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Research Article

Naimark-Type Results Using Frames

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In this article, a modified version of frame called frame associated with a sequence of scalars (FASS) is defined. This modified version of frame is used to study quantum measurements. Also, using FASS, some Naimark-type results are obtained. Finally, a formula to give the average probability of an incorrect measurement using FASS is obtained.

1. Introduction

Duffin and Schaeffer [1] formalised the definition of frames for Hilbert space in 1952 to examine some challenging issues involving nonharmonic Fourier series. In order to explore signal processing, Duffin and Schaeffer essentially abstracted the basic Gabor concept. However, until the seminal study by Daubechies et al. [2] in 1986, it did not seem that the concepts of Duffin and Schaeffer attracted much attention outside of the nonharmonic Fourier series. Although not to the level of the extremely quick growth of wavelets, the idea of frames started to be researched more extensively after this groundbreaking breakthrough. Frames have historically been utilised in sampling theory, data compression, image processing, and signal processing. The theory is now being used in a growing number of fields, including filterbanks, optics, signal detection, and the study of Besov spaces and Banach space theory.

Let $\mathscr{H}$ denote a separable Hilbert space equipped with inner product $\langle \cdot, \cdot \rangle$. A sequence $\{f_n\}_{n=1}^{\infty}$ of elements in $\mathscr{H}$ is called a frame for $\mathscr{H}$, if there exist positive constants C and D such that

$$C\|f\|^2 \le \sum_{n=1}^{\infty} |\langle f, f_n \rangle|^2 \le D\|f\|^2, \text{for all } f \in \mathscr{H}. \tag{1}$$

The scalars C and D are called frame bounds and they are not unique. If $C = D$, the frame $\{f_n\}_{n=1}^{\infty}$ is called a tight frame, whereas if $C = D = 1$, the frame $\{f_n\}$ is called a Parseval frame. For the frame $\{f_n\}_{n=1}^{\infty}$, the inequality in (1) is known as the frame inequality. The operator $\mathscr{T} : \ell^2(\mathbb{N}) \longrightarrow \mathscr{H}$ defined by

$$\mathscr{T}\{b_k\}_{n=1}^{\infty} = \sum_{k=1}^{\infty} b_k f_k \tag{2}$$

is called the preframe operator or the synthesis operator, and its adjoint operator $\mathscr{T}^* : \mathscr{H} \longrightarrow \ell^2(\mathbb{N})$ is called the analysis operator which is given by

$$\mathscr{T}^*(f) = \{\langle f, f_k \rangle\}, \text{for all } f \in \mathscr{H}. \tag{3}$$

Composing the operators $\mathscr{T}$ and $\mathscr{T}^*$, we obtain another operator called the *frame operator* $\mathscr{S} = \mathscr{T}\mathscr{T}^* : \mathscr{H} \longrightarrow \mathscr{H}$ which is given by

$$\mathscr{S}(f) = \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k \text{ for all } f \in \mathscr{H}. \tag{4}$$

The frame operator $\mathscr{S}$ is a positive, self-adjoint, and invertible operator on $\mathscr{H}$. Thus, the reconstruction formula for all $f \in \mathscr{H}$ is given by

$$f = \mathcal{S}\mathcal{S}^{-1}f = \sum_{k=1}^{\infty} \langle \mathcal{S}^{-1}f, f_k \rangle f_k \quad \left(= \sum_{k=1}^{\infty} \langle f, \mathcal{S}^{-1}f_k \rangle f_k \right). \quad (5)$$

For more details related to frames and some of their generalizations, one may refer to [3–8].

Eldar and Forney [9] investigated the connection between tight frames and rank-one quantum measurement. Additionally, they described frame matrices by comparing them to the measurement matrices of quantum mechanics. They extended tight frames to orthonormal bases utilising Neumark's theorem [10, 11]. They constructed the optimal tight frames by drawing inspiration from least squares measurement of quantum mechanics. In this study, we demonstrate various Naimark-type results and derive a formula to calculate the average probability of an incorrect measurement when utilising FASS.

2. Frames Associated with a Sequence of Scalars

We begin this section by defining a modified version of frame called frame associated with a sequence of scalars.

