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Research Article

A Modified Iterative Approach for Fixed Point Problem in Hadamard Spaces

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The role of iterative algorithms is vital in exploring the diverse domains of science and has proven to be a powerful tool for solving complex computational problems in the most trending branches of computer science. Taking motivation from this fact, we develop and apply a modified four-step iterative algorithm to solve the fixed point problem in the Hadamard spaces using a total asymptotic nonexpansive mapping. MATLAB R2018b is used for numerical experiments to ensure a better convergence rate of the proposed iterative algorithm with existing results.

1. Introduction

Iterative algorithms and fixed point problems are key concepts in numerical analysis and optimization that offer a powerful and flexible framework for solving diverse mathematical problems in computer science, engineering, and industry. Their applications continue to grow and motivate the development of new and more efficient algorithms and techniques that can tackle emerging challenges and applications. Therefore, metric fixed point theory has become a vital instrument for verifying procedures and algorithms using iterative schemes and functional equations in current emerging sciences, such as the field of artificial intelligence [1] and logic programming [2]. The subject has been studied for a long time using different principles of contraction [3]. Its usefulness mostly hinges on the availability of solutions to mathematical problems generated from systems engineering [4] and computer science [5]. Because of its novel development as a confluence of analysis [6, 7] and geometry [8], the theory of fixed points has also become a powerful and vital instrument for the study of nonlinear problems [9]. As such, a choice between several distinct iteration approaches must be made, taking important aspects into account. For example, simplicity and convergence speed are the two main factors that determine whether one iteration approach is more effective than the others. In situations like this, the following problems unavoidably come up: Which iteration method is speeding up convergence among these? It is thus shown in this article that our proposed iteration scheme converges faster than modified Picard, Picard-S, and Picard-Mann iterations.

One type of problem that can be addressed by iterative algorithms is the fixed point problem, which involves finding a point that remains unchanged when a mapping is iteratively applied to it. Specifically, given a nonlinear mapping $\eth : R^n \longrightarrow R^n$, we seek a fixed point $x_\star$ in R^n such that $x_\star = \eth(x_\star)$. If such a fixed point exists, it can be found by running an iterative algorithm that generates a sequence of points (x_k) that converges to $x_\star$, for example, by iterating the following updated rule: $x_{(k+1)} = g(x_k)$, where

$g(x) = x - \lambda\eth(x)$ for some scalar $\lambda > 0$. This description is known as the fixed point iteration or the Picard iteration method. Here and what follows, $(\aleph, d)$ is a $\mathbb{CAT}(0)$ space, $\mathscr{C}$ is nonempty closed convex subset of $(\aleph, d)$, $Q = \widehat{F}(\eth)$ is a set of fixed points of the mapping $\eth$, and $\widehat{N}$ is a set of natural numbers. Therefore, the following points are important for further development of this research. Let $\eth$ be a self-mapping defined on $\mathscr{C}$; then, $\eth$ is said to be as follows:

(1) Nonexpansive, if $d(\eth\widehat{u}, \eth\widehat{v}) \leq d(\widehat{u}, \widehat{v})$, $\widehat{u}, \widehat{v} \in \mathscr{C}$

(2) Asymptotically nonexpansive, if for the sequence $P_m \in [1,\infty)$ with $\lim_{m\longrightarrow\infty} P_m = 1$ such that $d(\eth^m\widehat{u}, \eth^m\widehat{v}) \leq P_m d(\widehat{u}, \widehat{v})$ for all $\widehat{u}, \widehat{v} \in \mathscr{C}$ and for all integers $m \geq 1$

(3) Uniformly L-Lipschitzian, if for constant $\widehat{L} \geq 0$, $d(\eth^m\widehat{u}, \eth^m\widehat{v}) \leq \widehat{L}d(\widehat{u}, \widehat{v}) \forall \widehat{u}, \widehat{v} \in \mathscr{C}$ and $m \geq 1$

(4) Total asymptotically nonexpansive [10], if there exist non-negative sequences $\{\zeta_m\}$, $\{Q_m\}$, and $m \geq 1$ with $\zeta_m, Q_m \longrightarrow 0$, as $m \longrightarrow \infty$ and strictly increasing and continuous function $\phi : [0,\infty) \longrightarrow [0,\infty)$ with $\phi(0) = 0$ such that

$$d(\eth^m\widehat{u}, \eth^m\widehat{v}) \leq d(\widehat{u}, \widehat{v}) + \zeta_m\phi d(\widehat{u}, \widehat{v}) + Q_m \quad \forall \widehat{u}, \widehat{v} \in \mathscr{C} \quad \text{and} \quad m \geq 1. \tag{1}$$

The last condition (4) contained the aforementioned conditions (1-3) such as $[\zeta_m = P_m - 1]$, $[Q_m = 0]$, $\forall m \geq 1$, and $\phi(t) = t$, $t \geq 0$. Additionally, every asymptotically nonexpansive mapping is an L-Lipschitzation mapping with $\widehat{L} = \sup_{m\in\widehat{N}}\{P_m\}$.

Mann [11], Ishikawa [12], and Halpern [13] are the fundamental iterative algorithms to approximate the fixed points of nonexpansive mappings. Following them, several new iterative algorithms were developed by Noor [14], Agarwal et al. [15], Garodia et al. [16], Abbas and Nazir [17], and Garodia and Uddin [18]. More specifically, the following iterative algorithm was defined using total asymptotically nonexpansive mappings in [19]:

$$\hbar_1 \in \mathscr{C}, \tag{2}$$

$$g_m = (1 - \sigma_m)\hbar_m \oplus \sigma_m\eth^m\hbar^m, \tag{3}$$

$$\hbar_{m+1} = \eth^m a_m \forall m \geq 1, \tag{4}$$

where C is a nonempty bounded closed and convex subset in a complete space and $\eta_m \in (0, 1)$. For further development in this direction, we refer the interested reader to [20–22].

