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Research Article

Simple Proofs for Bochner-Schoenberg-Eberlein and the Bochner-Schoenberg-Eberlein Module Properties on $\ell^p(X, A)$

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Let X be a nonempty set, A be a commutative Banach algebra, and $1 \leq p < \infty$. In this paper, we present a concise proof for the result concerning the BSE (Banach space extension) property of $\ell^p(X, A)$. Specifically, we establish that $\ell^p(X, A)$ possesses the BSE property if and only if X is finite and A is BSE. Additionally, we investigate the BSE module property on Banach $\ell^p(X, A)$-modules and demonstrate that a Banach space $\ell^p(X, Y)$ serves as a BSE Banach $\ell^p(X, A)$-module if and only if X is finite and Y represents a BSE Banach A-module.

1. Introduction

The abbreviation BSE refers to the well-known Bochner-Schoenberg-Eberlein theorem, which provides a characterization of the Fourier-Stieltjes transforms of bounded Borel measures on locally compact abelian groups. In essence, this theorem describes the BSE property of the group algebra $L^1(G)$ for a locally compact abelian group G. For further information, we refer to [1–4]. Additionally, for integrable functions on the positive real line, denoted as $L^1(\mathbb{R}^+)$, for the function space consisting of bounded complex-valued continuous functions on a locally compact Hausdorff space X, and for characterizing the Fourier and Fourier-Stieltjes transforms on locally compact abelian groups, consult references [5–7].

Let A be a commutative Banach algebra with the character space $\Delta(A)$ and X be a Banach A-module. We define a multiplier from A into X as an A-module morphism from A into X, denoted by $\mathscr{M}(A, X)$. For any $T \in \mathscr{M}(A, X)$, there exists a unique vector field $\widehat{T}$ on $\Delta(A)$ such that $\widehat{T(a)} = a\widehat{T}$ for all $a \in A$. When $X = A$, we denote $\mathscr{M}(A, A)$ by $\mathscr{M}(A)$, see [8], for more details.

A bounded continuous function σ on $\Delta(A)$ is called a BSE function if there exists a constant $C > 0$ such that for any finite number of $\varphi_1, \cdots, \varphi_n \in \Delta(A)$ and complex numbers $c_1, \cdots, c_n$, the inequality

$$\left|\sum_{i=1}^{n} c_i \sigma(\varphi_i)\right| \leq C \left\|\sum_{i=1}^{n} c_i \varphi_i\right\|_{A^*} \tag{1}$$

holds, where A^* is the first dual of A. The BSE norm of σ, denoted as $\|\cdot\|_{\mathrm{BSE}}$, is defined as the infimum of all such C. The set of all BSE functions is denoted by $C_{\mathrm{BSE}}(\Delta(A))$.

A Banach algebra A is called a BSE algebra if the BSE functions on $\Delta(A)$ are precisely the Gelfand transforms of the elements of $\mathscr{M}(A)$, i.e., $\widehat{\mathscr{M}(A)} = C_{\mathrm{BSE}}(\Delta(A))$. This notion was introduced by Takahasi and Hatori in [9] and characterized by Kaniuth and Ülger in [10]. There are many literatures that contain interesting results on BSE algebras; see [10–20] for more details.

Takahasi in [21] generalized the BSE property to Banach modules. Let A be a commutative Banach algebra with a bounded approximate identity and X be a symmetric Banach A-bimodule, i.e., $a \cdot x = x \cdot a$, for all $a \in A$ and $x \in X$. Let $\varphi \in \Delta(A)$. Denote $\ker\varphi$ by $M_\varphi = \{a \in A : \varphi(a) = 0\}$. There exists $e_\varphi \in A$ such that $\varphi(e_\varphi) = 1$. Now, define

$$X^\varphi = \overline{\mathrm{sp}}\left\{M_\varphi X + \left(1 - e_\varphi\right)X\right\}, \tag{2}$$

where $\overline{\mathrm{sp}}$ is the closed linear span. Note that X^φ is independent of the choice of e_φ. Then, X^φ becomes a Banach

A-submodule of X. Now, define $X_\varphi = X/X^\varphi$ and $\widehat{x}(\varphi) = x + X^\varphi$, for all $x \in X$. Hence, X_φ becomes a Banach A-bimodule. Let $\prod X_\varphi$ be the class of all functions σ defined on $\Delta(A)$ such that $\sigma(\varphi) \in X_\varphi$. An element of $\prod X_\varphi$ is called a *vector field* on $\Delta(A)$. The space $\prod X_\varphi$ is an A-module by the following action:

$$(a \cdot \sigma)(\varphi) = \varphi(a)\sigma(\varphi), \quad \left(a \in A, \varphi \in \Delta(A), \sigma \in \prod X_\varphi\right). \tag{3}$$

Set

$$\prod{}^b X_\varphi = \left\{\sigma \in \prod X_\varphi : \|\sigma\|_\infty = \sup_{\varphi \in \Delta(A)} \|\sigma(\varphi)\| < \infty\right\}. \tag{4}$$

For each $\varphi \in \Delta(A)$, define $\pi_\varphi(x) = \widehat{x}(\varphi)$, for all $x \in X$. A vector field $\sigma \in \prod X_\varphi$ is called *BSE* if there exists $\beta \in \mathbb{R}^+$ such that for any finite number of $\varphi_1, \cdots, \varphi_n \in \Delta(A)$ and the same number $f_1 \in (X_{\varphi_1})^*, \cdots, f_n \in (X_{\varphi_n})^*$, we have

$$\left|\sum_{i=1}^{n} \langle \sigma(\varphi_i), f_i \rangle\right| \le \beta \left\|\sum_{i=1}^{n} f_i \circ \pi_{\varphi_i}\right\|_{X^*}, \tag{5}$$

where $(X_{\varphi_i})^*$ denotes the dual space of the Banach space X_{φ_i}. Moreover, set

$$\prod_{\mathrm{BSE}} X_\varphi = \left\{\sigma \in \prod X_\varphi : \sigma \text{ is BSE}\right\}. \tag{6}$$

A vector field $\sigma \in \prod X_\varphi$ is called continuous if it is continuous at every $\varphi \in \Delta(A)$. The class of all continuous vector fields in $\prod X_\varphi$ is denoted by $\prod{}^c X_\varphi$ and set $\prod^c_{\mathrm{BSE}} X_\varphi = \prod_{\mathrm{BSE}} X_\varphi \bigcap \prod{}^c X_\varphi$. Let $\widehat{X} = \{\widehat{x} : x \in X\}$ and $\widehat{\mathscr{M}(A, X)} = \{\widehat{T} : T \in \mathscr{M}(A, X)\}$. A Banach A-module X is called *BSE* if $\widehat{\mathscr{M}(A, X)} = \prod^c_{\mathrm{BSE}} X_\varphi$, for all $\varphi \in \Delta(A)$. In [21], some examples of Banach algebras that have BSE module property such as group algebras on locally compact groups are given, and in [11], authors characterized module property of module extensions of Banach algebras.

Similar to $\prod X_\varphi$, we define $\prod X_\varphi$ as the class of all functions σ defined on $\Delta(A)$ such that $\sigma(\varphi) \in X^\varphi$. By the similar module action that we have defined for $\prod X_\varphi$, $\prod X_\varphi$ becomes an A-module.

Let X be a nonempty set and A be a commutative Banach algebra. Suppose that $1 \le p < \infty$ and define

$$\ell^p(X, A) = \left\{f : X \longrightarrow A : \sum_{x \in X} \|f(x)\|^p < \infty\right\}. \tag{7}$$

Then, $\ell^p(X, A)$ with the pointwise product and the following norm becomes a commutative Banach algebra:

$$\|f\|_p = \left(\sum_{x \in X} \|f(x)\|^p\right)^{1/p}, (f \in \ell^p(X, A)). \tag{8}$$

Some interesting results related to the maximal ideal space of $\ell^p(X, A)$ and BSE property on it are given in [16]. Moreover, some results such as the regularity of $\ell^p(X, A)$ and the existence of BED on $\ell^p(X, A)$ are obtained in [22].

