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Research Article

Relative Uniform Convergence of Sequence of Functions Related to ℓ_p-Spaces Defined by Orlicz Functions

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The Orlicz function-defined sequence spaces of functions by relative uniform convergence of sequences related to p-absolutely summable spaces are a new concept that is introduced in this article. We look at its various attributes, such as solidity, completeness, and symmetry. We also look at a few insertional connections involving these spaces.

1. Introduction

The spaces of all, convergent, null, p-absolutely summable, and limited sequences are identified by the symbols ω, c, c_0, ℓ_p, and ℓ_∞. Sargent [1] established the space $\check{\mathbf{m}}_\phi$ in 1960 after researching its many characteristics and determining its relationship to the sequence space $\ell_p, (p \geqslant 1)$. Following that, a large number of researchers looked into this sequence space from various angles. Tripathy and Sen [2] introduced the space $\check{\mathbf{m}}_\phi(p), p \geqslant 1$, which generalized the space $\check{\mathbf{m}}_\phi$; they also examined other properties.

The term "Orlicz function" refers to a continuous, nondecreasing, convex function that has the following characteristics: $\mathcal{O}(0) = 0$, $\mathcal{O}(x) > 0$, for $x > 0$, and $\mathcal{O}(x) \longrightarrow \infty$, for $x \longrightarrow \infty$.

If there is a constant $K > 0$ such that $\mathcal{O}(2x) \leqslant K\mathcal{O}(x)$, for all values of $x \geqslant 0$, then an Orlicz function $\mathcal{O}$ is considered to satisfy the δ_2-condition for all values of x.

Numerous authors, such as Bhardwaj and Singh [3], have introduced the concept of lacunary sequences defined by the Orlicz function and satisfied some basic property; Bilgin [4] has studied on difference sequence defined by the Orlicz function; Gungor et al. [5] have introduced the λ-convergence sequences defined by the Orlicz function and have initiated their different property; Tripathy and Mahanta [6] have introduced the $m(\phi)$-space in the setting vector valued Sargent type sequences defined by the Orlicz function and established their different algebraic and topological property; and Parashar and Choudhary [7] have extended the ℓ_M-space introduced and investigated by Lindenstrauss and Tzafriri to the setting of Paranormed sequence spaces defined by the Orlicz function. This motivated others to study different types of new sequence spaces defined by the Orlicz function.

The modulus function, first presented by Nakano [8], is the function that results when $\mathcal{O}(x+y) \leqslant \mathcal{O}(x) + \mathcal{O}(y)$ replaces the convexity of the Orlicz function.

The concept of relative uniform convergence of a set of functions with respect to a scale function was initially put forward by Moore [9].

Later, Chittenden [10] proposed the notion of relative uniform convergence of a sequence of functions.

2. Definition and Background

2.1. Relative Uniform Convergence

Definition 1. For every tiny positive number ε, there is an integer n_ε such that for every $n_\varepsilon \leqslant n$, the inequality

$$|f(x) - f_n(x)| \leqslant \varepsilon|\mu(x)| \tag{1}$$

holds. This sequence of functions is represented by (f_n), defined on the compact domain D.

Scale function describes the function μ of the previous equation.

Many others researchers like Demirci et al. [11], Demirci and Orhan [12], Sahin and Dirik [13], and Devi and Tripathy ([14–16]) have explored the idea further.

Presume that any subsets of natural number that limit the number of items to s fall under the category of x_{i_s}. ϕ_n stands for a sequence of real integers that does not decrease for any $n \in \mathbb{N}$ such that $(\phi_{n+1}/\phi_n) \leqslant (n + 1/n)$.

As established by Sargent [1], the sequence space $\check{\mathbf{m}}_\phi$ is defined as follows:

$$\check{\mathbf{m}}_\phi = \left\{ (x_n) \in \omega : \|x_n\|_{\check{\mathbf{m}}_\phi} = \sup_{s\geqslant 1, \sigma\in\xi_s} \frac{1}{\phi_s} \sum_{n\in\sigma} |x_n| < \infty \right\}. \tag{2}$$

The sequence space and the Orlicz function concepts were first introduced by Lindenstrauss and Tzafriri [17].

$$\ell_{\mathcal{O}} = \left\{ (x_n) \in \omega : \sum_{n=1}^{\infty} \mathcal{O}\left(\frac{|x_n|}{\nu}\right) < \infty, \quad \text{where} \quad \nu > 0 \right\}. \tag{3}$$