The modified Picard-S hybrid iterative process $\{\hbar_m\}$ introduced in [23] is defined as follows:

$$\hbar_1 \in \mathscr{C}, \tag{5}$$

$$g_m = (1 - \sigma_m)\hbar_m \oplus \sigma_m\eth^m\hbar_m, \tag{6}$$

$$a_m = (1 - \zeta_m)\hbar_m \oplus \zeta_m\eth^m g_m, \tag{7}$$

$$\hbar_{m+1} = \eth^m a_m(\sigma_m ; \zeta_m \in (0, 1)). \tag{8}$$

Another such iterative scheme introduced in [24] is stated as follows:

$$\hbar_1 \in \mathscr{C}, \tag{9}$$

$$g_m = \eth^m(1 - \sigma_m)\hbar_m \oplus \sigma_m\eth^m\hbar_m, \tag{10}$$

$$a_m = \eth^m(1 - \zeta_m)\hbar_m \oplus \zeta_m\eth^m g_m, \tag{11}$$

$$\hbar_{m+1} = \eth^m a_m ; (\sigma_m ; \zeta_m \in (0, 1)). \tag{12}$$

All of the aforementioned researchers focused on achieving a better convergence rate by minimizing the time needed to run their proposed iteration scheme. Taking motivation from the above discussion, we propose a novel modified four-step iterative algorithm as follows:

$$\hbar_1 \in \mathscr{C}, \tag{13}$$

$$g_m = \eth^m((1 - \sigma_m)\hbar_m \oplus \sigma_m\eth^m\hbar_m), \tag{14}$$

$$j_m = \eth^m(\eth^m g_m), \tag{15}$$

$$a_m = \eth^m(j_m), \tag{16}$$

$$\hbar_{m+1} = \eth^m a_m ; (m \geq 1 ; \sigma_m \in (0, 1)). \tag{17}$$

2. Preliminaries and Lemmas

This section contains some well-known concepts and results that are often used in this article.

Note: throughout the article, we use $\aleph$ for nonempty set, $(\aleph, d)$ for metric space, and $\widehat{N}$ for set of natural numbers.

Lemma 1 (see [25]). *Let $\widehat{u}, \widehat{v}, w \in (\aleph, d)$ and $\widehat{t} \in [0, 1]$; then,*

$$d\left(\widehat{t}\widehat{u} \oplus \left(1 - \widehat{t}\right)\widehat{v}, w\right) \leq \widehat{t}d(\widehat{u}, w) + \left(1 - \widehat{t}\right)d(\widehat{v}, w). \tag{18}$$

Consider a bounded sequence $\{\widehat{u}_m\}$ in $(\aleph, d)$ and for $\widehat{u} \in (\aleph, d)$

$$\widehat{r}(\widehat{u}, (\widehat{u}_m)) = \lim_{n\longrightarrow\infty} \sup d(\widehat{u}, \widehat{u}_m). \tag{19}$$

Then, the asymptotic radius $\widehat{r}(\{\widehat{u}_m\})$ is defined as

$$\widehat{r}(\widehat{u}, (\widehat{u}_m)) = \inf \{\widehat{r}(\widehat{u}, \widehat{u}_m) : \widehat{u} \in (\aleph, d)\}, \tag{20}$$

and the asymptotic center $\mathscr{A}(\{\widehat{u}_m\})$ of $(\{\widehat{u}_m\})$ is given by

$$\mathscr{A}(\{\widehat{u}_m\}) = [\widehat{u} \in \aleph : \widehat{r}(\widehat{u}, \widehat{u}_m) = \widehat{r}(\{\widehat{u}_m\})]. \tag{21}$$

Note that $\mathscr{A}(\{\widehat{u}_m\})$ has exactly one point in $(\aleph, d)$.

If $\widehat{u}$ is the distinct asymptotic center for each subsequence $(\widehat{z}_m)$ of $(\widehat{u}_m)$ in $(\aleph, d)$, then this sequence Δ-converges to $\widehat{u} \in (\aleph, d)$.

Lemma 2 (see [26]). *Consider the bounded sequence $\widehat{u}_m \in (\aleph, d)$. If $A(\widehat{u}_m) = p$ and $\widehat{z}_m$ is a subsequence of $\widehat{u}_m$ such that $A(\widehat{z}_m) = \widehat{z}$ and $d(\widehat{u}_m, \widehat{z})$ converges, then $p = \widehat{z}$.*

Karapinar et al. [27] demonstrated that the above result can be derived using the fixed point existence theorem and demiclosedness principle for those satisfying in $(\aleph, d)$.

Lemma 3 (see [27]). *Let $\mathscr{C} \subset (\aleph, d)$ and self-mapping $\eth : \mathscr{C} \longrightarrow \mathscr{C}$ be a total asymptotically nonexpansive and uniformly continuous mapping. Moreover, if the set of fixed points $\widehat{F}(\eth)$ is convex and closed, then $\eth$ has a fixed point.*

Lemma 4 (see [27]). *Consider a self-mapping $\eth$ on a complete metric space $(\aleph, d)$ and let $\eth : \mathscr{C} \longrightarrow \mathscr{C}$ be a total asymptotically nonexpansive mapping that is uniformly continuous. Then, it follows that $\lim_{m\longrightarrow\infty} d(\hbar_m, \eth\hbar_m) = 0$ and $\lim_{m\longrightarrow\infty} \hbar_m = q$ imply that $\eth q = q$.*

Lemma 5 (see [28]). *Let $(\aleph, d)$ be a metric space and $\hbar \in (\aleph, d)$, where $\{\hbar_m\}, \{\widehat{v}_m\}$ are sequences in $(\aleph, d)$ and assume that $\{\widehat{t}_m\}$ is a sequence in $[\widehat{b}, \widehat{c}]$ for some $\widehat{b}, \widehat{c} \in (0, 1)$ such that $\limsup_{m\longrightarrow\infty} d(\hbar_m, \hbar) \le \widehat{r}$, $\limsup_{m\longrightarrow\infty} d(\widehat{v}_m, \hbar) \le \widehat{r}$, and $\limsup_{m\longrightarrow\infty} d(1-\widehat{t}_m)\hbar_m \oplus \widehat{t}_m\widehat{v}_m, \hbar) = \widehat{r}$ for some $\widehat{r} \ge 0$. Then,*

$$\lim_{m\longrightarrow\infty} d(\hbar_m, \widehat{v}_m) = 0. \tag{22}$$

Lemma 6 (see [29]). *Let the positive number sequences $\{\widehat{\alpha}_m\}$, $\{\widehat{\beta}_m\}$, and $\{\widehat{\gamma}_m\}$ be such that*

$$\widehat{\alpha}_{m+1} \le \left(1 + \widehat{\beta}_m\right)\widehat{\alpha}_m + \widehat{\gamma}_m, \forall m \ge 1. \tag{23}$$

If $\sum_{m=1}^{\infty} \widehat{\beta}_m < \infty$ and $\sum_{m=1}^{\infty} \widehat{\gamma}_m < \infty$, then $\lim_{m\longrightarrow\infty} \widehat{\alpha}_m$ exists. However, if there exists a subsequence $\{\widehat{\alpha}_{m_n}\} \subseteq \{\widehat{\alpha}_m\}$ such that $\widehat{\alpha}_{m_n} \longrightarrow 0$ as $n \longrightarrow \infty$, then $\lim_{m\longrightarrow\infty} \widehat{\alpha}_m = 0$.