In the subsequent section, our initial focus is on examining the BSE property within the context of the ℓ^p-direct sum of Banach algebras. Through the insights gained from these investigations, we are able to provide a streamlined proof for the primary outcome established in [16]. This streamlined proof offers a clear and concise demonstration that can enhance comprehension of the BSE concept as it pertains to $\ell^p(X, A)$. Moving forward to Section 3, we delve into exploring the BSE module property concerning Banach modules over $\ell^p(X, A)$. Our key finding in this section establishes that for any $\ell^p(X, A)$-module, denoted as $\ell^p(X, Y)$, to exhibit the BSE property, it is a necessary and sufficient condition that the module Y itself possesses the BSE attribute as an A-module gate in the BSE module property on Banach modules over $\ell^p(X, A)$, and as a main result, we prove that for any $\ell^p(X, A)$-module, $\ell^p(X, Y)$ is BSE if and only if Y is BSE as an A-module.

2. A Simple Proof for BSE Property on $\ell^p(X, A)$

Kamali and Abtahi proved the following result in [16].

Theorem 1. *Let X be a set and A be a commutative semisimple Banach algebra. Then, $\ell^p(X, A)$ is a BSE algebra if and only if X is finite and A is a BSE algebra.*

In this section, we present an alternative and straightforward demonstration of the aforementioned outcome. Initially, we provide pertinent details pertaining to the direct sum of Banach algebras (modules). Let A and B be two commutative Banach algebras and $1 \le p, q \le \infty$ such that $1/p + 1/q = 1$. Consider the ℓ^p-direct sum algebra $A \oplus_p B$ of A and B with the coordinate wise sum and product, i.e.,

$$\begin{aligned}(a, b) + \left(a', b'\right) &= \left(a + a', b + b'\right),\\ (a, b) \cdot \left(a', b'\right) &= \left(aa', bb'\right),\end{aligned} \tag{9}$$

for all $(a, b), (a', b') \in A \times B$, where $A \times B$ is the Cartesian product of A and B. For $1 \le p < \infty$, we equip $A \oplus_p B$ by the following norm:

$$\|(a, b)\|_p = \left(\|a\|_A^p + \|b\|_B^p\right)^{1/p}, (a, b) \in A \oplus_p B. \tag{10}$$

Moreover, for $p=\infty$, *we equip* $A\oplus_\infty B$ *by the following norm:*

$$\|a,b\|_\infty = \max\left\{\|a\|_A, \|b\|_B\right\}, (a,b)\in A\oplus_\infty B. \tag{11}$$

Throughout this section, the above notations are supposed to hold. Note that for $p=1$, $A\oplus_1 B$ is the direct sum of A and B and we refer [17], for more details. To achieve our goal in this section, we need some results related to a direct sum of Banach algebras. We summarize some properties of the above structures as follows.

Proposition 2.

(i) $(A\oplus_p B, \|\cdot\|_p)$, *for* $1\le p<\infty$, *and* $(A\oplus_\infty B, \|\cdot\|_\infty)$ *are commutative Banach algebras*

(ii) The first dual of $(A\oplus_p B, \|\cdot\|_p)$, *for* $1\le p<\infty$ *and* $1<q\le\infty$, *is* $(A^*\oplus_q B^*, \|\cdot\|_q)$; *this holds for any Banach spaces A and B*

(iii) Let $E=\{(\varphi,0)\colon \varphi\in\Delta(A)\}$ *and* $F=\{(0,\psi)\colon \psi\in\Delta(B)\}$. *Then,* $\Delta(A\oplus_p B)=E\cup F$, *for* $1\le p<\infty$

(iv) $A\oplus_p B$ *has a bounded approximate identity if and only if A and B have bounded approximate identity*

Proof. (i) and (ii) are clear. (iii) is Lemma 2.1 in[17]. Let $1\le p<\infty$ and $(E_\alpha)_\alpha=(e_\alpha,f_\alpha)_\alpha\subseteq A\oplus_p B$ be a bounded approximate identity for $A\oplus_p B$. Hence, there exists $0<C<\infty$ such that $\|E_\alpha\|_p\le C$. This implies that $\|e_\alpha\|_A, \|f_\alpha\|_B\le C$. Moreover,

$$\|ae_\alpha-a\|_A^p=\|(ae_\alpha,0)-(a,0)\|_p^p=\|(a,0)E_\alpha-(a,0)\|_p^p\longrightarrow 0, \tag{12}$$

for all $a\in A$. Thus, $(e_\alpha)_\alpha\subseteq A$ is a bounded approximate identity for A. Similarly, $(f_\alpha)_\alpha$ becomes a bounded approximate identity for B. For the case $p=\infty$, similar to (12), we have

$$\|ae_\alpha-a\|_A=\|(ae_\alpha,0)-(a,0)\|_\infty=\|(a,0)E_\alpha-(a,0)\|_\infty\longrightarrow 0, \tag{13}$$

for all $a\in A$. The rest of the proof is similar.

Conversely, suppose that $(e_\alpha)_\alpha$ and $(f_\beta)_\beta$ are bounded approximate identities for A and B, respectively. Then, there exist $0<C_1, C_2<\infty$ such that $\|e_\alpha\|_A\le C_1$ and $\|f_\beta\|_B\le C_2$. Now, we set $E_{\alpha,\beta}=(e_\alpha,f_\beta)$ and $C=\max\{C_1,C_2\}$. Then,

$$\|E_\alpha\|_p^p=\|e_\alpha\|_A^p+\left\|f_\beta\right\|_B^p\le C_1^p+C_2^p\le 2C^p. \tag{14}$$

This means that $(E_{\alpha,\beta})_{\alpha,\beta}$ is bounded net in $A\oplus_p B$. For the case $A\oplus_\infty B$, the proof is similar. For any $(a,b)\in A\oplus_p B$,

$$\begin{aligned}\left\|(a,b)E_{\alpha,\beta}-(a,b)\right\|_p^p &= \left\|(a,b)\left(e_\alpha,f_\beta\right)-(a,b)\right\|_p^p\\ &= \left\|\left(ae_\alpha,bf_\beta\right)-(a,b)\right\|_p^p\\ &= \left\|\left(ae_\alpha-a,bf_\beta-b\right)\right\|_p^p\\ &= \|ae_\alpha-a\|_A^p+\left\|bf_\beta-b\right\|_B^p\longrightarrow 0.\end{aligned} \tag{15}$$

Thus, $(E_{\alpha,\beta})_{\alpha,\beta}$ is bounded approximate identity for $A\oplus_p B$. For the case $A\oplus_\infty B$, we have

$$\begin{aligned}\left\|(a,b)E_{\alpha,\beta}-(a,b)\right\|_\infty &= \left\|(a,b)\left(e_\alpha,f_\beta\right)-(a,b)\right\|_\infty\\ &= \left\|\left(ae_\alpha,bf_\beta\right)-(a,b)\right\|_\infty\\ &= \left\|\left(ae_\alpha-a,bf_\beta-b\right)\right\|_\infty\\ &= \max\left\{\|ae_\alpha-a\|_A,\left\|bf_\beta-b\right\|_B\right\}\longrightarrow 0.\end{aligned} \tag{16}$$

This completes the proof. □

The BSE property on direct sums of Banach algebras is investigated in [17], where the authors showed that for commutative semisimple Banach algebras A and B, $A\oplus B$ is BSE if and only if A and B are BSE of Theorem 2.4 in [17]. Similarly, we have the following.