Definition 1. Let $\mathbb{N} = \bigcup_{n=1}^{\infty} P_n$, where P_i are finite subsets of $\mathbb{N}$ with $P_i \cap P_j = \phi$ for all $i \neq j$, and let $\{\mu_n\}$ be any sequence of scalars. A sequence $\{x_n\}$ in Hilbert space $\mathcal{H}$ is called a frame associated with a sequence of scalars with respect to (P_n, μ_n), if there exist constants A_l and $A_u (0 < A_l \leq A_u < \infty)$ such that

$$A_l \|x\|^2 \leq \sum_{n=1}^{\infty} \sum_{j \in P_n} \left|\mu_j\right|^2 \left|\langle x, x_j \rangle\right|^2 \leq A_u \|x\|^2, \text{for all } x \in \mathcal{H}. \quad (6)$$

If $A_l = A_u$, then $\{x_n\}$ is called a tight frame associated with a sequence of scalars with respect to (P_n, μ_n). If $A_l = A_u = 1$, then $\{x_n\}$ is called a Parseval frame associated with a sequence of scalars with respect to (P_n, μ_n).

It should be noted that $\{x_n\}$ is a FASS for $\mathcal{H}$ associated with a sequence of scalars with respect to (P_n, μ_n) if and only if $\{\{\mu_i x_i\}_{i \in P_n}\}_{n=1}^{\infty}$ is a frame for $\mathcal{H}$.

Definition 2. Let $\{P_n\}_{n=1}^{\infty}$ be a sequence of subsets of $\mathbb{N}$ as defined in Definition 1. For an orthonormal basis $\{e_n\}$ of ℓ^2, let us consider

$$\ell^2_{P_n} = \left\{ \sum_{j \in P_n} \alpha_j e_j : \alpha_j \text{ are scalars and } n \in \mathbb{N} \right\},$$
$$\left(\sum_{n \in \mathbb{N}} \oplus \ell^2_{P_n} \right)_{\ell^2} = \left\{ \{u_n\} : u_n \in \ell^2_{P_n} \text{ and } \sum_{n=1}^{\infty} \|u_n\|^2 < \infty \right\}. \quad (7)$$

One may observe that $\ell^2_{P_n}, n \in \mathbb{N}$ are subspaces of ℓ^2. An inner product defined on $(\sum_{n \in \mathbb{N}} \oplus \ell^2_{P_n})_{\ell^2}$ is given by

$$\langle \{u_n\}, \{v_n\} \rangle = \sum_{n=1}^{\infty} \langle u_n, v_n \rangle, \text{for } \{u_n\}, \{v_n\} \in \left(\sum_{n \in \mathbb{N}} \oplus \ell^2_{P_n} \right)_{\ell^2}. \quad (8)$$

One can handily corroborate that $(\sum_{n \in \mathbb{N}} \oplus \ell^2_{P_n})_{\ell^2}$ is a Hilbert space.

It can easily be substantiated that the operator $\mathcal{T}_B : (\sum_{n \in \mathbb{N}} \oplus \ell^2_{P_n})_{\ell^2} \longrightarrow \mathcal{H}$ given by

$$\mathcal{T}_B \left(\left\{ \sum_{i \in P_n} \mu_i e_i \right\}_{n=1}^{\infty} \right) = \sum_{n=1}^{\infty} \sum_{i \in P_n} \mu_i x_i \quad (9)$$

is bounded and is called the synthesis operator of the frame $\{x_n\}$ associated with a sequence of scalars $\{\mu_n\}$. Also, the bounded operator $\mathcal{T}_B^* : \mathcal{H} \longrightarrow (\sum_{n \in \mathbb{N}} \oplus \ell^2_{P_n})_{\ell^2}$ given by

$$\mathcal{T}_B^*(x) = \left\{ \sum_{i \in P_n} \langle x, \mu_i x_i \rangle e_i \right\}_{n=1}^{\infty} \quad (10)$$

is called the analysis operator of the frame associated with a sequence of scalars $\{\mu_n\}$. By composing operators $\mathcal{T}_B$ and $\mathcal{T}_B^*$, we obtain frame operator $\mathcal{S}_B = \mathcal{T}_B \mathcal{T}_B^* : \mathcal{H} \longrightarrow \mathcal{H}$ given by

$$\mathcal{S}_B(x) = \sum_{n=1}^{\infty} \sum_{i \in P_n} \langle x, \mu_i x_i \rangle \mu_i x_i = \sum_{n=1}^{\infty} \sum_{i \in P_n} |\mu_i|^2 \langle x, x_i \rangle x_i. \quad (11)$$

