

Research Article

Existence, Decay, and Blow-up of Solutions for a Weighted *m*-**Biharmonic Equation with Nonlinear Damping and Source Terms**

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In this paper, we consider the weighted *m*-biharmonic equation with nonlinear damping and source terms. We proved the global existence of solutions. Later, the decay of the energy is established by using Nakao's inequality. Finally, we proved the blow-up of solutions in finite time.

1. Introduction

In this work, we study the following weighted *m*-biharmonic equation with initial-boundary value:

$$\begin{cases} z_{tt} + \Delta(k(x)|\Delta z|^{m-2}\Delta z) + \Delta^2 z_t + |z_t|^{p-1}z_t = |z|^{q-1}z, & x \in \Omega, t > 0, \\ z(x,t) = \Delta z(x,t) = 0, & x \in \partial\Omega, t > 0, \\ z(x,0) = z_0(x), z_t(x,0) = z_1(x), & x \in \Omega, \end{cases}$$
(1)

where, $\Omega \subset R^n (n \ge 1)$ is a domain with smooth boundary $\partial \Omega$ in R^n . $p, q \ge 1, m \ge 2$ and the coefficient k(x) are assumed a strictly continuous and positive differentiable function in Ω .

Freitas and Zuazua [1] considered the linear wave equation with indefinite damping of the form.

$$z_{tt} - z_{xx} + 2a(x)z_t = 0.$$
 (2)

He proved the stability results.

In [2], Yu investigated the equation with constant coefficients.

$$z_{tt} - \Delta z - \Delta z_t + |z_t|^{p-2} z_t = |z|^{q-2} z.$$
 (3)

He showed globality, boundedness, blow-up, convergence up to a subsequence towards the equilibria, and exponential stability. Gerbi and Said-Houari [3] proved the exponential decay of solutions (3) for p = 2.

Huang and Chen [4] considered the nonlinear Klein-Gordon equation with damping term.

$$z_{tt} - \Delta z + T(x)z + z + |z_t|^{m-2}z_t = |z|^{p-2}z,$$
(4)

Using potential well argument, they obtain global solutions and blow-up result in finite time.

Tahamtani [5] discussed with nonlinear hyperbolic equation with the Lewis function.

$$\alpha(x)z_{tt} + \rho\Delta z_t - \operatorname{div}\left(|\nabla z|^{m-2}\nabla z\right) = f(z). \tag{5}$$

He considered a blow-up result.

Pişkin and Fidan [6] considered the variable coefficient wave equation.

$$z_{tt} - \Delta z - \Delta z_t + \mu_1(t) |z_t|^{p-2} z_t = \mu_2(t) |z|^{q-2} z.$$
 (6)

They proved the blow-up of solutions.

Al-Gharabli and Al-Mahdi [7] investigated the following nonlinear plate equation.

$$z_{tt} + \Delta^2 z + \alpha(t)g(z_t) = z|z|^{\beta}.$$
(7)

They proved the local existence using the Faedo-Galerkin method.

Zheng et al. [8] considered the Petrovsky equation:

$$z_{tt} + \Delta^2 z + k_1(t) |z_t|^{m-2} z_t = k_2(t) |z|^{p-2} z,$$
(8)

in a bounded domain. They proved the blow-up of solutions.

Guo and Li [9] considered the Petrovsky equation with a strong damping term.

$$z_{tt} + \Delta^2 z - \Delta z - \Delta z_t + \alpha(t) z_t = |z|^{p-2} z.$$
(9)

They utilized an energy estimation technique to derive the minimum possible blow-up time.

Wu [10] considered with variable coefficients

$$z_{tt} + \Delta^2 z - \Delta z - \omega \Delta z_t + \alpha(t) z_t = |z|^{p-2} z, \qquad (10)$$

and obtained the blow-up result with lower and upper boundedness.

Messaoudi [11] studied the following problem:

$$z_{tt} - \text{div} \left(|\nabla z|^{m-2} \nabla z \right) - \Delta z_t + |z_t|^{q-1} z_t = |z|^{p-1} z.$$
(11)

He studied the decay of solutions of the problem (11). Then, the problem (11) was studied by Wu and Xue [12] and Pişkin [13] under different conditions.

Boonaama et al. [14] have studied blow-up, decay, and existence of solutions of the following equation:

$$z_{tt} - \operatorname{div} \left(\varphi(x) |\nabla z|^{p-2} \Delta z \right) - \Delta z_t + |z_t|^{q-1} u_t = \xi |z|^{m-1} z.$$
 (12)

Later, the same authors [15] studied the following equation:

$$z_{tt} - |\nabla z| \operatorname{div} \left(\alpha(x) \frac{\nabla z}{|\nabla z|} \right) - \Delta z_t + |z_t|^{q-1} z_t = |z|^{p-1} z. \quad (13)$$

They established global existence.