Theorem 3. *Let A and B be two commutative semisimple Banach algebras and* $1\le p<\infty$. *Then,* $A\oplus_p B$ *is BSE if and only if A and B are BSE.*

Proof. The proof for $p=1$ is the proof of Theorem 2.4 in [17]. Now, let $1<p<\infty$, A, and B be BSE. Similar discussion in the proof of Theorem 2.4 in [17] implies that $\mathscr{M}(A\oplus_p B)\subseteq C_{\mathrm{BSE}}(\Delta(A\oplus_p B))$, and moreover, for any $\sigma\in C_b(\Delta(A\oplus_p B))\cap(A\oplus_p B)^{**}|_{\Delta(A\oplus_p B)}=C_b(\Delta(A\oplus_p B))\cap A^{**}\oplus_p B^{**}|_{\Delta(A\oplus_p B)}$, there exist $\sigma_A\in A^{**}$ and $\sigma_B\in B^{**}$ such that $\sigma_A\in C_b(\Delta(A))\cap A^{**}|_{\Delta(A)}$, $\sigma_B\in C_b(\Delta(B))\cap B^{**}|_{\Delta(B)}$, and $\sigma=(\sigma_A,\sigma_B)|_{\Delta(A\oplus_p B)}$. By $\sigma\in C_{\mathrm{BSE}}(\Delta(A\oplus_p B))$, there exists $C>0$ such that for any finite number of $c_1,\cdots,c_n\in\mathbb{C}$ and $(\varphi_1,\psi_1),\cdots,(\varphi_n,\psi_n)\in\Delta(A\oplus_p B)$, we have

$$\left|\sum_{i=1}^n c_i\sigma(\varphi_i,\psi_i)\right|\le C\left\|\sum_{i=1}^n c_i(\varphi_i,\psi_i)\right\|_{A^*\oplus_q B^*}. \tag{17}$$

Then, by (17) and letting $\psi_i = 0$, for all $1 \le i \le n$, we have

$$\left|\sum_{i=1}^{n} c_i \sigma_A(\varphi_i)\right|^q = \left|\sum_{i=1}^{n} c_i \sigma(\varphi_i, 0)\right|^q \le C^q \left\|\sum_{i=1}^{n} c_i(\varphi_i, 0)\right\|_{A^* \oplus_q B^*}^q = C^q \left\|\sum_{i=1}^{n} c_i \varphi_i\right\|_{A^*}^q. \quad (18)$$

This implies that

$$\left|\sum_{i=1}^{n} c_i \sigma_A(\varphi_i)\right| \le C \left\|\sum_{i=1}^{n} c_i \varphi_i\right\|_{A^*}. \quad (19)$$

Thus, $\sigma_A \in C_{\mathrm{BSE}}(\Delta(A))$ and so is in $\mathcal{M}(A)$. Similarly, by letting $\varphi_i = 0$, for all $1 \le i \le n$ in (17), we obtain that $\sigma_B \in \mathcal{M}(B)$. These together imply that $\sigma = (\sigma_A, \sigma_B) \in \mathcal{M}(A \oplus_p B)$. This means that $C_{\mathrm{BSE}}(\Delta(A \oplus_p B)) \subseteq \mathcal{M}(A \oplus_p B)$. Hence, $A \oplus_p B$ is BSE. The converse is similar to the converse case of Theorem 2.4 in [17]. □

As an immediate result, we have the following result that plays an important role in the proof of the main result of this section.

Corollary 4. *Let $A_1 \cdots, A_n$ be commutative semisimple Banach algebras and $1 \le p < \infty$. Then, $\oplus_{i=1}^{n} A_i$ is BSE if and only if each A_i is BSE, for all $1 \le i \le n$.*

Lemma 5. *Let X be a nonempty set and A be a commutative Banach algebra. If X is finite, then $\ell^p(X, A)$ is a ℓ^p-direct sum of $|X|$ copies of A, where $|X|$ is the cardinal number of X.*

Proof. If X is finite, then there exists $n \in \mathbb{N}$ such that $|X|$ is isomorphic to $\mathbb{N}_n = \{1, \cdots, n\}$. Hence, without loss of generality, we can see $\ell^p(X, A)$ as follows:

$$\ell^p(\mathbb{N}_n, A) = \left\{ (a_i)_{i \in \mathbb{N}_n} : \sum_{i=1}^{n} \|a_i\|_A^p < \infty \right\}. \quad (20)$$

This implies that

$$\ell^p(X, A) = \ell^p(\mathbb{N}_n, A) = \underbrace{A \oplus_p \cdots \oplus_p A}_{n \text{ times}}. \quad (21)$$

□

If X is a finite set and we show its cardinal number by $|X|$, then we denote

$$\underbrace{A \oplus_p \cdots \oplus_p A}_{|X| \text{ times}} \quad (22)$$

by $\underset{|X|}{\oplus_p} A$. A bounded net $(e_\alpha)_\alpha \subseteq A$ is called a Δ-weak approximate identity for A, if $\varphi(e_\alpha) \longrightarrow 1$, for all $\varphi \in \Delta(A)$. Clearly, every bounded approximate identity for a Banach algebra is a Δ-weak approximate identity. We now investigate the existence of bounded approximate identity for $\ell^p(X, A)$. We recall the following result.

Theorem 6 (see [16], Theorem 2.6). *Let X be a set, A be a commutative Banach algebra, and $1 \le p < \infty$. Then, $\ell^p(X, A)$ has a Δ-weak approximate identity if and only if X is finite and A has a Δ -weak approximate identity.*

Now, we are ready to give a simple proof for Theorem 1.

Proof of Theorem 1. Let $\ell^p(X, A)$ be BSE. Then, by Corollary 5 in [9], it has a Δ-weak approximate identity. Thus, Theorem 6 implies that X is finite and A has Δ-weak approximate identity. Then, by employing Lemma 5, we see that $\ell^p(X, A)$ is as $\underset{|X|}{\oplus_p} A$. Then, by applying Corollary 4, we conclude that A is BSE. The converse holds, clearly, because if X is finite and A is BSE, again by Corollary 4, $\underset{|X|}{\oplus_p} A$ is BSE and Lemma 5 implies that $\ell^p(X, A)$ is BSE.

3. BSE Module Property of $\ell^p(X, A)$-Modules

The main aim of this section is the investigation of the existence of BSE module property on $\ell^p(X, A)$-modules, where A is a commutative Banach algebra.

Lemma 7. *Let X be a nonempty set, A be a commutative Banach algebra, and Y be in Banach A -module. Then, $\ell^p(X, Y)$ is a Banach $\ell^p(X, A)$-module.*

Proof. We define the left module action of $\ell^p(X, A)$ on $\ell^p(X, Y)$ as follows:

$$(f \cdot g)(x) = f(x) g(x), f \in \ell^p(X, A), g \in \ell^p(X, Y), x \in X. \quad (23)$$

It is easy to verify that the above-defined left action is well defined and

$$\begin{aligned} \|f \cdot g\|_p^p &= \sum_{x \in X} \|(f \cdot g)(x)\|_Y^p \\ &= \sum_{x \in X} \|f(x) g(x)\|_Y^p \\ &\le \sum_{x \in X} \|f(x)\|_A^p \|g(x)\|_Y^p \\ &\le \left(\sum_{x \in X} \|f(x)\|_A^p\right)\left(\sum_{x \in X} \|g(x)\|_Y^p\right) \\ &= \|f\|_p^p \|g\|_p^p. \end{aligned} \quad (24)$$

Thus, $\ell^p(X, Y)$ is a Banach $\ell^p(X, A)$-module.

Our main result in this section is the following result, and we give a short and simple proof for it. □

Theorem 8. *Let X be a nonempty set, A be a commutative Banach algebra, and Y be in Banach A-module. Then, $\ell^p(X, Y)$ is a BSE Banach $\ell^p(X, A)$-module if and only if X is finite and Y is a BSE Banach A-module.*

Before proving the above result, we give some results related to p-direct sums of Banach algebra and their module properties. Let X and Y be Banach A- and B-modules, respectively. For $1 \le p < \infty$, we consider the Banach space $X \oplus_p Y$ with the norm $\|\cdot\|_p$. Then, by the following action, $X \oplus_p Y$ becomes a Banach $A \oplus_p B$-module:

$$(a, b) \cdot (x, y) = (a \cdot x, b \cdot y), (a, b) \in A \oplus_p B, (x, y) \in X \oplus_p Y. \tag{25}$$

From now on, we suppose that A and B have bounded approximate identities.