Let $\{x_n\}$ be a frame associated with a sequence of scalars $\{\mu_n\}$. Let $\mathcal{T}_B$ and $\mathcal{T}_B^*$ be synthesis and analysis operators, respectively, and let $\mathcal{S}_B$ be the frame operator of the frame $\{x_n\}$. One may promptly observe that

$$\mathcal{S}_B^{-1} \mathcal{S}_B = I_{\mathcal{H}} = \mathcal{S}_B^{-1} \mathcal{T}_B \mathcal{T}_B^*. \quad (12)$$

Therefore, we have $\mathcal{T}_B^* = \mathcal{T}_B^* \mathcal{S}_B^{-1} \mathcal{T}_B \mathcal{T}_B^*$. So, $\mathcal{S}_B^{-1} \mathcal{T}_B$ is the pseudoinverse of $\mathcal{T}_B^*$, and $\mathcal{T}_B^* \mathcal{S}_B^{-1} \mathcal{T}_B$ is a projection from $(\sum_{n \in \mathbb{N}} \oplus \ell^2_{P_n})_{\ell^2}$ onto $\mathcal{T}_B^*(\mathcal{H})$.

If $\{x_n\}$ is a Parseval frame for $\mathcal{H}$ with respect to (P_n, μ_n), then $\mathcal{T}_B \mathcal{T}_B^* = I_{\mathcal{H}}$; that is, $\mathcal{T}_B^*$ is isometry.

Given a Parseval frame $\{x_n\}$ for $\mathcal{H}$ with respect to (P_n, μ_n), the following result establishes that there exist a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and an orthonormal frame $\{y_n\}$ for $\mathcal{H}$ with respect to (P_n, μ_n) such that the orthogonal projection of y_n onto $\mathcal{H}$ is x_n for each n. One may observe that (Proposition 1.1 in [12]) a classic result in frame theory is closely related to Theorem 3.

Theorem 3. *Let $\{x_n\}$ be a Parseval frame associated with a sequence of scalars for $\mathcal{H}$ with respect to (P_n, μ_n), where scalars μ_n are nonzero. Then, there exist a Hilbert space $\mathcal{K}$ with $\mathcal{H}$ as a subspace of $\mathcal{K}$ and an orthonormal frame $\{y_n\}$ associated with a sequence of scalars for $\mathcal{K}$ with respect to (P_n, μ_n)*

such that $P_{\mathcal{H}}(y_n) = x_n$, for all $n \in \mathbb{N}$, where $P_{\mathcal{H}}$ is a projection from $\mathcal{K}$ onto $\mathcal{H}$.

Proof. Let $\mathcal{K} = \mathcal{H} \oplus ker\mathcal{T}_B$ and $P_{k\mathcal{T}_B}$ be orthogonal projection from $(\sum_{n\in\mathbb{N}} \oplus \ell^2_{P_n})_{\ell^2}$ onto $ker\mathcal{T}_B$. Define $y_i = x_i + (1/\mu_i) P_{ker\mathcal{T}_B}(e_i) \in \mathcal{K} = \mathcal{H} \oplus ker\mathcal{T}_B$, $x \in \mathcal{H}$, and $\nu = \{\sum_{i\in P_n} \nu_i e_i\} \in ker\mathcal{T}_B$. Further, define $\mathcal{L}_B : \mathcal{K} \longrightarrow (\sum_{n\in\mathbb{N}} \oplus \ell^2_{P_n})_{\ell^2}$ as

$$\mathcal{L}_B(x \oplus \nu) = \left\{ \sum_{i\in P_n} \langle x \oplus \nu, \mu_i y_i \rangle \right\}. \tag{13}$$

This gives

$$\begin{aligned} \|\mathcal{L}_B(x\oplus\nu)\|^2 &= \sum_{n=1}^{\infty}\sum_{i\in P_n} |\mu_i|^2 |\langle x, x_i\rangle|^2 + \sum_{n=1}^{\infty}\sum_{i\in P_n} |\nu_i|^2 \\ &= \|\mathcal{T}_B^*(x)\|^2 + \|\nu\|^2 \\ &= \|x\|^2 + \|\nu\|^2 \\ &= \|x\oplus\nu\|^2. \end{aligned} \tag{14}$$