The $\ell_{\mathcal{O}}$-space with respect to the norm

$$\|x_n\|_{\ell_{\mathcal{O}}} = \inf \left\{ \nu > 0 : \sum_{n=1}^{\infty} \mathcal{O}\left(\frac{|x_n|}{\nu}\right) \leqslant 1 \right\} \tag{4}$$

becomes an Orlicz sequence space, a Banach space. It is a little close to the ℓ_p-space, an Orlicz sequence space with the formula $\mathcal{O}(x) = x^p$ for $1 \leqslant p < \infty$.

The following sequence of functions are used in this article:

$$\begin{aligned}
{}_{ru}\ell_p^{\mathcal{O}} &= \left\{ (f_n) \in \omega : \sum_{n=1}^{\infty} \mathcal{O}\left(\frac{|f_n(x)|}{\nu}\right)^p < \tau|\mu(x)|, \text{for some } \nu > 0 \right\}, \\
{}_{ru}\ell_\infty^{\mathcal{O}} &= \left\{ (f_n) \in \omega : \sup_{n\geq 1} \mathcal{O}\left(\frac{|f_n(x)|}{\nu}\right) < \tau|\mu(x)|, \text{for some } \nu > 0 \right\}, \\
{}_{ru}c_0^{\mathcal{O}} &= \left\{ (f_n) \in \omega : \mathcal{O}\left(\frac{|f_n(x)|}{\nu|\mu(x)|}\right) \longrightarrow 0, n \longrightarrow \infty, \text{for some } \nu > 0 \right\}, \\
{}_{ru}c^{\mathcal{O}} &= \left\{ (f_n) \in \omega : \mathcal{O}\left(\frac{|f_n(x)| - L}{\nu|\mu(x)|}\right) \longrightarrow 0, \text{as } n \longrightarrow \infty, \text{for some } \nu > 0 \right\}.
\end{aligned} \tag{5}$$

We introduced the sequence of function space in this article:

$$\begin{aligned}
{}_{ru}\check{\mathbf{m}}_\phi^{\mathcal{O}} = \Bigg\{ (f_n) \in \omega : \sup_{s\geqslant 1, \sigma\in\xi_s, x\in D} \frac{1}{\phi_s} \sum_{n\in\sigma} \mathcal{O}\left(\frac{|f_n(x)|}{\nu}\right) \\
< \tau|\mu(x)|, \text{for some } \nu > 0 \Bigg\},
\end{aligned} \tag{6}$$

where $\mu(x)$ represents the scaling function on a compact region D.

The space ${}_{ru}\check{\mathbf{m}}_\phi^{\mathcal{O}}$ is normed by

$$\|f_n(x)\|_{{}_{ru}\check{\mathbf{m}}_\phi^{\mathcal{O}}} = \inf \left\{ \nu > 0 : \sup_{s\geqslant 1, \sigma\in\xi_s, x\in D} \frac{1}{\phi_s} \sum_{n\in\sigma} \mathcal{O}\left(\frac{|f_n(x)|}{\nu|\mu(x)|}\right) \leqslant 1 \right\}, \tag{7}$$

where the scale function is represented by $\mu(x)$.

Example 2. Let $\phi_n = 1$ for all $n \in \mathbb{N}$.

Consider an Orlicz function $\mathcal{O} : [0,\infty) \longrightarrow [0.\infty)$ defined by

$$\mathcal{O}(x) = x^2, \text{for all } x \in \mathbb{R}. \tag{8}$$

Let a sequence of function (f_n) defined by

$$f_n(x) = \begin{cases} \dfrac{x}{n}, & \text{for } x \neq 0, \\ 0, & \text{for } x = 0. \end{cases} \tag{9}$$

The above sequence of functions is relative uniform convergence with respect to the scale function $\mu(x)$, where

$$\mu(x) = \begin{cases} x, & \text{for } x \neq 0, \\ 0, & \text{for } x = 0. \end{cases} \tag{10}$$

Then, we get, for $\nu = 1$,

$$\sup_{s\geqslant 1, \sigma\in\xi_s, x\in D} \sum_{n\in\sigma} \frac{1}{n^2} < \tau. \tag{11}$$

3. Main Results

In this section, we formulate the hypothesis of the results of this article and establish them.

