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Research Article

A New Method for Estimating General Coefficients to Classes of Bi-univalent Functions

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This study establishes a new method to investigate bounds of $|a_k|$; $(k \geq n)$, for certain general classes of bi-univalent functions. The results include a number of improvements and generalizations for well-known estimations. We also discuss bounds of $|na_n^2 - a_{2n-1}|$ and consider several corollaries, remarks, and consequences of the results presented in this paper.

1. Introduction and Preliminary

In the usual notation, let $\mathscr{A}$ denote the class of functions $f(\zeta)$ in the form

$$f(\zeta) = \zeta + \sum_{k=2}^{\infty} a_k \zeta^k, \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. We also denote by $\mathscr{S}$ the subclass of $\mathscr{A}$ consisting of univalent (one-to-one) functions in $\mathbb{U}$. Further, the class $\mathscr{P}$ is consisting of analytic functions $p(\zeta) = 1 + \sum_{n=1}^{\infty} c_n \zeta^n$ satisfying $p(0) = 1$ and $\operatorname{Re}\{p(\zeta)\} > 0$, $(\zeta \in \mathbb{U})$. Note that $|c_n| \leq 2$, for $n \geq 1$, by the Carathéodory lemma. It is well known that every function $f \in \mathscr{S}$, in the form (1), has an inverse function f^{-1} defined by $f^{-1}(f(\zeta)) = \zeta$, $(\zeta \in \mathbb{U})$ and $f(f^{-1}(w)) = w$, $(|w| < 1/4)$, according to the Koebe one-quarter theorem (see [1]). In fact, the inverse function $g = f^{-1}$ is given by

$$\begin{aligned} g(w) = f^{-1}(w) &= w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots \\ &=: w + \sum_{k=2}^{\infty} b_k w^k, \quad \left(|w| < r_0(f), r_0(f) \geq \frac{1}{4}\right). \end{aligned} \tag{2}$$

A function $f \in \mathscr{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. The class of bi-univalent functions in $\mathbb{U}$ is denoted by Σ. As can be observed from studies [2–4], research on the class of bi-univalent functions started some time ago, probably around the year 1967. Indeed, the bounds for the coefficients of functions in Σ were first investigated by Lewin [2], where he showed that $|a_2| < 1.51$. Later, Brannan and Clunie [4] conjectured that $|a_2| \leq \sqrt{2}$, and Netanyahu [3] proved that $|a_2| \leq 4/3$ for functions of Σ whose images cover the open unit disk. The best known estimate for $|a_2|$ is 1.485 given by Tan [5]. Brannan and Taha [6] have introduced nonsharp estimates for the first two coefficients of the strongly bistarlike, bistarlike, and biconvex function classes. In the recent years, interest in the subject has returned; for example, numerous publications have been published since 2010 [7–27].

Since the condition of bi-univalency renders the behavior of the higher coefficients unpredictable, determining the boundaries for $|a_n|$; $(n \geq 4)$ is a notable problem in geometric function theory. Ali et al. [28] also proclaimed it to be an open topic. Accordingly, numerous writers have estimated the initial nonzero coefficient (a_n) for a variety of Σ subclasses using the Faber polynomials (see [29–36]). Al-Refai and Ali, on the other hand, recently developed an alternative approach to estimating a_n. They have in fact validated the following intriguing theorem.

Theorem 1 (see [10]). *Let the function*

$$f(\zeta) = \zeta + \sum_{k=n}^{\infty} a_k \zeta^k \, ; (n \geq 2) \tag{3}$$

be univalent in $\mathbb{U}$ *with* $f^{-1}(w) = w + \sum_{k=n}^{\infty} b_k w^k$; $(|w| < r_0(f)$, $r_0(f) \geq 1/4)$. *Then,*

$$b_{2n-1} = n a_n^2 - a_{2n-1}, b_k = -a_k, \quad \text{for } (n \leq k \leq 2n-2). \tag{4}$$

This yields $|a_n| \leq \sqrt{4 - 2/n}$ and $|n a_n^2 - a_{2n-1}| \leq 2n - 1$, for $f \in \Sigma$. Motivated by Theorem 1, bounds of $|a_k|$; $(k \geq n)$ and $|n a_n^2 - a_{2n-1}|$ can be investigated for various subclasses of Σ. In this paper, we investigate such bounds for the following general interesting subclass of Σ and for some of its special cases. Some improvements and generalizations for well-known results will be obtained.