Thus, $\mathcal{L}_B$ is an isometry that is $\mathcal{L}_B^*\mathcal{L}_B = I_{\mathcal{K}}$. Also, we know that $Q_B = I - \mathcal{T}_B^*\mathcal{T}_B$ is a projection from $(\sum_{n\in\mathbb{N}} \oplus \ell^2_{P_n})_{\ell^2}$ onto $ker\mathcal{T}_B$. And we have

$$\begin{aligned} \left\langle \mu_i y_i, \mu_j y_j \right\rangle &= \left\langle \mu_i\left(x_i + \frac{1}{\mu_i} Q_B(e_i)\right), \mu_j\left(x_j + \frac{1}{\mu_j} Q_B(e_j)\right) \right\rangle \\ &= \left\langle \mu_i x_i, \mu_j x_j \right\rangle + \left\langle Q_B(e_i), Q_B(e_j) \right\rangle \\ &= \left\langle \mu_i x_i, \mu_j x_j \right\rangle + \left\langle e_i, Q_B(e_j) \right\rangle \\ &= \left\langle \mu_i x_i, \mu_j x_j \right\rangle + \left\langle e_i, (I - \mathcal{T}_B^*\mathcal{T}_B)(e_j) \right\rangle \\ &= \left\langle \mu_i x_i, \mu_j x_j \right\rangle + \left\langle e_i, e_j \right\rangle - \left\langle e_i, \mathcal{T}_B^*\mathcal{T}_B e_j \right\rangle \\ &= \left\langle \mu_i x_i, \mu_j x_j \right\rangle + \left\langle e_i, e_j \right\rangle - \left\langle \mathcal{T}_B e_i, \mathcal{T}_B e_j \right\rangle \\ &= \left\langle e_i, e_j \right\rangle \\ &= \delta_{ij}. \end{aligned} \tag{15}$$

□

3. Quantum Measurement

According to the well-known spectral theorem, the projection-valued measures (PVMs) or spectral measures correlate one to one with the self-adjoint operators. In conventional quantum mechanics, quantum observables are represented by PVMs. PVMs are defined in [9, 13–16] as follows.

3.1. Projection-Valued Measure (PVM). A PVM on Hilbert space $\mathcal{H}$ is any set of operators $\{E_n\}$ on $\mathcal{H}$ which satisfies the following:

(i) Each operator E_n is a self-adjoint projection for all $n \in \mathbb{N}$

(ii) $E_i E_j = 0$, $i \neq j$

(iii) $\sum_{n=1}^{\infty} E_n = I_{\mathcal{H}}$

Let $\mathcal{B}(X)$ be a σ-algebra of the subsets of a locally compact space X and $\mathcal{L}(\mathcal{H})$ be the set of bounded operators on Hilbert space $\mathcal{H}$. A positive operator-valued measure (POVM) is a function $\Pi : \mathcal{B}(X) \longrightarrow \mathcal{L}(\mathcal{H})$ such that

(i) For all $U \in \mathcal{B}(X)$, $\Pi(U)$ is a positive self-adjoint operator

(ii) $\Pi(\phi) = 0$

(iii) For all disjoint subsets $\{U_i\}_{i=1}^{\infty} \subset \mathcal{B}(X)$, we have

$$\Pi\left(\bigcup_{i=1}^{\infty} U_i\right) = \sum_{i=1}^{\infty} \Pi(U_i). \tag{16}$$

(iv) $\Pi(X) = I_{\mathcal{H}}$

The representation of quantum observables by POVMs is found to be more appropriate than by spectral measures. In 1940s, POVMs were defined to study some extensions of operators (symmetric). Later, around 1970s, POVMs were used as a tool to describe the quantum measurements. It was observed that POVMs are an extension of quantum observable that are embodied by a spectral measure (PVMs). Presently, POVMs are used as a basic tool in the study of quantum information theory [17] and quantum optics. In [18], Ali studied certain geometrical properties of POVM, defined on the Borel sets of locally compact space X, taking values in the set of all bounded operators on a separable Hilbert space. In terms of POVM for observables, a thorough examination of the basic ideas of quantum theory as well as current experiments connected to it is presented in [19]. The broad statistical (convex) approach framework in [20] presents a purely statistical characterization of measurements of observables (characterized by spectral measures in standard quantum mechanics formalism). In [21], Prugovečki studied the stochastic quantum mechanics. The necessity of using non-normalized POVM is also described in order to understand the idea of quantum localization in space-time. POVM is precisely defined in [13, 14, 22] as follows.