3. Main Result

Theorem 7. *Let $\mathscr{C}$ be a closed convex and bounded subset of $(\aleph, d)$. Consider $\eth : \mathscr{C} \longrightarrow \mathscr{C}$ be a total asymptotically nonexpansive, which is uniformly L-Lipschitzian. Moreover, let $\{\mathfrak{I}_m\}$, $\{\Psi_m\}$, and $m \ge 1$ be non-negative sequences with $\mathfrak{I}_m, \Psi_m \longrightarrow 0$, as $m \longrightarrow \infty$ and strictly increasing continuous function $\phi : [0,\infty) \longrightarrow [0,\infty)$ with $\phi(0) = 0$ satisfying the following conditions:*

A1. $\sum_{m=1}^{\infty} \mathfrak{I}_m < \infty$ and $\sum_{m=1}^{\infty} \Psi_m < \infty$

A2. For constants $\widehat{n}, \widehat{m}$, with $0 < \widehat{m} \le \sigma_m \le \widehat{n} < 1$ for each $\widehat{n} \in \widehat{N}$

A3. $\phi(\mu) \le R_1\mu$ for each $\mu \ge 0$ and R_1 is a constant

Then, the sequence $(\hbar_m)$ generated by (13) Δ-converges to an element of $\aleph$.

Proof. Let us use Lemma 3, which implies that $\widehat{F}(\eth) \ne \varnothing$. Here, on the first hand, we will prove that $\lim_{m\longrightarrow\infty} d(\hbar_m, p)$ exists for any $p \in \widehat{F}(\eth)$, where $\hbar_m$ is defined by (13), and let $p \in \widehat{F}(\eth)$; then, we have

$$\begin{aligned}
d(g_m, p) &= d(\eth^m((1-\sigma_m)\hbar_m \oplus \sigma_m\eth^m\hbar_m), p) \\
&\le d(((1-\sigma_m)\hbar_m \oplus \sigma_m\eth^m\hbar_m), p) \\
&\quad + \mathfrak{I}_m\phi d(((1-\sigma_m)\hbar_m \oplus \sigma_m\eth^m\hbar_m), p) \\
&\quad + \Psi_m \le (1 + \mathfrak{I}_m R_1) d(((1-\sigma_m)\hbar_m \oplus \sigma_m\eth^m\hbar_m), p) \\
&\quad + \Psi_m \le (1 + \mathfrak{I}_m R_1)(1-\sigma_m) d(\hbar_m, p) \oplus \sigma_m\eth^m d(\hbar_m, p) \\
&\quad + \Psi_m \le (1 + \mathfrak{I}_m R_1)[(1 + \mathfrak{I}_m R_1) d(\hbar_m, p) + \Psi_m] \\
&\quad + \Psi_m \le (1 + \mathfrak{I}_m R_1)^2 d(\hbar_m, p) + (2 + \mathfrak{I}_m R_1)\Psi_m.
\end{aligned} \tag{24}$$

Moreover, for each $m \in \widehat{N}$, we have

$$\begin{aligned}
d(j_m, p) &= d(\eth^m(\eth^m g_m), p) \le d(\eth^m g_m, p) + u_n\phi d(\eth^n g_n, p) \\
&\quad + Q_n \le d(\eth^m g_m, p) + \mathfrak{I}_m\phi(d(\eth^m g_m, p)) \\
&\quad + \Psi_m \le (1 + \mathfrak{I}_m R_1) d(\eth^m g_m, p) + \Psi_m \\
&\le (1 + \mathfrak{I}_m R_1)[d(g_m, p) + \mathfrak{I}_m\phi d(g_m, p) + \Psi_m] \\
&\quad + \Psi_m \le (1 + \mathfrak{I}_m R_1)^2 d(g_m, p) + (2 + \mathfrak{I}_m R_1)\Psi_m \\
&\le (1 + \mathfrak{I}_m R_1)^2 \left[(1 + \mathfrak{I}_m R_1)^2 d(\hbar_m, p) + (2 + \mathfrak{I}_m R_1)\Psi_m\right] \\
&\quad + (2 + \mathfrak{I}_m R_1)\Psi_m \le (1 + \mathfrak{I}_m R_1)^4 d(\hbar_m, p) \\
&\quad + (1 + (1 + \mathfrak{I}_m R_1))^2 (2 + \mathfrak{I}_m R_1)\Psi_m.
\end{aligned} \tag{25}$$

Similarly,

$$\begin{aligned}
d(a_m, p) &= d(\eth^m j_m, p) \le d(j_m, p) + \mathfrak{I}_m\phi d(j_m, P) + \Psi_m \\
&\le (1 + \mathfrak{I}_m R_1)\left[(1 + \mathfrak{I}_m R_1)^4 d(\hbar_m, p)\right. \\
&\quad \left. + \left(1 + (1 + \mathfrak{I}_m R_1)^2\right)(2 + \mathfrak{I}_m R_1)\Psi_m\right] + \Psi_m \\
&\le (1 + \mathfrak{I}_m R_1)^5 d(\hbar_m, p) + \left[(1 + \mathfrak{I}_m R_1)^2 (1 + \mathfrak{I}_m R_1)\right. \\
&\quad \left. \cdot (2 + \mathfrak{I}_m R_1)\Psi_m\right] + \Psi_m \le (1 + \mathfrak{I}_m R_1)^5 d(\hbar_m, p) \\
&\quad + \left[1 + \left(1 + (1 + \mathfrak{I}_m R_1)^2\right)(1 + \mathfrak{I}_m R_1)(2 + \mathfrak{I}_m R_1)\right]\Psi_m.
\end{aligned} \tag{26}$$

Finally, we obtain

$$\begin{aligned}
d(\hbar_{m+1}, p) &= d(\eth^m a_m, p) \le d(a_m, p) + \mathfrak{I}_m\phi(a_m, p) + \Psi_m \\
&\le (1 + \mathfrak{I}_m R_1) d(a_m, p) + \Psi_m(1 + \mathfrak{I}_m R_1) \\
&\quad \cdot (2 + \mathfrak{I}_m R_1)]\Psi_m] + \Psi_m \le (1 + \mathfrak{I}_m R_1)^6 d(\hbar_m, p) \\
&\quad + \left[1 + \left(1 + (1 + \mathfrak{I}_m R_1)^2\right)(1 + \mathfrak{I}_m R_1)^2\right](2 + \mathfrak{I}_m R_1)\Psi_m,
\end{aligned} \tag{27}$$

where

$$\begin{aligned}
\mathfrak{Q}_m &= (1 + \mathfrak{I}_m R_1)^6, \\
\gamma_m &= \left[1 + \left(1 + (1 + \mathfrak{I}_m R_1)^2\right)(1 + \mathfrak{I}_m R_1)^2\right](2 + \mathfrak{I}_m R_1),
\end{aligned} \tag{28}$$

and as stated earlier

$$\sum_{m=1}^{\infty} \varrho_m < \infty, \quad \sum_{m=1}^{\infty} \gamma_m < \infty. \tag{29}$$

Also, using Lemma 6 as well as the inequalities (26) and (27), we compute that the limit $\lim_{m\longrightarrow\infty} d(\hbar_m, p)$ exists.

