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Research Article

Scalability of Generalized Frames for Operators

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In this paper, the Parseval K-g-frames are constructed from a given K-g-frame by scaling the elements of the K-g-frame with the help of diagonal operators, and these frames are named scalable K-g-frames. Also, we prove some properties of scalable K-g-frames and construct new scalable K-g-frames from a given K-g-frame. The necessary and sufficient conditions for a K-g-frame to be scalable are given. Further, equivalent conditions for the scalability of K-g-frames and the K-frames induced by K-g-frames are obtained. Finally, it is shown that the direct sum of two scalable K-g-frames is again a scalable K-g-frame for some suitable bounded linear operator K.

1. Introduction

Duffin and Schaeffer [1] initiated the study of frames in Hilbert spaces in 1952 while working on some profound problems in non-Harmonic Fourier series and came up with the following definition of frames.

A sequence $\{f_j\}_{j\in J}$ in a separable Hilbert space $\mathscr{H}$ is said to be a frame for $\mathscr{H}$, if there exist constants $0 < A \le B < \infty$ such that for all $x \in \mathscr{H}$,

$$A\|x\|^2 \le \sum_{j\in J}\left|\left\langle x, f_j\right\rangle\right|^2 \le B\|x\|^2. \tag{1}$$

The constants A and B are called lower and upper frame bounds, respectively. If $A = B$, then the frame is said to be a tight frame. If $A = B = 1$, then the frame is said to be a Parseval frame. If only the right hand inequality in (1) holds, then $\{f_j\}_{j\in J}$ is known as a Bessel sequence. In 1986, Daubechies et al. [2] presented their remarkable work in which they proved that frames can also be used to provide series expansion of functions in $L^2(R)$. Since then, frames have been a point of fascination among researchers. For more information on frames, see [3, 4].

Tight frames are more flexible than the orthonormal basis in the sense that every vector can be expressed as a linear combination of frame elements where frame coefficients need not be unique; that is, if $\{f_j\}_{j\in J}$ is a tight frame for $\mathscr{H}$ with bound A, then every vector $x \in \mathscr{H}$ can be expressed as $x = (1/A)\sum_{j\in J}\langle x, f_j\rangle f_j$. Therefore, tight frames have direct applications in expanding data as they admit optimally stable reconstruction and hence have numerous applications in sampling theory, signal processing, filtering, smoothing, compression, image processing, etc. With this objective, Kutyniok et al. [5] generated a tight frame from a given general frame by scaling the frame vectors with the help of nonnegative scalars and termed such frames as scalable frames. They proved several important properties of scalable frames and provided geometrical interpretation of scalability in canonical surfaces.

In recent years, several generalizations of frames in Hilbert and Banach spaces have been proposed and studied like fusion frames in Hilbert spaces [6], g-frames by Sun [7, 8], fusion Banach frames in Banach spaces by Kaushik and Kumar [9–11], K-frames by Găvruta [12, 13], K-g-frames by Xiao et al. [14], and scalable K-frames by Ramesan and Ravindran [15]. The concept of K-frames was put forward to study the atomic systems with respect to a bounded linear operator K. These frames allowed to reconstruct elements from the range of bounded linear operator K and were more flexible than the classical frames. It was observed that in the case of frames, the frame operator turned out to be invertible on $\mathscr{H}$, whereas in the case of K-frames, the K-frame operator need not be invertible on $\mathscr{H}$. But, if the operator K has a closed range, then the K-frame operator is invertible on $R(K)$. Further, it has been seen that K-g-frames are the generalization of g-frames and have better practical applications. For more information on K-g-frames, one may refer to [14, 16–18]. Tight g-frames are similar to tight frames and are helpful in reconstructing signals. Recently, Ahmadi and Rahimi [19] came up with the idea of scalability of g-frames based on nonnegative diagonal operators instead of nonnegative scalars for scale change and obtained its various characterizations.

The Parseval K-g-frames are the generalization of the Parseval g-frames and help in restoring data loss in signal processing. Motivated by the work in [5, 15, 19], in this paper, we study the problem of determining whether a K-g-frame for $K \in B(\mathscr{H})$ is scalable or not, like g-frames. Therefore, we construct a Parseval K-g-frame from a given K-g-frame by scaling the K-g-frame elements with the help of nonnegative diagonal operators and introduce the notion of scalable K-g-frames. Some examples are provided to show the existence of such frames. Further, we determine its various characterizations and construct new scalable K-g-frames from a given scalable K-g-frame. The necessary and sufficient conditions for a K-g-frame to be scalable are proven. At last, we prove a result relating the direct sum of two scalable K-g-frames.

2. Preliminaries

Throughout this paper, $\mathscr{H}$ is used to denote a separable Hilbert space. J and I are countable indexing sets, and $\{\mathscr{V}_j : j \in J\}$ is a sequence of separable Hilbert spaces. $B(\mathscr{H}, \mathscr{V}_j)$ denotes the collection of all bounded linear operators from $\mathscr{H}$ into $\mathscr{V}_j$. $B(\mathscr{H})$ is the collection of all bounded linear operators on $\mathscr{H}$. For $T \in B(\mathscr{H})$, $\mathrm{Dom}(T)$, $R(T)$, $I_{R(T)}$, and T^* denote the domain of T, range of T, identity operator on the range of T, and adjoint of T, respectively. An operator $T \in B(\mathscr{H})$ is said to be nonnegative if $\langle Tx, x\rangle \geq 0$, for all $x \in \mathscr{H}$, and positive if $\langle Tx, x\rangle > 0$, for all $0 \neq x \in \mathscr{H}$. For more details on positive operators, see [20]. Further, the sequence space

$$l^2(\oplus \mathscr{V}_j) = \left\{ \{x_j\}_{j\in J} : x_j \in \mathscr{V}_j, j \in J, \sum_{j\in J} \|x_j\|^2 < \infty \right\} \quad (2)$$

with the inner product given by $\langle \{x_j\}_{j\in J}, \{y_j\}_{j\in J}\rangle = \sum_{j\in J} \langle x_j, y_j\rangle_{\mathscr{V}_j}$ is a separable Hilbert space.

