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Research Article

Global Existence and Blow-up of Solutions for a Class of Singular Parabolic Equations with Viscoelastic Term

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In this paper, we consider the initial boundary value problem for a class of singular parabolic equations with viscoelastic term and logarithmic term. By using the technique of cut-off and the method of Faedo-Galerkin approximation, the local existence of the weak solution is established. Based on the potential well method, the global existence of the weak solution is derived. Furthermore, we prove that the weak solution blows up in finite time by taking the concavity analysis method.

1. Introduction

In this paper, we consider the initial boundary value problem for a class of singular and viscoelastic nonlinear parabolic equations with logarithmic source.

$$\begin{cases} |x|^{-s}u_t - \Delta u + \int_0^t g(t-s)\Delta u(s)ds = |u|^{q-2}u \ln |u|, & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $R^N (N > 2)$ with a smooth boundary $\partial\Omega$, $u_0(x) \in X = H_0^1(\Omega) \setminus \{0\}$, $2 < q < 2(1 + 2/N)$, and $0 \leq s \leq 2$ is a constant. It is well known that many physical phenomena, such as viscoelasticity and quantum mechanics, can be described by differential equations. For the structure of the solution space of second-order half-linear differential equations, the author discussed various classifications regarding the asymptotics of solutions in [1]. In recent years, the problems related to partial differential equations with logarithmic source and viscoelastic term have been widely concerned by numerous scholars, see [2–6] and references therein. Chen and Tian in [7] considered the following initial boundary value problem for a class of semilinear pseudo-parabolic equations with logarithmic nonlinearity:

$$\begin{cases} u_t - \Delta u - \Delta u_t = u \ln |u|, & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{2}$$

where $u_0(x) \in H_0^1(\Omega)$, $T \in (0,+\infty]$, and $\Omega \subset R^N (N \geq 1)$ is a bounded domain with a smooth boundary $\partial\Omega$. By using the logarithmic Sobolev inequality (see [8, 9]) and a family of potential wells, they obtained the existence of global solution and blow-up at $+\infty$. Besides, they also discussed the asymptotic behavior of solutions. Their result showed that polynomial nonlinearity is important for the solutions to blow up in finite time. Peng and Zhou in [10] investigated the following initial boundary value problem for a semilinear heat equation with logarithmic nonlinearity:

$$\begin{cases} u_t - \Delta u = |u|^{p-2}u \ln |u|, & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{3}$$

where $u_0(x) \in H_0^1(\Omega)$ and $\Omega \subset R^N (N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, $2 < p < 2^*$. By using the potential well method first proposed by Payne and Sattinger et al. [11, 12], the existence of global solutions and finite time blow-up solutions was proved. Moreover, they obtained the upper bound of blow-up time under suitable assumptions. In their work [13], Deng and Zhou studied the following semilinear heat equation with singular potential and logarithmic nonlinearity:

$$\begin{cases} |x|^{-s}u_t - \Delta u = u \ln |u|, & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{4}$$

where $\Omega \subset B_1(0) \subset R^N (N > 2)$ is a bounded domain with smooth boundary $\partial\Omega$, $0 \le s < 2$. Under some appropriate initial-boundary value conditions, they made use of the logarithmic Sobolev inequality to treat the difficulties caused by the nonlinear logarithmic term. By virtue of a family of potential wells, the global existence and infinite time blow-up of the solutions were obtained. Besides, for the study of parabolic equations with singular terms, we also refer the reader to [14, 15] and references therein.

Regarding the initial boundary value problem for a class of viscoelastic parabolic equations with logarithmic terms,

$$\begin{cases} u_t - \Delta u - \Delta u_t + \int_0^t g(t-\tau)\Delta u(\tau)d\tau = |u|^{p-2}u \ln |u|, & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{5}$$

where $u_0(x) \in H_0^1(\Omega)$, $p > 2$. The authors in [16] studied the existence of local solution to problem (5) by using the principle of contraction mapping. Besides, they derived the blow-up property of the weak solution of the problem under the assumption of appropriate $g(\cdot)$ and initial energy, and they also gave the life interval estimation of the solution.

Inspired by the literature [7, 10, 13, 16], a natural question is, what are the properties of the weak solution for a nonlinear parabolic equation with singular term and viscoelastic term? This is the main problem in this paper.

The paper is planned as follows. In Section 2, we collect preliminary results for proving our main theorems. In Section 3, we prove the existence and uniqueness of local solution. In Section 4, we prove the existence of global solution. In Section 5, the blow-up phenomenon of weak solution is discussed.

2. Preliminaries

In this section, we present some preliminaries to prove the main results. We denote the conventional notation $L^q(\Omega)$ $(1 \le q \le \infty)$ for the usual Lebesgue space equipped with $\|\cdot\|_{L^q(\Omega)}$ norm. For simplicity, we write $\|\cdot\|_q$ for $\|\cdot\|_{L^q(\Omega)}$ and we denote the inner product by $(\cdot,\cdot)$. In this paper, C is an arbitrary positive number which may be different from line to line. For problem (1), assume that q and $g(\cdot)$ satisfy the following conditions:

(A_1) $2 < q < 2(1 + 2/N)$, $N > 2$,

(A_2) $g \in C^1(R^+, R^+)$ satisfying $g(s) \ge 0$, $g'(s) \le 0$, $l = 1 - \int_0^\infty g(s)ds > 0$.

Multiplying equation (1) by u_t and integrating over $\Omega \times [0,t)$, we have

$$\begin{aligned} \int_0^t \||x|^{-s/2}u_\tau\|_2^2 d\tau &+ \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 \\ &- \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u_0\|_2^2 \\ &+ \frac{1}{2}\int_0^t g(\tau)\|\nabla u\|_2^2 d\tau - \frac{1}{2}\int_0^t \left(g' \circ \nabla u\right)(\tau)d\tau \\ &+ \frac{1}{2}(g \circ \nabla u)(t) - \frac{1}{2}(g \circ \nabla u_0)(t) = \frac{1}{q}\int_\Omega |u|^q \ln |u| dx \\ &- \frac{1}{q}\int_\Omega |u_0|^q \ln |u_0| dx - \frac{1}{q^2}\|u\|_q^q + \frac{1}{q^2}\|u_0\|_q^q. \end{aligned} \tag{6}$$

Motivated by the calculation above, we define the following functionals:

$$\begin{aligned} J(u) = \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 - \frac{1}{q}\int_\Omega |u|^q \ln |u| dx \\ + \frac{1}{2}(g \circ \nabla u)(t) + \frac{1}{q^2}\|u\|_q^q, \end{aligned} \tag{7}$$

$$I(u) = \left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 - \int_\Omega |u|^q \ln |u| dx + (g \circ \nabla u)(t), \tag{8}$$

$$\begin{aligned} E(t) = \int_0^t \||x|^{-s/2}u_\tau\|_2^2 d\tau + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 \\ - \frac{1}{q}\int_\Omega |u|^q \ln |u| dx + \frac{1}{2}(g \circ \nabla u)(t) + \frac{1}{q^2}\|u\|_q^q, \end{aligned} \tag{9}$$

where $(g \circ \nabla u)(t) = \int_0^t g(t-s)\|\nabla u(t) - \nabla u(s)\|_2^2 ds$.

