\partial\xi^2}\right|\right\}\\&+\limsup_{i\longrightarrow\infty}\sup_{|\psi(\xi)|>\gamma_N}\left(1-|\xi|^2\right)^\beta|\psi'(\xi)|^2(\gamma_i)^2\\&\times\left\{\left|\frac{\partial^2 f(\gamma_i\psi(\xi))}{\partial\bar{\xi}^2}\right|+\left|\frac{\partial^2 f(\gamma_i\psi(\xi))]}{\partial\bar{\xi}^2}\right|\right\}\\={}&R_1+R_2+R_3+R_4.\end{aligned} \tag{91}$$

To find estimates of the quantities R_1, R_2, R_3, and R_4, we define

$$\begin{aligned}G_x(\xi)&=F^\lambda_{x,1}(\xi)-\frac{2\lambda+3}{\lambda+2}F^\lambda_{x,2}(\xi)+\frac{\lambda+1}{\lambda+2}F^\lambda_{x,3}(\xi),\\K_x(\xi)&=F^\lambda_{x,1}(\xi)-2F^\lambda_{x,2}(\xi)+F^\lambda_{x,3}(\xi).\end{aligned} \tag{92}$$

Because $\|f\|_{\mathscr{B}^\lambda_H}\le 1$ and $\beta^{\lambda,n}_H(f)\le\|f\|_{\mathscr{B}^\lambda_H}$, for all $f\in\mathscr{B}^\lambda_H$ and $n\ge 2$ and by Lemma 2, we have

$$\begin{aligned}&\sup_{|\psi(\xi)|>\gamma_N}\left(1-|\xi|^2\right)^\beta|\psi''(\xi)|\left\{\left|\frac{\partial f(\psi(\xi))}{\partial\xi}\right|+\left|\frac{\partial f(\psi(\xi))}{\partial\bar{\xi}}\right|\right\},\\&\quad\le\frac{1}{\gamma_N}\|f\|_{\mathscr{B}^\lambda_H}\sup_{|\psi(\xi)|>\gamma_N}\left(1-|\xi|^2\right)^\beta|\psi''(\xi)|\frac{(2+\lambda)^{-1}|\psi(\xi)|}{\left(1-|\psi(\xi)|^2\right)^\lambda}\\&\quad\le\sup_{|x|>\gamma_N}\left\|C_\psi G_x\right\|_{\mathscr{Z}^\beta_H}\\&\quad\le\sup_{|x|>\gamma_N}\left\|C_\psi F^\lambda_{x,1}\right\|_{\mathscr{Z}^\beta_H}+\frac{2\lambda+3}{\lambda+2}\sup_{|x|>\gamma_N}\left\|C_\psi F^\lambda_{x,2}\right\|_{\mathscr{Z}^\beta_H}\\&\quad+\frac{\lambda+1}{\lambda+2}\sup_{|x|>\gamma_N}\left\|C_\psi F^\lambda_{x,3}\right\|_{\mathscr{Z}^\beta_H}.\end{aligned} \tag{93}$$

Consequently,

$$R_1\le\sum_{\alpha=1}^{3}\limsup_{|x|\longrightarrow 1}\left\|C_\psi F^\lambda_{x,\alpha}\right\|_{\mathscr{Z}^\beta_H}. \tag{94}$$

Similarly, we see that

$$R_2\le\sum_{\alpha=1}^{3}\limsup_{|x|\longrightarrow 1}\left\|C_\psi F^\lambda_{x,\alpha}\right\|_{\mathscr{Z}^\beta_H}. \tag{95}$$