We state the following results without proof, which can be established using standard techniques.

Theorem 3. *A normed linear space based on the norm established by the space* ${}_{ru}\check{\mathbf{m}}_{\phi}^{\mathcal{O}}$ *is*

$$\|f_n\|_{{}_{ru}\check{\mathbf{m}}_{\phi}^{\mathcal{O}}} = \inf \left\{ \nu > 0 : \sup_{s\geq 1,\sigma\in\xi_s,x\in D} \frac{1}{\phi_s} \sum_{n\in\sigma} \mathcal{O}\left(\frac{|f_n(x)|}{\nu|\mu(x)|}\right) \leq 1 \right\}. \quad (12)$$

Remark 4. The spaces ${}_{ru}\ell_p^{\mathcal{O}}$ and ${}_{ru}\ell_\infty^{\mathcal{O}}$ are normed spaces, normed by

$$\|f\|_{{}_{ru}\ell_p^{\mathcal{O}}} = \inf \left\{ \nu > 0 : \sup_{x\in D} \sum_{n=1}^{\infty} \mathcal{O}\left(\frac{|f_n(x)|}{\nu|\mu(x)|}\right)^p \leq 1 \right\},$$
$$\|f\|_{{}_{ru}\ell_\infty^{\mathcal{O}}} = \inf \left\{ \nu > 0 : \sup_{n\geq 1,x\in D} \mathcal{O}\left(\frac{|f_n(x)|}{\nu|\mu(x)|}\right) \leq 1 \right\}, \quad (13)$$

respectively, where $\mu(x)$ is the scale function.

Theorem 5. *The class of sequences* ${}_{ru}\check{\mathbf{m}}_{\phi}^{\mathcal{O}}(p)$*, where* $1 > p > 0$*, is a p-normed space defined by*

$$\|f\|_{{}_{ru}\check{\mathbf{m}}_{\phi}^{\mathcal{O}}(p)} = \inf \left\{ \nu > 0 : \sup_{s\geq 1,\sigma\in\xi_s,x\in D} \sum_{n\in\sigma} \mathcal{O}\left(\frac{|f_n(x)|}{\nu|\mu(x)|}\right)^p \leq 1, \text{where } 1 > p > 0 \right\}, \quad (14)$$

where $f = (f_n) \in {}_{ru}\check{\mathbf{m}}_{\phi}^{\mathcal{O}}(p)$.

Proof. First, we establish that ${}_{ru}\check{\mathbf{m}}_{\phi}^{\mathcal{O}}(p)$ is a normed space for p.

(1) Clearly, $\|f\|_{{}_{ru}\check{\mathbf{m}}_{\phi}^{\mathcal{O}}(p)} = 0$, if and only if $f_n = \theta$, the null operator, for all $n \in \mathbb{N}$. Hence, $(f_n) = \bar{\theta}$, the null sequence of functions

(2) We have, for $f = (f_n) \in {}_{ru}\check{\mathbf{m}}_{\phi}^{\mathcal{O}}(p)$ and λ be any scalar

$$\|\lambda f\|_{{}_{ru}\check{\mathbf{m}}_{\phi}^{\mathcal{O}}(p)} = \sup_{s\geq 1,\sigma\in\xi_s,x\in D} \sum_{n\in\sigma} \mathcal{O}\left(\frac{|\lambda f_n(x)|}{\nu|\mu(x)|}\right)^p = |\lambda|^p \|f\|. \quad (15)$$

Let $(f_n), (g_n) \in {}_{ru}\check{\mathbf{m}}_{\phi}^{\mathcal{O}}(p)$. Then,

$$\begin{aligned} &\sup_{s\geq 1,\sigma\in\xi_s,x\in D} \frac{1}{\phi_s} \sum_{n\in\sigma} \mathcal{O}\left(\frac{|f_n(x) + g_n(x)|}{\nu|\mu(x)|}\right)^p \\ &\leq \sup_{s\geq 1,\sigma\in\xi_s,x\in D} \sum_{n\in\sigma} \mathcal{O}\left(\frac{|f_n(x)|}{\nu|\mu(x)|}\right)^p \\ &\quad + \sup_{s\geq 1,\sigma\in\xi_s,x\in D} \sum_{n\in\sigma} \mathcal{O}\left(\frac{|g_n(x)|}{\nu|\mu(x)|}\right)^p. \end{aligned} \quad (16)$$

Therefore,

$$\|f + g\|_{{}_{ru}\check{\mathbf{m}}_{\phi}^{\mathcal{O}}(p)} \leq \|f\|_{{}_{ru}\check{\mathbf{m}}_{\phi}^{\mathcal{O}}(p)} + \|g\|_{{}_{ru}\check{\mathbf{m}}_{\phi}^{\mathcal{O}}(p)}. \quad (17)$$

As a result, ${}_{ru}\check{\mathbf{m}}_{\phi}^{\mathcal{O}}(p)$, where $1 > p > 0$, is a normed space of type p.