Further, we will show that $\lim_{m\longrightarrow\infty} d(\hbar_m, \eth\hbar_m) = 0$; therefore, we consider

$$\lim_{m\longrightarrow\infty} d(\hbar_m, p) = \mathbb{J} \geq 0, \tag{30}$$

and from (24), we have

$$\lim_{m\longrightarrow\infty} \sup d(g_m, p) \leq \mathbb{J}. \tag{31}$$

According to the definition of $\eth$, we get [24].

$$\begin{aligned} d(\eth^m g_m, p) &\leq d(g_m, p) + \mathfrak{I}_m \phi d(g_m, p) + \Psi_m \\ &\leq (1 + \mathfrak{I}_m R_1) d(g_m, p) + \Psi_m. \end{aligned} \tag{32}$$

Then, from (30) and (31), we obtain

$$\lim_{m\longrightarrow\infty} \sup d(\eth^m g_m, p) \leq \mathbb{J}. \tag{33}$$

Similarly, we compute

$$\lim_{m\longrightarrow\infty} \sup d(\eth^m \hbar_m, p) \leq \mathbb{J}. \tag{34}$$

Since

$$\begin{aligned} d(\hbar_{m+1}, p) \leq{}& (1 + \mathfrak{I}_m R_1)^6 d(\hbar_m, p) \\ &+ \left[1 + \left(1 + (1 + \mathfrak{I}_m R_1)^2\right)(1 + \mathfrak{I}_m R_1)^2\right](2 + \mathfrak{I}_m R_1)\Psi_m, \end{aligned} \tag{35}$$

therefore, by taking $\lim_{m\longrightarrow\infty} \inf$ on both sides, we obtain

$$\mathbb{J} \leq \lim_{m\longrightarrow\infty} \inf d(g_m, p). \tag{36}$$

Continuing in this way, we obtain the following from expressions (30) and (34):

$$\mathbb{J} = \lim_{m\longrightarrow\infty} \sup d(g_m, p) = \lim_{m\longrightarrow\infty} \sup d(\eth^m((1 - \sigma_m)\hbar_m \oplus \sigma_m \eth^m \hbar_m, p), \tag{37}$$

$$\begin{aligned} &d(\eth^m((1 - \sigma_m)\hbar_m \oplus \sigma_m \eth^m \hbar_m), p) \\ &\quad \leq (1 + \mathfrak{I}_m R_1) d((1 - \sigma_m)\hbar_m \oplus \sigma_m \eth^m \hbar_m, p) + \Psi_m. \end{aligned} \tag{38}$$

Next, applying $\lim_{m\longrightarrow\infty} \sup$ on both sides, we get

$$\mathbb{J} \leq lim_{m\longrightarrow\infty} \sup d((1 - \sigma_m)\hbar_m \oplus \sigma_m \eth^m \hbar_m, p). \tag{39}$$

Similarly, using (29) and (33), we obtain

$$d((1 - \sigma_m)\hbar_m \oplus \sigma_m \eth^m \hbar_m, p) \leq (1 - \sigma_m) d(\hbar_m, p) \oplus \sigma_m d(\eth^m \hbar_m, p). \tag{40}$$

Then, applying lim sup

$$\lim_{m\longrightarrow\infty} \sup d((1 - \sigma_m)\hbar_m \oplus \sigma_m \eth^m \hbar_m, p) \leq \mathbb{J}, \tag{41}$$

as well as using (36) and (37), we get

$$\lim_{m\longrightarrow\infty} \sup d((1 - \sigma_m)\hbar_m \oplus \sigma_m \eth^m \hbar_m, p) = \mathbb{J}. \tag{42}$$

Next, by making use of (29), (33), (41), and Lemma 5, we obtain

$$\lim_{m\longrightarrow\infty} d(\hbar_m, \eth^m \hbar_m) = 0. \tag{43}$$

Similarly, by making use of (24)–(26), we obtain

$$d(\hbar_{m+1}, p) \leq (1 + \mathfrak{I}_m R_1) d(a_m, p) + \Psi_m \leq (1 + \mathfrak{I}_m R_1) d(j_m, p) + \Psi_m. \tag{44}$$

Next, by applying the lim inf on both sides

$$\mathbb{J} \leq \lim_{m\longrightarrow\infty} \inf d(j_m, p), \tag{45}$$

and using (24), we obtain

$$d(j_m, p) \leq (1 + \mathfrak{I}_m R_1)^4 d(\hbar_m, p) + (2 + \mathfrak{I}_m R_1)\left(1 + (1 + \mathfrak{I}_m R_1)^2\right)\Psi_m. \tag{46}$$

Continuing in this way, we apply $\lim_{m\longrightarrow\infty}$ sup on the both sides

$$\lim_{m\longrightarrow\infty} \sup d(j_m, p) \leq \mathbb{J}, \tag{47}$$

and we use (42) and (43) to get

$$\begin{aligned} \mathbb{J} &= \lim_{m\longrightarrow\infty} \sup d(j_m, p) = \lim_{m\longrightarrow\infty} \sup d(\eth^m(\eth^m g_m, p)), \\ d(\eth^m(\eth^m g_m, p)) &\leq d(\eth^m d_m, p) + \mathfrak{I}_m R_1 d(\eth^m g_m, p) + \Psi_m \\ &\leq (1 + \mathfrak{I}_m R_1) d(\eth^m g_m, p) + \Psi_m \\ &\leq (1 + \mathfrak{I}_m R_1)[d(g_m, p) + \mathfrak{I}_m R_1 d(g_m, p) + \Psi_m] + \Psi_m \\ &\leq (1 + \mathfrak{I}_m R_1)[(1 + \mathfrak{I}_m R_1) d(g_m, p) + \Psi_m] + \Psi_m \\ &\leq (1 + \mathfrak{I}_m R_1)^2 d(g_m, p) + (2 + \mathfrak{I}_m R_1)\Psi_m. \end{aligned} \tag{48}$$