By direct calculation, $\beta^{\lambda,2}_H(f)\le\|f\|_{\mathscr{B}^\lambda_H}$, for all $u\in\mathscr{B}^\lambda_H$. Because $\|f\|_{\mathscr{B}^\lambda_H}\le 1$,

$$\begin{aligned}&\sup_{|\psi(\xi)|>\gamma_N}\left(1-|\xi|^2\right)^\beta|\psi'(\xi)|^2\left\{\left|\frac{\partial^2 f(\psi(\xi))}{\partial\xi^2}\right|+\left|\frac{\partial^2 f(\psi(\xi))}{\partial\xi^2}\right|\right\}\\&\quad\le\|f\|_{\mathscr{B}^\lambda_H}\sup_{|\psi(\xi)|>\gamma_N}\left(1-|\xi|^2\right)^\beta|\psi'(\xi)|^2\frac{2|\psi(\xi)|^2}{3\left(1-|\psi(\xi)|^2\right)^{\lambda+1}}\\&\quad\le\sup_{|x|>\gamma_N}\left\|C_\psi K_x\right\|_{\mathscr{Z}^\beta_H}\\&\quad\le\sup_{|x|>\gamma_N}\left\|C_\psi F^\lambda_{x,1}\right\|_{\mathscr{Z}^\beta_H}+2\sup_{|x|>\gamma_N}\left\|C_\psi F^\lambda_{x,2}\right\|_{\mathscr{Z}^\beta_H}\\&\qquad+\sup_{|x|>\gamma_N}\left\|C_\psi F^\lambda_{x,3}\right\|_{\mathscr{Z}^\beta_H}.\end{aligned} \tag{96}$$

Thus, we obtain

$$R_3\le\sum_{\alpha=1}^{3}\limsup_{|x|\longrightarrow 1}\left\|C_\psi F^\lambda_{x,\alpha}\right\|_{\mathscr{Z}^\beta_H}. \tag{97}$$

Similarly, we see that

$$R_4 \preceq \sum_{\alpha=1}^{3} \limsup_{|x|\longrightarrow 1} \left\| C_\psi F_{x,\alpha}^{\lambda} \right\|_{\mathscr{Z}_H^\beta}. \tag{98}$$

By the inequalities (94)–(98), we obtain

$$J_{\psi,i} \preceq \max_{1\le\alpha\le3} \left\{ \limsup_{|x|\longrightarrow 1} \left\| C_\psi F_{x,\alpha}^{\lambda} \right\|_{\mathscr{Z}_H^\beta} \right\}. \tag{99}$$

Hence, by applying (90) and (99), we determine that

$$\limsup_{i\longrightarrow\infty} \left\| \left(C_\psi - C_\psi \mathscr{T}_{\gamma_i} \right\|_{\mathscr{B}_H^\lambda \longrightarrow \mathscr{Z}_H^\beta} \preceq \max_{1\le\alpha\le3} \left\{ \limsup_{|x|\longrightarrow 1} \left\| C_\psi F_{x,\alpha}^{\lambda} \right\|_{\mathscr{Z}_H^\beta} \right\}. \tag{100}$$

Finally, we prove that

$$\left\| C_\psi \right\|_{e,\mathscr{B}_H^\lambda \longrightarrow \mathscr{Z}_H^\beta} \preceq \max \left\{ B_1, B_2 \right\}. \tag{101}$$

Now, we only need to prove that

$$\limsup_{i\longrightarrow\infty} \left\| C_\psi - C_\psi \mathscr{T}_{\gamma_i} \right\|_{\mathscr{B}_H^\lambda \longrightarrow \mathscr{Z}_H^\beta} \preceq \max \left\{ B_1, B_2 \right\}. \tag{102}$$

From (93), we see that

$$R_1 \preceq \limsup_{|\psi(\xi)|\longrightarrow 1} \left(1 - |\xi|^2\right)^\beta \left|\psi''(\xi)\right| \frac{|\psi(\xi)|}{\left(1 - |\psi(\xi)|^2\right)^\lambda} = B_2. \tag{103}$$

Similarly,

$$R_2 \preceq B_2. \tag{104}$$

Further, for (96), we see that

$$R_3 \preceq \limsup_{|\psi(\xi)|\longrightarrow 1} \left(1 - |\xi|^2\right)^\beta \left|\psi'(\xi)\right|^2 \frac{2|\psi(\xi)|^2}{3\left(1 - |\psi(\xi)|^2\right)^{\lambda+1}} = B_1. \tag{105}$$