From (7) and (8), we obtain

$$\begin{aligned} J(u) = \frac{1}{q}I(u) + \frac{q-2}{2q}\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 \\ + \frac{q-2}{2q}(g \circ \nabla u)(t) + \frac{1}{q^2}\|u\|_q^q. \end{aligned} \tag{10}$$

Let

$$\mathcal{N} = \left\{u \in H_0^1(\Omega) \setminus \{0\} \,|\, I(u) = 0\right\}, \tag{11}$$

then

$$d = \inf_{u\in\mathcal{N}} J(u),$$
$$W = \{u \in X | J(u) < d, I(u) > 0\}, \quad (12)$$
$$V = \{u \in X | J(u) < d, I(u) < 0\}.$$

We give the definition of the weak solution to problem (1) as follows:

Definition 1. If $T > 0$, a function $u \in L^\infty(0, T; H_0^1(\Omega))$ with $|x|^{-s/2} u_t \in L^2(0, T; L^2(\Omega))$satisfying the following conditions:

(i) For any $\varphi \in H_0^1(\Omega)$, such that

$$\langle |x|^{-s} u_t, \varphi\rangle + \langle \nabla u, \nabla\varphi\rangle - \int_0^t g(t-s)\langle \nabla u(s), \nabla\varphi\rangle ds = \left\langle |u|^{q-2} u \ln |u|, \varphi\right\rangle. \quad (13)$$

(ii) $u(x, 0) = u_0(x)$ in $H_0^1(\Omega)$

We say that u is a weak solution of problem (1) in the interval $\Omega \times [0, T)$.

Definition 2 (see [17]) (Maximal existence time). Let $u(x, t)$ be a weak solution of problem (1), we define the maximal existence time T^* as follows:

(i) If $u(x, t)$ exists for $0 \le t < \infty$, then $T^* = +\infty$

(ii) If there exists a $t_0 \in (0,\infty)$ such that $u(x, t)$ exists for $0 \le t < t_0$, but does not exist at $t = t_0$, then $T = t_0$

Definition 3 (see [17]) (Finite time blow-up). Let $u(x, t)$ be a weak solution of problem (1), $u(x, t)$ is called finite time blow-up if the maximal existence time $T^* < +\infty$ and

$$\lim_{t\longrightarrow T^{*-}} \left\| |x|^{-s/2} u(x, t)\right\|_2^2 = +\infty. \quad (14)$$

Lemma 4. *Assume that α is a positive number, then we can obtain the following inequalities:*

$$s^q \ln s \le (e\alpha)^{-1} s^{q+\alpha}, \text{for all } s \ge 1,$$
$$s^q |\ln s| \le (eq)^{-1}, \text{for all } 0 < s < 1. \quad (15)$$

Lemma 5 (see [18]). *(i) For any function $u \in W_0^{1,p}(\Omega)$, we have the inequality*

$$\|u\|_q \le B_1 \|\nabla u\|_p, \quad (16)$$

for all $1 \le q \le p^*$, where $p^* = Np/N - p$ if $N > p$ and $p^* = \infty$ if $N \le p$. The best constant B_1 depends only on Ω, N, p, and q.

(ii) For any $u \in W_0^{1,p}(\Omega)$, $p \ge 1, r \ge 1$, the inequality

$$\|u\|_q \le C \|\nabla u\|_p^\theta \|u\|_r^{1-\theta} \quad (17)$$

is valid, where

$$\theta = \left(\frac{1}{r} - \frac{1}{q}\right)\left(\frac{1}{N} - \frac{1}{q} + \frac{1}{r}\right)^{-1}, \quad (18)$$

(i) for $p \ge N = 1, r \le q \le \infty$

(ii) for $N > 1$ and $p < N$, $q \in [r, p^*]$ if $r \le p^*$ and $q \in [p^*, r]$ if $r \ge p^*$

(iii) for $p = N > 1, r \le q < \infty$

(iv) for $p > N > 1, r \le q \le \infty$

Here, the constant C depends on N, p, q, and r.

Lemma 6 (see [16]). *Suppose that (A_1) and (A_2) hold, for any $u \in X$, then*

(i) $\lim_{\lambda\longrightarrow 0^+} J(\lambda u) = 0, \lim_{\lambda\longrightarrow +\infty} J(\lambda u) = -\infty$

(ii) There exists a unique $\lambda^* > 0$ such that $d/d\lambda J(\lambda u)|_{\lambda=\lambda^*} = 0$; $J(\lambda u)$ increases on interval $(0, \lambda^*)$, decreases on interval $(\lambda^*, +\infty)$, and attains the maximum at $\lambda = \lambda^*$

(iii) $I(\lambda u) > 0$, for $0 < \lambda < \lambda^*$, $I(\lambda u) < 0$, for $\lambda^* < \lambda < +\infty$, and $I(\lambda^* u) = 0$

Proof. (i) By the definition of $J(u)$, we have

$$J(\lambda u) = \frac{1}{2}\lambda^2\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 - \frac{\lambda^q}{q}\ln\lambda\|u\|_q^q - \frac{\lambda^q}{q}\int_\Omega |u|^q \ln|u| dx + \frac{1}{2}\lambda^2 (g\circ\nabla u)(t) + \frac{\lambda^q}{q^2}\|u\|_q^q, \quad (19)$$

where $\lambda > 0$, then clearly the conclusion of (i) holds.

(ii) Taking derivative of $J(\lambda u)$ in λ, we gain

$$\begin{aligned}\frac{d}{d\lambda} J(\lambda u) &= \lambda\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 - \lambda^{q-1}\ln\lambda\|u\|_q^q - \lambda^{q-1}\int_\Omega |u|^q \ln|u| dx + \lambda (g\circ\nabla u)(t) \\ &= \lambda\left[\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 - \lambda^{q-2}\ln\lambda\|u\|_q^q - \lambda^{q-2}\int_\Omega |u|^q \ln|u| dx + (g\circ\nabla u)(t)\right].\end{aligned} \quad (20)$$

Let $K(\lambda u)=\lambda^{-1}(d/d\lambda)J(\lambda u)$, then

$$\begin{aligned}\frac{d}{d\lambda}K(\lambda u)&=-(q-2)\lambda^{q-3}\ln\lambda\|u\|_q^q-\lambda^{q-3}\|u\|_q^q\\&\quad-(q-2)\lambda^{q-3}\int_\Omega|u|^q\ln|u|dx\\&=-\lambda^{q-3}\left[(q-2)\ln\lambda\|u\|_q^q+\|u\|_q^q+(q-2)\int_\Omega|u|^q\ln|u|dx\right].\end{aligned}\tag{21}$$

Hence, by taking

$$\lambda_1=\exp\left[\frac{\|u\|_q^q+(q-2)\int_\Omega|u|^q\ln|u|dx}{(2-q)\|u\|_q^q}\right]>0,\tag{22}$$

such that $(d/d\lambda)K(\lambda u)>0$ on $\lambda\in(0,\lambda_1)$, $(d/d\lambda)K(\lambda u)<0$ on $\lambda\in(\lambda_1,+\infty)$, and $(d/d\lambda)K(\lambda_1 u)=0$. Since $K(\lambda u)|_{\lambda=0}=(1-\int_0^t g(s)ds)\|\nabla u\|_2^2+(g\circ\nabla u)(t)>0$ and $\lim_{\lambda\longrightarrow+\infty}K(\lambda u)=-\infty$, there exists $\lambda^*>0$ such that $K(\lambda^* u)=0, K(\lambda u)>0$ on $\lambda\in(0,\lambda^*)$ and $K(\lambda u)<0$ on $\lambda\in(\lambda^*,+\infty)$. So, $(d/d\lambda)J(\lambda u)$ is positive on $(0,\lambda^*)$, $(d/d\lambda)J(\lambda u)$ is negative on $(\lambda^*,+\infty)$, and $(d/d\lambda)J(\lambda^* u)=0$. Thus, the conclusion of (ii) holds.