Definition 2. A function $f(\zeta) = \zeta + \sum_{k=n}^{\infty} a_k \zeta^k \in \Sigma$, $(n \geq 2)$ is said to be in the class $\mathscr{B}_{\Sigma,n}^{h,p}(\eta) \quad (\eta \geq 1)$, if the following conditions are satisfied:

$$(1-\eta)\frac{f(\zeta)}{\zeta} + \eta f'(\zeta) \in h(\mathbb{U}), \quad (\zeta \in \mathbb{U} \, ; \eta \geq 1), \tag{5}$$

$$(1-\eta)\frac{g(w)}{w} + \eta g'(w) \in p(\mathbb{U}), \quad (w \in \mathbb{U} \, ; \eta \geq 1), \tag{6}$$

where $h, p \in \mathscr{P}$ and

$$g(w) = f^{-1}(w) = w + \sum_{k=n}^{\infty} b_k w^k. \tag{7}$$

The functions $h(\zeta)$ and $p(\zeta)$ can be specialized to provide interesting subclasses of analytic functions. If we set

$$h(\zeta) = 1 + \sum_{k=n}^{\infty} h_k \zeta^{k-1}, p(\zeta) = 1 + \sum_{k=n}^{\infty} p_k \zeta^{k-1}, \quad (\zeta \in \mathbb{U}, 0 \leq \beta < 1, n \geq 2), \tag{8}$$

where $|h_k| \leq 2 - 2\beta$ and $|p_k| \leq 2 - 2\beta$, for every $k \geq n$, and $\operatorname{Re}\{h(\zeta)\} > \beta$, $\operatorname{Re}\{p(\zeta)\} > \beta$, and $(\zeta \in \mathbb{U})$, then the hypotheses of Definition 2 are satisfied, and we have the following subclass of bi-univalent functions.

Definition 3. A function $f \in \Sigma$, in the form (3), is said to be in the class $\mathscr{B}_{\Sigma,n}(\beta, \eta)$, $(n \geq 2)$, if the following conditions are satisfied:

$$\begin{aligned} &\operatorname{Re}\left((1-\eta)\frac{f(\zeta)}{\zeta} + \eta f'(\zeta)\right) > \beta, \quad (\zeta \in \mathbb{U}, 0 \leq \beta < 1, \eta \geq 1), \\ &\operatorname{Re}\left((1-\eta)\frac{g(w)}{w} + \eta g'(w)\right) > \beta, \quad (w \in \mathbb{U}, 0 \leq \beta < 1, \eta \geq 1), \end{aligned} \tag{9}$$

where the function g is given by (7).

Similarly, it can be verified that the hypotheses of Definition 2 are satisfied for the choice

$$h(\zeta) = (\phi(\zeta))^{\alpha}, p(\zeta) = (\psi(\zeta))^{\alpha}, \quad (0 < \alpha \leq 1), \tag{10}$$

where the functions ϕ and ψ are defined by

$$\phi(\zeta) = 1 + \sum_{k=n}^{\infty} c_k \zeta^{k-1}, \psi(\zeta) = 1 + \sum_{k=n}^{\infty} d_k \zeta^{k-1}, \quad (\zeta \in \mathbb{U}, n \geq 2), \tag{11}$$

such that $|c_k| \leq 2$ and $|d_k| \leq 2$, for every $k \geq n$. This provides the following subclass of bi-univalent functions.