Proposition 9. *$T \in \mathcal{M}(A \oplus_p B, X \oplus_p Y)$ if and only if there exist $T_{A,X} \in \mathcal{M}(A, X)$ and $T_{B,Y} \in \mathcal{M}(B, Y)$ such that*

$$T(a, b) = (T_{A,X}(a), T_{B,Y}(b)) \quad (a, b) \in A \oplus_p B. \tag{26}$$

Proof. Let $T \in \mathcal{M}(A \oplus_p B, X \oplus_p Y$. Define $\imath_A : A \longrightarrow A \oplus_p B$ by $\imath_A(a) = (a, 0)$, $\imath_B : A \longrightarrow A \oplus_p B$ by $\imath_B(b) = (0, b)$, $\pi_X : X \oplus_p Y \longrightarrow X$ by $\pi_X(x, y) = x$, and $\pi_Y : X \oplus_p Y \longrightarrow Y$ by $\pi_Y(x, y) = y$, for all $a \in A$, $b \in B$, $x \in X$, and $y \in Y$. These maps are linear and bounded. Finally, we define $T_{A,X} \coloneqq \pi_X \circ T \circ \imath_A$, $T_{A,Y} \coloneqq \pi_Y \circ T \circ \imath_A$, $T_{B,X} \coloneqq \pi_X \circ T \circ \imath_B$, and $T_{B,Y} \coloneqq \pi_Y \circ T \circ \imath_B$. Clearly, these maps are linear and bounded, and we have

$$T(a, b) = (T_{A,X}(a) + T_{B,X}(b), T_{A,Y}(a) + T_{B,Y}(b)) \quad (a, b) \in A \oplus_p B. \tag{27}$$

For any $(a, b), (a', b') \in A \oplus_p B$, (27) implies that

$$(a, b) \cdot T\left(a', b'\right) = \left(a \cdot T_{A,X}\left(a'\right) + a \cdot T_{B,X}\left(b'\right), b \cdot T_{A,Y}\left(a'\right) + b \cdot T_{B,X}\left(b'\right)\right), \tag{28}$$

$$T\left((a, b)\left(a', b'\right)\right) = \left(T_{A,X}\left(aa'\right) + T_{B,X}\left(bb'\right), T_{A,Y}\left(aa'\right) + T_{B,X}\left(bb'\right)\right). \tag{29}$$

Letting $a = 0$ in (28) and (29) and possessing a bounded approximate identity for B together imply that $T_{B,X} = 0$. Similarly, by letting $b = 0$, we obtain that $T_{A,Y} = 0$. Moreover, we have $a \cdot T_{A,X}(a') = T_{A,X}(aa')$ and $b \cdot T_{B,X}(b') = T_{B,X}(bb')$, for all $a, a' \in A$ and $b, b' \in B$. These show that $T_{A,X} \in \mathcal{M}(A, X)$ and $T_{B,Y} \in \mathcal{M}(B, Y)$. The converse can be verified easily. So the proof holds. □

Proposition 10. *Let $\varphi \in \Delta(A)$, $\psi \in \Delta(B)$, X be a Banach A-module, and Y be Banach B-module. Then,*

$$M_{(\varphi,\psi)} = \left(M_\varphi, 0\right) \cup \left(0, M_\psi\right). \tag{30}$$

(i) $(X \oplus_p Y)^{(\varphi,0)} = X^\varphi \oplus_p Y$ and $(X \oplus_p Y)^{(0,\psi)} = X \oplus_p Y^\psi$

(ii) $(X \oplus_p Y)_{(\varphi,0)} \cong X_\varphi$ and $(X \oplus_p Y)_{(0,\psi)} \cong Y_\psi$ as Banach spaces

(iii) $\prod_{BSE}^c (X \oplus_p Y)_\Phi = \{(\sigma_X, \sigma_Y): \sigma_X \in \prod_{BSE}^c X_\varphi, \sigma_Y \in \prod_{BSE}^c Y_\psi\}$, where $\Phi \in \Delta(A \oplus_p B)$ such that $\sigma(\varphi, 0) = \sigma_X(\varphi)$ and $\sigma(0, \psi) = \sigma_Y(\psi)$

Proof.

(i) Let $a \in M_\varphi$ and $b \in M_\psi$. Then, $\varphi(a) + \psi(b) = 0$. This means that $(a, b) \in M_{(\varphi,\psi)}$. Hence, $(M_\varphi, 0) \cup (0, M_\psi) \subseteq M_{(\varphi,\psi)}$. Now, suppose that $(a, b) \in M_{(\varphi,\psi)}$. Thus, $\varphi(a) + \psi(b) = 0$. We claim that $a \in M_\varphi$ and $b \in M_\psi$. Assume towards a contradiction, $\varphi(a) \neq 0$. This implies that $\psi(b) \neq 0$. These facts say that $\varphi \neq 0$ implies that $\psi \neq 0$. For any $(a, b), (a', b') \in A \oplus_p B$, similar to proof of Lemma 2.1 in [17], we have

$$\begin{aligned}(\varphi, \psi)\left((a, b)\left(a', b'\right)\right) &= [(\varphi, \psi)(a, b)]\left[(\varphi, \psi)\left(a', b'\right)\right] \\ &= (\varphi(a) + \psi(b))\left(\varphi\left(a'\right) + \psi\left(b'\right)\right) \\ &= \varphi(a)\varphi\left(a'\right) + \varphi(a)\psi\left(b'\right) \\ &\quad + \psi(b)\varphi\left(a'\right) + \psi(b)\psi\left(b'\right).\end{aligned} \tag{31}$$

On the other hand, for any $(a, b), (a', b') \in A \oplus_p B$,

$$\begin{aligned}(\varphi, \psi)\left((a, b)\left(a', b'\right)\right) &= (\varphi, \psi)\left(aa', bb'\right) \\ &= \varphi\left(aa'\right) + \psi\left(bb'\right) \\ &= \varphi(a)\varphi\left(a'\right) + \psi(b)\psi\left(b'\right).\end{aligned} \tag{32}$$

So (31) and (32) imply that

$$\varphi(a)\psi\left(b'\right) + \psi(b)\varphi\left(a'\right) = 0, \tag{33}$$

for all $a, a' \in A$ and $b, b' \in B$. If $\varphi \neq 0$, by letting $a' \in M_\varphi$ and $a \notin M_\varphi$, we conclude that $\psi(b')\varphi(a) = 0$, for all $b \in B$. Hence, $\psi(b') = 0$, for all $b' \in B$. This means that $\psi = 0$, a contradiction. Thus, $a \in M_\varphi$. Similarly, this holds for M_ψ.

Hence, if $(a,b)\in M_{(\varphi,\psi)}$, then $a\in M_\varphi$ and $b\in M_\psi$. This shows that $M_{(\varphi,\psi)}\subseteq (M_\varphi,0)\cup(0,M_\psi)$.