(iii) By the definition of $I(u)$, we have

$$\begin{aligned}I(\lambda u)&=\lambda^2\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2-\lambda^q\ln\lambda\|u\|_q^q\\&\quad-\lambda^q\int_\Omega|u|^q\ln|u|dx+\lambda^2(g\circ\nabla u)(t)\\&=\lambda\left[\lambda\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2-\lambda^{q-1}\ln\lambda\|u\|_q^q\right.\\&\quad\left.-\lambda^{q-1}\int_\Omega|u|^q\ln|u|dx+\lambda(g\circ\nabla u)(t)\right]=\lambda\frac{d}{d\lambda}J(\lambda u),\end{aligned}\tag{23}$$

where $\lambda>0$. Combining with (ii), the conclusion of (iii) holds. □

Lemma 7. *Let (A_1) and (A_2) hold and $u\in X$ satisfy $I(u)<0$. Then, there exists a $\lambda^*\in(0,1)$ such that $I(\lambda^* u)=0$.*

Proof. For $\forall\lambda>0$, we have

$$I(\lambda u)=\lambda^2\left[\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2+(g\circ\nabla u)(t)-\phi(\lambda)\right],\tag{24}$$

where

$$\phi(\lambda)=\lambda^{q-2}\int_\Omega|u|^q\ln|u|dx+\lambda^{q-2}\ln\lambda\|u\|_q^q.\tag{25}$$

By $I(u)<0$, we can get

$$\int_\Omega|u|^q\ln|u|dx>\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2+(g\circ\nabla u)(t).\tag{26}$$

By (24) and (26), we obtain

$$\phi(1)=\int_\Omega|u|^q\ln|u|dx>\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2+(g\circ\nabla u)(t)>0,\tag{27}$$

$$\phi(\lambda)=\lambda^{q-2}\int_\Omega|u|^q\ln|u|dx+\lambda^{q-2}\ln\lambda\|u\|_q^q\longrightarrow0,\ as\ \lambda\longrightarrow0^+.\tag{28}$$

Combining (24), (27), and the equality above, we can derive that there exists $\lambda^*\in(0,1)$ such that $\phi(\lambda^*)=(1-\int_0^t g(s)ds)\|\nabla u\|_2^2+(g\circ\nabla u)(t)$ and $I(\lambda^* u)=0$. The proof is completed. □

Lemma 8. *Suppose that (A_1) and (A_2) hold and $u(x,t)$ be a weak solution of problem (1). Then, $E(t)$ is nonincreasing function, that is*

$$E'(t)\le0.\tag{29}$$

Proof. Multiplying problem (1) by u_t and integrating on Ω, we have

$$\begin{aligned}&\||x|^{-s/2}u_t\|_2^2+\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|_2^2-\int_0^t g(t-s)\int_\Omega\nabla u(s)\nabla u_t dxds\\&=\int_\Omega|u(t)|^{q-2}u(t)u_t\ln|u(t)|dx.\end{aligned}\tag{30}$$

Through direct calculation, it can be seen that

$$\begin{aligned}&\int_0^t g(t-s)\int_\Omega\nabla u(s)\nabla u_t dxds\\&=\frac{d}{dt}\left[-\frac{1}{2}(g\circ\nabla u)(t)+\frac{1}{2}\int_0^t g(s)ds\|\nabla u(t)\|_2^2\right]\\&\quad+\left[\frac{1}{2}\left(g'\circ\nabla u\right)(t)-\frac{1}{2}g(t)\|\nabla u(t)\|_2^2\right],\end{aligned}\tag{31}$$

$$\int_\Omega|u(t)|^{q-2}u(t)u_t\ln|u(t)|dx=\frac{1}{q}\frac{d}{dt}\int_\Omega|u(t)|^q\ln|u(t)|dx-\frac{1}{q^2}\frac{d}{dt}\|u(t)\|_q^q.\tag{32}$$

Inserting (31) and (32) into (30), we have

$$\begin{aligned}&\||x|^{-s/2}u_t\|_2^2+\frac{d}{dt}\left[\frac{1}{2}\left(1-\int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2\right.\\&\quad+\frac{1}{2}(g\circ\nabla u)(t)-\frac{1}{q}\int_\Omega|u(t)|^q\ln|u(t)|dx\\&\quad\left.+\frac{1}{q^2}\|u(t)\|_q^q\right]=\frac{1}{2}\left(g'\circ\nabla u\right)(t)-\frac{1}{2}g(t)\|\nabla u(t)\|_2^2\le0.\end{aligned}\tag{33}$$

The proof is completed. □

Lemma 9. *Assume that (A_1) and (A_2) hold and $u_0 \in X$. Then,*

(i) the solution u of problem (1) with $u_0 \in W$ satisfies that $u(t) \in W$ for all $t \in [0, T^*]$

(ii) the solution u of problem (1) with $u_0 \in V$ satisfies that $u(t) \in V$ for all $t \in [0, T^*]$

Proof. (i) Let $u(t)$ be the weak solution of problem (1) with $u_0 \in W$, it means that $J(u_0) < d, I(u_0) > 0$. Integrating with respect to time variable t on $(0, t)$ on both sides of (30), we can get

$$J(u(t)) + \int_0^t \left\| |x|^{-s/2} u_\tau(\tau) \right\|_2^2 d\tau + \frac{1}{2}\int_0^t g(\tau)\|\nabla u\|_2^2 d\tau - \frac{1}{2}\int_0^t \left(g' \circ \nabla u\right)(\tau) d\tau = J(u_0). \quad (34)$$

By (34), we can get

$$J(u) < J(u_0) < d, \forall t \in [0, T^*]. \quad (35)$$

Next, we claim that $I(u(t)) > 0$ for all $t \in [0, T^*]$, which together with (35) implies that $u(x, t) \in W$. Otherwise, by continuity of $I(u)$, there would exist a $t_0 \in (0, T^*)$ such that $I(u(t)) > 0$ for $t \in [0, t_0)$ and $I(u(t_0)) = 0, u(t_0) \neq 0$. It means that $u(t_0) \in \mathcal{N}$. Recalling the definition of d, it is clear that $d \leq J(u(t_0))$ which contradicts with (35). Then, $u(t) \in W$ for all $t \in [0, T^*]$.

(ii) The proof is similar to that of part (i), so we omit it. □

Lemma 10 (see [19]) (Hardy-Sobolev Inequality). *Let $R^N : R^k \times R^{N-k}$, $2 \leq k \leq N$, and $x = (y, z) \in R^k \times R^{N-k}$. For given γ, s satisfying $1 < \gamma < N$, $0 \leq s \leq \gamma$, $s < k$, and $m(s, N, \gamma) = \gamma (N - s)/N - \gamma$, there exists a constant $H = H(s, N, \gamma, k) > 0$ such that*

$$\int_{R^N} |y|^{-s}|u(x)|^m dx \leq H\left(\int_{R^N} |\nabla u(x)|^\gamma dx\right)^{N-s/N-\gamma}, \forall u \in W_0^{1,\gamma}(\Omega). \quad (36)$$

Remark 11. Setting $m = 2$, then the inequality above becomes

$$\int_\Omega |x|^{-s}|u(x)|^2 dx \leq H\left(\int_\Omega |\nabla u(x)|^{(2N/N-s+2)} dx\right)^{N-s+2/N}. \quad (37)$$

It follows from $0 \leq s \leq 2$ and $N > 2$, then by Hölder inequality, we can get

$$\int_\Omega |x|^{-s}|u(x)|^2 dx \leq H\left(\int_\Omega |\nabla u(x)|^{(2N/N-s+2)} dx\right)^{N-s+2/N} \leq H|\Omega|^{(N-s+2/N)-1}\|\nabla u\|_2^2 = H_N\|\nabla u\|_2^2. \quad (38)$$

3. Local Existence

Theorem 12. *Let (A_1) and (A_2) hold and $u_0(x) \in H_0^1(\Omega)$. Then, there exists a constant $T > 0$ such that problem (1) admits a unique weak solution.*

$$u(x, t) \in L^\infty\left(0, T; H_0^1(\Omega)\right), |x|^{-s/2}u_t \in L^2\left(0, T; L^2(\Omega)\right). \quad (39)$$

Proof. The proof of Theorem 12 is divided into 4 steps.

Step 1. Approximate problem.