Pişkin and Fidan [16] are concerned with the following problem:

$$z_{tt} - \operatorname{div} \left(|\nabla z|^{m-2} \nabla z \right) + \Delta^2 z + \mu_1(t) |z_t|^{p-2} z_t = \mu_2(t) |z|^{q-2} z.$$
(14)

They prove the blow-up of solutions for finite time with negative initial energy.

Then, Mokeddem [17] studied the global solutions and decay rate estimate for the energy of the following equation:

$$z_{tt} - \operatorname{div}\left(|\nabla z|^{p-2}\nabla z\right) + \sigma(t)(z_t - \Delta z_t) + \omega|z|^{m-2}z = |z|^{r-2}z.$$
(15)

Motivated by the above-mentioned papers, in this paper, we investigate to prove the global existence, decay, and blowup of solutions for problem (1), which was not previously studied, where we study weighted m-biharmonic equation with nonlinear damping and source terms.

The rest of the work is as follows. In Section 2, we give some assumptions needed in this work. In Section 3, we prove the global existence theorem. In Section 4, we prove the decay of solutions by Nakao's inequality. In Section 5, the blow-up result is proved for p = 1.

2. Preliminaries

In this part, we present certain lemmas and assumptions required for the formulation and proof of our results. Let $\|.\|,\|.\|_p$ and $\|.\|_{\mathscr{D}^{2,m}_0(\Omega)}$ indicate the typical $L^2(\Omega)$, $L^p(\Omega)$, and $\mathscr{D}^{2,m}_0(\Omega, \varphi)$ norms (see [18, 19]).

To investigate eq. (1), we define the weighted Sobolev space $\mathscr{D}_0^{2,m}(\Omega, \varphi)$ as the closure of $C_0^{\infty}(\Omega)$ in the norm:

$$||z||_{\mathcal{D}_0^{2,m}(\Omega)} = \left(\int_{\Omega} k(x) |\Delta z|^m \mathrm{d}x\right)^{1/m}.$$
 (16)

Lemma 1 (see [20]). Let $\phi(t)$ be a nonincreasing and nonnegative function defined on [0, T], T > 1, satisfying

$$\phi^{1+w_2}(t) \le w_0(\phi(t) - \phi(t+1)), t \in [0, T].$$
(17)

 w_2 is a nonnegative constant, and w_0 is a positive constant. Then, we get, for each $t \in [0, T]$,

$$\begin{cases} \phi(t) \le \phi(0)e^{-w_1[t-1]^+}, & w_2 = 0, \\ \phi(t) \le \left(\phi(0)^{-w_2} + w_0^{-1}w_2[t-1]^+\right)^{-(1/w_2)}, & w_2 > 0, \end{cases}$$
(18)

where $[t-1]^+ = \max \{t-1, 0\}$ and $w_1 = \ln (w_0/(w_0 - 1))$.

Lemma 2 (see [21]). Let $B(t): \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a \mathbb{C}^2 -function satisfying

$$B''(t) - 4(\delta + 1)B'(t) + 4(\delta + 1)B(t) \ge 0.$$
(19)

If

$$B'(0) > r_2 B(0) + K_0, \tag{20}$$

then B'(t) > 0 for t > 0, where

$$r_2 = 2(\delta+1) - 2\sqrt{(\delta+1)\delta} \tag{21}$$

is the smallest root of the equation

$$r^{2} - 4(\delta + 1)r + 4(\delta + 1) = 0.$$
⁽²²⁾

Lemma 3 (see [21]). If H(t) be a nonincreasing function on $[t_0, \infty]$ and supplies the differential inequality

$$\left[H'(t)\right] \ge \alpha + \beta \left[H(t)\right]^{2+(1/\beta)} \text{ for } t \ge t_0, \tag{23}$$

Here, $\alpha > 0$ *and* $\beta \in R$ *exist a finite time* T^* *:*

$$\lim_{t \longrightarrow T^{*-}} H(t) = 0.$$
(24)

The upper limits for T^* are estimated as given below:

(i) If $\beta < 0$ and $H(t_0) < \min\{1, \sqrt{-(\alpha/\beta)}\}$

$$T^* \leq t_0 + \frac{1}{\sqrt{-\beta}} \ln \frac{\sqrt{-(\alpha/\beta)}}{\sqrt{-(\alpha/\beta)} - H(t_0)}.$$
 (25)

(ii) If $\beta = 0$

$$T^* \le t_0 + \frac{H(t_0)}{H'(t_0)}.$$
(26)

Next, we prove the local existence theorem which may be proved by [22, 23].