Definition 1 (see [12]). Let $K \in B(\mathscr{H})$. A sequence $\{f_j\}_{j\in J}$ in $\mathscr{H}$ is said to be a K-frame for $\mathscr{H}$, if there exist constants $0 < A \leq B < \infty$ such that for all $x \in \mathscr{H}$,

$$A\|K^*x\|^2 \leq \sum_{j\in J} \left|\left\langle x, f_j \right\rangle\right|^2 \leq B\|x\|^2. \quad (3)$$

The constants A and B are called lower and upper frame bounds of K-frame $\{f_j\}_{j\in J}$, respectively. In particular, if $A \|K^*x\|^2 = \sum_{j\in J} |\langle x, f_j\rangle|^2$, for all $x \in \mathscr{H}$, then the K-frame is said to be a tight K-frame for $\mathscr{H}$. If $A = 1$, then it is called a Parseval K-frame for $\mathscr{H}$.

Definition 2 (see [15]). Let $K \in B(\mathscr{H})$. A K-frame $\{f_j\}_{j\in J}$ for $\mathscr{H}$ is said to be a scalable K -frame for $\mathscr{H}$ if there exist nonnegative scalars$\{a_j\}_{j\in J}$, such that $\{a_j f_j\}_{j\in J}$ is a Parseval K -frame for $\mathscr{H}$.

Definition 3 (see [7]). A sequence $\{\Lambda_j \in B(\mathscr{H}, \mathscr{V}_j) : j \in J\}$ is said to be a g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in J}$, if there exist constants $0 < A \leq B < \infty$ such that for all $x \in \mathscr{H}$,

$$A\|x\|^2 \leq \sum_{j\in J} \|\Lambda_j x\|^2 \leq B\|x\|^2. \quad (4)$$

We call the constants A and B to be the lower and upper frame bounds of g-frame, respectively. A g-frame $\{\Lambda_j\}_{j\in J}$ is said to be a tight g-frame if $A = B$ in (4) and a Parseval g-frame, if $A = B = 1$. If only the right hand inequality in (4) holds, then $\{\Lambda_j\}_{j\in J}$ is called a g-Bessel sequence for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in J}$.

The synthesis operator for g-frame $\Lambda = \{\Lambda_j\}_{j\in J}$ is given by $T_\Lambda : l^2(\oplus \mathscr{V}_j) \longrightarrow \mathscr{H}$ such that $T_\Lambda(\{x_j\}_{j\in J}) = \sum_{j\in J} \Lambda_j^* x_j$, and the analysis operator is given by $T_\Lambda^* : \mathscr{H} \longrightarrow l^2(\oplus \mathscr{V}_j)$ such that $T_\Lambda^*(x) = \{\Lambda_j x\}_{j\in J}$. The g-frame operator given by $S_\Lambda(x) = \sum_{j\in J} \Lambda_j^* \Lambda_j x$, for all $x \in \mathscr{H}$ is bounded, self-adjoint, and invertible.

Definition 4 (see [14]). Let $K \in B(\mathscr{H})$. A sequence $\{\Lambda_j \in B(\mathscr{H}, \mathscr{V}_j) : j \in J\}$ is said to be a K - g frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in J}$, if there exist constants $0 < A \leq B < \infty$ such that for all $x \in \mathscr{H}$,

$$A\|K^*x\|^2 \leq \sum_{j\in J} \|\Lambda_j x\|^2 \leq B\|x\|^2. \quad (5)$$

We call the constants A and B to be the lower and upper frame bounds of the K-g-frame $\{\Lambda_j\}_{j\in J}$, respectively, and $\{\Lambda_j\}_{j\in J}$ called a g-Bessel sequence if only the right hand in (5) holds.

If $A\|K^*x\|^2 = \sum_{j\in J} \|\Lambda_j x\|^2$, for all $x \in \mathscr{H}$, then the K-g-frame $\{\Lambda_j\}_{j\in J}$ is called a tight K-g-frame; moreover, if $A = 1$, then it is called a Parseval K-g-frame.

Remark 5. For $K = I_{\mathscr{H}}$, K - g -frames are just the ordinary g-frames.

Definition 6 (see [16]). Let $K \in B(\mathscr{H})$. Suppose $\{\Lambda_j \in B(\mathscr{H}, \mathscr{V}_j): j \in J\}$ and $\{\Theta_j \in B(\mathscr{H}, \mathscr{V}_j): j \in J\}$ be two K - g frames. Then, $\{\Theta_j\}_{j\in J}$ is a K - g-dual of $\{\Lambda_j\}_{j\in J}$ if $T_\Lambda T_\Theta^* = I_{R(K)}$.

Definition 7 (see [19]). An operator D defined on a closed linear span of basis $\{e_j\}_{j\in J}$ in a normed space $\mathscr{X}$ is called a diagonal operator, whenever $De_j = \lambda_j e_j$, where $j \geq 1$ and λ_j's are complex numbers. If D is a continuous operator, then

$$\sup_{j\geq 1} |\lambda_j| < \infty, \\ \|D\| = \sup_{j\geq 1} |\lambda_j| < \infty. \tag{6}$$

Since a positive diagonal operator $D \in B(\mathscr{H})$ is invertible, so there exist two constants A' and A such that for all $x \in \mathscr{H}$,

$$A' \|x\| \leq \|Dx\| \leq A\|x\|. \tag{7}$$

Definition 8 (see [16]). Assume that D_j is a diagonal operator on $\mathscr{V}_j$ for $j \in J$. We say that the operator D on $l^2(\oplus \mathscr{V}_j)$ is a block diagonal operator with D_j as its diagonal whenever $D(\{v_j\}_{j\in J}) = \{D_j v_j\}_{j\in J}$, $\{v_j\}_{j\in J} \in \mathrm{Dom}(D)$, where $\mathrm{Dom}(D) = \{\{v_j\}_{j\in J} \in l^2(\oplus \mathscr{V}_j)$ such that $\{D_j v_j\}_{j\in J} \in l^2(\oplus \mathscr{V}_j)\}$. For the block diagonal operator D, $\|D\| = \sup_{j\in J} \|D_j\|$.

Definition 9 (see [19]). A sequence of positive diagonal operators $\{D_j \in B(\mathscr{V}_j)\}_{j\in J}$ is called seminormalized, whenever

$$0 < \inf_{j\in J} A_j' \leq \sup_{j\in J} A_j < \infty, \tag{8}$$

where A_j' and A_j are the lower and upper bounds of D_j, respectively.