Due to the singular potential existing in problem (1), the following cut-off function is introduced to deal with it:

$$\rho_n(x) = \min\{|x|^{-s}, n\}, \forall n \in Z^+. \quad (40)$$

For $\forall n \in Z^+$, problem (1) has a corresponding solution u_n satisfying

$$\begin{cases} \rho_n(x)u_{nt} - \Delta u_n + \int_0^t g(t-s)\Delta u_n(s) ds = |u_n|^{q-2}u_n \ln|u_n|, & x \in \Omega, t > 0, \\ u_n(x, t) = 0, & x \in \partial\Omega, t > 0, \\ u_n(x, 0) = u_{n0}(x), & x \in \Omega. \end{cases} \quad (41)$$

Let $\{\omega_j\}_{j=1}^\infty$ be a completed orthogonal basis of $H_0^1(\Omega)$ which is the standard orthogonal basis in $L^2(\Omega)$. Set

$$-\Delta\omega_j = \lambda_j\omega_j, \quad \left(\omega_i, \omega_j\right) = \delta_{ij}, \quad (42)$$

for all $i, j \in Z^+$, where $\lambda_j \in R$ and δ_{ij} is the Kronecker's delta. Let $\{u_{n0}\}_{n\in Z^+} \subset C_0^\infty(\Omega)$ be such that $u_{n0}(x) \longrightarrow u_0(x)$ in $H_0^1(\Omega)$asn $\longrightarrow +\infty$. We define the finite-dimensional space $W_h = \text{span}\{\omega_1, \omega_2, \cdots, \omega_h\}, h \in Z^+$ and construct the approximate solution

$$u_n^h(x, t) = \sum_{j=1}^h \xi_{nj}^h(t)\omega_j(x), \xi_{nj}^h \in C^1([0, T]), \quad (43)$$

solving the problem

$$\left(\rho_n u_{nt}^h, \omega_j\right) + \left(\nabla u_n^h, \nabla\omega_j\right) - \int_0^t g(t-s)\left(\nabla u_n^h(s), \nabla\omega_j\right) ds = \left(\left|u_n^h\right|^{q-2}u_n^h \ln\left|u_n^h\right|, \omega_j\right), \quad (44)$$

$$u_n^h(x,0)=\sum_{j=1}^{h}\xi_{nj}^h(0)w_j(x)=u_{n0}^h\longrightarrow u_0(x)\quad\text{in}\quad H_0^1(\Omega),\tag{45}$$

as $h\longrightarrow+\infty$, $n\longrightarrow+\infty$. We can obtain

$$\left(\rho_n u_{nt}^h,\omega_j\right)=\sum_{j=1}^{h}\left(\int_\Omega\rho_n(x)\omega_j(x)\omega_j dx\right)\left[\xi_{nj}^h(t)\right]_t=\sum_{j=1}^{h}a_{ij}\left[\xi_{nj}^h(t)\right]_t.\tag{46}$$

Furthermore, one has

$$\begin{aligned}&\left(\nabla u_n^h,\nabla\omega_j\right)=\left(\sum_{j=1}^{h}\xi_{nj}^h(t)\lambda_j\omega_j(x),\omega_j\right)=\lambda_j\xi_{nj}^h(t),\\&\left(\left|u_n^h\right|^{q-2}u_n^h\ln\left|u_n^h\right|,\omega_j\right)+\int_0^t g(t-s)\left(\nabla u_n^h(s),\nabla\omega_j\right)ds\\&=\left(\left|\sum_{j=1}^{h}\xi_{nj}^h(t)\omega_j(x)\right|^{q-2}\sum_{j=1}^{h}\xi_{nj}^h(t)\omega_j(x)\ln\left|\sum_{j=1}^{h}\xi_{nj}^h(t)\omega_j(x)\right|,\omega_j\right)\\&\quad+\lambda_j\int_0^t g(t-s)\xi_{nj}^h(s)ds=G_{nj}^h(t).\end{aligned}\tag{47}$$

Hence, $\{\xi_{nj}^h\}_{j=1}^h$ is determined by the following Cauchy problem:

$$\begin{aligned}&\sum_{j=1}^{h}a_{ij}\left[\xi_{nj}^h(t)\right]_t+\lambda_j\xi_{nj}^h(t)=G_{nj}^h(t),\\&\sum_{j=1}^{h}\xi_{nj}^h(0)=\int_\Omega u_{n0}^h\omega_j dx.\end{aligned}\tag{48}$$

A standard result on ODE systems now confirms the existence of a unique solution $\xi_{nj}^h\in C^1([0,T])$ to (48) and thus $u_n^h(x,t)\in C^1(0,T;H_0^1(\Omega))$.

Step 2. Priori estimates.

Multiplying (44) by $\xi_{nj}^h(t)$ and summing on $j=1,2,\cdots h$, we can obtain

$$\begin{aligned}&\left(\rho_n u_{nt}^h,u_n^h\right)+\left(\nabla u_n^h,\nabla u_n^h\right)-\int_0^t g(t-s)\\&\cdot\left(\nabla u_n^h(s),\nabla u_n^h\right)ds=\left(\left|u_n^h\right|^{q-2}u_n^h\ln\left|u_n^h\right|,u_n^h\right).\end{aligned}\tag{49}$$

Integrating with respect to time variable t on $(0,t)$ on both sides of (49), we know that

$$\begin{aligned}S_n^h(t)\le S_n^h(0)&+\int_0^t\int_0^\tau\int_\Omega g(\tau-s)\nabla u_n^h(x,s)\nabla u_n^h(x,\tau)dxdsd\tau\\&+\int_0^t\int_\Omega\left|u_n^h(x,\tau)\right|^q\ln\left|u_n^h(x,\tau)\right|dxd\tau,\end{aligned}\tag{50}$$

where

$$S_n^h(t)=\frac{1}{2}\left\||\rho_n(x)|^{1/2}u_n^h(t)\right\|_2^2+\int_0^t\left\|\nabla u_n^h(\tau)\right\|_2^2d\tau.\tag{51}$$

By Hölder inequality and Young inequality, we have

$$\begin{aligned}&\int_0^t\int_0^\tau\int_\Omega g(\tau-s)\nabla u_n^h(x,s)\nabla u_n^h(x,\tau)dxdsd\tau\\&\le\frac{1}{2}\int_0^t\left\|\nabla u_n^h(\tau)\right\|_2^2d\tau+\frac{1}{2}\int_0^t\left(\int_0^\tau g(\tau-s)\left\|\nabla u_n^h(s)\right\|_2 ds\right)^2d\tau\\&<\frac{1}{2}\int_0^t\left\|\nabla u_n^h(\tau)\right\|_2^2d\tau+\frac{1}{2}(1-l)\int_0^t\int_0^\tau g(\tau-s)\left\|\nabla u_n^h(s)\right\|_2^2dsd\tau\\&<\left(1-\frac{l}{2}\right)\int_0^t\left\|\nabla u_n^h(\tau)\right\|_2^2d\tau\le\left(1-\frac{l}{2}\right)S_n^h(t).\end{aligned}\tag{52}$$

On the other hand, from Lemma 4, we can get

$$\begin{aligned}&\int_\Omega\left|u_n^h(x,t)\right|^q\ln\left|u_n^h(x,t)\right|dx\\&=\int_{\Omega_1}\left|u_n^h(x,t)\right|^q\ln\left|u_n^h(x,t)\right|dx\\&\quad+\int_{\Omega_2}\left|u_n^h(x,t)\right|^q\ln\left|u_n^h(x,t)\right|dx\\&\le\int_{\Omega_1}\left|u_n^h(x,t)\right|^q\ln\left|u_n^h(x,t)\right|dx\le(e\alpha)^{-1}\left\|u_n^h(t)\right\|_{q+\alpha}^{q+\alpha},\end{aligned}\tag{53}$$

where $\Omega_1=\{x\in\Omega;|u_n^h(x,t)|\ge1\}$ and $\Omega_2=\{x\in\Omega;|u_n^h(x,t)|<1\}$. By Lemma 5 and Young inequality, we can choose $0<\alpha<2(1+(2/N))-q$ to obtain

$$\begin{aligned}&\int_\Omega\left|u_n^h(x,t)\right|^q\ln\left|u_n^h(x,t)\right|dx\\&\le(e\alpha)^{-1}\left\|u_n^h(t)\right\|_{q+\alpha}^{q+\alpha}\le(e\alpha)^{-1}C\left\|\nabla u_n^h(t)\right\|_2^{\theta(q+\alpha)}\left\|u_n^h(t)\right\|_2^{(1-\theta)(q+\alpha)}\\&\le\varepsilon\left\|\nabla u_n^h(t)\right\|_2^2+C(\varepsilon)\left\|u_n^h(t)\right\|_2^{2(1-\theta)(q+\alpha)/2-\theta(q+\alpha)},\end{aligned}\tag{54}$$

where $\varepsilon\in(0,l/2)$ and $\theta=((1/2)-(1/q+\alpha))N=N(q+\alpha-2)/2(q+\alpha)$. Now, we set

$$\beta=\frac{(1-\theta)(q+\alpha)}{2-\theta(q+\alpha)},\tag{55}$$

then $\beta>1$, since $2<q<2(1+(2/N))$.