We now show that ${}_{ru}\check{\mathbf{m}}_{\phi}^{\mathcal{O}}(p)$ is a whole p-normed space.

If (f_n^i) is a Cauchy sequence in ${}_{ru}\check{\mathbf{m}}_{\phi}^{\mathcal{O}}(p)$, then $\mu(x)$ is a scale function for and $\sup_i \mu^i(x)$ exists over $x \in D$, where $\sup_i \mu^i(x) = \mu(x)$.

Let $f^i = (f_n^i)_{n=1}^{\infty}$, for each fixed $i \in \mathbb{N}$.

After that, there is a positive integer $n_0(\varepsilon) > 0$ such that for a given $\varepsilon > 0$,

$$\left\| \frac{f_n^i - f_n^j}{\nu\mu} \right\|_{{}_{ru}\check{\mathbf{m}}_{\phi}^{\mathcal{O}}(p)} < \varepsilon, \quad (18)$$

$$\Longrightarrow \inf \left\{ \nu > 0 : \sup_{s\geq 1,\sigma\in\xi_s,x\in D} \frac{1}{\phi_s} \sum_{n\in\sigma} \mathcal{O}\left(\frac{|f_n^i(x) - f_n^j(x)|}{\nu|\mu(x)|}\right)^p \leq 1 \right\} < \varepsilon, \quad (19)$$

$$\Longrightarrow \inf \left\{ \nu > 0 : \sup_{s\geq 1,\sigma\in\xi_s,x\in D} \sum_{n\in\sigma} \mathcal{O}\left(\frac{|f_n^i(x) - f_n^j(x)|}{\nu|\mu(x)|}\right)^p \leq 1 \right\} < \varepsilon\phi_1, \quad (20)$$

for every $n_0 \leq i, j$ and $n \in \mathbb{N}$.

Taking $s = 1$ varying σ of ξ_s.

Let $(\phi_1\varepsilon)^{(1/p)} < \varepsilon_1$ then, we have from (20)

$$\inf \left\{ \nu > 0 : \sup_{s\geq 1,\sigma\in\xi_s,x\in D} \sum_{n\in\sigma} \mathcal{O}\left(\frac{|f_n^i(x) + f_n^j(x)|}{\nu|\mu(x)|}\right) \leq 1 \right\} < \varepsilon_1, \quad (21)$$

for every $n_0 \leq i, j$ and $n \in \mathbb{N}$.

Therefore,

$$\left(\frac{|f_n^i(x) - f_n^j(x)|}{\nu|\mu(x)|}\right) < \varepsilon_1. \quad (22)$$

Since C is complete, $(f_n^i)_{n=1}^{\infty}$ is convergence in C for $x \in D$, with regard to the scaling function $\mu(x)$. Therefore, for any fixed n, $(f_n^i)_{n=1}^{\infty}$ is a Cauchy sequence in C.

Let $f_n^i \longrightarrow f_n$ as $n \longrightarrow \infty$.

Now, we show that $(f_n) \in {}_{ru}\check{\mathbf{m}}_{\phi}^{\mathcal{O}}(p)$ and $f_n \longrightarrow f$, as $n \longrightarrow \infty$, for each, *fix* s.

$$\sum_{n\in\sigma} \mathcal{O}\left(\frac{|f_n^i - f_n^j|}{\nu|\mu(x)|}\right)^p \leq \varepsilon\phi_s, \text{for all } i, j \geq n_0,$$
$$\Longrightarrow \inf \left\{ \nu > 0 : \sup_{s\geq 1,\sigma\in\xi_s,x\in D} \sum_{n\in\sigma} \mathcal{O}\left(\frac{|f_n^i(x) - f_n(x)|}{\nu|\mu(x)|}\right)^p \leq 1 \right\} < \varepsilon_1, \quad (23)$$

meaning that for any $i \geqslant n_0$ related to the scale function $\mu(x)$, $(f_n^i - f_n) \in {}_{ru}\check{\mathbf{m}}_{\phi}^{\mathcal{O}}(p)$.