Definition 10 (see [19]). A g-frame $\{\Lambda_j \in B(\mathscr{H}, \mathscr{V}_j)\}_{j\in J}$ is said to be a scalable g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in J}$, whenever there exists a sequence of nonnegative diagonal operators $\{D_j \in B(\mathscr{V}_j)\}_{j\in J}$ such that $\{D_j \Lambda_j\}_{j\in J}$ is a Parseval g-frame. Further, if D_j's are positive operators, then the g-frame $\{\Lambda_j\}_{j\in J}$ is called a strictly scalable g-frame.

Definition 11 (see [21]). An operator $S : \mathscr{H} \longrightarrow \mathscr{H}$ is said to be bounded below if there exists $c > 0$ such that $\|Sx\| \geq c\|x\|$, for all $x \in \mathscr{H}$.

Theorem 12 (see [22] (Douglas Factorization Theorem)). *Let $T_1 \in B(\mathscr{H}_1, \mathscr{H})$ and $T_2 \in B(\mathscr{H}_2, \mathscr{H})$. Then, the following are equivalent:*

(i) $R(T_1) \subseteq R(T_2)$

(ii) there exists $\lambda > 0$ such that $T_1 T_1^* \leq \lambda T_2 T_2^*$

(iii) there exists $X \in B(\mathscr{H}_1, \mathscr{H}_2)$ such that $T_1 = T_2 X$

3. Some Properties of Scalable K-g-Frames

In this section, we introduce the concept of scalable K-g-frames and discuss some of their properties.

Definition 13. Let $K \in B(\mathscr{H})$. A K - g-frame $\Lambda = \{\Lambda_j\}_{j\in J}$ of $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in J}$ is called a scalable K - g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in J}$ whenever there exists a sequence of nonnegative diagonal operators $\{D_j \in B(\mathscr{V}_j)\}_{j\in J}$ such that $\{D_j \Lambda_j\}_{j\in J}$ is a Parseval K - g-frame, i.e., for all $x \in \mathscr{H}$,

$$\|K^* x\|^2 = \sum_{j\in J} \|D_j \Lambda_j x\|^2. \tag{9}$$

If D_j's are positive operators, then the K-g-frame $\Lambda = \{\Lambda_j\}_{j\in J}$ is called strictly scalable K-g-frame.

It is obvious that every K-g-frame is not scalable as is evident from the following example.

Example 14. Let $\mathscr{H}$ be a separable Hilbert space with an orthonormal basis $\{e_j\}_{j\in J}$ and define $\mathscr{V}_1 = \mathrm{span}\{e_1, e_2\}$, $\mathscr{V}_j = \mathrm{span}\{e_{j+1}\}$ for $j \geq 2$. For each $j \in J$, define $\Lambda_j : \mathscr{H} \longrightarrow \mathscr{V}_j$ as $\Lambda_1 x = \langle x, 2e_1 + e_2\rangle e_1 + \langle x, 3e_2\rangle e_2$, $\Lambda_j x = \langle x, e_{j+1}\rangle e_{j+1}$ for $j \geq 2$ and $K : \mathscr{H} \longrightarrow \mathscr{H}$ as $Ke_1 = e_1$, $Ke_2 = e_2$ and $Ke_j = 0$, for $j \geq 3$. Then, $K^* e_1 = e_1$, $K^* e_2 = e_2$, $K^* e_j = 0$, for $j \geq 3$. Then, $\{\Lambda_j\}_{j\in J}$ is a K - g-frame which is not scalable because there does not exist a sequence of nonnegative diagonal operators $\{D_j \in B(\mathscr{V}_j)\}_{j\in J}$ such that $\|K^* x\|^2 = \sum_{j\in J} \|D_j \Lambda_j x\|^2$.

The next example illustrates the existence of scalable K-g-frames.

Example 15. Let $\mathscr{H} = \mathbb{C}^3$, $\{e_j\}_{j=1}^3$ be its standard orthonormal basis and $\mathscr{V}_1 = \mathscr{V}_2 = \mathscr{V}_3 = \mathrm{span}\{e_1, e_2\}$. Define $K : \mathscr{H} \longrightarrow \mathscr{H}$ as $Kx = 2\langle x, e_1\rangle e_1 + \langle x, e_2\rangle e_2 + \sqrt{2}\langle x, e_3\rangle e_3$, $\Lambda_1 : \mathscr{H} \longrightarrow \mathscr{V}_1$ as $\Lambda_1 x = -2\langle x, e_1\rangle e_1 + 2\langle x, e_1\rangle e_2$, $\Lambda_2 : \mathscr{H} \longrightarrow \mathscr{V}_2$ as $\Lambda_2 x = \langle x, e_2\rangle e_1 + \langle x, e_2\rangle e_2$, and $\Lambda_3 : \mathscr{H} \longrightarrow \mathscr{V}_3$ as $\Lambda_3 x = \langle x, e_3\rangle e_1 - \langle x, e_3\rangle e_2$. Then,

$$\Lambda_1 = \begin{bmatrix} -2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}; \Lambda_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}; \Lambda_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}; K = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}. \quad (10)$$

Consider the operators $D_j : \mathcal{V}_j \longrightarrow \mathcal{V}_j$ for $j = 1, 2, 3$ given by

$$D_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}; D_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}; D_3 = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}. \quad (11)$$

Then for each $x \in \mathcal{H}$, $\|K^*x\|^2 \le \sum_{j=1}^3 \|\Lambda_j x\|^2 \le 8\|x\|^2$ and $\|K^*x\|^2 = \sum_{j=1}^3 \|D_j \Lambda_j x\|^2$. Therefore, $\{D_j \Lambda_j\}_{j=1}^3$ is a Parseval K-g-frame, and hence, $\{\Lambda_j\}_{j=1}^3$ is a scalable K-g-frame.

Since scaling of a K-g-frame by a sequence of diagonal operators does not need to always result in a K-g-frame, but if we scale the K-g-frame by a sequence of seminormalized diagonal operators, then it always gives rise to a new K-g-frame as shown in the next theorem.