Definition 4. A function $f \in \Sigma$, in the form (3), is said to be in the class $\mathscr{A}_{\Sigma,n}(\alpha, \eta)$, $(n \geq 2)$, if the following conditions are satisfied:

$$\begin{aligned} &\left|\arg\left((1-\eta)\frac{f(\zeta)}{\zeta} + \eta f'(\zeta)\right)\right| \leq \frac{\alpha\pi}{2}, \quad (\zeta \in \mathbb{U}, 0 < \alpha \leq 1, \eta \geq 1), \\ &\left|\arg\left((1-\eta)\frac{g(w)}{w} + \eta g'(w)\right)\right| \leq \frac{\alpha\pi}{2}, \quad (w \in \mathbb{U}, 0 < \alpha \leq 1, \eta \geq 1), \end{aligned} \tag{12}$$

where the function g is given by (7).

For the special case when $n = 2$, the class $\mathscr{B}_{\Sigma,n}^{h,p}(\eta)$ reduces to the class $\mathscr{B}_{\Sigma}^{h,p}(\eta)$, which was introduced and studied by Xu et al. [37]. However, for $n = 2$ and $\eta = 1$, the class $\mathscr{B}_{\Sigma,2}^{h,p}(1)$ was studied earlier by Xu et al. [38], and Definitions 3 and 4 have been defined by Frasin and Aouf [7], for the case when $n = 2$.

The following example shows that the subclasses $\mathscr{A}_{\Sigma,n}(\alpha, \eta), \mathscr{B}_{\Sigma,n}(\beta, \eta)$ of the general class $\mathscr{B}_{\Sigma,n}^{h,p}(\eta)$ are not empty.

Example 5. Consider the function

$$f(\zeta) = -\log(1-\zeta) = \zeta + \sum_{k=2}^{\infty} \frac{1}{k} \zeta^k. \tag{13}$$

Then, its inverse is given by

$$f^{-1}(w)=1-e^{-w}=w+\sum_{k=2}^{\infty}\frac{(-1)^{k+1}}{k!}w^k. \quad (14)$$

Now, $\mathrm{Re}\,\{f'(\zeta)\}=\mathrm{Re}\,\{1/(1-\zeta)\}>1/2$ and $\mathrm{Re}\,\{(f^{-1})'(w)\}=\mathrm{Re}\,\{e^{-w}\}>1/e$ imply that $f\in\mathscr{B}_{\Sigma,2}(1/e,1)$. Note that $\mathscr{B}_{\Sigma,2}(1/e,1)\subset\mathscr{B}_{\Sigma,2}(0,1):=\mathscr{A}_{\Sigma,2}(1,1)$.

With regard to specialization of the parameters of $\mathscr{A}_{\Sigma,n}(\alpha,\eta)$ and $\mathscr{B}_{\Sigma,n}(\beta,\eta)$ which gives other examples of functions that show that those classes are not empty, see Adegani et al. [39].

2. Main Results

First, we find general coefficient bounds for functions in the class $\mathscr{B}^{h,p}_{\Sigma,n}(\eta)$.

Theorem 6. *Let $f(\zeta)=\zeta+\sum_{k=n}^{\infty}a_kz^k$; $(n\geq 2)$ be in the class $\mathscr{B}^{h,p}_{\Sigma,n}(\eta)$ $(\eta\geq 1)$. Then,*

$$|a_n|\leq\min\left\{\sqrt{\frac{|h^{(2n-2)}(0)+p^{(2n-2)}(0)|}{(2n-2)![1+(2n-2)\eta]n}},\frac{\left|h^{(n-1)}(0)\right|}{(n-1)![1+(n-1)\eta]}\right\}, \quad (15)$$

$$|a_k|\leq\frac{\left|h^{(k-1)}(0)\right|}{(k-1)![1+(k-1)\eta]},\quad (k\geq n+1), \quad (16)$$

$$\left|na_n^2-a_{2n-1}\right|\leq\frac{\left|p^{(2n-2)}(0)\right|}{(2n-2)![1+(2n-2)\eta]}. \quad (17)$$