Hence, $f(x) = f^i(x) + f(x) - f^i(x) f \in {}_{ru}\check{\mathbf{m}}_{\phi}^{\mathcal{O}}(p)$ as ${}_{ru}\check{\mathbf{m}}_{\phi}^{\mathcal{O}}(p)$ is a linear space.

Also,

$$\left\| \frac{f_n - f}{\mu} \right\|_{{}_{ru}\check{\mathbf{m}}_{\phi}^{\mathcal{O}}(p)} < \varepsilon, \text{for all } n_0 \leqslant n. \tag{24}$$

Therefore, for $1 > p > 0$, ${}_{ru}\check{\mathbf{m}}_{\phi}^{\mathcal{O}}(p)$ is the whole p-normed space. □

Theorem 6. *The space* ${}_{ru}\check{\mathbf{m}}_{\phi}^{\mathcal{O}}(p) \subseteq {}_{ru}\check{\mathbf{m}}_{\psi}^{\mathcal{O}}$ *if and only if* $\sup_{s \geqslant 1} (\phi_s/\psi_s) < \infty$.

Proof. Suppose $\sup_{s \geqslant 1} (\phi_s/\psi_s) = L(<\infty)$ and $(f_n) \in {}_{ru}\check{\mathbf{m}}_{\phi}^{\mathcal{O}}$.

Then,

$$\sup_{s\geqslant 1, \sigma\in\xi_s, x\in D} \frac{1}{\phi_s} \sum_{n\in\sigma} \mathcal{O}\left(\frac{|f_n(x)|}{\nu}\right) < \tau|\mu(x)|, \text{in some } \nu > 0. \tag{25}$$

Now,

$$\begin{aligned} &\sup_{s\geqslant 1, \sigma\in\xi_s, x\in D} \frac{1}{\psi_s} \sum_{n\in\sigma} \mathcal{O}\left(\frac{|f_n(x)|}{\rho}\right) \\ &\leqslant \left(\sup_{s\geqslant 1} \frac{\phi_s}{\psi_s}\right) \left\{ \sup_{s\geqslant 1, \sigma\in\xi_s, x\in D} \frac{1}{\phi_s} \sum_{n\in\sigma} \mathcal{O}\left(\frac{|f_n(x)|}{\nu}\right) \right\} \\ &< L.\tau|\mu(x)| < \tau'|\mu(x)|, \text{where}, L\tau = \tau' \Longrightarrow (f_n) \in {}_{ru}\check{\mathbf{m}}_{\psi}^{\mathcal{O}}. \end{aligned} \tag{26}$$

Hence, ${}_{ru}\check{\mathbf{m}}_{\phi}^{\mathcal{O}} \subseteq {}_{ru}\check{\mathbf{m}}_{\psi}^{\mathcal{O}}$.

Conversely, suppose that

$${}_{ru}\check{\mathbf{m}}_{\phi}^{\mathcal{O}} \subseteq {}_{ru}\check{\mathbf{m}}_{\psi}^{\mathcal{O}}. \tag{27}$$

Next, suppose

$$\sup_{s\geqslant 1} \frac{\phi_s}{\psi_s} = \infty. \tag{28}$$

Then, there exists a natural s_i sequence such that $(\phi_{s_i}/\psi_{s_i}) = \infty \text{as } i \longrightarrow \infty$.

Let $(f_n) \in {}_{ru}\check{\mathbf{m}}_{\phi}^{\mathcal{O}}$.

Consequently, $\nu > 0$ exists in such a way that

$$\begin{aligned} \sup_{s\geqslant 1, \sigma\in\xi_s, x\in D} \frac{1}{\psi_s} \sum_{n\in\sigma} \mathcal{O}\left(\frac{|f_n(x)|}{\nu}\right) &= \left(\sup_{i\geqslant 1} \frac{\phi_{s_i}}{\psi_{s_i}}\right) \\ &\cdot \left\{ \sup_{s\geqslant 1, \sigma\in\xi_s, x\in D} \frac{1}{\phi_{s_i}} \sum_{n\in\sigma} \mathcal{O}\left(\frac{|f_n(x)|}{\nu}\right) \right\} = \infty. \end{aligned} \tag{29}$$

Therefore, $(f_n) \notin {}_{ru}\check{\mathbf{m}}_{\phi}^{\mathcal{O}}$.