In the next step, we apply lim sup

$$\lim_{m\longrightarrow\infty} \sup d(\eth^m(\eth^m g_m, p)) \leq \lim_{m\longrightarrow\infty} \sup d(g_m, p), \tag{49}$$

and use

$$\begin{aligned} d(g_m, p) &= d((1-\sigma_m)\hbar_m \oplus \sigma_m \eth^m \hbar_m) \\ &\le d(1-\sigma_m) d(\hbar_m, p) \oplus \sigma_m d(\eth^m \hbar_m, p), \\ \lim\ \sup\ d((1-\sigma_m)\hbar_m \oplus \sigma_m \eth^m \hbar_m) &\le \mathbb{J}. \end{aligned} \tag{50}$$

By applying Lemma 5, we obtain

$$\lim_{m\longrightarrow\infty} d(g_m, \eth^m g_m) = 0, \tag{51}$$

$$d(\hbar_{m+1}, p) \le (1 + \mathfrak{I}_m R_1) d(a_m, p) + \Psi_m. \tag{52}$$

Next, by using lim inf on both sides, we obtain

$$\mathbb{J} \le \lim_{m\longrightarrow\infty} \inf\ d(a_m, p), \tag{53}$$

and using (25), we get

$$\begin{aligned} d(a_m, p) &\le (1 + \mathfrak{I}_m R_1)^5 d(a_m, p) \\ &\quad + \Psi_m \left[\left(1 + (1 + \mathfrak{I}_m R_1)^2\right)(2 + \mathfrak{I}_m R_1)(1 + \mathfrak{I}_m R_1) \right]. \end{aligned} \tag{54}$$

Let us apply lim sup on both sides to get

$$\lim_{m\longrightarrow\infty} \sup\ d(a_m, p) \le \mathbb{J}. \tag{55}$$

Next, using (51) and (53), we obtain

$$\begin{aligned} \mathbb{J} &= \lim_{m\longrightarrow\infty} \sup\ d(a_m, p) = \lim_{m\longrightarrow\infty} \sup\ d(\eth^m j_m, p), \\ d(\eth^m{}_m, p) &\le d(j_m, p) + \mathfrak{I}_m R_1 d(j_m, p) + \Psi_m \le (1 + \mathfrak{I}_m R_1) d(j_m, p) + \Psi_m, \end{aligned} \tag{56}$$

and applying lim sup on both sides, we get

$$\lim_{m\longrightarrow\infty} \sup\ d(\eth^m j_m, p) \le \lim_{m\longrightarrow\infty} \sup\ d(j_m, p), \tag{57}$$

$$\mathbb{J} \le \lim_{m\longrightarrow\infty} \sup\ d(j_m, p). \tag{58}$$

Moreover,

$$\begin{aligned} &d((j_m, p)) \le (1 + \mathfrak{I}_m R_1)^2 d(g_m, p) + (2 + \mathfrak{I}_m R_1)\Psi_m, \\ &d((j_m, p)) \le d(g_m, p), \\ &d((j_m, p)) \le d(\hbar_m, p), \\ &\lim_{m\longrightarrow\infty} \sup\ d(j_m, p) \le \mathbb{J}. \end{aligned} \tag{59}$$

By using (45), (55), (57), and Lemma 5, we get

$$\lim_{m\longrightarrow\infty} d(j_m, \eth^m j_m) = 0. \tag{60}$$

Since $\eth^m$ is $(\mathfrak{I}_m, \Psi_m, \phi)$ is a total asymptotically nonexpansive mapping, therefore

$$\begin{aligned} d(\eth^m g_m, \eth^m \hbar_m) &\le d(g_m, \hbar_m) + \mathfrak{I}_m \phi d(g_m, \hbar_m) + \Psi_m \\ &\le (1 + \mathfrak{I}_m R_1) d(g_m, \hbar_m) + \Psi_m \\ &\le (1 + \mathfrak{I}_m R_1) d(\eth^m((1-\sigma_m)\hbar_m \oplus \sigma_m \eth^m \hbar_m), \hbar_m) + \Psi_m \\ &\le (1 + \mathfrak{I}_m R_1)[d(\eth^m((1-\sigma_m)\hbar_m \oplus \sigma_m \eth^m \hbar_m), \hbar_m) \\ &\quad + d(\eth^m \hbar_m, \hbar_m)] + \Psi_m \le (1 + \mathfrak{I}_m R_1)^2 [\sigma_m d(\eth^m \hbar_m, \hbar_m)] \\ &\quad + d(\eth^m \hbar_m, \hbar_m)(1 + \mathfrak{I}_m R_1) + (2 + \mathfrak{I}_m R_1)\Psi_m. \end{aligned} \tag{61}$$

By taking limit $m \longrightarrow \infty$ and using

$$\lim_{m\longrightarrow\infty} d(\eth^m g_m, \eth^m \hbar_m) = 0, \tag{62}$$

we obtain

$$\begin{aligned} d(\eth^m j_m, \eth^m g_m) &\le d(j_m, g_m) + \mathfrak{I}_m \phi d(j_m, g_m) + \Psi_m \\ &\le (1 + \mathfrak{I}_m R_1) d(j_m, g_m) + \Psi_m \\ &\le (1 + \mathfrak{I}_m R_1) d(\eth^m(\eth^m d_m, g_m) + \Psi_m \\ &\le (1 + \mathfrak{I}_m R_1)[d(\eth^m(\eth^m g_m), g_m) + d(\eth^m g_m, g_m)] \\ &\quad + \Psi_m \le (1 + \mathfrak{I}_m R_1)[d(\eth^m(\eth^m g_m), g_m) \\ &\quad + (1 + \mathfrak{I}_m R_1)] d(\eth^m g_m, g_m) + \Psi_m \\ &\le (1 + \mathfrak{I}_m R_1)[d(\eth^m g_m, g_m) + \mathfrak{I}_m R_1 d(\eth^m g_m, g_m) + \Psi_m] \\ &\quad + (1 + \mathfrak{I}_m R_1) d(\eth^m g_m, g_m) + \Psi_m \\ &\le (1 + \mathfrak{I}_m R_1)[(1 + \mathfrak{I}_m R_1) d(\eth^m g_m), g_m) + \Psi_m] \\ &\quad + (1 + \mathfrak{I}_m R_1) d((\eth^m g_m), g_m) + \Psi_m \\ &\le (1 + \mathfrak{I}_m R_1)^3 d(g_m, g_m) + (1 + \mathfrak{I}_m R_1) d(\eth^m g_m, g_m) \\ &\quad + (1 + \mathfrak{I}_m R_1)^2 \Psi_m + (2 + \mathfrak{I}_m R_1)\Psi_m. \end{aligned} \tag{63}$$