3.2. Positive Operator-Valued Measure (POVM). A POVM on Hilbert space $\mathcal{H}$ is any set of operators $\{E_n\}$ on $\mathcal{H}$ which satisfies the following conditions:

(i) Each operator E_n is positive, for all $n \in \mathbb{N}$

(ii) $\sum_{n=1}^{\infty} E_n = I_{\mathcal{H}}$

The quantum observables delineated by POVMs are like a generalization of the basic or standard quantum observables. So POVMs are generally called unsharp observables or generalized observables. A very important result which discusses the interconnection between POVMs and PVMs is the Naimark dilation theorem. However, its interpretation from physical perspective is not very clear, and so there is some awkwardness in interpreting the Hilbert space $\mathscr{H}$.

It is well known that a Parseval frame defines a POVM (see, for example, [23]). For the convenience of the reader, we state and prove this result in the particular case of Parseval frames associated with a sequence of scalars.

Theorem 4. *Let $\{x_n\}$ be Parseval frame associated with a sequence of scalars for $\mathscr{H}$ with respect to (P_n, μ_n), and let $E_n(x) = \sum_{i\in P_n}\langle x, \mu_i x_i\rangle \mu_i x_i$, for all $n \in \mathbb{N}$. Then, $\{E_n\}$ is a POVM on $\mathscr{H}$.*

Proof. First, we show that each E_n is a positive operator. For $x \in \mathscr{H}$, we obtain

$$\langle E_n(x), x\rangle = \left\langle \sum_{i\in P_n}\langle x, \mu_i x_i\rangle \mu_i x_i, x \right\rangle = \sum_{i\in P_n} |\langle x, \mu_i x_i\rangle|^2 \geq 0. \tag{17}$$

To establish the completeness relation, let $x \in \mathscr{H}$. Then,

$$\sum_{n=1}^{\infty} E_n(x) = \sum_{n=1}^{\infty}\sum_{i\in P_n}\langle x, \mu_i x_i\rangle \mu_i x_i = \sum_{n=1}^{\infty}\sum_{i\in P_n}|\mu_i|^2\langle x, x_i\rangle x_i = x. \tag{18}$$

In the following result, we show that POVM in a Hilbert space can give rise to a Parseval frame associated with a sequence of scalars for $\mathscr{H}$. ☐

Theorem 5. *Let Π be a POVM on a Hilbert space $\mathscr{H}$. Then, there exist a disjoint partition $\{P_n\}$ of $\mathbb{N}$ with P_n finite for all $n \in \mathbb{N}$ and a sequence of scalars $\{\mu_n\}$ and a sequence $\{x_n\}$ in $\mathscr{H}$ such that $\{x_n\}$ is a Parseval frame associated with a sequence of scalars for $\mathscr{H}$ with respect to (P_n, μ_n).*

Proof. Let $\{P_n\}$ be disjoint partitions of $\mathbb{N}$ with P_n finite for all $n \in \mathbb{N}$. Note that $\Pi(P_n)$ is positive and self-adjoint operator. So, in view of the spectral theorem of positive operator, there exists an orthonormal set $\{v_j\}_{j\in P_n}$ in $\mathscr{H}$ and positive numbers $\{\xi_j\}_{j\in P_n}$ such that for all $x \in \mathscr{H}$, we have

$$\Pi(P_n)(x) = \sum_{j\in P_n}\xi_j\langle x, v_j\rangle v_j = \sum_{j\in P_n}\left\langle x, \sqrt{\xi_j}v_j\right\rangle\sqrt{\xi_j}v_j. \tag{19}$$

But $\mathbb{N} = \bigcup_{n=1}^{\infty} P_n$, and so, we obtain

$$x = \Pi(\mathbb{N})(x) = \Pi\left(\bigcup_{n=1}^{\infty} P_n\right)(x) = \sum_{n=1}^{\infty}\Pi(P_n)(x). \tag{20}$$

Taking $x_i = v_i$ and $\mu_i = \sqrt{\xi_i}$, for $i \in \mathbb{N}$, we get

$$x = \sum_{n=1}^{\infty}\sum_{i\in P_n}\langle x, \mu_i x_i\rangle \mu_i x_i = \sum_{n=1}^{\infty}\sum_{i\in P_n}|\mu_i|^2\langle x, x_i\rangle x_i, \text{ for all } x \in \mathscr{H}. \tag{21}$$

We shall now prove Naimark-type results using frames associated with a sequence of scalars. More precisely, we prove that an orthonormal frame associated with a sequence of scalars represents projection-valued measure in Hilbert spaces. ☐