Similarly,

$$\limsup_{i\longrightarrow\infty} R_4 \preceq B_1. \tag{106}$$

Therefore, by the inequalities (103)–(106), we get

$$\left\| C_\psi \right\|_{e,\mathscr{B}_H^\lambda \longrightarrow \mathscr{Z}_H^\beta} \preceq \max \left\{ B_1, B_2 \right\}. \tag{107}$$

The proof now is complete. □

Theorem 7. *For $\psi : \mathbb{U} \longrightarrow \mathbb{U}$, let $C_\psi : \mathscr{B}_H^\lambda \longrightarrow \mathscr{Z}_H^\beta$ be bounded. Then,*

$$\left\| C_\psi \right\|_{e,\mathscr{B}_H^\lambda \longrightarrow \mathscr{Z}_H^\beta} \approx \limsup_{i\longrightarrow\infty} \left\| i^{\lambda-1} \left(\psi^i + \bar{\psi}^i \right) \right\|_{\mathscr{Z}_H^\beta}. \tag{108}$$

Proof. First, we prove that

$$\left\| C_\psi \right\|_{e,\mathscr{B}_H^\lambda \longrightarrow \mathscr{Z}_H^\beta} \succeq \limsup_{i\longrightarrow\infty} \left\| i^{\lambda-1} \left(\psi^i + \bar{\psi}^i \right) \right\|_{\mathscr{Z}_H^\beta}. \tag{109}$$

Recall that the sequence $p_i(z) = i^{\lambda-1}(z^i + \bar{z}^i)$, for $z \in \mathbb{U}$ and when $i \in \mathbb{N}_0$. Then, $\|p_i\|_{\mathscr{B}_H^\lambda} \approx 1$, and p_i converges uniformly to 0 on $\mathbb{G}$. Therefore, by Lemma 4, we see that

$$\lim_{i\longrightarrow\infty} \left\| \mathscr{T} p_i \right\|_{\mathscr{Z}_H^\beta} = 0. \tag{110}$$

Hence,

$$\left\| C_\psi - \mathscr{T} \right\|_{\mathscr{B}_H^\lambda \longrightarrow \mathscr{Z}_H^\beta} \succeq \limsup_{i\longrightarrow\infty} \left\| \left(C_\psi - \mathscr{T} \right) p_i \right\|_{\mathscr{Z}_H^\beta} \succeq \limsup_{i\longrightarrow\infty} \left\| C_\psi p_i \right\|_{\mathscr{Z}_H^\beta}. \tag{111}$$

Therefore,

$$\left\| C_\psi \right\|_{e,\mathscr{B}_H^\lambda \longrightarrow \mathscr{Z}_H^\beta} \succeq \limsup_{i\longrightarrow\infty} \left\| C_\psi p_i \right\|_{\mathscr{Z}_H^\beta} = \limsup_{i\longrightarrow\infty} \left\| i^{\lambda-1} \left(\psi^i + \bar{\psi}^i \right) \right\|_{\mathscr{Z}_H^\beta}. \tag{112}$$

Next, we prove that

$$\left\| C_\psi \right\|_{e,\mathscr{B}_H^\lambda \longrightarrow \mathscr{Z}_H^\beta} \preceq \limsup_{i\longrightarrow\infty} \left\| i^{\lambda-1} \left(\psi^i + \bar{\psi}^i \right) \right\|_{\mathscr{Z}_H^\beta}. \tag{113}$$

Since $C_\psi : \mathscr{B}_H^\lambda \longrightarrow \mathscr{Z}_H^\beta$ is bounded, then by Theorem 2

$$L := \sup_{i\ge0} \left\| i^{\lambda-1} \left(\psi^i + \bar{\psi}^i \right) \right\|_{\mathscr{Z}_H^\beta} < \infty. \tag{114}$$