Next, by taking lim $m \longrightarrow \infty$ and using (51), we obtain

$$\lim_{m\longrightarrow\infty} d(\eth^m j_m, \eth^m g_m) = 0. \tag{64}$$

Hence, we obtain the following:

$$\begin{aligned} d(\eth^m a_m, \eth^m j_m) &\le d(a_m, j_m) + \mathfrak{I}_m \phi d(a_m, j_m) + \Psi_m \\ &\le (1 + \mathfrak{I}_m R_1) d(a_m, j_m) + \Psi_m \\ &\le (1 + \mathfrak{I}_m R_1) d(\eth^m(j_m, j_m) + \Psi_m \\ &\le (1 + \mathfrak{I}_m R_1)[d(\eth^m j_m, j_m) + d(\eth^m j_m, j_m)] + \Psi_m \\ &\le (1 + \mathfrak{I}_m R_1)^2 d(\eth^m j_m, j_m) + \Psi_m. \end{aligned} \tag{65}$$

Then, by taking limit $m \longrightarrow \infty$

$$\lim_{m\longrightarrow\infty} d(\eth^m a_m, \eth^m j_m) = 0, \tag{66}$$

and using (42), (60), (62), and (64), we get

$$\begin{aligned} d(\hbar_m, \hbar_{m+1}) = d(\hbar_m, \eth^m a_m &\le d(\hbar_m, \eth^m \hbar_m) \\ &\quad + d(\eth^m \hbar_m, \eth^m g_m) + d(\eth^m g_m, \eth^m j_m) \\ &\quad + d(\eth^m j_m, \eth^m a_m) \longrightarrow 0 asm \longrightarrow \infty. \end{aligned} \tag{67}$$

Since $\eth$ is nonexpansive and uniformly L-Lipschitzian, therefore we obtain

Table 1: Numerical values of the sequence $\sigma_l ; \zeta_l$ with the initial point (0.5).

Total iterations (l)	Numeric outcomes Mann (2)	Picard-S (5)	Modified-S (9)	Proposed (13)
1	0.500000000000000	0.500000000000000	0.500000000000000	0.500000000000000
2	1.723928538741106	1.886255449358858	1.976613028097837	1.996145420286175
3	1.950092442436139	1.988108110779719	1.999553814382810	1.999985260725864
4	1.991717174248763	1.998714262800699	1.999992020382923	1.999999941419748
5	1.998712485903244	1.999859435104046	1.999999864826100	1.999999999763727
6	1.999809279065809	1.999984550416402	1.999999997803722	1.999999999999040
7	1.999972754626556	1.999998296729063	1.999999999965455	2.000000000000000
8	1.999996215929949	1.999999811857257	1.999999999999470	2.000000000000000
9	1.999999486114128	1.999999979190913	1.999999999999992	2.000000000000000
10	1.999999931481887	1.999999997696364	1.999999999999998	2.000000000000000
11	1.999999991002672	1.999999999744810	2.000000000000000	2.000000000000000
12	1.999999998833680	1.999999999971717	2.000000000000000	2.000000000000000
13	1.999999999850472	1.999999999996864	2.000000000000000	2.000000000000000
14	1.999999999981012	1.999999999999652	2.000000000000000	2.000000000000000
15	1.999999999997609	1.999999999999961	2.000000000000000	2.000000000000000
16	1.999999999999701	1.999999999999996	2.000000000000000	2.000000000000000
17	1.999999999999963	1.999999999999999	2.000000000000000	2.000000000000000
18	1.999999999999995	2.000000000000000	2.000000000000000	2.000000000000000
19	1.999999999999999	2.000000000000000	2.000000000000000	2.000000000000000
20	2.000000000000000	2.000000000000000	2.000000000000000	2.000000000000000

Table 2: Numerical values of the sequence $\sigma_m ; \zeta_m$ with the initial point (0.5).

Total iterations (m)	Numeric outcomes Picard-Mann	Picard-S	Modified Picard	Proposed algorithm
1	0.500000000000000	0.500000000000000	0.500000000000000	0.500000000000000
2	1.723928538741106	1.905366735526605	1.985210450750278	1.997551030817435
3	1.950092442436139	1.992893963797876	1.999840830715960	1.999994722348261
4	1.991717174248763	1.999487151289271	1.999998427982258	1.999999988416683
5	1.998712485903244	1.999964058581792	1.999999985369638	1.999999999974333
6	1.999809279065809	1.999997531809481	1.999999999869551	1.999999999999943
7	1.999972754626556	1.999999832979629	1.999999999998874	2.000000000000000
8	1.999996215929949	1.999999988823132	1.999999999999991	2.000000000000000
9	1.999999486114128	1.999999999258555	1.999999999999999	2.000000000000000
10	1.999999931481887	1.999999999951159	2.000000000000000	2.000000000000000
11	1.999999991002672	1.999999999996801	2.000000000000000	2.000000000000000
12	1.999999998833680	1.999999999999792	2.000000000000000	2.000000000000000
13	1.999999999850472	1.999999999999987	2.000000000000000	2.000000000000000
14	1.999999999981012	1.999999999999999	2.000000000000000	2.000000000000000
15	1.999999999997609	2.000000000000000	2.000000000000000	2.000000000000000
16	1.999999999999701	2.000000000000000	2.000000000000000	2.000000000000000
17	1.999999999999963	2.000000000000000	2.000000000000000	2.000000000000000
18	1.999999999999995	2.000000000000000	2.000000000000000	2.000000000000000
19	1.999999999999999	2.000000000000000	2.000000000000000	2.000000000000000
20	2.000000000000000	2.000000000000000	2.000000000000000	2.000000000000000

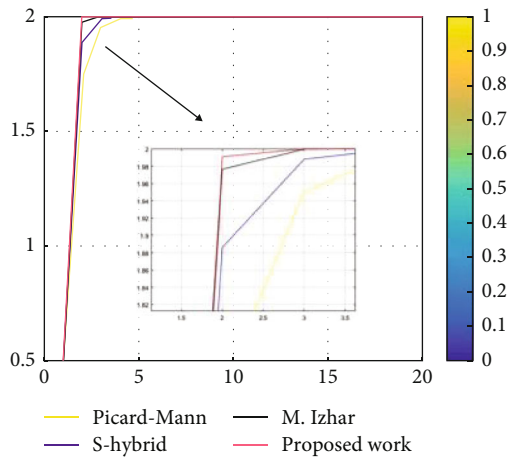

Figure 1: Graphical comparison between different algorithms for the sequence $\sigma_l\,;\zeta_l$.