Theorem 4 (local existence). We assume that $2 \le m < q + 1$ $< nm/(n - m + \gamma) < n, z_0 \in \mathcal{D}_0^{2,m}(\Omega)$, and $z_1 \in L^2(\Omega)$; then, problem (1) is a unique local solution.

$$z \in C([0, T); \mathcal{D}_0^{2,m}(\Omega)),$$

$$z_t \in C([0, T); L^2(\Omega)) \cap L^{p+1}([0, T) \times \Omega).$$
(27)

3. Global Existence

In this part, we show the global existence of the solution for problem (1). We define the following functionals:

$$J(t) = \frac{1}{m} \int_{\Omega} k(x) |\Delta z|^m dx - \frac{1}{q+1} ||z||_{q+1}^{q+1},$$
(28)

$$I(t) = \int_{\Omega} k(x) |\Delta z|^m dx - ||z||_{q+1}^{q+1}.$$
 (29)

The functional *E* of problem (1) is as follows:

$$E(t) = \frac{1}{2} ||z_t||^2 + \frac{1}{m} \int_{\Omega} k(x) |\Delta z|^m dx - \frac{1}{q+1} ||z||_{q+1}^{q+1}, \quad (30)$$

and we denote the Nehari set

$$W = \left\{ z \in \mathcal{D}_0^{2,m}(\Omega), I(z) > 0 \right\} \cup \{0\}.$$
(31)

Lemma 5. Assume that z is a solution to problem (1). Then, the energy of problem (1) defined by (30) satisfies and

$$E'(t) = -\left(\|\Delta z_t\|^2 + \|z_t\|_{p+1}^{p+1} \right) \le 0.$$
(32)

Proof. Multiply eq. (1) by z_t , integrate it over Ω , and apply Green's formula,

$$E(t) - E(0) = -\int_0^t ||\Delta z_\tau||^2 d\tau - \int_0^t ||z_\tau||_{p+1}^{p+1} d\tau, \text{ for } t \ge 0.$$
(33)

Lemma 6. Suppose that $z_0 \in W$, $z_1 \in L^2(\Omega)$, q + 1 > m, and

$$\beta = C^* \left(\frac{m(q+1)}{q+1-m} E(0) \right)^{(q+1-m)/m} < 1.$$
(34)

Then, for each $t \ge 0, z \in W$.

Proof. Since I(0) > 0 and due to the continuity of z(t), it follows that I(t) > 0 for some interval near t = 0. Let $T_n > 0$ be the maximum time for which eq. (29) holds on the interval $[0, T_n]$.

Thus, from (28) and (29),

$$J(t) = \frac{1}{m} \int_{\Omega} k(x) |\Delta z|^m dx - \frac{1}{q+1} ||z||_{q+1}^{q+1}$$

= $\frac{1}{q+1} I(t) + \frac{q+1-m}{m(q+1)} \int_{\Omega} k(x) |\Delta z|^m dx.$ (35)

From I(t) > 0, we get

$$J(t) \ge \frac{q+1-m}{m(q+1)} \|z\|_{\mathscr{D}_{0}^{2,m}(\Omega)}^{m},$$
(36)

Then, using the definition E(t) and E'(t), we have

$$\|z\|_{D_0^{2m}(\Omega)}^m \le \frac{m(q+1)}{q+1-m} J(t) \le \frac{m(q+1)}{q+1-m} E(t) \le \frac{m(q+1)}{q+1-m} E(0).$$
(37)

Thanks to Lemma 6 and (37), we obtain

$$\begin{split} \|z\|_{q+1}^{q+1} &\leq C^* \, \|z\|_{\mathscr{D}_{0}^{2,m}(\Omega)}^{q+1} = C^* \, \|z\|_{\mathscr{D}_{0}^{2,m}(\Omega)}^{q+1-m} \|z\|_{\mathscr{D}_{0}^{2,m}(\Omega)}^{m} \\ &\leq C^* \left(\frac{m(q+1)}{q+1-m} E(0)\right)^{(q+1-m)/m} \|z\|_{\mathscr{D}_{0}^{2,m}(\Omega)}^{m} \qquad (38) \\ &= \beta \|z\|_{\mathscr{D}_{0}^{2,m}(\Omega)}^{m} < \|z\|_{\mathscr{D}_{0}^{2,m}(\Omega)}^{m}, \forall t \in [0, T_n]. \end{split}$$

We can conclude that I(t) > 0 based on reference (21), $\forall t \in [0, T_n]$. When by repeating the procedure, T_n is extended to T.