Proof. Let $g(w)=w+\sum_{k=n}^{\infty}b_kw^k$; $(n\geq 2)$ be the inverse function of f. According to conditions (5) and (6), we have

$$\begin{aligned}h(\zeta)&=(1-\eta)\frac{f(\zeta)}{\zeta}+\eta f'(\zeta)=1+\sum_{k=n}^{\infty}[1+(k-1)\eta]a_k\zeta^{k-1},\quad(\zeta\in\mathbb{U}),\\ p(w)&=(1-\eta)\frac{g(w)}{w}+\eta g'(w)=1+\sum_{k=n}^{\infty}[1+(k-1)\eta]b_kw^{k-1},\quad(w\in\mathbb{U}),\end{aligned} \quad (18)$$

where h and p satisfy the hypotheses of Definition 2. A computation shows, for $j\geq n$, that

$$h^{(j-1)}(\zeta)=\sum_{k=j}^{\infty}(k-1)(k-2)\cdots(k-j+1)[1+(k-1)\eta]a_k\zeta^{k-j},\quad(\zeta\in\mathbb{U}), \quad (19)$$

$$p^{(j-1)}(w)=\sum_{k=j}^{\infty}(k-1)(k-2)\cdots(k-j+1)[1+(k-1)\eta]b_kw^{k-j},\quad(w\in\mathbb{U}). \quad (20)$$

Substituting $z=0$ in (19) and $w=0$ in (20) yields

$$a_j=\frac{h^{(j-1)}(0)}{(j-1)![1+(j-1)\eta]},\quad(j\geq n), \quad (21)$$

$$b_j=\frac{p^{(j-1)}(0)}{(j-1)![1+(j-1)\eta]},\quad(j\geq n). \quad (22)$$

It follows that

$$a_n=\frac{h^{(n-1)}(0)}{(n-1)![1+(n-1)\eta]}, \quad (23)$$

$$a_{2n-1}=\frac{h^{(2n-2)}(0)}{(2n-2)![1+(2n-2)\eta]}, \quad (24)$$

$$b_{2n-1}=\frac{p^{(2n-2)}(0)}{(2n-2)![1+(2n-2)\eta]}. \quad (25)$$

In view of Theorem 1, (24) and (25) yield that

$$a_n^2=\frac{a_{2n-1}+b_{2n-1}}{n}=\frac{h^{(2n-2)}(0)+p^{(2n-2)}(0)}{(2n-2)![1+(2n-2)\eta]n}. \quad (26)$$

Thus, (23) in conjunction with (26) yields (15). Since $b_{2n-1}=na_n^2-a_{2n-1}$, (25) yields (17). Finally, (16) follows from (21). This completes the proof of Theorem 6. ☐

Setting $n=2$ in Theorem 6 gives the following corollary.

Corollary 7. *Let $f(z)=z+\sum_{k=2}^{\infty}a_kz^k$ be in the class $\mathscr{B}^{h,p}_{\Sigma,2}(\eta)$ $(\eta\geq 1)$. Then,*

$$|a_2|\leq\min\left\{\sqrt{\frac{|h''(0)+p''(0)|}{4(1+2\eta)}},\frac{|h'(0)|}{1+\eta}\right\}, \quad (27)$$

$$|a_k|\leq\frac{\left|h^{(k-1)}(0)\right|}{(k-1)![1+(k-1)\eta]},\quad(k\geq 3), \quad (28)$$

$$\left|2a_2^2-a_3\right|\leq\frac{|p''(0)|}{2(1+2\eta)}. \quad (29)$$

Remark 8. The estimate (27) improves that given by Xu et al. ([37], Theorem 3). For $k=3$, (28) reduces to estimate of $|a_3|$ given by Theorem 3 of [37]. Indeed, for $\eta=1$, (27) improves the bound of $|a_2|$ and is identical to the bound of $|a_3|$ given in Theorem 3 of [38].