Theorem 6. *Let $\{x_n\}$ be an orthonormal frame associated with a sequence of scalars for $\mathscr{H}$ with respect to (P_n, μ_n). Also, let $E_n(x) = \sum_{i\in P_n}\langle x, \mu_i x_i\rangle \mu_i x_i$, for $n \in \mathbb{N}$. Then, $\{E_n\}$ is a projection-valued measure on $\mathscr{H}$.*

Proof. It is clear that E_n is self-adjoint for $n \in \mathbb{N}$ and

$$x = \sum_{n=1}^{\infty} E_n(x) = \sum_{n=1}^{\infty}\sum_{i\in P_n}\langle x, \mu_i x_i\rangle \mu_i x_i, \text{ for all } x \in \mathscr{H}. \tag{22}$$

We are now left to show that E_n is a projection for $n \in \mathbb{N}$. Let $x \in \mathscr{H}$. Then, we have

$$\begin{aligned} E_n^2(x) &= \sum_{i\in P_n}\left\langle E_n^2(x), \mu_i x_i\right\rangle \mu_i x_i \\ &= \sum_{i\in P_n}\sum_{j\in P_n}\left\langle x, \mu_j x_j\right\rangle\left\langle \mu_j x_j, \mu_i x_i\right\rangle \mu_i x_i \\ &= \sum_{i\in P_n}\langle x, \mu_i x_i\rangle \mu_i x_i \\ &= E_n(x). \end{aligned} \tag{23}$$

Finally, we show that a Parseval frame associated with a sequence of scalars can also give a PVM through dilation theorem. ☐

Theorem 7. *Let $\{x_n\}$ be a frame associated with a sequence of scalars for $\mathscr{H}$ with respect to (P_n, μ_n). Then, there exist a Hilbert space $\mathscr{K}$ with $\mathscr{H}$ as a subspace of $\mathscr{K}$, $\{y_n\} \subseteq \mathscr{K}$ with $P_{\mathscr{H}}(y_n) = x_n$, for all $n \in \mathbb{N}$, and a sequence of operators $\{F_n\}$ on $\mathscr{K}$ such that $\{F_n\}$ is a projection-valued measure on $\mathscr{K}$, where $P_{\mathscr{H}}$ is a projection from $\mathscr{K}$ onto $\mathscr{H}$ and $F_n(y) = \sum_{i\in P_n}\langle y, \mu_i y_i\rangle \mu_i y_i$, for $y \in \mathscr{K}$.*

Proof. Proof follows from Theorem 3 and Theorem 6. ☐

A quantum system in a pure state is characterized by a normalized vector ψ in a Hilbert space $\mathscr{H}$. Information about a quantum system is extracted by subjecting the system to a measurement. In quantum theory, the outcome of a measurement is inherently probabilistic, with the probabilities of the outcomes of any conceivable measurement determined by the state vector $\psi \in \mathscr{H}$. Now, we will show how Parseval frame associated with a sequence of scalars can be used in quantum measurement. Let $\{x_n\}$ be Parseval frame associated with a sequence of scalars for $\mathscr{H}$ with respect to (P_n, μ_i). Take $E_n(x) = \sum_{i\in P_n} \langle x, \mu_i x_i\rangle \mu_i x_i$, for $x \in \mathscr{H}$. Suppose the measurement is performed upon a quantum system in a pure state ψ. Then, the probability of the n outcome is given by

$$p(n) = \langle \psi, E_n\psi\rangle = \left\langle \psi, \sum_{i\in P_n} \langle \psi, \mu_i x_i\rangle \mu_i x_i \right\rangle = \sum_{i\in P_n} |\langle \psi, \mu_i x_i\rangle|^2. \tag{24}$$

Moreover, one can notice that

$$\sum_{n=1}^{\infty} p(n) = \sum_{n=1}^{\infty} \sum_{i\in P_n} |\langle \psi, \mu_i x_i\rangle|^2 = \|\psi\|^2 = 1. \tag{25}$$

Let $\{\psi_i\}$ be unit normed states in Hilbert space $\mathscr{H}$ with corresponding probabilities $\{\rho_i\}$ that sum to 1. If the state of the system is ψ_i, for $i \in \mathbb{N}$, then the measurement provides us the information that the system is in the i^{th} state with high probability of $p(j)$, given by

$$p(j) = \langle \psi_i, E_j\psi_i\rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{26}$$