Lemma 7. If the conditions of Lemma 6 are satisfied, then there exists $\eta = 1 - \beta$ such that the

$$||z||_{q+1}^{q+1} \le (1-\eta) ||z||_{\mathscr{D}_{0}^{2,m}(\Omega)}^{m}.$$
(39)

Proof. We get

$$\|z\|_{q+1}^{q+1} \le \beta \|z\|_{\mathscr{D}_{0}^{2,m}(\Omega)}^{m}, \tag{40}$$

and when set $\eta = 1 - \beta$, we may deduce that

$$\|z\|_{\mathscr{D}^{2,m}_{0}(\Omega)}^{m} \leq \frac{1}{\eta}I(t).$$
(41)

Theorem 8. Assume that $z_0 \in W$ by Lemma 6, and $2 \le m < q + 1 < nm/(n - m + \gamma) = m_{\gamma}^*$, n > m. Then, the solution for (1) is global.

Proof. We get

$$E(0) \geq E(t) = \frac{1}{2} ||z_t||^2 + \frac{1}{m} \int_{\Omega} k(x) |\Delta z|^m dx - \frac{1}{q+1} ||z||_{q+1}^{q+1}$$

$$= \frac{1}{2} ||z_t||^2 + \frac{q+1-m}{m(q+1)} \int_{\Omega} k(x) |\Delta z|^m dx + \frac{1}{q+1} I(t)$$

$$\geq \frac{1}{2} ||z_t||^2 + \frac{q+1-m}{m(q+1)} ||z||_{\mathscr{D}_0^{2,m}(\Omega)}^m,$$

(42)

by $I(t) \ge 0$, then $||z_t||^2 + ||z||_{\mathfrak{D}_0^{2m}(\Omega)}^m \le CE(0)$; here, $C = \max \{2, (m(q+1))/(q+1-m)\}$. From Theorem 4, we get the global existence result.

4. Decay

In this part, we show the decay of the solution for problem (1).

Theorem 9. Assume that $z_0 \in W, z_1 \in L^2(\Omega)$. We assume that $2 \le m < q + 1 < nm/(n - m + \gamma)$, n > m, and $\beta = C^*$ $((m(q + 1))/(q + 1 - m)E(0))^{(q+1-m)/m}$. Thus, we have the following decay:

$$E(t) = \begin{cases} E(0)e^{-w_0[t-1]^+}, & \text{if } p = 1, \\ \left(E(0)^{-\lambda} + w_1\lambda[t-1]^+\right)^{-(1/\lambda)}, & \text{if } p > 1, \end{cases}$$
(43)

where w_0 and w_1 and λ are positive constants.

Proof. Integrating E'(t) over [t, t + 1], t > 0, we obtain

$$E(t) - E(t+1) = D^{p+1}(t),$$

$$D^{p+1}(t) = \int_{t}^{t+1} \left(\|\Delta z_{\tau}\|^{2} + \|z_{\tau}\|_{p+1}^{p+1} \right) d\tau.$$
(44)

Therefore, by using $D^{p+1}(t)$ and Hölder's inequality, we get

$$\int_{t}^{t+1} \int_{\Omega} |z_t|^2 \mathrm{d}x \mathrm{d}t \le C_{\Omega} D^2(t), \tag{45}$$

where $C_{\Omega} > 0$.

Then, there exist $t_1 \in [t, t + (1/4)]$ and $t_2 \in [t + (3/4), t + 1]$ so that

$$||z_t(t_i)|| \le CD(t), i = 1, 2.$$
 (46)

Multiply the first equation of (1) by z, integrate it over $\Omega \times [t_1, t_2]$, and apply Green's formula; we get

$$\int_{t}^{t+1} I(t) dt = -\left[\int_{t}^{t+1} \int_{\Omega} zz_{tt} dx dt + \int_{t}^{t+1} \int_{\Omega} \Delta z \Delta z_{t} dx dt + \int_{t}^{t+1} \int_{\Omega} |z_{t}|^{p-1} z_{t} z dx dt\right].$$

$$(47)$$

Now, we use the Cauchy-Schwarz inequality and Hölder inequality, and we get

$$\int_{t}^{t+1} I(t) dt \leq ||z_{t}(t_{1})|| ||z(t_{1})|| + ||z_{t}(t_{2})|| ||z(t_{2})|| + \int_{t}^{t+1} ||z_{t}(t)||^{2} dt + \int_{t}^{t+1} ||\Delta z_{t}|| ||\Delta z|| dt \qquad (48) - \int_{t}^{t+1} \int_{\Omega} |z_{t}|^{p-1} z_{t} z dx dt.$$

By using the Hölder inequality from the last term, we have

$$\int_{t}^{t+1} \int_{\Omega} |z_{t}|^{p-1} z_{t} z dx dt \leq \int_{t}^{t+1} ||z_{t}(t)||_{p+1}^{p} ||z(t)||_{p+1} dt.$$
(49)