(ii) Suppose that $e_\varphi\in A$ and $e_\psi\in B$ such that $\varphi(e_\varphi)=1$ and $\psi(e_\psi)=1$. Let $x\in X^\varphi$ and $y\in Y^\psi$. Then, there exist $a_1,\cdots,a_n\in M_\varphi$, $x_1,\cdots,x_n,x_{1'},\cdots,x_{m'}\in X$, $b_1,\cdots,b_t\in M_\psi$, and $y_1,\cdots,y_t,y_1,\cdots,y_l\in Y$ such that, for every $\varepsilon>0$, we have

$$\left\|x-\left(\sum_{i=i}^{n}a_ix_i+\left(1-e_\varphi\right)\sum_{j=1}^{m}x_{j'}\right)\right\|_X<\varepsilon, \tag{34}$$

$$\left\|y-\left(\sum_{i=1}^{t}b_iy_i+\left(1-e_\psi\right)\sum_{j=1}^{l}y_{j'}\right)\right\|_Y<\varepsilon. \tag{35}$$

Then, for any $x'\in X$ and $y'\in Y$, (34) and (35) imply that

$$\begin{aligned}
&\left\|\left(x,y'\right)-\left(\sum_{i=1}^{n}(a_i,0)\left(x_i,y'\right)+\left((1,1)-(e_\varphi,0)\right)\sum_{j=1}^{m}\left(x_{j'},\frac{1}{m}y'\right)\right)\right\|_{X\oplus_p Y}\\
&=\left\|\left(x-\left(\sum_{i=1}^{n}a_ix_i+\left(1-e_\varphi\right)\sum_{j=1}^{m}x_{j'}\right),0\right)\right\|_{X\oplus_p Y}\\
&=\left\|x-\left(\sum_{i=i}^{n}a_ix_i+\left(1-e_\varphi\right)\sum_{j=1}^{m}x_{j'}\right)\right\|_X<\varepsilon,\\
&\left\|\left(x',y\right)-\left(\sum_{i=i}^{t}(0,b_i)\left(x',y_i\right)+\left((1,1)-(0,e_\psi)\right)\sum_{j=1}^{l}\left(\frac{1}{l}x',y_{j'}\right)\right)\right\|_{X\oplus_p Y}\\
&=\left\|\left(0,y-\left(\sum_{i=i}^{N}b_iy_i+\left(1-e_\psi\right)\sum_{j=1}^{L}y_{j'}\right)\right)\right\|_{X\oplus_p Y}\\
&=\left\|y-\left(\sum_{i=1}^{t}b_iy_i+\left(1-e_\psi\right)\sum_{j=1}^{l}y_{j'}\right)\right\|_Y<\varepsilon.
\end{aligned} \tag{36}$$

These show that $X^\varphi\oplus_p Y\subseteq (X\oplus_p Y)^{(\varphi,0)}$ and $X\oplus_p Y^\psi\subseteq (X\oplus_p Y)^{(0,\psi)}$. Now, let $(x,y)\in(X\oplus_p Y)^{(\varphi,0)}$. Then, from (i), there exist $(a_1,0),\cdots,(a_n,0)\in M_{(\varphi,\psi)}$ and $(x_1,),\cdots,(x_n,y)$, $(x_{1'},y),\cdots,(x_{m'},y)\in X\oplus_p Y$ such that, for every $\varepsilon>0$,

$$\begin{aligned}
\varepsilon&>\left\|(x,y)-\left(\sum_{i=1}^{n}(a_i,0)(x_i,y)+\left((1,1)-(e_\varphi,0)\right)\sum_{j=1}^{m}\left(x_{j'},\frac{1}{m}y\right)\right)\right\|_{X\oplus_p Y}\\
&=\left\|\left(x-\left(\sum_{i=1}^{n}a_ix_i+\left(1-e_\varphi\right)\sum_{j=1}^{m}x_{j'}\right),0\right)\right\|_{X\oplus_p Y}\\
&=\left\|x-\left(\sum_{i=i}^{n}a_ix_i+\left(1-e_\varphi\right)\sum_{j=1}^{m}x_{j'}\right)\right\|_X.
\end{aligned} \tag{37}$$

Thus, (37) implies that $x\in X^\varphi$. This implies that $(x,y)\in X^\varphi\oplus_p Y$, and so, we have $(X\oplus_p Y)^{(\varphi,0)}\subseteq X^\varphi\oplus_p Y$. This implies that $(X\oplus_p Y)^{(\varphi,0)}=X^\varphi\oplus_p Y$. Similarly, one can verify that $X\oplus_p Y^\psi=(X\oplus_p Y)^{(0,\psi)}$. Thus, (ii) holds.

(iii) Define $\Theta: X\oplus_p Y\longrightarrow X_\varphi\oplus Y$ by $\Theta(x,y)=(x+X^\varphi, y)$, for all $(x,y)\in X\oplus_p Y$. It is easy to verify that Θ is a continuous homomorphism between Banach spaces. Moreover,

$$\ker\Theta=\left\{(x,y)\in X\oplus_p Y:\Theta(x,y)=0_{X_\varphi\oplus Y}=X^\varphi\oplus_p Y\right\}=X^\varphi\oplus_p Y. \tag{38}$$

This implies that

$$\left(X\oplus_p Y\right)_{(\varphi,0)}=\frac{X\oplus_p Y}{\left(X\oplus_p Y\right)^{(\varphi,0)}}\cong X_\varphi. \tag{39}$$

Similarly, define $\Phi: X\oplus_p Y\longrightarrow X\oplus Y_\psi$ by $\Theta(x,y)=(x, y+Y^\psi)$, for all $(x,y)\in X\oplus_p Y$. We have

$$\ker\Phi=\left\{(x,y)\in X\oplus_p Y:\Phi(x,y)=0_{X\oplus Y_\psi}=X\oplus_p Y^\psi\right\}=X\oplus_p Y^\psi. \tag{40}$$

Then,

$$\left(X\oplus_p Y\right)_{(0,\psi)}=\frac{X\oplus_p Y}{\left(X\oplus_p Y\right)^{(0,\psi)}}\cong Y_\psi. \tag{41}$$

(iv) Define $\pi_\varphi^X(x)=\widehat{x}(\varphi)$, $\pi_\psi^Y(y)=\widehat{y}(\psi)$, and $\pi_{(\varphi,\psi)}(x,y)=(\pi_\varphi^X(x),\pi_\psi^Y(y))$, for all $x\in X$ and $y\in Y$. Let $\sigma_X\in\prod_{\mathrm{BSE}}^c X_\varphi$ and $\sigma_Y\in\prod_{\mathrm{BSE}}^c Y_\psi$; then, for any $\varphi_1,\cdots,\varphi_n\in\Delta(A)$ and $\psi_1,\cdots,\psi_m\in\Delta(B)$, there exist $\beta_1,\beta_2\in\mathbb{R}^+$, $f_1\in(X_{\varphi_1})^*,\cdots,f_n\in(X_{\varphi_n})^*$, and $g_1\in(Y_{\psi_1})^*,\cdots,g_m\in(Y_{\psi_n})^*$ such that

$$\left|\sum_{i=1}^{n}\langle\sigma_X(\varphi_i),f_i\rangle\right|\le\beta_1\left\|\sum_{i=1}^{n}f_i\circ\pi_{\varphi_i}^X\right\|_{X^*}, \tag{42}$$

$$\left|\sum_{i=1}^{m}\langle\sigma_Y(\psi_i),g_i\rangle\right|\le\beta_2\left\|\sum_{i=1}^{m}g_i\circ\pi_{\psi_i}^Y\right\|_{Y^*}. \tag{43}$$

Let $\Phi_1,\cdots,\Phi_t\in\Delta(A\oplus_p B)$ and $F_1\in(X\oplus_p Y)^*_{\Phi_1},\cdots,F_t\in(X\oplus_p Y)^*_{\Phi_t}$, where $t=m+n$. Then, by Proposition 2 (iii), $\Phi_i=(\varphi_i,0)$ or $=(0,\psi_i)$. Thus, rearrange Φ_i's as follows:

$$\Phi_i=\begin{cases}(\varphi_i,0), & 1\le i\le n,\\ (0,\psi_i), & n+1\le i\le t.\end{cases} \tag{44}$$