Theorem 16. *Let $K \in B(\mathcal{H})$ and $\{\Lambda_j\}_{j\in J}$ be a K - g-frame for $\mathcal{H}$ with respect to $\{\mathcal{V}_j\}_{j\in J}$ with K - g-dual $\{\Theta_j\}_{j\in J}$. If $\{D_j \in B(\mathcal{V}_j)\}_{j\in J}$ is a sequence of seminormalized positive diagonal operators, then $\{D_j \Lambda_j\}_{j\in J}$ is a K - g-frame for $\mathcal{H}$ with respect to $\{\mathcal{V}_j\}_{j\in J}$ with $\{D_j^{-1}\Theta_j\}_{j\in J}$ as its K - g -dual.*

Proof. Let A and B be bounds of the K-g-frame $\{\Lambda_j\}_{j\in J}$, and A_j' and A_j be the lower and upper bounds, respectively, for the diagonal operators D_j, for $j \in J$. Then, for each $x \in \mathcal{H}$,

$$\begin{aligned} \sum_{j\in J} \|D_j \Lambda_j x\|^2 &\le \sum_{j\in J} \|D_j\|^2 \|\Lambda_j x\|^2 \le \left(\sup_{j\in J} A_j\right)^2 B\|x\|^2, \\ \sum_{j\in J} \|D_j \Lambda_j x\|^2 &\ge \left(\inf_{j\in J} A_j'\right)^2 A\|K^*x\|^2. \end{aligned} \quad (12)$$

Also, for each $x \in R(K)$,

$$x = \sum_{j\in J} \Lambda_j^* \Theta_j x = \sum_{j\in J} \Lambda_j^* D_j D_j^{-1} \Theta_j x = \sum_{j\in J} (D_j \Lambda_j)^* \left(D_j^{-1}\Theta_j\right) x. \quad (13)$$

Hence, it follows that $\{D_j \Lambda_j\}_{j\in J}$ is a K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{V}_j\}_{j\in J}$ with $\{D_j^{-1}\Theta_j\}_{j\in J}$ as its K-g-dual. □

Next, we examine the relationship between a scalable K-g-frame and a scalable g-frame. For $K = I_{\mathcal{H}}$, every scalable K-g-frame is a scalable g-frame. In the following example, we observe that a scalable K-g-frame can be a scalable g-frame even if $K \ne I_{\mathcal{H}}$.

Example 17. Let $\mathcal{H} = \mathbb{C}^3$, $\{e_j\}_{j=1}^3$ be its standard orthonormal basis and $\mathcal{V}_1 = \text{span}\{e_1, e_2\}$, $\mathcal{V}_2 = \text{span}\{e_1, e_3\}$, and $\mathcal{V}_3 = \text{span}\{e_2, e_3\}$. Define $K : \mathcal{H} \longrightarrow \mathcal{H}$ as $Kx = (1/\sqrt{2})(\langle x, e_1\rangle + \langle x, e_2\rangle)e_1 + (1/\sqrt{2})(\langle x, e_1\rangle - \langle x, e_2\rangle)e_2 + \langle x, e_3\rangle e_3$, $\Lambda_1 : \mathcal{H} \longrightarrow \mathcal{V}_1$ as $\Lambda_1 x = \langle x, e_1\rangle e_1 + \langle x, e_2\rangle e_2$, $\Lambda_2 : \mathcal{H} \longrightarrow \mathcal{V}_2$ as $\Lambda_2 x = \langle x, e_1\rangle e_1 + \langle x, e_3\rangle e_3$, and $\Lambda_3 : \mathcal{H} \longrightarrow \mathcal{V}_3$ as $\Lambda_3 x = \langle x, e_2\rangle e_2 + \langle x, e_3\rangle e_3$, and for $j \in \{1, 2, 3\}$, $D_j \in B(\mathcal{V}_j)$ as

$$D_j = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}. \quad (14)$$

Here

$$\begin{aligned} \Lambda_1 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \\ \Lambda_2 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ \Lambda_3 &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned} \quad (15)$$

Then, $\sum_{j=1}^3 \|\Lambda_j x\|^2 = 2\|x\|^2$ and $\sum_{j=1}^3 \|D_j \Lambda_j x\|^2 = \|x\|^2 = \|K^*x\|^2$. Thus, $\{\Lambda_j\}_{j\in J}$ is a scalable g-frame and scalable K - g-frame for $\mathcal{H}$ with respect to $\{\mathcal{V}_j\}_{j\in J}$, where $K \ne I_{\mathcal{H}}$.

In the next theorem, we prove the necessary and sufficient condition on the operator $K \in B(\mathcal{H})$ under which a scalable K-g-frame becomes a scalable g-frame. Further, we find a relationship between the $R(K)$ and $R(T_{D\Lambda})$, where $T_{D\Lambda}$ denotes the synthesis operator of $\{D_j \Lambda_j\}_{j\in J}$.

Theorem 18. *Let $K \in B(\mathcal{H})$ and $\Lambda = \{\Lambda_j\}_{j\in J}$ be a scalable K - g -frame for $\mathcal{H}$ with respect to $\{\mathcal{V}_j\}_{j\in J}$ with nonnegative diagonal operators $\{D_j \in B(\mathcal{V}_j)\}_{j\in J}$. Then, the following holds:*

(i) $\{\Lambda_j\}_{j\in J}$ is a scalable g-frame with nonnegative diagonal operators $\{D_j \in B(\mathcal{V}_j)\}_{j\in J}$ if and only if $KK^* = I_{\mathcal{H}}$

(ii) $R(K) \subseteq R(T_{D\Lambda})$, where $T_{D\Lambda}$ denotes the synthesis operator of $\{D_j \Lambda_j\}_{j\in J}$

Proof.

(i) By definition of scalable g-frame and scalable K-g-frame, for all $x \in \mathscr{H}$, $\|x\|^2 = \sum_{j\in J}\|D_j\Lambda_j x\|^2$ and $\|K^*x\|^2 = \sum_{j\in J}\|D_j\Lambda_j x\|^2$. Thus, we get $\|x\|^2 = \|K^*x\|^2$, for each $x \in \mathscr{H}$. Hence, $KK^* = I_{\mathscr{H}}$. Conversely, if $KK^* = I_{\mathscr{H}}$, then $\|K^*x\|^2 = \|x\|^2$ and as $\{\Lambda_j\}_{j\in J}$ is a scalable K-g-frame for $\mathscr{H}$, we get $\|x\|^2 = \sum_{j\in J}\|D_j\Lambda_j x\|^2$

(ii) As $\{\Lambda_j\}_{j\in J}$ is a scalable K-g-frame, and $\{D_j \in B(\mathscr{V}_j)\}_{j\in J}$ is a sequence of nonnegative diagonal operators; therefore, $\{D_j\Lambda_j\}_{j\in J}$ is a g-Bessel sequence, and so its synthesis operator $T_{D\Lambda}$ is well defined. For each $x \in \mathscr{H}$

$$\langle KK^*x, x\rangle = \sum_{j\in J}\left\|D_j\Lambda_j x\right\|^2 = \langle T_{D\Lambda}T^*_{D\Lambda}x, x\rangle. \quad (16)$$

Using Theorem 12, we get $R(K) \subseteq R(T_{D\Lambda})$. □

Next, we construct new scalable K-g-frames from a given scalable K-g-frame.