Then, by (37), we have

$$\begin{split} & \overset{t+1}{t} \|z_{t}(t)\|_{p+1}^{p} \|z(t)\|_{p+1} \mathrm{d}t \\ & \leq C_{*} \int_{t}^{t+1} \|z_{t}\|_{p+1}^{p} \|z\|_{\mathscr{D}_{0}^{2,m}(\Omega)} \mathrm{d}t \\ & \leq C_{*} \int_{t}^{t+1} \|z_{t}\|_{p+1}^{p} \|z\|_{\mathscr{D}_{0}^{2,m}(\Omega)}^{m} \mathrm{d}t \\ & \leq C_{*} \left(\frac{m(q+1)}{q+1-m}\right)^{1/m} \int_{t}^{t+1} \|z_{t}\|_{p+1}^{p} E^{1/m}(s) \mathrm{d}t \\ & \leq C_{*} \left(\frac{m(q+1)}{q+1-m}\right)^{1/m} \sup_{t_{1} \leq s \leq t_{2}} E^{1/m}(s) D^{p}(t). \end{split}$$

Next, we calculate the fourth term of the right-hand side of (48), and we obtain

$$\begin{split} \int_{t}^{t+1} \|\Delta z_{t}(t)\| \|\Delta z(t)\| dt \\ &\leq C_{*} \int_{t}^{t+1} \|\Delta z_{t}\| \|z\|_{\mathcal{D}_{0}^{2m}(\Omega)}^{m} dt \\ &\leq C_{*} \left(\frac{m(q+1)}{q+1-m}\right)^{1/m} \int_{t}^{t+1} \|\Delta z_{t}\| E^{1/m}(s) dt \\ &\leq C_{*} \left(\frac{m(q+1)}{q+1-m}\right)^{1/m} \sup_{t_{1} \leq s \leq t_{2}} E^{1/m}(s) \int_{t}^{t+1} \|\Delta z_{t}\| dt. \end{split}$$
(51)

Later

$$\int_{t}^{t+1} \|\Delta z_{t}\| dt \leq \left(\int_{t}^{t+1} 1 dt\right)^{1/2} \left(\int_{t}^{t+1} \|\Delta z_{t}\|^{2} dt\right)^{1/2} \leq k \mathcal{D}(t).$$
(52)

Then,

$$\int_{t}^{t+1} \|\Delta z_{t}(t)\| \|\Delta z(t)\| dt \le kC_{*} \left(\frac{m(q+1)}{q+1-m}\right)^{1/m} \sup_{t_{1} \le s \le t_{2}} E^{1/m}(s) D^{p}(t).$$
(53)

We have

$$\|z_t(t_i)\| \|z(t_i)\| \le C_1 D(t) \sup_{t_1 \le s \le t_2} E^{1/m}(s),$$
 (54)

where $C_1 = kC_* (m(q+1)/(q+1-m))^{1/m}$. Therefore,

$$\begin{split} \int_{t_1}^{t_2} I(t) \mathrm{dt} &\leq C_1 D(t) \sup_{t_1 \leq s \leq t_2} E^{1/m}(s) + D^2(t) \\ &+ \mathrm{kC}_* \left(\frac{m(q+1)}{q+1-m} \right)^{1/m} \sup_{t_1 \leq s \leq t_2} E^{1/m}(s) D(t) \qquad (55) \\ &+ \mathrm{kC}_* \left(\frac{m(q+1)}{q+1-m} \right)^{1/m} \sup_{t_1 \leq s \leq t_2} E^{1/m}(s) D^p(t). \end{split}$$

Moreover, we get

$$E(t) \le \frac{1}{2} ||z_t||^2 + C_2 I(t).$$
(56)

Here, $C_2 = (1/\eta)((q+1-m)/m(q+1)) + (1/q+1)$. Integrating over $[t_1, t_2]$, we get

$$\int_{t_1}^{t_2} E(t) dt \le \frac{1}{2} \int_{t_1}^{t_2} ||z_t||^2 dt + C_2 \int_{t_1}^{t_2} I(t) dt.$$
 (57)