By (iii) and Proposition 2 (ii), we have $(X \oplus_p Y)^*_{(\varphi_i,0)} \cong X^*_{\varphi_i} \oplus_q Y^*$ and $(X \oplus_p Y)^*_{(0,\psi_i)} \cong X^* \oplus_q Y^*_{\psi_i}$, for any $1 \le i \le t$. Thus, there exist $f_1 \in (X_{\varphi_1})^*, \cdots, f_n \in (X_{\varphi_n})^*$ and $g_{n+1} \in (Y_{\psi_{n+1}})^*, \cdots, g_t \in (Y_{\psi_t})^*$ such that

$$F_i = \begin{cases} (f_i, 0), & 1 \le i \le n, \\ (0, g_i), & n+1 \le i \le t. \end{cases} \tag{45}$$

First, we suppose that $p \neq 1$, so $1 < q < \infty$. Then, by employing (42) and (43) and the fact that $(r+s)^q \le 2^{q-1}(r^q + s^q)$, for all $r, s \in \mathbb{R}^+$, we have

$$\begin{aligned}
\left|\sum_{i=1}^{t} \langle (\sigma_X, \sigma_Y)(\Phi_i), F_i \rangle\right|^q &= \left|\sum_{i=1}^{n} \langle (\sigma_X, \sigma_Y)(\Phi_i), F_i \rangle + \sum_{i=n+1}^{t} \langle (\sigma_X, \sigma_Y)(\Phi_i), F_i \rangle\right|^q \\
&= \left|\sum_{i=1}^{n} \langle (\sigma_X, \sigma_Y)(\varphi_i, 0), (f_i, 0) \rangle \right. \\
&\quad \left. + \sum_{i=n+1}^{t} \langle (\sigma_X, \sigma_Y)(0, \psi_i), (0, g_i) \rangle\right|^q \\
&= \left|\sum_{i=1}^{n} \langle \sigma_X(\varphi_i), f_i \rangle + \sum_{i=n+1}^{t} \langle \sigma_Y(\psi_i), g_i \rangle\right|^q \\
&\le 2^{q-1} \left(\left|\sum_{i=1}^{n} \langle \sigma_X(\varphi_i), f_i \rangle\right|^q + \left|\sum_{i=1}^{m} \langle \sigma_Y(\psi_i), g_i \rangle\right|^q \right) \\
&\le 2^{q-1} \left(\beta_1^q \left\|\sum_{i=1}^{n} f_i \circ \pi^X_{\varphi_i}\right\|^q_{X^*} + \beta_2^q \left\|\sum_{i=1}^{m} g_i \circ \pi^Y_{\psi_i}\right\|^q_{Y^*} \right) \\
&\le \max\left\{2^{q-1}\beta_1^q, 2^{q-1}\beta_2^q\right\} \left(\left\|\sum_{i=1}^{n} f_i \circ \pi^X_{\varphi_i}\right\|^q_{X^*} \right. \\
&\quad \left. + \left\|\sum_{i=1}^{m} g_i \circ \pi^Y_{\psi_i}\right\|^q_{Y^*} \right) \\
&= \beta^q \left\| \left(\sum_{i=1}^{n} f_i \circ \pi^X_{\varphi_i}, \sum_{i=1}^{m} g_i \circ \pi^Y_{\psi_i} \right) \right\|^q_{X^* \oplus_q Y^*} \\
&= \beta^q \left\| \sum_{i=1}^{t} F_i \circ \pi_{\Phi_i} \right\|^q_{X^* \oplus_q Y^*} \\
&= \beta^q \left\| \sum_{i=1}^{t} F_i \circ \pi_{\Phi_i} \right\|^q_{(X \oplus_p Y)^*},
\end{aligned} \tag{46}$$

where $\beta = \max\{2^{1/p}\beta_1, 2^{1/p}\beta_2\}$. Thus,

$$\left|\sum_{i=1}^{n} \langle (\sigma_X, \sigma_Y)(\Phi_i), F_i \rangle\right| \le \beta \left\|\sum_{i=1}^{n} F_i \circ \pi_{\Phi_i}\right\|_{(X \oplus_p Y)^*}. \tag{47}$$

Now, let $p = 1$; then, similar to the above discussion, we have

$$\begin{aligned}
\left|\sum_{i=1}^{t} \langle (\sigma_X, \sigma_Y)(\Phi_i), F_i \rangle\right| &= \left|\sum_{i=1}^{n} \langle (\sigma_X, \sigma_Y)(\Phi_i), F_i \rangle \right. \\
&\quad \left. + \sum_{i=n+1}^{t} \langle (\sigma_X, \sigma_Y)(\Phi_i), F_i \rangle\right| \\
&= \left|\sum_{i=1}^{n} \langle (\sigma_X, \sigma_Y)(\varphi_i, 0), (f_i, 0) \rangle \right. \\
&\quad \left. + \sum_{i=n+1}^{t} \langle (\sigma_X, \sigma_Y)(0, \psi_i), (0, g_i) \rangle\right| \\
&\le \left|\sum_{i=1}^{n} \langle \sigma_X(\varphi_i), f_i \rangle\right| + \left|\sum_{i=1}^{m} \langle \sigma_Y(\psi_i), g_i \rangle\right| \\
&\le \beta_1 \left\|\sum_{i=1}^{n} f_i \circ \pi^X_{\varphi_i}\right\|_{X^*} + \beta_2 \left\|\sum_{i=1}^{m} g_i \circ \pi^Y_{\psi_i}\right\|_{Y^*} \\
&\le 2\max\{\beta_1, \beta_2\} \max \\
&\quad \cdot \left\{ \left\|\sum_{i=1}^{n} f_i \circ \pi^X_{\varphi_i}\right\|_{X^*}, \left\|\sum_{i=1}^{m} g_i \circ \pi^Y_{\psi_i}\right\|_{Y^*} \right\} \\
&= \beta \left\| \left(\sum_{i=1}^{n} f_i \circ \pi^X_{\varphi_i}, \sum_{i=1}^{m} g_i \circ \pi^Y_{\psi_i} \right) \right\|_{X^* \oplus_\infty Y^*} \\
&= \beta \left\| \sum_{i=1}^{t} F_i \circ \pi_{\Phi_i} \right\|^q_{X^* \oplus_\infty Y^*} \\
&= \beta \left\| \sum_{i=1}^{t} F_i \circ \pi_{\Phi_i} \right\|_{(X \oplus_1 Y)^*},
\end{aligned} \tag{48}$$

where $\beta = 2\max\{\beta_1, \beta_2\}$. Hence,

$$\left|\sum_{i=1}^{n} \langle (\sigma_X, \sigma_Y)(\Phi_i), F_i \rangle\right| \le \beta \left\|\sum_{i=1}^{n} F_i \circ \pi_{\Phi_i}\right\|_{(X \oplus_1 Y)^*}. \tag{49}$$

Moreover, from the continuity of σ_X and σ_Y on $\Delta(A)$ and $\Delta(B)$, we have $(\sigma_X, \sigma_Y) \in \prod^c_{\mathrm{BSE}}(X \oplus_p Y)_{(\varphi,\psi)}$.