The probability that the output of our measurement device will be j is $\langle \psi_i, E_j\psi_i\rangle$ if the state of the system state is ψ_i. Consequently, $\langle \psi_i, E_i\psi_i\rangle$ is the probability of correct measurement. Since each ψ_j occurs with probability ρ_j, the average probability of a successful measurement is

$$\begin{aligned} \mathscr{E}(\text{success}) &= \mathscr{E}\left(\{\langle \psi_i, E_i(\psi_i)\rangle\}_{i=1}^{\infty}\right) \\ &= \sum_{i=1}^{\infty} \rho_i \langle \psi_i, E_i(\psi_i)\rangle \\ &= \sum_{i=1}^{\infty} \rho_i \sum_{k\in P_i} |\langle \psi_i, \mu_k x_k\rangle|^2. \end{aligned} \tag{27}$$

Thus, the average probability of incorrect measurement is given by

$$p_e = 1 - \sum_{i=1}^{\infty} \rho_i \langle \psi_i, E_i(\psi_i)\rangle = 1 - \sum_{i=1}^{\infty} \rho_i \sum_{k\in P_i} |\langle \psi_i, \mu_k x_k\rangle|^2. \tag{28}$$

Note that the probability that we measure the system erroneously to be ψ_j is $\langle \psi_i, E_j(\psi_i)\rangle$ if the state of the system is ψ_i for $i \in \mathbb{N}$ and $i \neq j$. Hence, the following relation yields the average probability of an inaccurate measurement:

$$\begin{aligned} \mathscr{E}(\text{incorrect}) &= \mathscr{E}\left(\{\psi_i, E_j(\psi_i)\}_{i\neq j}\right) \\ &= \sum_{i\neq j} \rho_i \left\langle \psi_i, \sum_{k\in P_j} \langle \psi_i, \mu_k x_k\rangle \mu_k x_k \right\rangle \\ &= \sum_{i\neq j} \rho_i \sum_{k\in P_j} |\langle \psi_i, \mu_k x_k\rangle|^2. \end{aligned} \tag{29}$$

Next, we use a frame associated with a sequence of scalars for $\mathscr{H}$ with respect to (P_n, μ_n) to provide the average probability of an inaccurate measurement.

Theorem 8. *Let $\{\psi_i\}$ be unit normed states in Hilbert space $\mathscr{H}$ with corresponding probabilities $\{\rho_i\}$ that sum to 1 and $\{x_n\}$ be Parseval frame associated with a sequence of scalars for $\mathscr{H}$ with respect to (P_n, μ_n). Then, the average probability of an incorrect measurement is given by*

$$\mathscr{E}(\mathit{incorrect}) = p_e = 1 - \sum_{i=1}^{\infty} \rho_i \sum_{k\in P_i} |\langle \psi_i, \mu_k x_k\rangle|^2. \tag{30}$$

Proof. We know that

$$\begin{aligned} \sum_{i\neq j} \rho_i \langle \psi_i, E_j(\psi_i)\rangle + \sum_{i\in\mathbb{N}} \rho_i \langle \psi_i, E_i(\psi_i)\rangle &= \sum_{i,j\in\mathbb{N}} \rho_i \langle \psi_i, E_j(\psi_i)\rangle \\ &= \sum_{i\in\mathbb{N}} \rho_i \left\langle \psi_i, \sum_{j\in\mathbb{N}} E_j(\psi_i) \right\rangle \\ &= \sum_{i\in\mathbb{N}} \rho_i \langle \psi_i, \psi_i\rangle = 1. \end{aligned} \tag{31}$$

Thus, we obtain

$$\mathscr{E}(\text{incorrect}) = p_e = 1 - \sum_{i=1}^{\infty} \rho_i \sum_{k\in P_i} |\langle \psi_i, \mu_k x_k\rangle|^2. \tag{32}$$

□

4. Conclusion

Positive operator-valued measures (POVMs) have long been the subject of the study. Later, POVMs were used as a tool to delineate the quantum measurements. According to the Naimark dilation theorem, POVMs are seen as an extension of a quantum observable that is represented by spectral measures (or PVMs). POVMs are currently employed in the research of quantum information theory and quantum optics as a basic tool. In this article, we proved various Naimark-type results using frames associated with a sequence of scalars. The average probability of an incorrect measurement is then obtained using a frame associated with a sequence of scalars.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Both authors contributed equally to this paper and read and approved the final manuscript.

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