Corollary 9. *Let $f(\zeta)=\zeta+\sum_{k=n}^{\infty}a_k\zeta^k$ be in the class $\mathscr{B}_{\Sigma,n}(\beta,\eta)$, $(n\geq 2)$. Then,*

$$
\begin{aligned}
|a_n| &\leq \begin{cases} \sqrt{\dfrac{4(1-\beta)}{n[1+2(n-1)\eta]}}, & \text{if } 0\leq\beta\leq 1-\dfrac{[1+(n-1)\eta]^2}{n[1+2(n-1)\eta]}, \\ \dfrac{2(1-\beta)}{1+(n-1)\eta}, & \text{if } 1-\dfrac{[1+(n-1)\eta]^2}{n[1+2(n-1)\eta]}\leq\beta<1, \end{cases} \\
|a_k| &\leq \frac{2(1-\beta)}{1+(k-1)\eta}, \quad (k\geq n+1), \\
\left|na_n^2-a_{2n-1}\right| &\leq \frac{2(1-\beta)}{1+2(n-1)\eta}.
\end{aligned} \tag{30}
$$

Proof. Let the functions h and p be defined as in (8). A computation shows, for $k\geq n$, that

$$\left|h^{(k-1)}(0)\right|=(k-1)!|h_k|\leq(k-1)!(2-2\beta), \tag{31}$$

$$\left|p^{(k-1)}(0)\right|=(k-1)!|p_k|\leq(k-1)!(2-2\beta). \tag{32}$$

By applying (31) and (32) to Theorem 6, we get the desired estimates.

The estimate of $|a_n|$ in Corollary 9 improves Corollary 3 in [32] and Theorem 1 in [33]. Now, for the special case when $n=2$, we have the following remark. □

Remark 10. If $f\in\mathscr{B}_{\Sigma,2}(\beta,\eta)$, then

$$
\begin{aligned}
|a_2| &\leq \begin{cases} \sqrt{\dfrac{2(1-\beta)}{1+2\eta}}, & \text{if } 0\leq\beta\leq\dfrac{1+2\eta-\eta^2}{2(1+2\eta)}, \\ \dfrac{2(1-\beta)}{1+\eta}, & \text{if } \dfrac{1+2\eta-\eta^2}{2(1+2\eta)}\leq\beta<1, \end{cases} \\
|a_k| &\leq \frac{2(1-\beta)}{1+(k-1)\eta}, \quad (k\geq 3), \\
\left|2a_2^2-a_3\right| &\leq \frac{2(1-\beta)}{1+2\eta}.
\end{aligned} \tag{33}
$$

Note that Remark 10, for $k=3$, reduces to Corollary 6 by Bulut [32]. The estimates of $|a_2|$ and $|a_3|$ are much better than those given by Xu et al. ([37], Corollary 2) and Frasin and Aouf ([7], Theorem 3.2). Moreover, the estimate of $|a_2|$ which gives the range of β corresponds to the suitable bound of $|a_2|$, which facilitates Corollary 11 in [40].

Now, for the case whenever $\eta=1$, Corollary 9 reduces to Theorem 3.2 in [10], for $p=1$, as follows.

Remark 11. If $f(\zeta)=\zeta+\sum_{k=n}^{\infty}a_k\zeta^k$ satisfies $\operatorname{Re}(f'(\zeta))>\beta$ and $\operatorname{Re}(g'(w))>\beta$, then

$$
|a_n|\leq\begin{cases} \sqrt{\dfrac{4(1-\beta)}{n(2n-1)}}, & \text{if } 0\leq\beta\leq\dfrac{n-1}{2n-1}, \\ \dfrac{2(1-\beta)}{n}, & \text{if } \dfrac{n-1}{2n-1}\leq\beta<1, \end{cases} \tag{34}
$$

$$|a_k|\leq\frac{2(1-\beta)}{k}, \quad (k\geq n+1), \tag{35}$$

$$\left|na_n^2-a_{2n-1}\right|\leq\frac{2(1-\beta)}{2n-1}. \tag{36}$$

The estimate (36) improves that given in Corollary 15 of [41]. Moreover, when $n=2$ and $k=3$, Remark 11 reduces to Corollary 7 by Bulut [32]. In fact, it improves the estimate of $|a_2|$ given in Theorem 2 by Srivastava et al. [8] and Corollary 2 by Xu et al. [38], as follows.