Let $\sigma \in \prod^c_{\mathrm{BSE}}(X \oplus_p Y)_{\Phi}$. Then, there exists $\beta \in \mathbb{R}^+$ such that for any $\Phi_1, \cdots, \Phi_t \in \Delta(A \oplus_p B)$ and $F_1 \in (X \oplus_p Y)^*_{\Phi_1}, \cdots, F_t \in (X \oplus_p Y)^*_{\Phi_t}$,

$$\left|\sum_{i=1}^{t} \langle \sigma(\Phi_i), F_i \rangle\right| \le \beta \left\|\sum_{i=1}^{t} F_i \circ \pi_{\Phi_i}\right\|_{(X \oplus_p Y)^*}. \tag{50}$$

Then, Φ_i's and F_i's are similar to (44) and (45). Moreover, for any $(\varphi, 0), (0, \psi) \in \Delta(A \oplus_p B)$, $\sigma(\varphi, 0) \in (X \oplus_p Y)_{(\varphi,0)}$ and $\sigma(0, \psi) \in (X \oplus_p Y)_{(0,\psi)}$. Then, by employing (iii), there exist $\sigma_X \in X_\varphi$ and $\sigma_Y \in Y_\psi$ such that $\sigma(\varphi, 0) = \sigma_X(\varphi)$ and $\sigma(0, \psi) = \sigma_Y(\psi)$. If for any $1 \le i \le n$, we suppose that $F_i = (f_i, 0)$, and for $n+1 \le i \le t$, $F_i = (0, 0)$; (50) implies that

$$\begin{aligned}\left|\sum_{i=1}^{n}\langle\sigma_X(\varphi_i), f_i\rangle\right| &= \left|\sum_{i=1}^{n}\langle(\sigma_X,\sigma_Y)(\Phi_i), (f_i, 0)\rangle\right| \\ &= \left|\sum_{i=1}^{t}\langle\sigma(\Phi_i), F_i\rangle\right| \le \beta\left\|\sum_{i=1}^{t} F_i\circ\pi_{\Phi_i}\right\|_{(X\oplus_p Y)^*} \\ &= \beta\left\|\sum_{i=1}^{t}(f_i,0)\circ\left(\pi_{\varphi_i}^X, \pi_{\psi_i}^Y\right)\right\|_{X^*\oplus_q Y^*} \\ &= \beta\left\|\sum_{i=1}^{n} f_i\circ\pi_{\varphi_i}^X\right\|_{X^*}.\end{aligned} \tag{51}$$

Thus, $\sigma_X \in \prod_{\mathrm{BSE}} X_\varphi$. Moreover, continuity of σ on $\Delta(A\oplus_p B)$ implies that σ_X is continuous and this means that $\sigma_X \in \prod_{\mathrm{BSE}}^c X_\varphi$. Similarly, by letting $F_i = (0,0)$, for $1\le i\le n$ and $F_i = (0, g_i)$, for all $i = 1+n,\cdots,t$, we conclude that $\sigma_Y \in \prod_{\mathrm{BSE}}^c Y_\psi$. □

Theorem 11. *Let X be a Banach A-module and Y be Banach B-module. Then, $X\oplus_p Y$ is a BSE Banach $A\oplus_p B$-module if and only if X is a BSE Banach A-module and Y is a BSE Banach B-module.*

Proof. Suppose that $X\oplus_p Y$ is a BSE Banach $A\oplus_p B$-module. Let $\sigma_X \in \prod_{\mathrm{BSE}}^c X_\varphi$ and $\sigma_Y \in \prod_{\mathrm{BSE}}^c Y_\psi$. Define $\sigma : \Delta(A\oplus_p B) \longrightarrow \bigcup_{\Phi\in E\cup F}(X\oplus_p Y)_\Phi$ by $\sigma(\varphi,0)=\sigma_X(\varphi)$ and $\sigma(0,\psi)=\sigma_Y(\psi)$, for all $\varphi\in\Delta(A)$ and $\psi\in\Delta(B)$. Since σ_X and σ_Y are BSE, similar to the proof of Proposition 10 (iv), we can conclude that $\sigma\in\prod_{\mathrm{BSE}}^c (X\oplus_p Y)_\Phi$. On the other hand, $X\oplus_p Y$ is a BSE Banach $A\oplus_p B$-module, so there exists $T\in\mathscr{M}(A\oplus_p B, X\oplus_p Y)$ such that $\sigma=\widehat{T}$. Moreover, $\widehat{T(a,b)} = (a,b)\widehat{T}$, for all $(a,b)\in A\oplus_p B$. By Proposition 9, there exist $T_{A,X}\in\mathscr{M}(A,X)$ and $T_{B,Y}\in\mathscr{M}(B,Y)$ such that

$$T(a,b) = (T_{A,X}(a), T_{B,Y}(b)) \quad (a,b)\in A\oplus_p B. \tag{52}$$

By letting $b=0$, in the above equation, we have $T(a,0) = ((T_{A,X}(a),0)$, for all $a\in A$. Then,

$$\begin{aligned}(a,0)\cdot\sigma = (a,0)\widehat{T} = \widehat{T(a,0)} &= (\widehat{T_{A,X}(a)}, 0) \\ &= \left(\widehat{T_{A,X}(a)}, 0\right) = \left(a\widehat{T_{A,X}}, 0\right).\end{aligned} \tag{53}$$

Moreover,

$$((a,0)\cdot\sigma)(\varphi,0) = \varphi(a)\sigma_X(\varphi) = (a\cdot\sigma_X)(\varphi) = (a\cdot\sigma_X, 0)(\varphi,0). \tag{54}$$

Thus, (53) and (54) imply that $a\cdot\sigma_X = a\widehat{T_{A,X}}$, for all $a\in A$. This implies that $\sigma_X = \widehat{T_{A,X}}$. This means that $\prod_{\mathrm{BSE}}^c X_\varphi \subseteq \mathscr{M}(A,X)$. Similarly, by letting $a=0$ in (52) and by the similar arguments as the above, we obtain that $\sigma_Y = \widehat{T_{B,Y}}$. Therefore, we have $\prod_{\mathrm{BSE}}^c Y_\psi \subseteq \mathscr{M}(B,Y)$.

Now, let $T\in\mathscr{M}(A\oplus_p B, X\oplus_p Y)$. Then, there exist $T_{A,X}\in\mathscr{M}(A,X)$ and $T_{B,Y}\in\mathscr{M}(B,Y)$ and (52) holds. Thus, there exists $\sigma\in\prod_{\mathrm{BSE}}^c (X\oplus_p Y)_\Phi$ such that $\sigma=\widehat{T}$. Then, by Proposition 10 (iv), there exist $\sigma_X\in\prod_{\mathrm{BSE}}^c X_\varphi$ and $\sigma_Y\in\prod_{\mathrm{BSE}}^c Y_\psi$ such that $\sigma(\varphi,0)=\sigma_X(\varphi)$ and $\sigma(0,\psi)=\sigma_Y(\psi)$. Choose an element $e_\varphi\in A$ such that $\varphi(e_\varphi)=1$. Then,

$$\begin{aligned}\sigma_X(\varphi) = \sigma(\varphi,0) = \widehat{T}(\varphi,0) &= \widehat{T(e_\varphi,0)}(\varphi,0) \\ &= (\widehat{T_{A,X}(e_\varphi)}, 0)(\varphi,0) = \left(\widehat{T_{A,X}(e_\varphi)}, 0\right)(\varphi,0) \\ &= \widehat{T_{A,X}(e_\varphi)}(\varphi) = \widehat{T_{A,X}}(\varphi).\end{aligned} \tag{55}$$

This implies that $\sigma_X = \widehat{T_{A,X}}$. Thus, $\mathscr{M}(A,X)\subseteq\prod_{\mathrm{BSE}}^c X_\varphi$. Hence, we obtain that $\prod_{\mathrm{BSE}}^c X_\varphi = \mathscr{M}(A,X)$. This means that X is a BSE Banach A-module. Similarly, if $e_\psi\in B$ satisfies $\psi(e_\psi)=1$, then we obtain $\sigma_Y(\psi) = \widehat{T_{B,Y}}(\psi)$. Thus, we have $\mathscr{M}(B,Y)\subseteq\prod_{\mathrm{BSE}}^c Y_\psi$ and so $\prod_{\mathrm{BSE}}^c Y_\psi = \mathscr{M}(B,Y)$. This implies that Y is BSE a Banach B-module.