$$\begin{aligned} d(\hbar_m, \eth\hbar_m) &\le d(\hbar_m, \hbar_{m+1}) + d\left(\hbar_{m+1}, \eth^{m+1}\hbar_{m+1}\right) \\ &\quad + d\left(\eth^{m+1}\hbar_{m+1}, \eth^{m+1}x_m\right) + d\left(\eth^{m+1}x_m, \eth^m x_m\right) \\ &\le d(\hbar_m, \hbar_{m+1}) + d\left(\hbar_{m+1}, \eth^{m+1}\hbar_{m+1}\right) + \mathbb{L}d(\hbar_{m+1}, x_m) \\ &\quad + \mathbb{L}d(\eth^m x_m, x_m) \longrightarrow 0 asm \longrightarrow \infty. \end{aligned} \tag{68}$$

Let $x \in W_\Delta(\hbar_m)$ and a subsequence $\{z_m\}$ of $\{\hbar_m\}$ with $\mathscr{A}(\{z_m\}) = \{x\}$ exists; then, Lemmas 3 and 4 will imply that there exists a subsequence $\{y_m\}$ of $\{z_m\}$ such that $\{y_m\}\Delta$-converges to $y \in C$ and $y \in \widehat{F}(\eth)$, respectively. Next, we verify that $W_\Delta(\hbar_m)$ contains only one point. For this, let $\{z_m\}$ be a subsequence of $(\hbar_m)$ with $\mathscr{A}(\{z_m\}) = \{x\}$ and $\mathscr{A}(\{\hbar_m\}) = \{\hbar\}$. We see that $x = y$ and $y \in \widehat{F}(\eth)$. Finally, since $\{d(\hbar_m, y)\}$ converges therefore by Lemma 2, we obtain $\hbar = y \in \widehat{F}(\eth)$. This shows that $W_\Delta(\hbar_m) = \{\hbar\}$. □

Theorem 8. *Consider $(\aleph, d)$, $\eth$, $\mathscr{C}$, (A1), (A2), (A3), σ_m, and $\mathfrak{I}_m$ be the same as defined in Theorem 7. Then, $\{\hbar_m\}$ defined in (13) converges strongly to a fixed point of*

$$\lim_{m\longrightarrow\infty} \inf d\left(\hbar_m, \widehat{F}(\eth^m)\right) = 0, \tag{69}$$

where $d(\hbar, \widehat{F}(\eth)) = \inf d(\hbar, p)$: $p \in \widehat{F}(\eth)$.

Senter and Dotson [30] defined condition (I) for mapping $\eth : \mathscr{C} \longrightarrow \mathscr{C}$ by following the same steps as followed by Thakur et al. [19]. Hence, we were able to obtain the following result.

Theorem 9. *Considering $\eth$ satisfies condition (I) and $(\aleph, d)$, $\mathscr{C}$, (A1), (A2), (A3), σ_m, and $\mathfrak{I}_m$ be the same as in Theorem 7, then $\{\hbar_m\}$ defined by (13) converge to a point of $\widehat{F}(\eth)$.*

By following to Karapinar et al. [27], the concept of semicompact mapping was introduced in [31]. Thereafter, we state the following theorem.

Theorem 10. *Considering $\eth : \mathscr{C} \longrightarrow \mathscr{C}$ satisfies property of semicompact and $(\aleph, d)$, $\mathscr{C}$, (A1), (A2), (A3), σ_m, and $\mathfrak{I}_m$ be the same as in Theorem 7, then, $\{\hbar_m\}$ defined by (13) converges to a point of Q.*

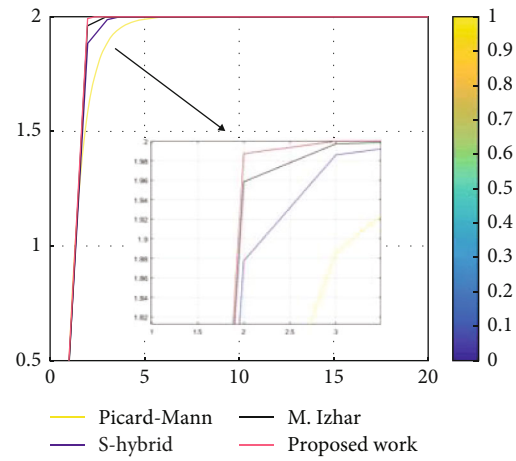

Figure 2: Graphical comparison for different algorithms for the sequence $\sigma_m\,;\zeta_m$.

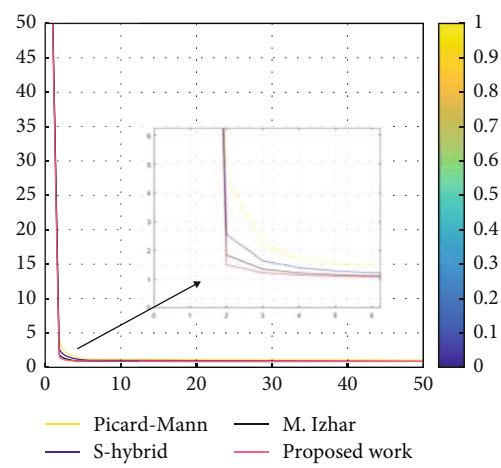

Figure 3: Comparison of the speed of convergence using proposed and existing algorithms.

Remark 11. If we choose $\eth^m = \eth$, then algorithms ((5)), (9), and (13) reduce to the following:

$$\begin{aligned} &\hbar_1 \in \mathscr{C}, \\ &g_m = (1-\sigma_m)\hbar_m \oplus \sigma_m \eth\hbar_m, \\ &a_m = (1-\zeta_m)\hbar_m \oplus \zeta_m \eth g_m, \\ &\hbar_{m+1} = \eth a_m, \end{aligned} \tag{70}$$

$$\begin{aligned} &\hbar_1 \in \mathscr{C}, \\ &g_m = \eth(1-\sigma_m)\hbar_m \oplus \sigma_m \eth\hbar_m, \\ &a_m = \eth(1-\zeta_m)\hbar_m \oplus \zeta_m \eth g_m, \\ &\hbar_{m+1} = \eth a_m, \end{aligned} \tag{71}$$

Table 3: Numerical values of the sequence $\sigma_n ; \zeta_n$ with the initial point (50).