Since, we arrive at a contradiction.

Hence,

$$\sup_{s\geqslant 1} \frac{\phi_s}{\psi_s} < \infty. \tag{30}$$

We make the following conclusion without providing any proof in light of the previous theorem. □

Remark 7. Let $0 < p < 1$, then ${}_{ru}\check{\mathbf{m}}_{\phi}^{\mathcal{O}}(p) \subseteq {}_{ru}\check{\mathbf{m}}_{\phi}^{\mathcal{O}}(p)$ if and only if $\sup_{s\geqslant 1} (\phi_s/\psi_s) < \infty$.

Proof. The previous remark's proof is identical to that of Theorem 6. □

Theorem 8. *Suppose* $\mathcal{O}, \mathcal{O}_1, \mathcal{O}_2$ *is an Orlicz function that satisfies the* δ_2*-condition.*

Then,

(1) ${}_{ru}\check{\mathbf{m}}_{\phi}^{\mathcal{O}_1} \subseteq {}_{ru}\check{\mathbf{m}}_{\phi}^{\mathcal{O}\circ\mathcal{O}_1}$.

(2) ${}_{ru}\check{\mathbf{m}}_{\phi}^{\mathcal{O}_1} \cap {}_{ru}\check{\mathbf{m}}_{\phi}^{\mathcal{O}_2} \subseteq {}_{ru}\check{\mathbf{m}}_{\phi}^{\mathcal{O}_1+\mathcal{O}_2}$.

Proof.

(1) Let $(f_n) \in {}_{ru}\check{\mathbf{m}}_{\phi}^{\mathcal{O}_1}$.

Consequently, there is $\nu > 0$ such that,

$$\sup_{s\geqslant 1, \sigma\in\xi_s, x\in D} \frac{1}{\phi_s} \sum_{n\in\sigma} \mathcal{O}_1\left(\frac{|f_n(x)|}{\nu|\mu(x)|}\right) < \tau. \tag{31}$$

Let $0 < \varepsilon < 1$ and η with $0 < \eta < 1$ such that $\mathcal{O}(t) < \eta$ for $0 \leqslant t < \eta$.

Now,

$$\begin{aligned} &\sup_{s\geqslant 1, \sigma\in\xi_s, x\in D} \frac{1}{\phi_s} \sum_{n\in\sigma} \mathcal{O}\left(\mathcal{O}_1\left(\frac{|f_n(x)|}{\nu|\mu(x)|}\right)\right) \\ &= \sup_{s\geqslant 1, \sigma\in\xi_s, x\in D} \frac{1}{\phi_s} \sum_{1} \mathcal{O}\left(\mathcal{O}_1\left(\frac{|f_n(x)|}{\nu|\mu(x)|}\right)\right) \\ &+ \sup_{s\geqslant 1, \sigma\in\xi_s, x\in D} \frac{1}{\phi_s} \sum_{2} \mathcal{O}\left(\mathcal{O}_1\left(\frac{|f_n(x)|}{\nu|\mu(x)|}\right)\right), \end{aligned} \tag{32}$$

where the summation $\sum_1$ is over $\mathcal{O}_1(|f_n(x)|/\nu|\mu(x)|)\leqslant\delta'$ and the summation $\sum_2$ is over $\mathcal{O}_1(|f_n(x)|/\nu|\mu(x)|)>\eta$.

Since $\mathcal{O}$ is continuous, then we have

$$\begin{aligned}&\sup_{s\geqslant 1,\sigma\in\xi_s,x\in D}\frac{1}{\phi_s}\sum_1\mathcal{O}\left(\mathcal{O}_1\left(\frac{|f_n(x)|}{\nu|\mu(x)|}\right)\right)\\&\leqslant\mathcal{O}(1)\sup_{s\geqslant 1,\sigma\in\xi_s,x\in D}\frac{1}{\phi_s}\sum_1\mathcal{O}_1\left(\frac{|f_n(x)|}{\nu|\mu(x)|}\right).\end{aligned}\tag{33}$$