Then, we get

$$\begin{split} \int_{t_1}^{t_2} & E(t) dt \leq \frac{1}{2} C_{\Omega} D^2(t) + C_2 \left[C_1 D(t) \sup_{t_1 \leq s \leq t_2} E^{1/m}(s) + D^2(t) \right. \\ & + k C_* \left(\frac{m(q+1)}{q+1-m} \right)^{1/m} \sup_{t_1 \leq s \leq t_2} E^{1/m}(s) D(t) \\ & + k C_* \left(\frac{m(q+1)}{q+1-m} \right)^{1/m} \sup_{t_1 \leq s \leq t_2} E^{1/m}(s) D^p(t) \right]. \end{split}$$

$$(58)$$

Next, integrating over $[t, t_2]$, we obtain

$$E(t) = E(t_2) + \int_t^{t_2} \left(\|\Delta z_\tau\|^2 + \|z_\tau\|_{p+1}^{p+1} \right) d\tau.$$
 (59)

Since $t_2 - t_1 \ge (1/2)$, we decide that

$$\int_{t_1}^{t_2} E(t) dt \ge (t_2 - t_1) E(t_2) \ge \frac{1}{2} E(t_2).$$
 (60)

Thus,

$$E(t_2) \le 2 \int_{t_1}^{t_2} E(t) dt.$$
 (61)

As a result,

$$E(t) \leq 2 \int_{t_1}^{t_2} E(t) dt + \int_{t}^{t_2} \left(\|\Delta z_{\tau}\|^2 + \|z_{\tau}\|_{p+1}^{p+1} \right) d\tau$$

= $2 \int_{t_1}^{t_2} E(t) dt + D^{p+1}(t).$ (62)

Hence,

$$E(t) \le \left(\frac{1}{2}C_{\Omega} + C_2\right)D^2(t) + D^{p+1}(t) + C_3[D(t) + D^p(t)]E^{1/m}(t).$$
(63)

Therefore,

$$E(t) \le C_4 \left[D^2(t) + D^{p+1}(t) + D^{m/(m-1)}(t) + D^{(m/(m-1))p}(t) \right].$$
(64)

From E(t) that is a nonincreasing function and $E(t) \ge 0$ on $[0, \infty)$, we have

$$D^{p+1}(t) = E(t) - E(t+1) \le E(0).$$
(65)

After that,

$$D(t) \le E^{1/(p+1)}(0).$$
 (66)

Therefore,

$$\begin{split} E(t) &\leq C_4 \left[D^2(t) + D^{p+1}(t) + D^{m/(m-1)}(t) + D^{(m/(m-1))p}(t) \right] \\ &\leq C_4 D^{m/(m-1)}(t) \left[D^{(m-2)/(m-1)}(t) - D^{p-(1/(m-1))}(t) \right. \\ &\quad + 1 + D^{m(p-1)/(m-1)}(t) \right] = C_5 D^{m/(m-1)}(t). \end{split}$$

$$\end{split}$$

We obtain

$$E^{1+(((m-1)p-1)/m)}(t) \le C_6 D^{p+1}(t).$$
(68)

Case 1. When p = 1 and m = 2

$$E(t) \le C_6 D^2(t) = C_6 [E(t) - E(t+1)], \tag{69}$$

and by Lemma 1, we obtain

$$E(t) \le E(0)e^{-w_0[t-1]^+},\tag{70}$$

where $w_0 = \ln (C_6/(C_6 - 1))$. *Case 2.* When (m - 1)p > 1

$$E(t) \le \left(E(0)^{-\lambda} + C_6^{-1}\lambda[t-1]^+ \right)^{-(1/\lambda)}, \tag{71}$$

where $\lambda = ((m-1)p - 1)/m$, which complete the proof of Theorem 9.

5. Blow-up

In this part, we prove our main blow-up results to problem (1) for p = 1.

Definition 10. A solution z to problem (1) is referred to as a blow-up if there exists a finite time T^* so that

$$\lim_{t \longrightarrow T^{*-}} \mathscr{B}(t) = \infty.$$
(72)

Then, we have

$$\mathscr{B}(t) = \int_{\Omega} z^2 \mathrm{d} \mathbf{x} + \int_0^t \int_{\Omega} \left(|z|^2 + |\Delta z|^2 \right) \mathrm{d} \mathbf{x} \mathrm{d} \tau \text{ for } \mathbf{t} \ge 0.$$
(73)

Lemma 11. We assume that m < q + 1 < nm, n > m, p = 1, and $m - 2 \le 4\delta \le q - 1$. Then

$$\mathcal{B}^{\prime\prime}(t) \ge (4\delta + 4) \int_{\Omega} z_t^2 dx - (8\delta + 4) E(0) + (8\delta + 4) \int_0^t (\|z_{\tau}\|^2 + \|\Delta z_{\tau}\|^2) d\tau.$$
(74)