Theorem 19. *Let $T, L, K \in B(\mathscr{H})$ and $\Lambda = \{\Lambda_j\}_{j\in J}$ be a scalable K - g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in J}$. Then, the following holds:*

(i) If T is an isometry that commutes with K^*, then $\{\Lambda_j T\}_{j\in J}$ is a scalable K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in J}$

(ii) $\{\Lambda_j L^*\}_{j\in J}$ is a scalable LK-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in J}$

(iii) $\{\Lambda_j\}_{j\in J}$ be a scalable $S^{1/2}_{D\Lambda}$-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in J}$, where $S_{D\Lambda}$ denote the frame operator of $\{D_j\Lambda_j\}_{j\in J}$

Proof. Since $\{\Lambda_j\}_{j\in J}$ is a scalable K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in J}$, therefore, there exists a sequence of nonnegative diagonal operators $\{D_j \in B(\mathscr{V}_j)\}_{j\in J}$ such that for all $x \in \mathscr{H}$,

$$\|K^*x\|^2 = \sum_{j\in J}\left\|D_j\Lambda_j x\right\|^2. \quad (17)$$

(i) Let T be an isometry that commutes with K^*. Then for all $x \in \mathscr{H}$, we have

$$\sum_{j\in J}\left\|D_j\Lambda_j Tx\right\|^2 = \|K^*Tx\|^2 = \|TK^*x\|^2 = \|K^*x\|^2. \quad (18)$$

(ii) Replacing x by L^*x in the definition of scalable K-g-frame, we obtain for all $x \in \mathscr{H}$, $\|(LK)^*x\|^2 = \|K^*L^*x\|^2 = \sum_{j\in J}\|D_j\Lambda_j L^*x\|^2$

(iii) By definition of scalable K-g-frame, for all $x \in \mathscr{H}$, $\|K^*x\|^2 = \sum_{j\in J}\|D_j\Lambda_j x\|^2$. Thus, $\langle S_{D\Lambda}x, x\rangle = \langle KK^*x, x\rangle$, for all $x \in \mathscr{H}$ and $\|(S^{1/2}_{D\Lambda})^*x\|^2 = \|K^*x\|^2$. Hence, $\{\Lambda_j\}_{j\in J}$ is a scalable $S^{1/2}_{D\Lambda}$-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in J}$

□

Theorem 20. *Let $K_1 \in B(\mathscr{H}_1)$ and $K_2 \in B(\mathscr{H}_2)$. If $\{\Lambda_j\}_{j\in J}$ is a scalable K_1 - g-frame for $\mathscr{H}_1$ with respect to $\{\mathscr{V}_j\}_{j\in J}$ and $T \in B(\mathscr{H}_2, \mathscr{H}_1)$ is an isometry satisying $K_1^*T = TK_2^*$, then $\{\Lambda_j T\}_{j\in J}$ is a scalable K_2 - g-frame for $\mathscr{H}_2$ with respect to $\{\mathscr{V}_j\}_{j\in J}$.*

Proof. By definition of scalable K_1-g-frame, there exists a sequence of nonnegative diagonal operators $\{D_j \in B(\mathscr{V}_j)\}_{j\in J}$ such that for all $x \in \mathscr{H}_1$, $\|K_1^*x\|^2 = \sum_{j\in J}\|D_j\Lambda_j x\|^2$. Hence, for $y \in \mathscr{H}_2$, $\|K_1^*Ty\|^2 = \sum_{j\in J}\|D_j\Lambda_j Ty\|^2$. As $K_1^*T = TK_2^*$ and T is an isometry, we have for all $y \in \mathscr{H}_2$, $\|K_2^*y\|^2 = \|TK_2^*y\|^2 = \sum_{j\in J}\|D_j\Lambda_j Ty\|^2$. Hence, $\{\Lambda_j T\}_{j\in J}$ is a scalable K_2-g-frame for $\mathscr{H}_2$ with respect to $\{\mathscr{V}_j\}_{j\in J}$. □

Theorem 21. *Let $K \in B(\mathscr{H})$ and $\{\Lambda_j\}_{j\in J}$ be a K - g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in J}$. If for $j \in J$, $\{\Gamma_{j,i}\}_{i\in I}$ is a scalable g-frame of $\mathscr{V}_j$ with respect to $\{\mathscr{M}_{j,i}\}_{i\in I}$ by a sequence of nonnegative diagonal operators $\{D_{j,i} \in B(\mathscr{M}_{j,i})\}_{i\in I}$. Then, $\{\Gamma_{j,i}\Lambda_j\}_{j\in J, i\in I}$ is a scalable K - g -frame for $\mathscr{H}$ with respect to $\{\mathscr{M}_{j,i}\}_{j\in J, i\in I}$ by the sequence of nonnegative diagonal operators $\{D_{j,i} \in B(\mathscr{M}_{j,i})\}_{j\in J, i\in I}$ if and only if $\{\Lambda_j\}_{j\in J}$ is a Parseval K - g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in J}$.*

Proof. For each $j \in J$, $\{\Gamma_{j,i}\}_{i\in I}$ is a scalable g-frame of $\mathscr{V}_j$ with respect to $\{\mathscr{M}_{j,i}\}_{i\in I}$ by a sequence of nonnegative diagonal operators $\{D_{j,i} \in B(\mathscr{M}_{j,i})\}_{i\in I}$, therefore for all $v \in \mathscr{V}_j$,

$$\|v\|^2 = \sum_{i\in I}\left\|D_{j,i}\Gamma_{j,i}v\right\|^2. \quad (19)$$

First, assume that $\{\Gamma_{j,i}\Lambda_j\}_{j\in J, i\in I}$ is a scalable K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{M}_{j,i}\}_{j\in J, i\in I}$ by the sequence of nonnegative diagonal operators $\{D_{j,i} \in B(\mathscr{M}_{j,i})\}_{j\in J, i\in I}$. Then, for each $x \in \mathscr{H}$

$$\|K^*x\|^2 = \sum_{j\in J, i\in I} \left\|D_{j,i}\Gamma_{j,i}\Lambda_j x\right\|^2. \quad (20)$$

As, for each $j \in J$, $\Lambda_j x \in \mathscr{V}_j$, therefore, using (19) and (20), we obtain

$$\|K^*x\|^2 = \sum_{j\in J} \left\|\Lambda_j x\right\|^2. \quad (21)$$

Thus, $\{\Lambda_j\}_{j\in J}$ is a Parseval K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in J}$.