For $\mathcal{O}_1(|f_n(x)|/\nu|\mu(x)|)>\eta$, we use the fact that

$$\mathcal{O}_1\left(\frac{|f_n(x)|}{\nu|\mu(x)|}\right)<\mathcal{O}_1\left(\frac{|f_n(x)|}{\nu|\mu(x)|}\right)\eta^{-1}\leqslant 1+\mathcal{O}_1\left(\frac{|f_n(x)|}{\nu|\mu(x)|}\right)\eta^{-1}.\tag{34}$$

Because $\mathcal{O}$ is convex and has a nondecreasing nature,

$$\begin{aligned}&\mathcal{O}\left(\mathcal{O}_1\left(\frac{|f_n(x)|}{\nu|\mu(x)|}\right)\right)<\mathcal{O}\left(1+\mathcal{O}_1\left(\frac{|f_n(x)|}{\nu|\mu(x)|}\right)\frac{1}{\eta}\right)\\&\leqslant\frac{1}{2}\mathcal{O}(2)+\frac{1}{2}\mathcal{O}\left(2\mathcal{O}_1\left(\frac{|f_n(x)|}{\nu|\mu(x)|}\right)\frac{1}{\eta}\right)\\&=\frac{1}{2}\mathcal{O}(2)+\frac{1}{2}\mathcal{O}(2)\left(\mathcal{O}_1\left(\frac{|f_n(x)|}{\nu|\mu(x)|}\right)\frac{1}{\eta}\right).\end{aligned}\tag{35}$$

There exists $K>0$ such that, given that $\mathcal{O}$ meets the requirement of δ_2,

$$\begin{aligned}&\leqslant\frac{1}{2}K\mathcal{O}(2)\mathcal{O}_1\left(\frac{|f_n(x)|}{\nu|\mu(x)|}\right)\frac{1}{\eta}+\frac{1}{2}K\mathcal{O}(2)\mathcal{O}_1\left(\frac{|f_n(x)|}{\nu|\mu(x)|}\right)\frac{1}{\eta}\\&\leqslant K\mathcal{O}(2)\mathcal{O}_1\left(\frac{|f_n(x)|}{\nu|\mu(x)|}\right)\frac{1}{\eta}.\end{aligned}\tag{36}$$

Hence,

$$\sup_{s\geqslant 1,\sigma\in\xi_s,x\in D}\frac{1}{\phi_s}\sum_2\mathcal{O}\left(\mathcal{O}_1\left(\frac{|f_n(x)|}{\nu|\mu(x)|}\right)\right),\tag{37}$$

$$\leqslant\max\left\{1,\left(K\mathcal{O}(2)\eta^{-1}\right)\right\}\sup_{s\geqslant 1,\sigma\in\phi_s,x\in D}\frac{1}{\phi_s}\sum_2\mathcal{O}_1\left(\frac{|f_n(x)|}{\nu|\mu(x)|}\right),\tag{38}$$

$$\leqslant\max\left\{1,\left(K\mathcal{O}(2)\eta^{-1}\right)\right\}\sup_{s\geqslant 1,\sigma\in\xi_s,x\in D}\frac{1}{\phi_s}\sum_{n\in\sigma}\mathcal{O}_1\left(\frac{|f_n(x)|}{\nu|\mu(x)|}\right).\tag{39}$$

From (38) and (39), it follows that $(f_n)\in^{\mathcal{O}\circ\mathcal{O}_1}\check{\mathbf{m}}_\phi(ru)$.

Hence, ${}_{ru}\check{\mathbf{m}}_\phi^{\mathcal{O}_1}\subseteq{}_{ru}\check{\mathbf{m}}_\phi^{\mathcal{O}\circ\mathcal{O}_1}$.

Let $(f_n)\in{}_{ru}\check{\mathbf{m}}_\phi^{\mathcal{O}_1}\cap{}_{ru}\check{\mathbf{m}}_\phi^{\mathcal{O}_2}$.

Therefore, $(f_n)\in{}_{ru}\check{\mathbf{m}}_\phi^{\mathcal{O}_1}$ and $(f_n)\in{}_{ru}\check{\mathbf{m}}_\phi^{\mathcal{O}_2}$.