Proof. By taking the first- and second-order derivative of (73), we get

$$\mathscr{B}'(t) = 2 \int_{\Omega} z z_t^2 d\mathbf{x} + ||z||^2 + ||\Delta z||^2,$$
(75)

$$\mathscr{B}''(t) = 2 \int_{\Omega} z_t^2 d\mathbf{x} + 2 \int_{\Omega} z z_{tt} d\mathbf{x} + 2 \int_{\Omega} \Delta z \Delta z_t d\mathbf{x} + 2 \int_{\Omega} |z| z_t d\mathbf{x}$$
$$= 2 ||z_t||^2 - 2 \int_{\Omega} k(x) |\Delta z|^m d\mathbf{x} + 2 ||z||_{q+1}^{q+1}.$$
(76)

By (76) and (30), we have

$$\mathcal{B}^{\prime\prime}(t) = (4\delta + 4) \int_{\Omega} z_t^2 d\mathbf{x} - (8\delta + 4) E(0) + (8\delta + 4) \int_0^t (||z_\tau||^2 + ||\Delta z_\tau||^2) d\tau + \left(\frac{8\delta + 4 - 2m}{m}\right) \int_{\Omega} k(x) |\Delta z|^m dx + \left(\frac{2(q+1) - (8\delta + 4)}{q+1}\right) ||z||_{q+1}^{q+1}.$$
(77)

Here, $m - 2 < 4\delta \le q - 1$, we have (74).

Lemma 12. Let m < q + 1 < (mn/(n - m)) and n > m, then

(i) If
$$E(0) < 0$$
, then $\mathscr{B}'(t) > ||z_0||^2$ for $t > t^*$, here $t_0 = t^*$

$$t^* = \max\left\{0, \frac{\mathscr{B}'(t) - ||z_0||^2}{(8\delta + 4)E(0)}\right\}.$$
 (78)

(ii) When E(0) = 0 and $\int_{\Omega} z_0 z_1 dx > 0$. Then, $\mathscr{B}'' \ge 0$ for $t \ge 0$. We get

$$\mathscr{B}' \ge ||z_0||^2, t \ge 0.$$
 (79)

Proof.

(i) When E(0) < 0 and for $t \ge 0$, then

$$\mathscr{B}''(t) \ge -(8\delta + 4)E(0)t,$$
 (80)

and by integration over [0, t], we get

$$\mathscr{B}'(t) \ge \mathscr{B}'(0) - (8\delta + 4)E(0)t, t \ge 0.$$
 (81)

Then, we get $\mathscr{B}'(t) > ||z_0||^2$ for $t > t^*$, with

$$t^* = \max\left\{0, \frac{\mathscr{B}'(t) - ||z_0||^2}{(8\delta + 4)E(0)}\right\}.$$
 (82)

(ii) When
$$E(0) = 0$$
 and $\int_{\Omega} z_0 z_1 dx > 0$. Then, $\mathscr{B}' \ge 0$ for $t \ge 0$. We get $\mathscr{B}' \ge ||z_0||^2$, $t \ge 0$

Theorem 13. Suppose that m < q + 1 < (mn/(n - m)) and m < n, we obtain Case 1 and Case 2.

Case 1. If E(0) < 0 and the solution z blows up in finite time T^* in the sense of $\lim_{t \to T^{*-}} \mathscr{B}(t) = \infty$ and

$$T^* \leq t_0 - \frac{\mathscr{L}(t_0)}{\mathscr{L}'(t_0)}.$$
(83)

Furthermore, if $\mathcal{L}(t_0) < \min \{1, (-(\alpha/\beta))^{1/2}\}$, we get

$$T^* \le t_0 + \frac{1}{\sqrt{-\beta}} \ln \frac{\sigma}{\sigma - \mathscr{L}(t_0)}, \tag{84}$$

where $\sigma = (-(\alpha/\beta))^{\alpha/\beta}$.

Case 2. If E(0) = 0 and $\int_{\Omega} z_0 z_1 dx > 0$. The solution z blows up in finite time T^* in the sense of $\lim_{t \to T^{*-}} \mathscr{B}(t) = \infty$ and

$$T^* \leq t_0 - \frac{\mathscr{L}(t_0)}{\mathscr{L}'(t_0)}. \tag{85}$$

With

$$\begin{split} \alpha &= \delta^2 \mathscr{L}^{2+(2/\delta)}(t_0) \left[\left(\mathscr{F}'(t_0) \right)^2 - 8E(0) \mathscr{L}^{-(1/\delta)}(t_0) \right] > 0, \\ \beta &= 8\delta^2 E(0). \end{split}$$

$$\end{split}$$

$$(86)$$

Proof. Set

$$\mathscr{L}(t) = \left[\mathscr{B}(t) + (T - t) \|z_0\|^2\right]^{-\delta}, t \in [0, T],$$
(87)

where

$$\mathscr{F}(t) = \mathscr{B}(t) + (T - t) ||z_0||^2,$$
(88)