Conversely, suppose that X and Y are BSE Banach A-module and B-module, respectively. Let $\sigma\in\prod_{\mathrm{BSE}}^c (X\oplus_p Y)_\Phi$, where $\Phi\in\Delta(A\oplus_p B)$. By Proposition 10 (iv), there exist $\sigma_X\in\prod_{\mathrm{BSE}}^c X_\varphi$ and $\sigma_Y\in\prod_{\mathrm{BSE}}^c Y_\psi$ such that $\sigma=(\sigma_X,\sigma_Y)$, $\sigma(\varphi,0)=\sigma_X(\varphi)$, and $\sigma(0,\psi)=\sigma_Y(\psi)$, for all $\varphi\in\Delta(A)$ and $\psi\in\Delta(B)$. Then, there exist $T_{A,X}\in\mathscr{M}(A,X)$ and $T_{B,Y}\in\mathscr{M}(B,Y)$ such that $\sigma_X=\widehat{T_{A,X}}$ and $\sigma_Y=\widehat{T_{B,Y}}$. Define $T : A\oplus_p B \longrightarrow X\oplus_p Y$ by $T(a,b) = (T_{A,X}(a), T_{B,Y}(b))$, for all $(a,b)\in A\oplus_p B$. Then, by Proposition 9, T is in $\mathscr{M}(A\oplus_p B, X\oplus_p Y)$. Assume that $e_\varphi\in A$ and $e_\psi\in B$ such that $\varphi(e_\varphi)=1$ and $\psi(e_\psi)=1$. Then,

$$\begin{aligned}\widehat{T}(\varphi,0) = (\widehat{T_{A,X}, T_{B,Y}})(\varphi,0) &= (\widehat{T_{A,X}, T_{B,Y}})(e_\varphi,0)(\varphi,0) \\ &= (\widehat{T_{A,X}(e_\varphi)}, 0)(\varphi,0) = \left(\widehat{T_{A,X}(e_\varphi)}, 0\right)(\varphi,0) \\ &= \left(\widehat{T_{A,X}}, 0\right)(\varphi,0) = \widehat{T_{A,X}}(\varphi) = \sigma_A(\varphi),\end{aligned} \tag{56}$$

$$\begin{aligned}\widehat{T}(0,\psi) = (\widehat{T_{A,X}, T_{B,Y}})(0,\psi) &= (\widehat{T_{A,X}, T_{B,Y}})(0,e_\psi)(0,\psi) \\ &= (0, \widehat{T_{B,Y}(e_\psi)})(0,\psi) = \left(0, \widehat{T_{B,Y}}(e_\psi)\right)(0,\psi) \\ &= \left(0, \widehat{T_{B,Y}}\right)(0,\psi) = \widehat{T_{B,Y}}(\psi) = \sigma_B(\psi).\end{aligned} \tag{57}$$

On the other hand, from Proposition 10 (iv), $\sigma=(\sigma_X,\sigma_Y)$ is in $\prod_{\mathrm{BSE}}^c (X\oplus_p Y)_\Phi$ such that $\sigma(\varphi,0)=\sigma_X(\varphi)$ and $\sigma(0,\psi)=\sigma_Y(\psi)$, for all $\varphi\in\Delta(A)$ and $\psi\in\Delta(B)$. Thus, (56) and (57) imply the $\sigma=\widehat{T}$, and consequently, $\prod_{\mathrm{BSE}}^c (X\oplus_p Y)_\Phi \subseteq \mathscr{M}(A\oplus_p B, X\oplus_p Y)$. Now, we show that $\mathscr{M}(A\oplus_p B, X\oplus_p Y)\subseteq\prod_{\mathrm{BSE}}^c (X\oplus_p Y)_\Phi$. Let $T\in\mathscr{M}(A\oplus_p B, X\oplus_p Y)$. Again, by

Proposition 9, there exist $T_{A,X} \in \mathscr{M}(A, X)$ and $T_{B,Y} \in \mathscr{M}(B, Y)$ satisfying (52). Thus, there exist $\sigma_X \in \prod^c_{\mathrm{BSE}} X_\varphi$ and $\sigma_Y \in \prod^c_{\mathrm{BSE}} Y_\psi$ such that $\widehat{T_{A,X}} = \sigma_X$ and $\widehat{T_{B,Y}} = \sigma_Y$. By similar argument in (56) and (57), we conclude that $\widehat{T} = (\sigma_X, \sigma_Y) \in \prod^c_{\mathrm{BSE}} (X \oplus_p Y)_\Phi$. Hence, $\mathscr{M}(A \oplus_p B, X \oplus_p Y) \subseteq \prod^c_{\mathrm{BSE}} (X \oplus_p Y)_\Phi$. This completes the proof. □

Example 1.

(i) Let A and B be two commutative C^*-algebras and I and J be closed ideals of A and B, respectively. From Theorem 3.1 in [21], I and J are BSE Banach A- and B-modules. Then, by Theorem 11, $I \oplus_p J$ is a BSE Banach $A \oplus_p B$-module

(ii) Let G and H be two compact abelian groups. Then, by Theorem 3.3 in [21], $L^r(G)$ and $L^s(H)$, where $1 \le r, s \le \infty$, are BSE Banach $L^1(G)$- and $L^1(H)$-modules. Hence, Theorem 11 implies that $L^r(G) \oplus_p L^s(H)$ is a BSE Banach $L^1(G) \oplus_p L^1(H)$-module

Proposition 12. *$\ell^p(X, A)$ has a bounded approximate identity if and only if A has a bounded approximate identity and X is finite.*

Proof. Suppose that $\ell^p(X, A)$ has a bounded approximate identity $(E_\alpha)_\alpha$ such that $\|E_\alpha\| < M$. This implies that it has a Δ-weak approximate identity. Then, by Theorem 6, we conclude that X is finite. For a fixed $x_0 \in X$ and any α, define $e_\alpha := E_\alpha(x_0)$. Thus, $(e_\alpha)_\alpha \subseteq A$ and

$$\|e_\alpha\|_A = \|E_\alpha(x_0)\|_A \le \left(\sum_{x\in X} \|E_\alpha(x_0)\|_A^p\right)^{1/p} \le \|E_\alpha\|_p < M. \tag{58}$$

This shows that $(e_\alpha)_\alpha$ is a bounded net in A. Moreover,

$$\begin{aligned} \lim_\alpha a e_\alpha &= \lim_\alpha a E_\alpha(x_0) = \lim_\alpha \delta_a^{x_0}(x_0) E_\alpha(x_0) \\ &= \lim_\alpha (\delta_a^{x_0} E_\alpha)(x_0) = \delta_a^{x_0}(x_0) = a. \end{aligned} \tag{59}$$

Hence, $(e_\alpha)_\alpha$ is a bounded approximate identity for A. Conversely, suppose that X is finite and A has a bounded approximate identity. Thus, by Lemma 5, $\ell^p(X, A)$ is as $\oplus_{p\,|X|} A$. Then, Proposition 2 (iv) implies $\oplus_{p\,|X|} A$, and consequently, $\ell^p(X, A)$ has a bounded approximate identity. □

Proof of Theorem 4. Since $\ell^p(X, A)$ has a bounded approximate identity, Proposition 12 implies that X is finite and A has a bounded approximate identity. Hence, by Lemma 5, one can see $\ell^p(X, A)$ as $\oplus_{p\,|X|} A$ and $\ell^p(X, Y)$ as $\oplus_{p\,|X|} Y$. Now, by employing Theorem 11, the proof holds. □

4. Conclusion

The BSE property and BSE module property in commutative Banach algebras and Banach modules are crucial for understanding the relationships between their multiplier spaces and maximal ideal spaces (character spaces). As mentioned earlier, the presence of these properties provides valuable insights into the structures of the spaces under investigation. In this study, we explore the BSE and BSE module properties of vector-valued functions belonging to $\ell^p(X, A)$. By examining the ℓ^p-direct sum of commutative Banach algebras, we present a concise and straightforward proof of the BSE property on $\ell^p(X, A)$, contrasting with the more complex proof presented in [16]. Furthermore, through the utilization of the ℓ^p-direct sum and the characterization of module multiplier spaces, we delve into the BSE module property of modules over $\ell^p(X, A)$. Our analysis reveals that $\ell^p(X, Y)$ is a BSE Banach $\ell^p(X, A)$-module if and only if X is finite and Y is a BSE Banach A-module.

Data Availability

The data supporting the findings of this study are available within the article.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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