Due to Ω is a bounded domain in R^N, so we can get

$$\begin{aligned}\int_\Omega \left|u_n^h(t)\right|^2 dx &= \int_\Omega \frac{1}{|\rho_n(x)|}|\rho_n(x)|\left|u_n^h(t)\right|^2 dx\\ &\le C(\Omega)\left\||\rho_n(x)|^{1/2}u_n^h(t)\right\|_2^2,\end{aligned} \tag{56}$$

where $C(\Omega)$ is related to Ω. Thus, by (54) and (56), we have

$$\begin{aligned}&\int_0^t\int_\Omega \left|u_n^h(x,\tau)\right|^q \ln\left|u_n^h(x,\tau)\right|dxd\tau\\ &\le \varepsilon\int_0^t\left\|\nabla u_n^h(\tau)\right\|_2^2 d\tau + C(\varepsilon)\int_0^t\left\|u_n^h(\tau)\right\|_2^{2\beta}d\tau\\ &\le \varepsilon S_n^h(t) + C(\varepsilon)\int_0^t\left(S_n^h(\tau)\right)^\beta d\tau.\end{aligned} \tag{57}$$

Combining with (50), (52), and (57), we obtain

$$S_n^h(t) \le C_1 + C_2\int_0^t\left(S_n^h(\tau)\right)^\beta d\tau, \tag{58}$$

where $C_1 = 2S_n^h(0)/l - 2\varepsilon$ and $C_2 = 2C(\varepsilon)/l - 2\varepsilon$. Through calculation, we have

$$S_n^h(t) \le C_T, \tag{59}$$

where C_T is independent of n and h, namely,

$$\frac{1}{2}\left\||\rho_n(x)|^{1/2}u_n^h(t)\right\|_2^2 + \int_0^t\left\|\nabla u_n^h(\tau)\right\|_2^2 d\tau \le C_T, \forall h, n \in Z^+. \tag{60}$$

Multiplying (44) by $[\xi_{nj}^h(t)]_t$, summing on $j = 1, 2, \cdots h$ and then integrating on $(0, t)$, we know that

$$\begin{aligned}&\int_0^t\left\||\rho_n(x)|^{1/2}u_{n\tau}^h\right\|_2^2 d\tau + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\left\|\nabla u_n^h\right\|_2^2\\ &\quad + \frac{1}{2}\left(g\circ\nabla u_n^h\right)(t) + \frac{1}{q^2}\left\|u_n^h\right\|_q^q\\ &\quad + \frac{1}{2}\int_0^t g(\tau)\left\|\nabla u_n^h\right\|_2^2 d\tau - \frac{1}{2}\int_0^t\left(g'\circ\nabla u_n^h\right)(\tau)d\tau\\ &\quad - \frac{1}{q}\int_\Omega\left|u_n^h\right|^q \ln\left|u_n^h\right|dx = J\left(u_{n0}^h\right), \forall t \in [0, T).\end{aligned} \tag{61}$$

By using the continuity of functional $J(u)$, we deduce that there exists a positive constant C such that

$$J\left(u_{n0}^h\right) \le C, \forall h, n \in Z^+. \tag{62}$$

Combining with (57), (59), (61), and (62), we can derive

$$\begin{aligned}C \ge J\left(u_{n0}^h\right) &\ge \int_0^t\left\||\rho_n(x)|^{1/2}u_{n\tau}^h\right\|_2^2 d\tau\\ &\quad + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\left\|\nabla u_n^h(t)\right\|_2^2 + \frac{1}{2}\left(g\circ\nabla u_n^h\right)(t)\\ &\quad + \frac{1}{q^2}\left\|u_n^h(t)\right\|_q^q - \frac{1}{q}\int_\Omega\left|u_n^h(t)\right|^q \ln\left|u_n^h(t)\right|dx\\ &\ge \int_0^t\left\||\rho_n(x)|^{1/2}u_{n\tau}^h\right\|_2^2 d\tau + \left(\frac{l}{2} - \frac{\varepsilon}{q}\right)\left\|\nabla u_n^h(t)\right\|_2^2\\ &\quad - \frac{C(\varepsilon)}{q}\left\|u_n^h(t)\right\|_2^{2\beta} + \frac{1}{2}\left(g\circ\nabla u_n^h\right)(t) + \frac{1}{q^2}\left\|u_n^h(t)\right\|_q^q\\ &\ge \int_0^t\left\||\rho_n(x)|^{1/2}u_{n\tau}^h\right\|_2^2 d\tau + \left(\frac{l}{2} - \frac{\varepsilon}{q}\right)\left\|\nabla u_n^h(t)\right\|_2^2\\ &\quad - \frac{C(\varepsilon)}{q}(2S_n(t))^\beta + \frac{1}{2}\left(g\circ\nabla u_n^h\right)(t) + \frac{1}{q^2}\left\|u_n^h(t)\right\|_q^q.\end{aligned} \tag{63}$$

Subsequently, we have

$$\begin{aligned}&\int_0^t\left\||\rho_n(x)|^{1/2}u_{n\tau}^h\right\|_2^2 d\tau + \left(\frac{l}{2} - \frac{\varepsilon}{q}\right)\left\|\nabla u_n^h(t)\right\|_2^2\\ &\quad + \frac{1}{2}\left(g\circ\nabla u_n^h\right)(t) + \frac{1}{q^2}\left\|u_n^h(t)\right\|_q^q \le C_T, \forall h, n \in Z^+.\end{aligned} \tag{64}$$

Let $\Omega_1' = \{x \in \Omega : |x|^{-s} \ge n\}$ and $\Omega_2' = \{x \in \Omega : |x|^{-s} < n\}$, and by (64), we can get

$$\begin{aligned}\int_0^t\int_\Omega (u_{n\tau})^2 dxd\tau &\le \int_0^t\int_{\Omega_1'}\frac{1}{\rho_n}\cdot\rho_n(u_{n\tau})^2 dxd\tau\\ &\quad + \int_0^t\int_{\Omega_2'}\frac{1}{\rho_n}\cdot\rho_n(u_{n\tau})^2 dxd\tau\\ &\le \frac{1}{n}\int_0^t\int_\Omega \rho_n(u_{n\tau})^2 dxd\tau\\ &\quad + \operatorname{diam}(\Omega)^s\int_0^t\int_\Omega \rho_n(u_{n\tau})^2 dxd\tau\\ &\le \left(1 + \operatorname{diam}(\Omega)^s\right)\int_0^t\int_\Omega \rho_n(u_{n\tau})^2 dxd\tau \le C.\end{aligned} \tag{65}$$

Step 3. Pass to the limit.