Conversely, assume that $\{\Lambda_j\}_{j\in J}$ is a Parseval K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in J}$. As for each $j \in J$, $\{\Gamma_{j,i}\}_{i\in I}$ is a scalable g-frame of $\mathscr{V}_j$ with respect to $\{\mathscr{M}_{j,i}\}_{i\in I}$ by a sequence of nonnegative diagonal operators $\{D_{j,i} \in B(\mathscr{M}_{j,i})\}_{i\in I}$. Therefore,

$$\sum_{j\in J, i\in I} \left\|D_{j,i}\Gamma_{j,i}\Lambda_j x\right\|^2 = \sum_{j\in J} \left\|\Lambda_j x\right\|^2 = \|K^*x\|^2. \quad (22)$$

□

Now, we prove a necessary and sufficient condition under which a scalable K-g-frame generates a scalable T K-g-frame for any $T \in B(\mathscr{H})$.

Theorem 22. *Let $T, K \in B(\mathscr{H})$. Then, $\{\Lambda_j\}_{j\in J}$ is a scalable K - g-frame for $R(T^*)$ if and only if $\{\Lambda_j T^*\}_{j\in J}$ is a scalable TK - g -frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in J}$.*

Proof. Suppose $\{\Lambda_j\}_{j\in J}$ is a scalable K-g-frame for $R(T^*)$, then there exists a sequence of nonnegative diagonal operators $\{D_j \in B(\mathscr{V}_j)\}_{j\in J}$ such that for all $y \in R(T^*)$,

$$\|K^*y\|^2 = \sum_{j\in J} \left\|D_j\Lambda_j y\right\|^2. \quad (23)$$

This implies that for $y = T^*x, x \in \mathscr{H}$, we have

$$\|(TK)^*x\|^2 = \sum_{j\in J} \left\|D_j\Lambda_j T^*x\right\|^2. \quad (24)$$

Hence, it follows that $\{\Lambda_j T^*\}_{j\in J}$ is a scalable TK-g -frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in J}$. The converse part can be proved in the similar manner. □

Next, we determine the conditions under which a K-g -frame can be rescaled to obtain another K-g-frame.

Theorem 23. *Let $K \in B(\mathscr{H})$ and $\Lambda = \{\Lambda_j\}_{j\in J}$ be a K - g -frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in J}$ with analysis operator T_Λ^*, and $\{D_j \in B(\mathscr{V}_j)\}_{j\in J}$ be a sequence of nonnegative diagonal operators. Then, the following conditions are equivalent:*

(i) $\Gamma = \{D_j\Lambda_j\}_{j\in J}$ is a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in J}$

(ii) There exists a diagonal operator D on $l^2(\oplus\mathscr{V}_j)$ such that $R(T_\Lambda^*) \subseteq Dom(D)$, and $D|_{R(T_\Lambda^*)}$ is bounded and $R(K) \subseteq R((DT_\Lambda^*)^*)$. In this case, the frame operator of Γ is given by $S_\Gamma = T_\Lambda D^2 T_\Lambda^*$

Proof. Suppose that $\Gamma = \{D_j\Lambda_j\}_{j\in J}$ is a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in J}$ with bounds A and B, respectively and synthesis operator T_Γ. Define $D : l^2(\oplus\mathscr{V}_j) \longrightarrow l^2(\oplus\mathscr{V}_j)$ as $D(\{v_j\}_{j\in J}) = \{D_j v_j\}_{j\in J}$. Then, D is a bounded block diagonal operator on $l^2(\oplus\mathscr{V}_j)$ with D_j as its diagonal blocks and for any $x \in \mathscr{H}$,

$$T_\Gamma^* x = \{D_j\Lambda_j x\}_{j\in J} = D\left(\{\Lambda_j x\}_{j\in J}\right) = DT_\Lambda^* x. \quad (25)$$

Thus, $T_\Gamma^* = DT_\Lambda^*$ and $R(T_\Lambda^*) \subseteq$ Dom(D). Also, for $y \in R(T_\Lambda^*)$, $y = T_\Lambda^* x$, for some $x \in \mathscr{H}$. Then, $\|Dy\|^2 = \|DT_\Lambda^* x\|^2 = \|T_\Gamma^* x\|^2 = \sum_{j\in J}\|D_j\Lambda_j x\|^2 \le B\|x\|^2 = B\left\|(T_\Lambda^*)^{-1}y\right\|^2 \le B\|T_\Lambda^*\|^{-2}\|y\|^2$. Therefore, $D|_{R(T_\Lambda^*)}$ is bounded. As, $A\|K^*x\|^2 \le \sum_{j\in J}\|D_j\Lambda_j x\|^2 = \|DT_\Lambda^* x\|^2$, therefore by Theorem 12, $R(K) \subseteq R((DT_\Lambda^*)^*)$.