Remark 12. If $f(\zeta)=\zeta+\sum_{k=2}^{\infty}a_k\zeta^k$ satisfies $\operatorname{Re}(f'(\zeta))>\beta$ and $\operatorname{Re}(g'(w))>\beta$, then

$$
\begin{aligned}
|a_2| &\leq \begin{cases} \sqrt{\dfrac{2(1-\beta)}{3}}, & \text{if } 0\leq\beta\leq\dfrac{1}{3}, \\ 1-\beta, & \text{if } \dfrac{1}{3}\leq\beta<1, \end{cases} \\
|a_k| &\leq \frac{2}{k}(1-\beta), \quad (k\geq 3), \\
\left|2a_2^2-a_3\right| &\leq \frac{2}{3}(1-\beta).
\end{aligned} \tag{37}
$$

Remark 13. Note that the function

$$f(\zeta)=-\log(1-\zeta)=\zeta+\sum_{k=2}^{\infty}\frac{1}{k}\zeta^k, \tag{38}$$

given in Example 5, satisfies the conclusions of Remark 12. Indeed, in view of Remark 12, we find that

$$
\begin{aligned}
|a_k| &= \frac{1}{k}\leq\frac{2}{k}\left(1-\frac{1}{e}\right), \quad (k\geq 2), \\
\left|2a_2^2-a_3\right| &= \frac{1}{6}\leq\frac{2}{3}\left(1-\frac{1}{e}\right).
\end{aligned} \tag{39}
$$

The following theorem introduces general coefficient bounds for functions in the class $\mathscr{A}_{\Sigma,n}(\alpha,\eta)$.

Theorem 14. *Let f, in form (3), be in the class $\mathscr{A}_{\Sigma,n}(\alpha,\eta)$, $(n \geq 2)$. Then,*

$$\begin{aligned} |a_n| &\leq \begin{cases} \dfrac{2\alpha}{\sqrt{n[1+2(n-1)\eta]}}, & \text{if } 1 \leq \eta \leq 1+\sqrt{\dfrac{n}{n-1}}, \\ \dfrac{2\alpha}{1+(n-1)\eta}, & \text{if } \eta \geq 1+\sqrt{\dfrac{n}{n-1}}, \end{cases} \\ |a_k| &\leq \frac{2\alpha}{1+(k-1)\eta}, \quad (n \leq k \leq 2n-2), \\ |a_{2n-1}| &\leq \frac{2\alpha^2}{1+2(n-1)\eta}, \\ \left|na_n^2 - a_{2n-1}\right| &\leq \frac{2\alpha^2}{1+2(n-1)\eta}. \end{aligned} \tag{40}$$

Proof. Let $h(\zeta) = (\phi(\zeta))^\alpha$ and $p(\zeta) = (\psi(\zeta))^\alpha$, $(0 < \alpha \leq 1)$, where ϕ and ψ are defined as in (11). It follows, for $j \geq n$, that

$$\phi^{(j-1)}(\zeta) = \sum_{k=j}^{\infty}(k-1)(k-2)\cdots(k-j+1)c_k\zeta^{k-j}. \tag{41}$$

A computation shows, for $n = 2$, that

$$h''(0) = \alpha\left[(\alpha-1)\left(\phi'(0)\right)^2 + \phi''(0)\right] = \alpha\left[(\alpha-1)(c_2)^2 + 2!c_3\right], \tag{42}$$

and, for $n = 3$, we have

$$\begin{aligned} h^{(4)}(0) &= \alpha\left[3(\alpha-1)\left(\phi''(0)\right)^2 + \phi^{(4)}(0)\right] \\ &= \alpha\left[3(\alpha-1)(2!c_3)^2 + 4!c_5\right]. \end{aligned} \tag{43}$$

Next, for $n = 4$, we get

$$\begin{aligned} h^{(6)}(0) &= \alpha\left[10(\alpha-1)\left(\phi'''(0)\right)^2 + \phi^{(6)}(0)\right] \\ &= \alpha\left[10(\alpha-1)(3!c_4)^2 + 6!c_7\right]. \end{aligned} \tag{44}$$