Total iterations (n)	Numeric outcomes Picard-Mann	Picard-S	Modified Picard	Propose iterations
1	50.000000000000000	50.000000000000000	50.000000000000000	50.000000000000000
5	1.525749884405584	1.278044778721721	1.151666956326911	1.101364278336563
10	1.446940787467405	1.112238701544899	1.062061334277530	1.042983468543593
15	1.448954616364597	1.069825417644027	1.038815704452426	1.027220837647252
20	1.450054039511293	1.050582730782256	1.028200159941627	1.019906529413124
25	1.450686892984754	1.039624731120415	1.022131578013160	1.015687061493346
30	1.451098379685536	1.032556692905615	1.018207109516873	1.012942077463268
35	1.451387493666156	1.027622414503705	1.015462252261662	1.011013983045187
40	1.451601803573477	1.023983655549045	1.013435182016224	1.009585504971188
45	1.451767034137480	1.021190037437011	1.011877167731332	1.008484796621000
50	1.451898321240493	1.018978097710047	1.010642430686560	1.007610696284585

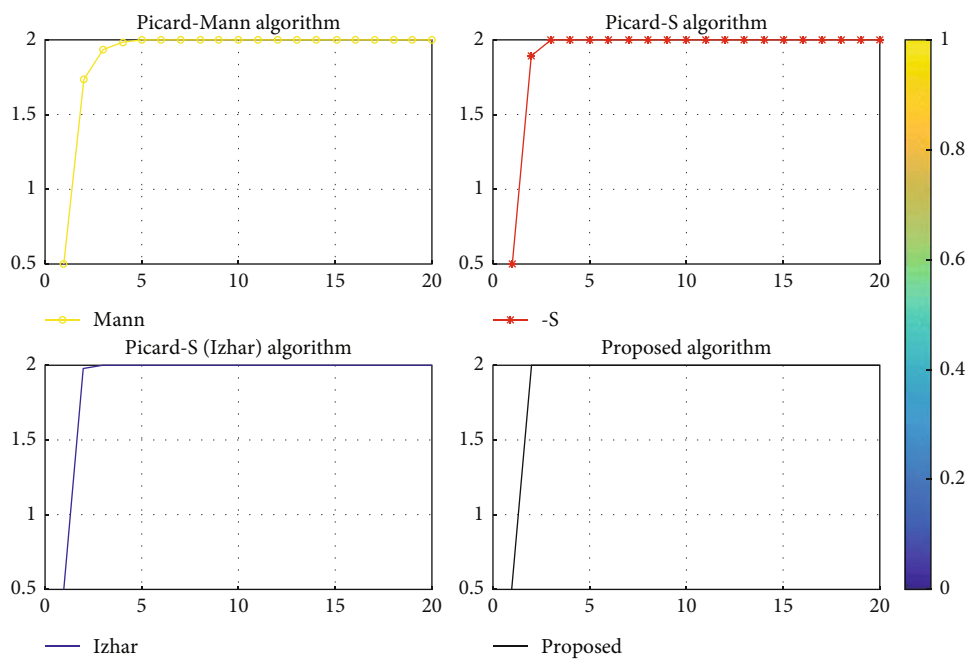

Figure 4: Graphical comparison using $\sigma_n ; \zeta_n$.

$$\begin{aligned} \hbar_1 &\in C, \\ g_m &= \eth((1-\sigma_m)\hbar_m \oplus \sigma_m \eth \hbar_m), \\ j_m &= \eth(\eth g_m), \\ a_m &= \eth(j_m), \\ \hbar_{m+1} &= \eth(a_m), \end{aligned} \tag{72}$$

respectively.

4. Application

In this section, we compare the numerical outcomes of the existing algorithms (2), (5), and (9) with our proposed algorithm (13). We ensure the fast convergence for our proposed iterative scheme (13) by considering Examples 1 and 2.

Example 1. Let $(\aleph, d) = (\mathbb{R}, d)$, $\mathscr{C} = [1, 20]$, and $\eth : \mathscr{C} \longrightarrow \mathscr{C}$ be a self-mapping as follows:

$$\eth(\hbar) = \sqrt[3]{\hbar^2 + 4}) \forall \hbar \in \mathscr{C}. \tag{73}$$

It was proved in [23] that such a class of mappings is a total asymptotically nonexpansive mapping. Particular conditions satisfied by the mapping $\eth$ were discussed in [24] by using the initial point $\hbar = 0.5$ and setting the stopping criteria

$$\|\hbar - 2\| \leq 10^{-15}. \tag{74}$$

We consider the following sequences:

$$\sigma_l = 1 - \frac{l}{\sqrt{l^2+1}}\,;\zeta_l = \frac{l}{l+1},$$
$$\sigma_m = 1 - \frac{m}{3m+1}\,;\zeta_m = \frac{m}{m+1}. \tag{75}$$

We apply iterative schemes (2), (5), (9), and (13). Hence, the corresponding numerical values are provided in Table 1 and Table 2, and their graphical comparison is provided in Figures 1 and 2, respectively.

Example 2. Let $(\aleph, d) = (\mathbb{R}, d)$, $\mathscr{C} = [1, 50]$, and $\eth : \mathbb{R} \longrightarrow \mathbb{R}$ be a self-mapping

$$\eth(\hbar) = \ln \hbar + 1. \tag{76}$$

Then, it is obvious that $\eth$ is continuous uniform L-Lipschitzian and $\widehat{F}(\eth) = 1$. Moreover, it was proved in [23] that this class of mappings is a total asymptotically nonexpansive. The graphical comparison using discussed iterative schemes is provided in Figure 3. We consider the following choice of sequences:

$$\sigma_n = 1 - \frac{n^2}{\sqrt{n^4+1}},$$
$$\zeta_n = \frac{n}{n+1}. \tag{77}$$

We apply iterative schemes (2), (5), (9), and (13). Hence, the corresponding numerical values are provided in Table 3, and the graphical comparison is provided in Figure 3. However, the graphical comparison of the speed of convergence among the proposed and existing algorithms is provided in Figure 4.

5. Conclusion

In this research article, we proposed a modern iterative algorithm and used it to obtain numerical results. These results proved that the proposed method is effective and can accelerate the convergence rate of existing methods for tackling the fixed point problems of the total asymptotically nonexpansive mapping. This research provides both theoretical and practical contributions to the study of fixed point theory and iterative algorithms in the Hadamard spaces. For future work, algorithm can be further modified to obtain better rate of convergence for different classes of mapping [32].

Data Availability

No underlying data was collected or produced in this study.

Conflicts of Interest

The authors declare no conflict of interest/competing interests.

Authors' Contributions

All have equally contributed to this manuscript in all stages, from conceptualization to the write-up of the final draft.

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