Now assume the test function $F_{x,\alpha}^\lambda$ with $x \in \mathbb{U}$ in (14), for $\alpha = 1, 2, 3$. By linearity of the composition operator, for any fixed positive integer $n \ge 2$, we obtain

$$\begin{aligned}
\left\| C_\psi F_{x,\alpha}^\lambda \right\|_{\mathscr{Z}_H^\beta} &\le \left(1 - |x|^2\right)^\alpha \sum_{i=0}^{\infty} \frac{\Gamma(i+\alpha+\lambda-1)}{i!\,\Gamma(\alpha+\lambda-1)} i^{1-\lambda} |x|^i \left\| C_\psi p_i \right\|_{\mathscr{Z}_H^\beta} \\
&\le \left(1 - |x|^2\right)^\alpha \left\{ \left[\sum_{i=\alpha-1}^{n+\alpha-2} + \sum_{i=n+\alpha-1}^{\infty} \right] \frac{\Gamma(i+\alpha+\lambda-1)}{i!\,\Gamma(\alpha+\lambda-1)} i^{1-\lambda} |x|^i \left\| C_\psi p_i \right\|_{\mathscr{Z}_H^\beta} \right. \\
&\preceq \left(1 - |x|^2\right)^\alpha L + \left(1 - |x|^2\right)^\alpha \sum_{i=n+\alpha-1}^{\infty} \frac{\Gamma(i+\alpha+\lambda-1)}{i!\,\Gamma(\alpha+\lambda-1)} i^{1-\lambda} |x|^i \left\| C_\psi p_i \right\|_{\mathscr{Z}_H^\beta} \\
&\preceq \left(1 - |x|^2\right)^\alpha L + \sup_{i\ge n} \left\| i^{\lambda-1} \left(\psi^i + \bar{\psi}^i \right) \right\|_{\mathscr{Z}_H^\beta}.
\end{aligned} \tag{115}$$

Then, for all positive integer $n \geq 2$ and $\alpha = 1, 2, 3$, we get

$$\begin{aligned}\limsup_{|x|\longrightarrow 1}\left\|C_\psi F_{x,\alpha}^\lambda\right\|_{\mathscr{Z}_H^\beta} &\leq \sup_{i\geq n}\left\|i^{\lambda-1}\left(\psi^i+\bar{\psi}^i\right)\right\|_{\mathscr{Z}_H^\beta} \\ &\leq \limsup_{i\longrightarrow\infty}\left\|i^{\lambda-1}\left(\psi^i+\bar{\psi}^i\right)\right\|_{\mathscr{Z}_H^\beta}.\end{aligned} \tag{116}$$

Hence,

$$\max_{1\leq\alpha\leq 3}\left\{\limsup_{|x|\longrightarrow 1}\left\|C_\psi F_{x,\alpha}^\lambda\right\|_{\mathscr{Z}_H^\beta}\right\} \leq \limsup_{i\longrightarrow\infty}\left\|i^{\lambda-1}\left(\psi^i+\bar{\psi}^i\right)\right\|_{\mathscr{Z}_H^\beta}. \tag{117}$$

Since $C_\psi : \mathscr{B}_H^\lambda \longrightarrow \mathscr{Z}_H^\beta$ is bounded, then we have

$$\begin{aligned}\left\|C_\psi\right\|_{e,\mathscr{B}_H^\lambda\longrightarrow\mathscr{Z}_H^\beta} &\leq \max_{1\leq\alpha\leq 3}\left\{\limsup_{|x|\longrightarrow 1}\left\|C_\psi F_{x,\alpha}^\lambda\right\|_{\mathscr{Z}_H^\beta}\right\} \\ &\leq \sup_{i\longrightarrow\infty}\left\|i^{\lambda-1}\left(\psi^i+\bar{\psi}^i\right)\right\|_{\mathscr{Z}_H^\beta}.\end{aligned} \tag{118}$$

By (112) and (118), we fulfilled the desired result. □

Data Availability

The research conducted in this paper does not make use of separate data.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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