By means of (60), (64), and (65), there exists a subsequence of $\{u_n^h\}_{h,n=1}^\infty$, which we still denote by $\{u_n^h\}_{h,n=1}^\infty$ for convenience. As $h \longrightarrow +\infty, n \longrightarrow +\infty$, we have that

$$u_n^h \overset{w^*}{\rightharpoonup} u, \text{ in } L^\infty\left(0, T; H_0^1(\Omega)\right), \tag{66}$$

$$u_n^h \overset{w}{\rightharpoonup} u, \text{ in } L^2\left(0, T; H_0^1(\Omega)\right), \tag{67}$$

$$|\rho_n(x)|^{1/2}u_{nt}^h \overset{w}{\rightharpoonup} \frac{u_t}{|x|}, \text{ in } L^2\left(0, T; L^2(\Omega)\right), \tag{68}$$

$$u_{nt}^h \overset{w}{\rightharpoonup} u_t, \text{in } L^2\left(0, T; L^2(\Omega)\right). \tag{69}$$

Since (66) and (69), it follows from Aubin-Lions Lemma (see [20], Corollary 4) that

$$u_n^h \longrightarrow u, \text{in } C\left(0, T; L^2(\Omega)\right), \tag{70}$$

as $h \longrightarrow +\infty, n \longrightarrow +\infty$. Thus, we have $u_n^h \longrightarrow u, a.e.(x, t) \in \Omega \times (0, T)$, which implies

$$\left|u_n^h\right|^{q-2} u_n^h \ln \left|u_n^h\right| \longrightarrow |u|^{q-2} u \ln |u|, a.e.(x, t) \in \Omega \times (0, T). \tag{71}$$

On the other hand, from Lemma 4 and Lemma 5, we have

$$\begin{aligned}
&\int_\Omega \left|\left|u_n^h(x,t)\right|^{q-2} u_n^h(x,t) \ln \left|u_n^h(x,t)\right|\right|^2 dx \\
&= \int_{\Omega_1} \left|\left|u_n^h(x,t)\right|^{q-2} u_n^h(x,t) \ln \left|u_n^h(x,t)\right|\right|^2 dx \\
&\quad + \int_{\Omega_2} \left|\left|u_n^h(x,t)\right|^{q-2} u_n^h(x,t) \ln \left|u_n^h(x,t)\right|\right|^2 dx \\
&\le \int_{\Omega_1} \left|\left|u_n^h(x,t)\right|^{q-1} \ln \left|u_n^h(x,t)\right|\right|^2 dx \\
&\quad + \int_{\Omega_2} \left|\left|u_n^h(x,t)\right|^{-\alpha} \ln \left|u_n^h(x,t)\right| \left|u_n^h(x,t)\right|^{q-1+\alpha}\right|^2 dx \\
&\le (e\alpha)^{-2} \left\|u_n^h(x,t)\right\|_{2(q-1+\alpha)}^{2(q-1+\alpha)} + [e(q-1)]^{-2} |\Omega| \\
&\le (e\alpha)^{-2} B_1 \left\|\nabla u_n^h(x,t)\right\|_2^{2(q-1+\alpha)} + [e(q-1)]^{-2} |\Omega| \le C,
\end{aligned} \tag{72}$$

where B_1 is the optimal constant of the embedding $H_0^1(\Omega) \longrightarrow L^{2(q-1+\alpha)}(\Omega)$. Here, we choose $0 < \alpha < 2(1 + (2/N)) + 1 - q, q - 1 < 2(1 + (2/N))$. Thus, from (70) and (72), we have

$$\left|u_n^h\right|^{q-2} u_n^h \ln \left|u_n^h\right| \overset{w^*}{\rightharpoonup} |u|^{q-2} u \ln |u|, \text{in} L^\infty\left(0, T; L^2(\Omega)\right). \tag{73}$$

From (70), we have $u_n^h(x, 0) \longrightarrow u(x, 0)$ in $L^2(\Omega)$. Combining (45) with $u_{n0}^h \longrightarrow u_0(x)$ in $H_0^1(\Omega)$, we observe that $u(x, 0) = u_0$ in $H_0^1(\Omega)$. By (66), (68), and (73), passing to the limit in (44) as $h \longrightarrow +\infty, n \longrightarrow +\infty$, we see that u satisfies

$$\langle |x|^{-s} u_t, \omega \rangle + \langle \nabla u, \nabla \omega \rangle - \int_0^t g(t-s) \langle \nabla u(s), \nabla \omega \rangle ds = \left\langle |u|^{q-2} u \ln |u|, \omega \right\rangle, \tag{74}$$

for all $\omega \in H_0^1(\Omega)$, and for a.e. $t \in [0, T)$.

Step 4. Uniqueness.

Now, we assume that u_1 and u_2 are two solutions to problem (1) which have the same initial condition, we can get

$$\begin{aligned}
&\langle |x|^{-s} u_{1t}, w \rangle + \langle \nabla u_1, \nabla w \rangle - \int_0^t g(t-s) \langle \nabla u_1(s), \nabla w \rangle ds = \left\langle |u_1|^{q-2} u_1 \ln |u_1|, w \right\rangle, \\
&\langle |x|^{-s} u_{2t}, w \rangle + \langle \nabla u_2, \nabla w \rangle - \int_0^t g(t-s) \langle \nabla u_2(s), \nabla w \rangle ds = \left\langle |u_2|^{q-2} u_2 \ln |u_2|, w \right\rangle.
\end{aligned} \tag{75}$$

By putting $v = u_1 - u_2$, we have $v(x, 0) = 0$. Then, by subtracting the above two equations, we obtain

$$\begin{aligned}
&\langle |x|^{-s} v_t, w \rangle + \langle \nabla v, \nabla w \rangle - \int_0^t g(\tau - s) \langle \nabla v(s), \nabla w \rangle ds \\
&= \left\langle |u_1|^{q-2} u_1 \ln |u_1| - |u_2|^{q-2} u_2 \ln |u_2|, w \right\rangle.
\end{aligned} \tag{76}$$

Taking $\omega = v$ and integrating the above equation on $(0, t)$, we have

$$\begin{aligned}
&\frac{1}{2} \left\| |x|^{-s/2} v \right\|_2^2 + \int_0^t \|\nabla v\|_2^2 d\tau - \int_0^t \int_0^\tau g(\tau - s) \int_\Omega \nabla v(s) \nabla v(\tau) dx ds d\tau \\
&= \int_0^t \int_\Omega \left(|u_1|^{q-2} u_1 \ln |u_1| - |u_2|^{q-2} u_2 \ln |u_2|\right) v dx d\tau.
\end{aligned} \tag{77}$$

Substituting (52) into the above equation, we can get

$$\begin{aligned}
&\int_0^t \int_\Omega \left(|u_1|^{q-2} u_1 \ln |u_1| - |u_2|^{q-2} u_2 \ln |u_2|\right) v dx d\tau \\
&= \frac{1}{2} \left\| |x|^{-s/2} v \right\|_2^2 + \int_0^t \|\nabla v\|_2^2 d\tau - \int_0^t \int_0^\tau g(\tau - s) \int_\Omega \nabla v(s) \nabla v(\tau) dx ds d\tau \\
&\ge \frac{1}{2} \left\| |x|^{-s/2} v \right\|_2^2 + \int_0^t \|\nabla v\|_2^2 d\tau - \left(1 - \frac{l}{2}\right) \int_0^t \|\nabla v\|_2^2 d\tau \\
&= \frac{1}{2} \left\| |x|^{-s/2} v \right\|_2^2 + \frac{l}{2} \int_0^t \|\nabla v\|_2^2 d\tau \ge \frac{1}{2} \left\| |x|^{-s/2} v \right\|_2^2,
\end{aligned} \tag{78}$$

then

$$\frac{1}{2} \left\| |x|^{-s/2} v \right\|_2^2 \le \int_0^t \int_\Omega \left(|u_1|^{q-2} u_1 \ln |u_1| - |u_2|^{q-2} u_2 \ln |u_2|\right) v dx d\tau. \tag{79}$$

We define $F(u)$: $R^+ \longrightarrow R^+$ and $F(u) = |u|^{q-2} u \ln |u|$. It implies that $F(u)$ is locally Lipschitz continue, so we have

$$\int_0^t \int_\Omega [F(u_1) - F(u_2)] v dx d\tau \le M_T \int_0^t \|v\|_2^2 d\tau. \tag{80}$$

Combining with (56), (79) and (80), we obtain

$$\|v\|_2^2 \le \frac{2}{C_\Omega} M_T \int_0^t \|v\|_2^2 d\tau. \tag{81}$$

The uniqueness follows from (81) by Gronwall inequality. The proof of Theorem 12 is completed. □

4. Global Existence

Theorem 13. *Let $u_0(x) \in H_0^1(\Omega)$, (A_1) and (A_2) hold. If $J(u_0) \le d$, $I(u_0) > 0$, then problem (1) admits a global solution $u \in L^\infty(0, T; H_0^1(\Omega))$ with $|x|^{-s/2} u_t \in L^2(0, T; L^2(\Omega))$.*

Proof. We divide the proof into 2 steps.