Consequently, there are $\nu_1>0$ and $\nu_2>0$ such that

$$\begin{aligned}&\sup_{s\geqslant 1,\sigma\in\xi_s,x\in D}\frac{1}{\phi_s}\sum_{n\in\sigma}\mathcal{O}_1\left(\frac{|f_n(x)|}{\nu_1|\mu(x)|}\right)<\frac{\tau}{2}.\\&\sup_{s\geqslant 1,\sigma\in\xi_s,x\in D}\frac{1}{\phi_s}\sum_{n\in\sigma}\mathcal{O}_2\left(\frac{|f_n(x)|}{\nu_2|\mu(x)|}\right)<\frac{\tau}{2}.\end{aligned}\tag{40}$$

Let $\nu_3=\max\{\nu_1,\nu_2\}$ then,

$$\begin{aligned}&\sup_{s\geqslant 1,\sigma\in\xi_s,x\in D}\frac{1}{\phi_s}\sum_{n\in\sigma}(\mathcal{O}_1+\mathcal{O}_2)\left(\frac{|f_n(x)|}{\nu_3|\mu(x)|}\right).\\&\leqslant\sup_{s\geqslant 1,\sigma\in\xi_s,x\in D}\frac{1}{\phi_s}\sum_{n\in\sigma}\mathcal{O}_1\left(\frac{|f_n(x)|}{\nu_1|\mu(x)|}\right)\\&\quad+\sup_{s\geqslant 1,\sigma\in\xi_s,x\in D}\frac{1}{\phi_s}\sum_{n\in\sigma}\mathcal{O}_2\left(\frac{|f_n(x)|}{\nu_2|\mu(x)|}\right)\\&<\frac{\tau}{2}+\frac{\tau}{2}=\tau.\end{aligned}\tag{41}$$

Therefore, $(f_n)\in{}_{ru}\check{\mathbf{m}}_\phi^{\mathcal{O}_1+\mathcal{O}_2}$.

Hence, ${}_{ru}\check{\mathbf{m}}_\phi^{\mathcal{O}_1}\cap{}_{ru}\check{\mathbf{m}}_\phi^{\mathcal{O}_2}\subseteq{}_{ru}\check{\mathbf{m}}_\phi^{\mathcal{O}_1+\mathcal{O}_2}$. □

Theorem 9. *${}_{ru}\check{\mathbf{m}}_\phi^{\mathcal{O}}$ is a solid space.*

Proof. Let $(f_n)\in{}_{ru}\check{\mathbf{m}}_\phi^{\mathcal{O}}$.

$$\sup_{s\geqslant 1,\sigma\in\xi_s,x\in D}\frac{1}{\phi_s}\sum_{n\in\sigma}\mathcal{O}\left(\frac{|f_n(x)|}{\nu|\mu(x)|}\right)<\tau.\tag{42}$$

Consider λ_n for all $n\in\mathbb{N}$, where $|\lambda_n|\leqslant 1$. Then,

$$\begin{aligned}&\sup_{s\geqslant 1,\sigma\in\xi_s,x\in D}\frac{1}{\phi_s}\sum_{n\in\sigma}\mathcal{O}\left(\frac{|\lambda_n f_n(x)|}{\nu|\mu(x)|}\right)\\&=\sup_{s\geqslant 1,\sigma\in\xi_s,x\in D}\frac{1}{\phi_s}\sum_{n\in\sigma}\mathcal{O}\left(\frac{|\lambda_n||f_n(x)|}{\nu|\mu(x)|}\right)\\&\leqslant\sup_{s\geqslant 1,\sigma\in\xi_s,x\in D}\frac{1}{\phi_s}\sum_{n\in\sigma}\mathcal{O}\left(\frac{|f_n(x)|}{\nu|\mu(x)|}\right)<\tau.\end{aligned}\tag{43}$$

Hence, $(\lambda_n f_n)\in{}_{ru}\check{\mathbf{m}}_\phi^{\mathcal{O}}$.

It follows that ${}_{ru}\check{\mathbf{m}}_\phi^{\mathcal{O}}$ is solid. □

We make the subsequent conclusion without providing any proof in the light of Theorem 9.

Remark 10. For the whole number $0<p<\infty$, the space ${}_{ru}\check{\mathbf{m}}_\phi^{\mathcal{O}}$ is solid.

Regarding the description of ${}_{ru}\check{\mathbf{m}}_\phi^{\mathcal{O}}$-space, we arrive at the following conclusion.

Remark 11. The space ${}_{ru}\check{\mathbf{m}}_\phi^{\mathcal{O}}$ is symmetric.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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