Conversely, suppose there exists a diagonal operator D on $l^2(\oplus\mathscr{V}_j)$ such that $R(T_\Lambda^*) \subseteq$ Dom(D), $D|_{R(T_\Lambda^*)}$ is bounded and $R(K) \subseteq R((DT_\Lambda^*)^*)$. By Theorem 12, we have for all $x \in \mathscr{H}$,

$$\|K^*x\|^2 \le \lambda\|DT_\Lambda^* x\|^2, \quad (26)$$

for some $\lambda > 0$. As $D|_{R(T_\Lambda^*)}$ is bounded, there exists some $\beta > 0$ such that $\|Dy\| \le \beta\|y\|$, for all $y \in R(T_\Lambda^*)$. Thus, for $x \in \mathscr{H}$, we have

$$\|DT_\Lambda^* x\|^2 \le \beta^2\|T_\Lambda^* x\|^2 \le \beta^2\|T_\Lambda^*\|^2\|x\|^2. \quad (27)$$

Thus, from equations (26) and (27), it follows that $\Gamma = \{D_j\Lambda_j\}_{j\in J}$ is a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in J}$. Also, $S_\Gamma = T_\Gamma T_\Gamma^* = T_\Lambda D^2 T_\Lambda^*$. □

Theorem 24. *Let $K \in B(\mathscr{H})$ and $\Lambda = \{\Lambda_j\}_{j\in J}$ be a K - g -frame for $\mathscr{H}$ with respect $\mathscr{V}_j$ such that Λ_j's are bounded below and* lim inf $\|\Lambda_j\| > 0$. *If $\{D_j \in B(\mathscr{V}_j)\}_{j\in J}$ is a sequence of nonnegative diagonal operators, then the following statements are equivalent:*

(i) $\Gamma = \{D_j\Lambda_j\}_{j\in J}$ is a K-g-frame

(ii) There exists a bounded diagonal operator D on $l^2(\oplus\mathscr{V}_j)$ with D_j's as its diagonal blocks and $R(K) \subseteq Dom((DT_\Lambda^*)^*)$

Proof. Suppose that Γ is a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in J}$ with bounds A and B, respectively. Since for each $j \in J$, Λ_j is bounded below, therefore there exist $c_j > 0$ such that $c_j\|x\|^2 \le \|\Lambda_j x\|^2$, for each $x \in \mathscr{H}$. As $\lim\inf \|\Lambda_j\| > 0$, so there exist $\delta > 0$ and $i \in J$ such that $c_j \ge \delta$, $\forall j \ge i$. Also, for each $x \in \mathscr{H}$,

$$\left\|\Lambda_j x\right\|^2 \le B\left\|D_j\right\|^{-2}\|x\|^2. \quad (28)$$

Therefore, we have $\|D_j\|^2 \le B\delta^{-1}$, for each $j > i$. Hence, there exists $M > 0$ such that $\|D_j\| \le M$, for all $j \in J$, and thus, D is bounded on $l^2(\oplus\mathscr{V}_j)$. Using Theorem 23, we obtain $R(K) \subseteq \operatorname{Dom}((DT_\Lambda^*)^*)$. Conversely, assume that the diagonal operator D on $l^2(\oplus\mathscr{V}_j)$ with D_j as its diagonal blocks is bounded and $R(K) \subseteq \operatorname{Dom}((DT_\Lambda^*)^*)$. By Theorem 23, Γ is a K-g-frame. □

In the following result, we present an equivalent condition for the scalability of K-g-frames.

Theorem 25. *Let $K \in B(\mathscr{H})$ and $\Lambda = \{\Lambda_j\}_{j\in J}$ be a K - g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in J}$ with analysis operator T_Λ^*. Then, the following are equivalent:*

(i) $\{\Lambda_j\}_{j\in J}$ is a scalable K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in J}$

(ii) There exists a nonnegative bounded block diagonal operator D on $l^2(\oplus\mathscr{V}_j)$ such that $KK^* = T_\Lambda D^2 T_\Lambda^*$

Proof. First, let us assume that $\{\Lambda_j\}_{j\in J}$ is a scalable K-g-frame. Then, there exists a sequence of nonnegative diagonal operators $\{D_j \in B(\mathscr{V}_j)\}_{j\in J}$ such that for all $x \in H$,

$$\|K^*x\|^2 = \sum_{j\in J}\left\|D_j\Lambda_j x\right\|^2. \quad (29)$$

Define $D : l^2(\oplus\mathscr{V}_j) \longrightarrow l^2(\oplus\mathscr{V}_j)$ as $D(\{v_j\}_{j\in J}) = \{D_j v_j\}_{j\in J}$.

Then, D is a bounded block diagonal operator on $l^2(\oplus\mathscr{V}_j)$ with D_j's as its diagonal blocks. By Theorem 23, the frame operator of $\{D_j\Lambda_j\}_{j\in J}$ is $T_\Lambda D^2 T_\Lambda^*$. So, we have $KK^* = T_\Lambda D^2 T_\Lambda^*$.

Conversely, assume that there exists a nonnegative bounded block diagonal operator D with diagonal blocks $\{D_j \in B(\mathscr{V}_j)\}_{j\in J}$ on $l^2(\oplus\mathscr{V}_j)$ such that $KK^* = T_\Lambda D^2 T_\Lambda^*$.

Then, for all $x \in \mathscr{H}$, we have

$$\begin{aligned}\langle KK^*x, x\rangle &= \left\langle T_\Lambda D^2 T_\Lambda^* x, x\right\rangle,\\ \|K^*x\|^2 &= \|DT_\Lambda^* x\|^2 = \sum_{j\in J}\left\|D_j\Lambda_j x\right\|^2.\end{aligned} \quad (30)$$

Hence, the result holds. □

Xiang [17] proved that if $\{\Lambda_j\}_{j\in J}$ is a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in J}$, then the K-frame induced by this K-g-frame takes over the desired properties of the K-g-frame. In the next result, we prove another property that the K-frame induced by K-g-frame inherits in terms of scalability.

Theorem 26. *Let $K \in B(\mathscr{H})$ and $\{\Lambda_j\}_{j\in J}$ be a K - g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in J}$. If $\{e_{j,i}\}_{j\in J, i\in I}$ is an orthonormal basis of $\mathscr{V}_j$, for each $j \in J$, then the following are equivalent:*

(i) $\{\Lambda_j\}_{j\in J}$ is a scalable K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in J}$

(ii) $\{\Lambda_j^* e_{j,i}\}_{j\in J, i\in I}$ is a scalable K-frame for $\mathscr{H}$

Proof. Let $\{\Lambda_j\}_{j\in J}$ be a scalable K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in J}$; then, there exists a sequence of nonnegative diagonal operators $\{D_j \in B(\mathscr{V}_j)\}_{j\in J}$ such that for all $x \in \mathscr{H}$, $\|K^*x\|^2 = \sum_{j\in J}\|D_j\Lambda_j x\|^2$. By definition of diagonal operators, there exist nonnegative scalars $\{\beta_{j,i}\}_{j\in J, i\in I}$ satisfying $D_j e_{j,i} = \beta_{j,i} e_{j,i}$, $j \in J$, $i \in I$, where $\{e_{j,i}\}_{j\in J, i\in I}$ is an orthonormal basis for $\mathscr{V}_j$. Then, we have for all $x \in \mathscr{H}$,

$$\|K^*x\|^2 = \sum_{j\in J}\left\|D_j\Lambda_j x\right\|^2 = \sum_{j\in J}\sum_{i\in I}\left|\left\langle x, \beta_{j,i}\Lambda_j^* e_{j,i}\right\rangle\right|. \quad (31)$$

Hence, $\{\Lambda_j^* e_{j,i}\}_{j\in J, i\in I}$ is a scalable K-frame for $\mathscr{H}$.