Step 1. $J(u_0) < d$ and $I(u_0) > 0$.

By (34), we know that

$$J(u(t)) + \int_0^t \left\| |x|^{-s/2} u(\tau) \right\|_2^2 d\tau + \frac{1}{2}\int_0^t g(\tau) \|\nabla u\|_2^2 d\tau - \frac{1}{2}\int_0^t \left(g' \circ \nabla u\right)(\tau) d\tau = J(u_0) < d, \forall t \in [0, T^*]. \tag{82}$$

Next, we prove that $T^* = +\infty$. From Lemma 9, we show that $u \in W, \forall t \in [0, T^*]$. Combining (10) and (82) to derive

$$\int_0^t \left\| |x|^{-s/2} u_\tau \right\|_2^2 d\tau + \frac{q-2}{2q}\left(1 - \int_0^t g(s) ds\right) \|\nabla u\|_2^2 + \frac{q-2}{2q} (g \circ \nabla u)(t) + \frac{1}{q^2} \|u\|_q^q < d, t > 0, \tag{83}$$

which implies

$$\int_0^t \left\| |x|^{-s/2} u_\tau \right\|_2^2 d\tau < d,$$

$$\|\nabla u\|_2^2 < \frac{2qd}{(q-2)l}, \tag{84}$$

$$(g \circ \nabla u)(t) < \frac{2qd}{q-2},$$

$$\|u\|_q^q < q^2 d. \tag{85}$$

Obviously, the constants on the right side of (83)–(85) are independent of T, we can choose $T^* = +\infty$ that $u(x, t)$ is the global weak solution of problem (1).

Step 2. $J(u_0) = d$ and $I(u_0) > 0$.

For $m = 2, 3, \cdots$, we define $\mu_m = 1 - (1/m)$ and $u_{m0} = \mu_m u_0$. The following problem is considered.

$$\begin{cases} |x|^{-s} u_t - \Delta u + \int_0^t g(t-s) \Delta u(s) ds = |u|^{q-2} u \ln |u|, & x \in \Omega, t > 0, \\ u(x, t) = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_{0m}(x), & x \in \Omega. \end{cases} \tag{86}$$

Since $I(u_0) > 0$, the constant $\lambda^* = \lambda^*(u_0)$ defined in Lemma 6 satisfies $\lambda^* \ge 1 > \mu_m$. Hence, we get $I(u_{0m}) = I(\mu_m u_0) > 0$and $J(u_{0m}) = J(\mu_m u_0) < J(u_0) = d$, which means $u_{0m} \in W$. From Step 1, it follows that for each m problem (86) admits global weak solution $u_m \in L^\infty(0, T; H_0^1(\Omega))$ with $|x|^{-s/2} u_{mt} \in L^2(0, T; L^2(\Omega))$ and $u_m \in W$.

The remainder of the proof can be processed similarly to the previous subsection. The proof of Theorem 13 is completed. □

5. Finite Time Blow-up

Lemma 14 (see [21]). *If $G(t)$ is a nonincreasing function on $[t_0, \infty)$ and satisfies the following differential equation*

$$G'(t)^2 \ge a + bG(t)^{2+(1/\delta)}, t \ge t_0, \tag{87}$$

where $a > 0$ and $b \in R$, then there exists a finite positive number T^* such that $\lim_{t \longrightarrow T^*} G(t) = 0$ and an upper bound of T^* can be estimated, respectively, in the following case

(i) *When $b < 0$ and $G(t_0) < \min\{1, (a/-b)^{1/2}\}$, $T^* \le t_0 + \sqrt{(1/-b)} \ln\left(\sqrt{(a/-b)}/\sqrt{(a/-b) - G(t_0)}\right)$*

(ii) *When $b = 0$, $T^* \le t_0 + (G(t_0)/\sqrt{a})$*

(iii) *When $b > 0$, $T^* \le t_0 + 2^{3\delta+1/2\delta}(\delta h/\sqrt{a})\{1 - [1 + hG(t_0)]^{-1/2\delta}\}$, where $h = (a/b)^{2+1/\delta}$*

Lemma 15. *Suppose that (A_1) and (A_2) hold and $u_0 \in V$, then we have*

$$(q-2)\left(1 - \int_0^t g(s) ds\right) \|\nabla u\|_2^2 + (q-2)(g \circ \nabla u)(t) + \frac{2}{q}\int_\Omega |u|^q dx \ge 2qd. \tag{88}$$

Proof. Since $u_0 \in V$, from Lemma 9, we get $u \in V$, *i.e.*$J(u(t)) < d, I(u(t)) < 0$. In view of Lemma 7, we know that there is $\lambda^* \in (0, 1)$, so that $I(\lambda^* u) = 0$. Recalling the definition of d, we can obtain

$$d \le J(\lambda^* u) = \frac{1}{q} I(\lambda^* u) + \frac{q-2}{2q}\left(1 - \int_0^t g(s) ds\right) \|\nabla \lambda^* u\|_2^2 + \frac{q-2}{2q} (g \circ \nabla \lambda^* u)(t) + \frac{1}{q^2} \|\lambda^* u\|_q^q. \tag{89}$$

Then

$$(q-2)\left(1 - \int_0^t g(s) ds\right) \|\nabla u\|_2^2 + (q-2)(g \circ \nabla u)(t) + \frac{2}{q}\int_\Omega |u|^q dx \ge 2qd. \tag{90}$$

Let

$$F(t) = \int_0^t \left\| |x|^{-s/2} u(\tau) \right\|_2^2 d\tau + (T - t) \left\| |x|^{-s/2} u_0 \right\|_2^2, \tag{91}$$

then

$$F'(t) = \left\| |x|^{-s/2} u(t) \right\|_2^2 - \left\| |x|^{-s/2} u_0 \right\|_2^2,$$
$$\begin{aligned} F''(t) &= 2\int_\Omega |x|^{-s} u(t) u_t dx = -2\|\nabla u\|_2^2 \\ &\quad + 2\int_0^t g(t-s)(\nabla u(s), \nabla u(t)) ds + 2\left(|u|^{q-2} u \ln |u|, u\right) \\ &= -2\|\nabla u\|_2^2 + 2\int_0^t g(t-s)(\nabla u(s), \nabla u(t)) ds + 2\int_\Omega |u|^q \ln |u| dx. \end{aligned} \tag{92}$$

□

Lemma 16. *Assume that $\int_0^\infty g(s)ds \le (q-3/q-2)$, then*

$$\begin{aligned} &F''(t) - 2q\int_0^t \left\| |x|^{-s/2} u_\tau \right\|_2^2 d\tau \ge -2qE(0) \\ &\quad + \alpha\left[(q-2)\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 + (q-2)(g \circ \nabla u)(t) + \frac{2}{q}\int_\Omega |u|^q dx\right], \end{aligned} \tag{93}$$

where $\alpha = 1 - (1/(q-2)l)$.