In general, for $n \geq 2$, we obtain

$$\begin{aligned} \left|h^{(2n-2)}(0)\right| &= \alpha\left|\frac{(2n-2)!}{2[(n-1)!]^2}(\alpha-1)\left(\phi^{(n-1)}(0)\right)^2 + \phi^{(2n-2)}(0)\right| \\ &= \alpha\left|\frac{(2n-2)!}{2[(n-1)!]^2}(\alpha-1)[(n-1)!c_n]^2 + (2n-2)!c_{2n-1}\right| \\ &\leq \frac{(2n-2)!}{2[(n-1)!]^2}(\alpha-1)\alpha[(n-1)!(2)]^2 + (2n-2)!(2\alpha) \\ &= (2n-2)!\left(2\alpha^2\right). \end{aligned} \tag{45}$$

Similarly,

$$\left|p^{(2n-2)}(0)\right| \leq (2n-2)!\left(2\alpha^2\right). \tag{46}$$

Note that $\phi(0) = 1$ and $\phi^{(j-1)}(0) = 0$, for all $j = 2, \cdots, n-1$ and $n \geq 3$. Therefore, for every $n \geq 2$, we obtain

$$\left|h^{(k-1)}(0)\right| = \left|\alpha\phi^{(k-1)}(0)\right| = (k-1)!|c_k|\alpha \leq (k-1)!(2\alpha), \quad (n \leq k \leq 2n-2). \tag{47}$$

By applying (45), (46), and (47) to Theorem 6, we obtain the desired estimates. This completes the proof of Theorem 14. □

Corollary 15. *If $f \in \mathscr{A}_{\Sigma,2}(\alpha,\eta)$, then*

$$\begin{aligned} |a_2| &\leq \begin{cases} \dfrac{2\alpha}{\sqrt{2[1+2\eta]}}, & \text{if } 1 \leq \eta \leq 1+\sqrt{2}, \\ \dfrac{2\alpha}{1+\eta} & \text{if } \eta \geq 1+\sqrt{2}, \end{cases} \\ |a_3| &\leq \frac{2\alpha^2}{1+2\eta}, \\ \left|2a_2^2 - a_3\right| &\leq \frac{2\alpha^2}{1+2\eta}. \end{aligned} \tag{48}$$

Remark 16. The estimates of $|a_2|$ and $|a_3|$ in Corollary 15 improve those given in Theorem 2.2 by Frasin and Aouf [7]. In particular, for $\eta = 1$, the bounds improve the given estimates in Theorem 1 by Srivastava et al. [8]. Also, the estimate of $|a_2|$ improves that given in Corollary 1 by Xu et al. [37] and Corollary 1 by Xu et al. [38].

Corollary 17. *If $f \in \mathscr{A}_{\Sigma,3}(\alpha,\eta)$, then*

$$\begin{aligned} |a_3| &\leq \begin{cases} \dfrac{2\alpha}{\sqrt{3(1+4\eta)}}, & \text{if } 1 \leq \eta \leq 1+\sqrt{\dfrac{3}{2}}, \\ \dfrac{2\alpha}{1+2\eta} & \text{if } \eta \geq 1+\sqrt{\dfrac{3}{2}}, \end{cases} \\ |a_4| &\leq \frac{2\alpha}{1+3\eta}, \\ |a_5| &\leq \frac{2\alpha^2}{1+4\eta}, \\ \left|3a_3^2 - a_5\right| &\leq \frac{2\alpha^2}{1+4\eta}. \end{aligned} \tag{49}$$

3. Conclusion

Geometric function theory is a branch of complex analysis with a rich history that studies various analytical tools to study the geometric features of complex-valued functions. Due to the major importance of the study of the coefficients which plays an important role in the theory of univalent functions, the primary goal of this work is to determine coefficient bounds for certain general classes of bi-univalent functions. Making use of Theorem 1 due to Al-Refai and Ali [10], a new method of estimating coefficients is applied, and interesting results that improve and generalize well-known estimates are obtained. The used technique may motivate other researchers to study other classes of bi-univalent functions and obtain new results.

Data Availability

No underlying data was collected or produced in this study.

Conflicts of Interest

The authors declare that there is no conflict of interest.

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