Conversely, if $\{\Lambda_j^* e_{j,i}\}_{j\in J, i\in I}$ is a scalable K-frame for $\mathscr{H}$, then there exists a sequence of nonnegative scalars $\{\lambda_{j,i}\}_{j\in J, i\in I}$ satisfying

$$\|K^*x\|^2 = \sum_{j\in J}\sum_{i\in I}\left|\left\langle x, \lambda_{j,i}\Lambda_j^* e_{j,i}\right\rangle\right|, \forall x \in \mathscr{H}. \quad (32)$$

For each $j \in J$, define $D_j : \mathscr{V}_j \longrightarrow \mathscr{V}_j$ as $D_j e_{j,i} = \lambda_{j,i} e_{j,i}$, $i \in I$. Then, $\{D_j \in B(\mathscr{V}_j)\}_{j\in J}$ is a sequence of nonnegative diagonal operators such that $\|K^*x\|^2 = \sum_{j\in J}\|D_j\Lambda_j x\|^2$, for all $x \in \mathscr{H}$. □

If $\mathscr{H}_1$ and $\mathscr{H}_2$ are any two Hilbert spaces, then the direct sum of $\mathscr{H}_1$ and $\mathscr{H}_2$ is denoted by $\mathscr{H}_1 \oplus H_2 = \{h_1 \oplus h_2 : h_1 \in \mathscr{H}_1, h_2 \in \mathscr{H}_2\}$ is a Hilbert space with pointwise operations and inner product given by

$$\left\langle (h_1, h_2), \left(h_1', h_2'\right) \right\rangle = \left\langle h_1, h_1' \right\rangle_{H_1} \left\langle h_2, h_2' \right\rangle_{H_2}, \forall h_1, h_1' \in \mathscr{H}_1, h_2, h_2' \in \mathscr{H}_2. \quad (33)$$

If $\mathscr{U}$ and $\mathscr{W}$ are Hilbert spaces and $T \in B(\mathscr{H}_1, \mathscr{U})$ and $S \in B(\mathscr{H}_2, \mathscr{W})$, then $T \oplus S \in B(\mathscr{H}_1, \mathscr{U}) \oplus B(\mathscr{H}_2, \mathscr{W})$ and $(T \oplus S)(h_1, h_2) = (Th_1, Sh_2), \forall h_1 \in \mathscr{H}_1, h_2 \in \mathscr{H}_2$. Next, for $K_1 \in B(\mathscr{H}_1), K_2 \in B(\mathscr{H}_2)$, we construct scalable $(K_1 \oplus K_2)$-g-frame on Hilbert space $\mathscr{H}_1 \oplus \mathscr{H}_2$. For more information on direct sums, one can refer to [23–26].

Theorem 27. *Let $K_1 \in B(\mathscr{H}_1)$ and $K_2 \in B(\mathscr{H}_2)$. If $\Lambda = \{\Lambda_j\}_{j \in J}$ is a scalable K_1 - g-frame for $\mathscr{H}_1$ with respect to $\{\mathscr{V}_{1j}\}_{j \in J}$ and $\Gamma = \{\Gamma_j\}_{j \in J}$ is a scalable K_2 - g -frame for $\mathscr{H}_2$ with respect to $\{\mathscr{V}_{2j}\}_{j \in J}$, then $\Lambda \oplus \Gamma = \{\Lambda_j \oplus \Gamma_j\}_{j \in J}$ is a scalable $(K_1 \oplus K_2)$ - g-frame for $\mathscr{H}_1 \oplus \mathscr{H}_2$ with respect to $\{\mathscr{V}_{1j} \oplus \mathscr{V}_{2j}\}_{j \in J}$.*

Proof. By definition of scalable K_i-g-frame, $i = 1, 2$, there exist nonnegative diagonal operators $D_j \in B(\mathscr{V}_{1j})$, $D_j' \in B(\mathscr{V}_{2j})$ for $j \in J$ such that

$$\|K_1^* h_1\|^2 = \sum_{j \in J} \left\| D_j \Lambda_j h_1 \right\|^2, \forall h_1 \in \mathscr{H}_1, \quad (34)$$

and

$$\|K_2^* h_2\|^2 = \sum_{j \in J} \left\| D_j' \Gamma_j h_2 \right\|^2, \forall h_2 \in \mathscr{H}_2. \quad (35)$$

For $j \in J$, define $T_j = \begin{pmatrix} D_j & 0 \\ 0 & D_j' \end{pmatrix}$. Then $\{T_j \in B(\mathscr{V}_{1j} \oplus \mathscr{V}_{2j}) : j \in J\}$ is a sequence of nonnegative diagonal operators and for all $(h_1, h_2) \in \mathscr{H}_1 \oplus \mathscr{H}_2$,

$$\begin{aligned} \sum_{j \in J} \left\| T_j \left(\Lambda_j \oplus \Gamma_j \right) (h_1, h_2) \right\|^2 &= \sum_{j \in J} \left\| D_j \Lambda_j h_1 \right\|^2 + \sum_{j \in J} \left\| D_j' \Gamma_j h_2 \right\|^2 \\ &= \|K_1^* h_1\|^2 + \|K_2^* h_2\|^2 \\ &= \|(K_1 \oplus K_2)^* (h_1, h_2)\|^2. \end{aligned} \quad (36)$$

□

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors have contributed equally to manuscript writing, editing, and conceptualization. All authors reviewed the manuscript and confirmed that it met the ICMJE criteria.

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