Proof. Utilizing Young inequality and Lemma 8, we can obtain

$$\begin{aligned} F''(t) &- 2q\int_0^t \left\| |x|^{-s/2} u_\tau \right\|_2^2 d\tau = -2\|\nabla u(t)\|_2^2 \\ &+ 2\int_0^t g(t-s)(\nabla u(s), \nabla u(t)) ds + 2\int_\Omega |u(t)|^q \ln |u(t)| dx \\ &- 2q\int_0^t \left\| |x|^{-s/2} u_\tau \right\|_2^2 d\tau \ge -2q\int_0^t \left\| |x|^{-s/2} u_\tau \right\|_2^2 d\tau \\ &- 2\left(1 - \int_0^t g(t-s)ds\right)\|\nabla u(t)\|_2^2 + 2\int_\Omega |u(t)|^q \ln |u(t)| dx \\ &- 2\left[(g \circ \nabla u)(t) + \frac{1}{4}\int_0^t g(t-s)ds\|\nabla u(t)\|_2^2\right] \ge -2qE(0) \\ &+ (q-2)\left(1 - \int_0^t g(t-s)ds\right)\|\nabla u(t)\|_2^2 \\ &- \frac{1}{2}\int_0^t g(t-s)ds\|\nabla u(t)\|_2^2 + \frac{2}{q}\int_\Omega |u(t)|^q dx \\ &+ (q-2)(g \circ \nabla u)(t) \ge -2qE(0) \\ &+ \left(1 - \frac{1}{(q-2)l}\right)\left[(q-2)\left(1 - \int_0^t g(\tau)d\tau\right)\|\nabla u(t)\|_2^2 \right. \\ &\left. + (q-2)(g \circ \nabla u)(t) + \frac{2}{q}\int_\Omega |u(t)|^q dx\right]. \end{aligned} \tag{94}$$

The proof of Lemma 16 is completed. □

Lemma 17. *Suppose that (A_1) and (A_2) hold and $u_0 \in V$, $u(x,t)$ is the solution of problem (1), if one of the following conditions is true*

$$E(0) < 0, (2) E(0) = 0, (3) 0 < E(0) < \alpha d. \tag{95}$$

Then, $F'(t) > 0$ for $t > 0$.

Proof. (1) If $E(0) < 0$, then by Lemma 16, we obtain

$$F'(t) \ge F'(0) - 2qE(0)t. \tag{96}$$

Thus, $F'(t) > 0$ for $t > 0$.

(1) If $E(0) = 0$, then by Lemma 16, we have $F''(t) > 0$ for $t > 0$, since $F'(0) = 0$ for $t > 0$, we get $F'(t) > 0$ for $t > 0$

(2) If $0 < E(0) < \alpha d$ and $I(u_0) < 0$, then combining with Lemma 15 and Lemma 16, we obtain

$$F''(t) \ge 2q(\alpha d - E(0)) > 0. \tag{97}$$

Integrating with respect to time variable t on $(0, t)$ on both sides of the above equation, we see that

$$F'(t) \ge F'(0) + 2q(\alpha d - E(0))t, t > 0. \tag{98}$$

Therefore, we get $F'(t) > 0$ for $t > 0$. □

Theorem 18. *Suppose that (A_1) and (A_2) hold and $u_0 \in V$, u is the weak solution of problem (1), if one of the following conditions is true*

(1) $E(0) < 0$, (2) $E(0) = 0$, (3) $0 < E(0) < \alpha d$.

Then, the weak solution $u(t)$ blow-up at a finite time T^* in the sense of

$$\lim_{t \longrightarrow T^{*-}} \left\| |x|^{-s/2} u(x,t) \right\|_2^2 = +\infty. \tag{99}$$

In case (1), $T^* \le t_* - (A(t_*)/A'(t_*))$ and if $A(t_*) < \min\{1, \sqrt{a/-b}\}$, then $T^* \le t_* + \sqrt{(a/-b)} \ln\left(\sqrt{a/-b}/\sqrt{(a/-b) - (A(t_*))}\right)$.

In case (2), $T^* \le t_* + A(t_*)/\sqrt{a}$.

In case (3), $T^* \le t_* - (A(t_*)/A'(t_*))$, and if $A(t_*) < \min\{1, \sqrt{a_1/-b_1}\}$, then $T^* \le t_* + \sqrt{(1/-b_1)} \ln\left(\sqrt{a_1/-b_1}/\sqrt{(a_1/-b_1) - (A(t_*))}\right)$.

Proof. Let

$$A(t) = F(t)^{-(q-2/2)}. \tag{100}$$

Then

$$A'(t) = -\frac{q-2}{2} F(t)^{-(q-2/2)-1} F'(t) = -\frac{q-2}{2} F(t)^{-q/2} F'(t), \tag{101}$$

$$A''(t) = -\frac{q-2}{2}A(t)^{1+(4/q-2)}\left[F''(t)F(t) - \frac{q}{2}\left(F'(t)\right)^2\right]. \tag{102}$$

By using Lemma 15, Lemma 16, and Hölder inequality, we get

$$\begin{aligned}
F''(t)F(t) - \frac{q}{2}\left(F'(t)\right)^2 &\geq \left\{2q\int_0^t \left\||x|^{-s/2}u_\tau\right\|_2^2 d\tau - 2qE(0)\right.\\
&\quad + \alpha\left[(q-2)\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2 + \frac{2}{q}\int_\Omega |u|^q dx\right.\\
&\quad \left.\left. + (q-2)(g\circ\nabla u)(t)\right]\right\}F(t) - \frac{q}{2}\left[4F(t)\int_0^t \left\||x|^{-s/2}u_\tau\right\|_2^2 d\tau\right]\\
&\geq \left\{-2qE(0) + \alpha\left[(q-2)\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2\right.\right.\\
&\quad \left.\left. + \frac{2}{q}\int_\Omega |u|^q dx + (q-2)(g\circ\nabla u)(t)\right]\right\}F(t)\\
&= \left\{-2qE(0) + \alpha\left[(q-2)\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2\right.\right.\\
&\quad \left.\left. + \frac{2}{q}\int_\Omega |u|^q dx + (q-2)(g\circ\nabla u)(t)\right]\right\}A(t)^{-(2/q-2)}\\
&\geq [2q\alpha d - 2qE(0)]A(t)^{-(2/q-2)}.
\end{aligned} \tag{103}$$

Now, we substitute (103) into (102) to obtain

$$A''(t) \leq q(q-2)(E(0)-\alpha d)A(t)^{1+(2/q-2)}. \tag{104}$$

If case (1) or case (2) holds, by (104), we have

$$A''(t) \leq q(q-2)E(0)A(t)^{1+(2/q-2)}. \tag{105}$$

By Lemma 17, multiplying (105) by $A'(t)$ and integrating on $[t_*, t]$, we arrive at

$$\left(A'(t)\right)^2 \geq a + bA(t)^{2+(2/q-2)}, t \geq t_*, \tag{106}$$

where

$$a = A'(t_*)^2 - \frac{q(q-2)^2}{q-1}E(0)A(t_*)^{2+(2/q-2)}, b = \frac{q(q-2)^2}{q-1}E(0). \tag{107}$$

If the case (3) holds, we can get

$$A''(t) \leq -q(q-2)(\alpha d - E(0))A(t)^{1+(2/q-2)}. \tag{108}$$

By employing identical reasoning as presented in (104), we know that

$$\left(A'(t)\right)^2 \geq a_1 + b_1 A(t)^{2+(2/q-2)}, t \geq t_*, \tag{109}$$

where

$$\begin{aligned}
a_1 &= A'(t_*)^2 - \frac{q(q-2)^2}{q-1}(E(0)-\alpha d)A(t_*)^{2+(2/q-2)},\\
b &= \frac{q(q-2)^2}{q-1}(E(0)-\alpha d).
\end{aligned} \tag{110}$$

Therefore, when $\delta = q-2/2$ and $t_0 = t_* > 0$, by Lemma 14, there exists a finite time T^* such that

$$\lim_{t\longrightarrow T^{*-}} A(t) = 0, \tag{111}$$

i.e.,

$$\lim_{t\longrightarrow T^{*-}} \left\||x|^{-s/2}u(x,t)\right\|_2^2 = +\infty. \tag{112}$$

This finished the proof of Theorem 18. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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