where T is a strictly positive constant that will defined later. Then, by taking the first- and the second-order derivative of $\mathcal{L}(t)$, we have

$$\mathscr{L}'(t) = -\delta \left[\mathscr{B}(t) + (T-t) \|z_0\|^2 \right]^{-\delta-1} \left[\mathscr{B}'(t) - \|z_0\|^2 \right]$$
$$= -\delta \mathscr{L}^{1+(1/\delta)}(t) \left[\mathscr{B}'(t) - \|z_0\|^2 \right],$$
$$\mathscr{L}''(t) = -\delta \mathscr{L}^{1+(2/\delta)}(t) \mathscr{B}''(t) \left[\mathscr{B}(t) - (T-t) \|z_0\|^2 \right]$$
$$+ \delta \mathscr{L}^{1+(2/\delta)}(t) (1+\delta) \left[\mathscr{B}'(t) - \|z_0\|^2 \right]^2,$$
$$\mathscr{L}''(t) = -\delta \mathscr{L}^{1+(2/\delta)}(t) \mathscr{C}(t),$$
(89)

where

$$\mathscr{C}(t) = \mathscr{B}''(t)\mathscr{F}(t) - (1+\delta)\left(\mathscr{F}'(t)\right)^2.$$
(90)

For simplicity of calculation, we define

$$A_{z} = \int_{\Omega} z^{2} d\mathbf{x},$$

$$B_{z} = \int_{\Omega} z_{t}^{2} d\mathbf{x},$$

$$C_{z} = \int_{0}^{t} ||z||^{2} d\mathbf{t},$$

$$D_{z} = \int_{0}^{t} ||z_{t}||^{2} d\mathbf{t}.$$
(91)

By (75) and Hölder's inequality, we get

$$\mathscr{B}'(t) = 2 \int_{\Omega} zz_t dx + ||z_0||^2 + 2 \int_0^t \int_{\Omega} zz_t dx dt$$

$$\leq 2 \left((B_z A_z)^{1/2} + (C_z D_z)^{1/2} \right) + ||z_0||^2.$$
(92)

By Case 1 and Lemma 12, we obtain

$$\mathscr{B}''(t) \ge -(8\delta + 4)E(0) + (4\delta + 4)(B_z + D_z).$$
(93)

Then, by (87), (90), and (93), we have

$$\begin{aligned} \mathscr{C}(t) &\geq \left[-(8\delta + 4)E(0) + (4\delta + 4)(B_z + D_z) \right] \mathscr{L}^{-(1/\delta)}(t) \\ &- (4\delta + 4) \left((B_z A_z)^{1/2} + (C_z D_z)^{1/2} \right)^2. \end{aligned} \tag{94}$$

By $\mathscr{B}(t)$, we get

$$\mathscr{B}(t) = \int_{\Omega} z^2 d\mathbf{x} + \int_0^t \int_{\Omega} z^2 dt ds = A_z + C_z$$
(95)

and from $\mathscr{L}(t)$, we get

$$\begin{aligned} \mathscr{C}(t) &\geq -(8\delta + 4)E(0)\mathscr{L}^{-(1/\delta)}(t) \\ &+ (4\delta + 4)\left[(B_z + D_z)(T - t)\|z_0\|^2 + \lambda(t)\right], \\ \lambda(t) &= (B_z + D_z)(A_z + C_z) - \left((B_z A_z)^{1/2} + (C_z D_z)^{1/2}\right)^2. \end{aligned}$$

$$\tag{96}$$

When $\lambda(t)$ is a nonnegative function, we obtain

$$\mathscr{C}(t) \ge -(8\delta + 4)E(0)\mathscr{L}^{-(1/\delta)}(t), \text{ for } t \ge t_0.$$
(97)

Therefore, from $\mathscr{B}''(t)$, we have

$$\mathscr{L}''(t) \le \left(4\delta + 8\delta^2\right) E(0) \mathscr{L}^{1+(1/\delta)}(t), \text{ for } t \ge t_0.$$
(98)

We obtain

$$\mathscr{L}'(t) < 0, t \ge t_0. \tag{99}$$

Multiplying of (98) by $\mathscr{B}'(t)$, integrate it over $[t_0, t]$, and we get

$$\mathscr{L}'2(t) \ge \alpha + \beta \mathscr{L}^{2+(1/\delta)}(t), \text{ for } t \ge t_0,$$
(100)

with α and β defined.

Finally, utilizing Lemma 3, there exists a T^* such that $\lim_{t \to T^{*-}} \mathscr{L}(t) = 0$ and is estimated based on the sign of E(0). Thus, equation (72) is satisfied.

Data Availability

There are no underlying data supporting the results of the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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